REACHING A BEQUEST GOAL WITH LIFE INSURANCE: AMBIGUITY ABOUT THE RISKY ASSET'S DRIFT AND MORTALITY'S HAZARD RATE

BY

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ABSTRACT

We determine the optimal robust strategy of an individual who seeks to maximize the (penalized) probability of reaching a bequest goal when she is uncertain about the drift of the risky asset and her hazard rate of mortality. We assume the individual can invest in a Black–Scholes market. We solve two optimization problems with ambiguity. The first is to maximize the penalized probability of reaching a bequest goal without life insurance in the market. In the second problem, in addition to investing in the financial market, the individual is allowed to purchase term life insurance to help her reach her bequest goal. As the individual becomes more ambiguity averse concerning the drift of the risky asset, she becomes more conservative with her investment strategy. Also, as she becomes more ambiguity averse about her hazard rate of mortality, numerical work indicates she is more likely to buy life insurance when the ambiguity towards the return of the risky asset is not too large.

KEYWORDS

Bequest motive, optimal investment, term life insurance, ambiguity aversion, stochastic control, robust control, penalized probability.

JEL codes: C61, C73, D81, G22.

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1. Introduction

We determine the optimal robust strategy of an individual who seeks to maximize the probability of reaching a bequest goal when she is uncertain about the drift of the risky asset and her hazard rate of mortality. We assume the individual can invest in a financial market that consists of one riskless asset with constant rate of return and one risky asset whose price process follows a geometric Brownian motion, that is, the financial market is a Black–Scholes market. We solve two optimization problems with ambiguity. The first is to maximize the probability of reaching a bequest goal while investing only in the financial market. In the second problem, in addition to investing in the financial market, we allow the individual to purchase term life insurance to help her reach her bequest goal. In both cases, we assume that the individual's consumption needs are supplied by her income, separate from the wealth we consider here. We, thereby, extend Bayraktar and Young (2016) by allowing for ambiguity in the financial market and for ambiguity in the individual's hazard rate of mortality. As the individual becomes more ambiguity averse concerning the drift of the risky asset, she becomes more conservative with her investment strategy. As she becomes more ambiguity averse about her hazard rate of mortality, numerical work indicates she is more likely to buy life insurance when the ambiguity towards the return of the risky asset is not too large.

The work in this paper combines two areas of research. One area is that of maximizing the probability of reaching a goal. The seminal work of this area begins with Dubins and Savage (1965, 1976) and continues with the work of Pestien and Sudderth (1985), Sudderth and Weerasinghe (1989), and Browne (1997, 1999a,b). In the context of non-life insurance, researchers focus on minimizing the probability of ruin from the perspective of an insurance company. See Schmidli (2002) and Promislow and Young (2005) for early papers in this area. In the life-insurance and life-annuity literature, researchers are more concerned about the individual's point of view, and commonly, they consider two goal-seeking criteria. One is to minimize the probability of lifetime ruin, which was first introduced in Milevsky and Robinson (2000) and analyzed by Young (2004). A number of papers extend Young's work by adding borrowing constraints, stochastic consumption, stochastic volatility, and purchasing life annuities to cover consumption. The other goal-seeking criterion is to maximize the probability of reaching a bequest goal. Bayraktar *et al.* (2014) compute the optimal strategy for purchasing life insurance in order to reach a bequest goal, in which the individual does not consume from the investment account and only invests in the riskless asset. Bayraktar and Young (2016) determine the optimal strategy for investing in a Black–Scholes market to reach a bequest goal when life insurance is not available in the market. Bayraktar *et al.* (2016) extend these two works by finding the optimal strategies for both purchasing term life insurance and investing in a Black–Scholes market in order to reach a bequest goal. They also allow the individual to consume from her investment account.

The second area is related to model uncertainty and ambiguity aversion. It is well recognized in the portfolio-choice literature that decision-makers need to take model misspecification into account when determining optimal strategies. There are good estimates of the volatility of the stock price, but estimating its drift is almost impossible; see Section 4.2 in Rogers (2013). It would require centuries of data to obtain a good estimate of the drift. A number of papers focus on model uncertainty in the framework of portfolio choice. Anderson *et al.* (2003) consider a continuous-time asset pricing model with model misspecification. Maenhout (2004) extends Anderson *et al.* (2003) to a dynamic portfolio choice problem with power utility of terminal wealth. Jaimungal and Sigloch (2012) compute indifference prices in a hybrid model of default. Yi *et al.* (2013) study the robust optimal reinsurance-investment problem under the Heston model for an ambiguity-averse insurer.

Recently, a few papers have included ambiguity in the jump intensity (or hazard rate). In the context of an insurer who seeks a mean-variance equilibrium, Zeng *et al.* (2016) find the optimal proportional reinsurance policy while allowing for ambiguity in both the drift of the risky asset in a financial market and the jump intensity of the insurance claim process. Gu *et al.* (2017) and Li *et al.* (2018) find the optimal robust excess-of-loss reinsurance and investment strategies for an insurer that is uncertain about the jump intensity of its claim process; the criterion in both papers is to maximize the expected exponential utility of terminal wealth of the insurer. Hu *et al.* (2018) consider a principal-agent problem between an insurer and a reinsurer; in this problem the reinsurer is uncertain about the jump intensity of the claim process for which it is offering reinsurance.

Although research on robust optimization problems has been greatly increasing in recent years, few of these contributions deal with *goal-seeking* problems under model ambiguity. Bayraktar and Zhang (2015) analyze the optimal robust investment strategy to minimize the probability of lifetime ruin under ambiguity. Young and Zhang (2016) consider the same objective, but they assume the individual's hazard rate of mortality is ambiguous. Li and Young (2019) determine the optimal per-loss reinsurance for an insurer facing ambiguity in the Poisson rate of claim occurrence; the optimization criterion is to minimize the expectation of the discounted time of ruin. Luo *et al.* (2019) study an optimal robust investment-reinsurance problem to maximize the probability of reaching a given level of surplus before ruin. They allow the insurer to purchase proportional reinsurance and to invest in a Black–Scholes market; they include both ambiguity in the drift of the risky asset, as we do, and ambiguity of the drift of the claim process, which is assumed to follow Brownian motion with drift, as in Promislow and Young (2005).

In this paper, we introduce model uncertainty into the bequest-goal problem and solve two versions of that problem. We call the model considered in Bayraktar and Young (2016) the *reference* model, and we assume the individual is uncertain about the drift of the stock price process and about her own hazard rate of mortality. As in the robust control approach developed by

Maenhout (2004), we penalize the probability that the individual's wealth at death reaches her bequest goal, in which the penalization is based on the relative entropy of the alternative measure with respect to the reference measure. By solving the corresponding Hamilton–Jacobi–Bellman (HJB) equation, we derive an explicit expression for the penalized probability and the associated optimal investment strategy when life insurance is *not* available in the market, the first version of our problem.

For the second version of our problem, we assume life insurance *is* available in the market, and we show that the penalized probability is related to the solution of an Abel equation of the second kind. We show that it is optimal to purchase life insurance only when wealth is greater than some level, as in Bayraktar and Young (2016), and we obtain semi-explicit expressions for the penalized probability and optimal strategies. For both problems (with and without life insurance in the market), as the ambiguity-aversion parameter for the drift of the risky asset increases, the penalized probability and the optimal amount invested in the risky asset decrease. Also, as the ambiguity-aversion parameter for the hazard rate of mortality increases, numerical work indicates she is more likely to buy life insurance when the ambiguity towards the return of the risky asset is not too large.

The rest of the paper is organized as follows. In Section 2, we present the financial and insurance market in which the individual invests, and we also introduce the definition of model uncertainty. We formalize the robust problems of maximizing the penalized probability of reaching a bequest goal with and without life insurance. In Section 3, we solve the robust problem of maximizing the penalized probability of reaching a bequest goal without life insurance. Section 4 parallels Section 3 when life insurance is available in the market. We also provide some sensitivity analysis and a numerical example in this section.

2. Model formulation

In this section, we first describe the financial market for the individual. Then, we formulate the optimization problems our individual faces.

2.1. Financial market

We assume the individual can invest in a financial market consisting of one riskless and one risky asset, whose prices, respectively, evolve according to the dynamics

$$dR_t = rR_t dt \quad \text{and} \quad dS_t = \mu S_t dt + \sigma S_t dB_t,$$

in which $\mu > r > 0$ and $\sigma > 0$ are constants. Here, $B = (B_t)_{t\geq 0}$ is a standard Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. In other words, the individual invests in a Black–Scholes market.

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Let τ_d denote the time of death of the individual, which is assumed to be exponentially distributed with parameter λ, that is, $\mathbb{E}\tau_d = 1/\lambda$. Although a constant hazard rate of mortality is unrealistic, we do not feel that this time independence is a drawback of our model. Indeed, in Section 3, within our relatively simple market, we obtain an explicit expression for the optimal investment strategy. Working with simple models helps researchers determine what properties might be true more generally. Under more realistic models, we fully expect that our qualitative results to hold, and perhaps the simple strategy we find will be nearly optimal. For example, Moore and Young (2006) observe that when minimizing the probability that an individual financially ruins before dying, investment strategies computed by assuming that the hazard rate is constant are nearly optimal for the case of an exponentially increasing hazard rate.

The time of death τ_d is defined on the same probability space as the stock price and is assumed independent of B. We use $D_t := \mathbf{1}_{\{\tau_d \le t\}}$ to form the death indicator process; $D = (D_t)_{t\ge 0}$ jumps from 0 to 1 when the individual dies. Let $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ be the progressive enlargement of the filtration $\mathbb{F}$ to include the information generated by D, specifically, $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(D_u : 0 \le u \le t)$. Assume $\mathbb{F}$ and $\mathbb{G}$ have been augmented to satisfy the usual conditions of completeness and right continuity. Under the measure $\mathbb{P}$, D has jump rate $\lambda \mathbf{1}_{\{D_t = 0\}}$, and $M_t^{\mathbb{P}} := D_t - \int_0^t \lambda \mathbf{1}_{\{D_u = 0\}} du$ forms a $(\mathbb{P}, \mathbb{G})$-martingale. We also define the "coffin state" Δ, which represents the death of the individual; set $\Delta + w = \Delta$ for all $w \in \mathbb{R}$, and all functions evaluated at Δ equal zero.

Let π_t denote the dollar amount invested in the risky asset at time t. In one version of our problem, we also allow the individual to purchase instantaneous term life insurance which pays a benefit at time τ_d. The insurance premium is paid continuously at the rate of h per unit of death benefit. Let I_t denote the amount of death benefit payable at time t if death occurs then. With continuously paid premium for instantaneous term life insurance, the individual's wealth process $W = (W_t)_{t\ge 0}$ follows the dynamics

$$\begin{cases} dW_t = \big(rW_t + (\mu - r)\pi_t - hI_t\big)dt + \sigma\pi_t dB_t, & 0 \le t < \tau_d, \\ W_{\tau_d} = W_{\tau_d -} + I_{\tau_d}. \end{cases}$$

Also, without life insurance, the individual's wealth process follows

$$\begin{cases} dW_t = \big(rW_t + (\mu - r)\pi_t\big)dt + \sigma\pi_t dB_t, & 0 \le t < \tau_d, \\ W_{\tau_d} = W_{\tau_d -}, \end{cases}$$

that is, no death benefit is payable at time τ_d. In both cases, we assume that the individual receives income, separate from W, which supplies all her consumption needs; such could be the case for a retiree who has amassed wealth that she wants to invest, with her consumption covered by Social Security and pension income. If we include consumption in the investment account, then the solution is more complicated and is not available in closed form; see, for example, Sections 4 and 5 in Bayraktar and Young (2016).

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2.2. Ambiguity

Bayraktar and Young (2016) maximize the probability of reaching a bequest goal for an individual who believes the model, described in the previous section, with certainty; we call this model the *reference model*, and the probability $\mathbb{P}$ is the *reference measure*. In this paper, we incorporate model ambiguity, or uncertainty, into the investor's problem. We assume the investor is ambiguous about her hazard rate of mortality, namely, the investor's future life expectancy is unknown. We also allow for ambiguity in the financial market; specifically, the individual is uncertain about the drift of the stock price process. Thus, instead of optimizing under the reference measure $\mathbb{P}$, the individual considers alternative models, which are characterized by a class $\mathcal{Q}$ of probability measures $\mathbb{Q}$ equivalent to $\mathbb{P}$.

A probability measure $\mathbb{Q}$ is in $\mathcal{Q}$ if it satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = L_\infty = \lim_{t\to\infty} L_t,$$

in which

$$L_t = \exp\bigg(-\int_0^t \varphi_u dB_u - \frac{1}{2}\int_0^t \varphi_u^2 du + \int_0^t \ln(\phi_{u-}+1) dM_u^{\mathbb{P}} + \int_0^t \lambda \mathbf{1}_{\{D_u=0\}}\big(\ln(\phi_u+1)-\phi_u\big)du \bigg),$$

or

$$dL_t = -L_{t-}\varphi_t dB_t + L_{t-}\phi_{t-} dM_t^{\mathbb{P}}, \qquad L_0 = 1.$$

Here, $(\varphi, \phi) = (\varphi_t, \phi_t)_{t\ge 0}$ are $\mathbb{G}$-progressively measurable processes, under which $L = (L_t)_{t\ge 0}$ is a $(\mathbb{P}, \mathbb{G})$-martingale. Under the measure $\mathbb{Q}$, $B_t^{\mathbb{Q}} := B_t + \int_0^t \varphi_u du$ forms a $(\mathbb{Q}, \mathbb{G})$-Brownian motion, and $M_t^{\mathbb{Q}} := D_t - \int_0^t \lambda(\phi_u+1) \mathbf{1}_{\{D_u=0\}} du$ forms a $(\mathbb{Q}, \mathbb{G})$-martingale.

From the expression for $M_t^{\mathbb{Q}}$ we see that the effect of the process ϕ is to replace reference hazard rate for mortality λ with $\lambda(\phi_t+1)$, a type of proportional hazards transform. An individual who is risk averse will act as if her hazard rate is at least as large as its actual value, that is, $\lambda(\phi_t+1) \ge \lambda$, or equivalently, $\phi_t \ge 0$. Thus, to model ambiguity in the hazard rate for the problem of reaching a bequest goal, we restrict ϕ so that $\phi_t \ge 0$ for all $t \ge 0$.

We say the investment strategy $\pi = (\pi_t)_{t\ge 0}$ is *admissible* if it is $\mathbb{G}$-progressively measurable and satisfies $\int_0^t \pi_u^2 du < \infty$, $\mathbb{P}$-almost surely for all $t \ge 0$, and an insurance strategy $I = (I_t)_{t\ge 0}$ is *admissible* if it is a nonnegative, $\mathbb{G}$-progressively measurable process, independent of τ_d. Note that we require $I_t \ge 0$ for all $t \ge 0$. If we were to allow $I_t < 0$, then the resulting product could be thought of as an instantaneous life annuity with a lump sum payable to the insurance company upon the death of the individual, as is the case for a reverse

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annuity; see, for example, Pirvu and Zhang (2012). However, because we are focusing on reaching a bequest goal and do not include income in our model, we feel it is reasonable to consider only life insurance and, thereby, require $I_t \geq 0$ for all $t \geq 0$.

We use $\mathcal{A}^{\pi}$ to denote the set of all admissible investment strategies and $\mathcal{A}^{(\pi,I)}$ to denote the set of all admissible investment and insurance strategies. The dynamics of W under $\mathbb{Q}$ without and with life insurance follow the respective processes

$$\begin{cases} dW_t = \big(rW_t + (\mu - \sigma\varphi_t - r)\pi_t\big)dt + \sigma\pi_t dB_t^{\mathbb{Q}}, & 0 \leq t < \tau_d, \\ W_{\tau_d} = W_{\tau_d -}, \end{cases}$$

and

$$\begin{cases} dW_t = (rW_t + (\mu - \sigma\varphi_t - r)\pi_t - hI_t)dt + \sigma\pi_t dB_t^{\mathbb{Q}}, & 0 \leq t < \tau_d, \\ W_{\tau_d} = W_{\tau_d -} + I_{\tau_d}. \end{cases}$$

Note that the effect of the process φ is to replace the reference drift μ with $\mu - \sigma\varphi_t$.

Under $\mathbb{Q}$, we rewrite L_t as follows:

$$L_t = \exp\bigg(-\int_0^t \varphi_u dB_u^{\mathbb{Q}} + \frac{1}{2}\int_0^t \varphi_u^2 du + \int_0^t \ln(\phi_{u-} + 1) dM_u^{\mathbb{Q}} + \int_0^t \lambda \mathbf{1}_{\{D_u = 0\}}\big((\phi_u + 1)\ln(\phi_u + 1) - \phi_u\big)du\bigg).$$

By assuming additional conditions so that the stochastic integral terms are $\mathbb{Q}$-martingales, the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ is given by

$$\mathbb{E}^{\mathbb{Q}}\left[\ln \frac{d\mathbb{Q}_{\tau_d}}{d\mathbb{P}_{\tau_d}}\right] = \mathbb{E}^{\mathbb{Q}}\Big[\ln L_{\tau_d}\Big] = \mathbb{E}^{\mathbb{Q}}\left[\frac{1}{2}\int_0^{\tau_d} \varphi_u^2 du + \int_0^{\tau_d} \lambda \mathbf{1}_{\{D_u = 0\}}\big((\phi_u + 1)\ln(\phi_u + 1) - \phi_u\big)du\right]. \tag{2.1}$$

We use the relative entropy of $\mathbb{Q}$ with respect to $\mathbb{P}$ in our measure of ambiguity, as described below.

In this paper, we solve two optimization problems with ambiguity. The first is to maximize the probability of reaching a bequest goal while investing *only* in the Black–Scholes market; the second allows the individual to buy instantaneous term life insurance to help her reach that bequest goal, in addition to investing in the Black–Scholes market. Let $b > 0$ denote the bequest goal of the individual, and define the time of ruin by $\tau_0 = \inf\{t \geq 0 : W_t \leq 0\}$. The individual invests in order to maximize the probability of having wealth at death of at

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least b without ruining beforehand penalized for the entropic distance between the measure $\mathbb{Q}$ and the reference measure $\mathbb{P}$. More specifically, we analyze the following robust problem:

$$\Phi(w) = \sup_{\pi \in \mathcal{A}^\pi} \inf_{\mathbb{Q} \in \mathcal{Q}} \left\{ \mathbb{Q}_w(W_{\tau_d \wedge \tau_0} \geq b) + \frac{1}{\alpha_1} \mathbb{E}_w^{\mathbb{Q}} \int_0^{\tau_d \wedge \tau_0} \frac{\varphi_s^2}{2} \Phi(W_s)\, ds \right. \\ \left. + \frac{1}{\alpha_2} \mathbb{E}_w^{\mathbb{Q}} \int_0^{\tau_d \wedge \tau_0} \lambda \big[(\phi_s + 1) \ln (\phi_s + 1) - \phi_s \big] \Phi(W_s)\, ds \right\}, \tag{2.2}$$

in which $\mathbb{Q}_w$ and $\mathbb{E}_w^{\mathbb{Q}}$ denote the probability and expectation, respectively, under $\mathbb{Q}$ conditional on $W_0 = w$. Here, α_1 and α_2 are positive constants that measure the degree of the individual's ambiguity concerning the drift of the risky asset and her hazard rate of mortality. When $\alpha_1 = 0 = \alpha_2$, we obtain the ambiguity-neutral problem of Bayraktar and Young (2016), and Φ is a true probability. As α_1 or α_2 increases, Φ decreases but is always uniformly bounded below by 0; thus, Φ takes values in [0, 1] and can be viewed as a penalized probability, with the penalty controlled by α_1 and α_2.

In the second version of this problem, we include life insurance in the market; denote the corresponding penalized probability by Φ^i (i for **i**nsurance) and, in its definition, replace $\sup_{\pi \in \mathcal{A}^\pi}$ with $\sup_{(\pi, I) \in \mathcal{A}^{(\pi, I)}}$.

Remark 2.1. *There are three meaningful factors in the integrands of the second and third terms in* (2.2)*. The first factor is* $1/\alpha_1$ *or* $1/\alpha_2$*; the parameters* α_1 *and* α_2 *are positive constants that measure the level of ambiguity aversion. A larger value of* α_1 *or* α_2 *means the individual is more ambiguity averse and believes less in the reference model. As stated above, as* α_1 *and* α_2 *approach* 0*, the problem in* (2.2) *becomes the ambiguity-neutral problem of Bayraktar and Young* (2016)*. As* α_1 *and* α_2 *approach* ∞*, the problem corresponds to the worst-case approach, that is, the individual believes equally in all candidate measures and optimizes against the worst-case scenario. The second factor is* $\varphi_s^2/2$ *or* $\lambda\big[(\phi_s + 1) \ln (\phi_s + 1) - \phi_s\big]$*, which equals an integrand in the relative entropy* (2.1)*, a measure of how far the measure* $\mathbb{Q}$ *lies from the reference measure* $\mathbb{P}$*. Finally, the third factor is* $\Phi(W_s)$*, a scaling by the penalized probability between now and when the first of death or ruin occurs. This scaling results in the penalized probability having the same functional form as the maximum probability of reaching a bequest goal, which we will see in Corollary 3.4. As Maenhout* (2004) *argues, with this scaling, "robustness will no longer wear off as wealth rises." Also, see pages* 64 *and* 65 *in Jaimungal and Sigloch* (2012) *for further discussion of this penalty.* □

In Section 3, we solve for Φ and the corresponding optimal investment strategy; in Section 4, we solve for Φ^i and the corresponding optimal investment and life-insurance strategies.

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3. Reaching a Bequest Goal Without Life Insurance

In this section, we consider the problem of maximizing the probability of reaching a bequest goal without life insurance in the market. We first provide a verification theorem that we use to compute the penalized probability and corresponding optimal investment strategy. We omit the proof of the verification theorem because it closely follows the proofs of Theorem 6.1 in Bayraktar and Zhang (2015) and of Lemma 2.1 in Bayraktar and Young (2016).

For any triplet $(\pi, \varphi, \phi) \in \mathbb{R}^3$, define the differential operator $\mathcal{L}^{\pi,\varphi,\phi}$ through its action on a test function f by

$$\mathcal{L}^{\pi,\varphi,\phi} f = \big(rw + (\mu - \sigma\varphi - r)\pi\big) f_w + \frac{1}{2}\sigma^2\pi^2 f_{ww} - \lambda(\phi+1)\left(f - \mathbf{1}_{\{w \ge b\}}\right) + \frac{\varphi^2}{2\alpha_1} f + \frac{\lambda\big[(\phi+1)\ln(\phi+1) - \phi\big]}{\alpha_2} f.$$

If wealth at time t is greater than or equal to b, then the individual can invest all her wealth in the riskless asset so that $W_{\tau_d} \ge W_t \ge b$, and nature can choose $\varphi_s = 0$ and $\phi_s = 0$ for all $s \ge t$. Thus, $\Phi(w) = 1$ if $w \ge b$. Also, if wealth reaches 0, then the individual has ruined, and the "game" ends, which implies that $\Phi(0) = 0$. It follows that we only need to determine $\Phi(w)$ and the optimal controls for $0 < w < b$.

Theorem 3.1 (Verification theorem). *Suppose* $v : [0, b] \to [0, 1]$, $\pi^* : (0, b) \to \mathbb{R}$, $\varphi^* : \mathbb{R} \times (0, b) \to \mathbb{R}$, *and* $\phi^* : \mathbb{R} \times (0, b) \to \mathbb{R}^+$ *are measurable functions satisfying the following conditions:*

(*i*) $v \in \mathcal{C}^2([0, b])$ *is increasing.*
(*ii*) v *is a classical solution of*

$$\begin{cases} \sup_{\pi\in\mathbb{R}} \inf_{\varphi\in\mathbb{R},\, \phi\ge 0} \mathcal{L}^{\pi,\varphi,\phi} v(w) = 0, & 0 < w < b, \\ v(0) = 0, \quad v(b) = 1. \end{cases} \tag{3.1}$$

(*iii*) $\pi^*(w)$ *attains the supremum in* (*ii*) *for each* $w \in (0, b)$, $\varphi^*(\pi, w)$ *and* $\phi^*(\pi, w)$ *attain the infimum in* (*ii*) *for each* $\pi \in \mathbb{R}$ *and* $w \in (0, b)$.
(*iv*) π^*, φ^* *and* ϕ^* *are bounded, and* π^* *is Lipschitz continuous.*

Then,

$$\Phi = v,$$

on $[0, b]$, *and* $\pi^*(\cdot)$, $\varphi^*(\pi^*(\cdot), \cdot)$, *and* $\phi^*(\pi^*(\cdot), \cdot)$ *are optimal feedback controls.* □

In the following, we use Theorem 3.1 to calculate Φ. According to Theorem 3.1, if we find an increasing solution of the following boundary-value problem (BVP) on $[0, b]$, then that solution equals the maximum probability of reaching the bequest goal Φ, as long as the corresponding optimal controls satisfy Condition (iv) of that theorem: For $0 < w < b$,

$$\sup_{\pi} \inf_{\varphi, \phi} \left\{ (\mu - \sigma\varphi - r)\pi v_w + \frac{1}{2}\sigma^2\pi^2 v_{ww} - \lambda(\phi+1)v + \frac{\varphi^2}{2\alpha_1} v + \frac{\lambda\left[(\phi+1)\ln(\phi+1) - \phi\right]}{\alpha_2} v \right\} + rwv_w = 0, \tag{3.2}$$

with boundary conditions $v(0) = 0$ and $v(b) = 1$. From the first-order necessary conditions in (3.2), we obtain

$$\varphi^*(\pi, w) = \alpha_1\sigma\pi \frac{v_w}{v}, \tag{3.3}$$

$$\phi^*(\pi, w) = e^{\alpha_2} - 1 > 0. \tag{3.4}$$

Because α_1 and α_2 are positive, the expression in curly brackets in (3.2) is convex with respect to φ and ϕ; thus, φ^* and ϕ^* in (3.3) and (3.4), respectively, yield the global minimum.

By substituting the expression for φ^* and ϕ^* into (3.2), we obtain

$$\lambda\delta v = rwv_w + \sup_{\pi} \left\{ (\mu - r)\pi v_w + \frac{1}{2}\sigma^2\pi^2 \left(v_{ww} - \alpha_1 \frac{v_w^2}{v} \right) \right\}, \tag{3.5}$$

in which

$$\delta = \frac{e^{\alpha_2} - 1}{\alpha_2} > 1. \tag{3.6}$$

The first-order necessary condition for π in (3.5) yields

$$\pi^*(w) = -\frac{\mu - r}{\sigma^2} \frac{v_w}{v_{ww} - \alpha_1 v_w^2 / v}. \tag{3.7}$$

This expression for π^* gives the global maximum if $vv_{ww} - \alpha_1 v_w^2 < 0$, which we verify after computing v. By substituting π^* in (3.7) into (3.5), we obtain the following nonlinear differential equation:

$$\lambda\delta v = rwv_w - m \frac{v_w^2}{v_{ww} - \alpha_1 v_w^2 / v}, \tag{3.8}$$

in which

$$m = \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2. \tag{3.9}$$

We conjecture that the solution of (3.8) has the form

$$v(w) = \left(\frac{w}{b}\right)^q,$$

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for some positive constant q. Substituting this ansatz into (3.8) yields

$$q = \begin{cases} \dfrac{1}{2r(1-\alpha_1)}\left[\left(r+m+\lambda\delta(1-\alpha_1)\right) - \sqrt{\left(r+m+\lambda\delta(1-\alpha_1)\right)^2 - 4\lambda\delta r(1-\alpha_1)}\right], & \text{if } \alpha_1 \neq 1, \\ \dfrac{\lambda\delta}{r+m}, & \text{if } \alpha_1 = 1, \end{cases} \tag{3.10}$$

with $q > 0$ and $q(1-\alpha_1) < 1$, which implies that $v v_{ww} - \alpha_1 v_w^2 < 0$.

Based on the above discussion, we present the penalized probability Φ in the next theorem along with the optimal strategies. Because the proof is a straightforward check of the conditions of Theorem 3.1, we omit it.

Theorem 3.2. *The maximum penalized probability of reaching the bequest goal equals*

$$\Phi(w) = \left(\frac{w}{b}\right)^q, \qquad 0 \leq w \leq b, \tag{3.11}$$

in which q is given in (3.10). If wealth equals $w \in (0, b)$, then the optimal amount invested in the risky asset is given by

$$\pi^*(w) = \frac{\mu - r}{\sigma^2} \frac{w}{1-(1-\alpha_1)q}, \tag{3.12}$$

and the minimizing measure is given by the constants

$$\varphi^* = \frac{\mu - r}{\sigma} \frac{\alpha_1 q}{1-(1-\alpha_1)q}, \tag{3.13}$$

$$\phi^* = e^{\alpha_2} - 1. \tag{3.14}$$

Moreover, ϕ^* is a positive constant, and because $1-(1-\alpha_1)q > 0$, π^* is a positive constant proportion of wealth, and φ^* is a positive constant. □

The optimal investment strategy is not continuous at $w = b$; indeed, $\lim_{w \to b-} \pi^*(w) > 0$, but if wealth equals b, then it is optimal to invest all wealth in the riskless asset to maintain one's assets. Also, the optimal investment strategy is independent of the bequest goal b as in Bayraktar and Young (2016), except that b determines the range of values of w.

In the first corollary, we deduce that Φ's game has an equilibrium value if q is less than 1 (or equivalently, if Φ is concave), that is, if we switch sup and inf in (2.2), the penalized probability Φ does not change. In other words, it does not matter which "player" acts first, the individual choosing an investment strategy $\{\pi_t\}$ or nature choosing a measure $\mathbb{Q}$ via $\{\varphi_t\}$ and $\{\phi_t\}$. In this case, the optimal strategies given in Theorem 3.2 are Nash equilibrium strategies for this

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zero-sum, continuous-time differential game. The corollary follows by showing that the solution in Theorem 3.2 also solves the following BVP:

$$\begin{cases} \inf_{\varphi,\phi} \sup_{\pi} \mathcal{L}^{\pi,\varphi,\phi}\, v(w) = 0, & 0 < w < b, \\ v(0) = 0, \qquad v(b) = 1. \end{cases} \tag{3.15}$$

See Appendix A for the proof of this corollary.

Corollary 3.1. *If q in (3.10) is less than 1, then the game embodied in (2.2) has a value, that is, Φ also equals*

$$\Phi(w) = \inf_{\mathbb{Q}\in\mathcal{Q}} \sup_{\pi\in\mathcal{A}^\pi} \left\{ \mathbb{Q}_w(W_{\tau_d\wedge\tau_0} \geq b) + \frac{1}{\alpha_1}\, \mathbb{E}^{\mathbb{Q}}_w \int_0^{\tau_d\wedge\tau_0} \frac{\varphi_s^2}{2}\, \Phi(W_s)\, ds \right.$$
$$\left. + \frac{1}{\alpha_2}\, \mathbb{E}^{\mathbb{Q}}_w \int_0^{\tau_d\wedge\tau_0} \lambda\big[(\phi_s + 1)\ln(\phi_s + 1) - \phi_s\big]\, \Phi(W_s)\, ds \right\}. \tag{3.16}$$

Moreover, the optimal controls are given in feedback form in (3.12)–(3.14). □

Remark 3.1. *As observed in Bayraktar and Young (2016), the investment strategy in (3.12) is* identical *to the one employed by an investor who maximizes the expected discounted utility of her wealth at death under the utility function $u(w) = w^q$, in which q is given in (3.10) with $q < 1$, so that the utility function is concave. Specifically, the maximum-utility problem is $\sup_\pi \mathbb{E}^w\big[e^{-\rho\tau_d}\,(W_{\tau_d})^q\big]$, for some $\rho > 0$. Thus, if we were to observe an individual investing a constant proportion of her wealth in a risky asset, then we could say she is maximizing the expected discounted power utility of her wealth at death or maximizing the penalized probability that her wealth at death exceeds a specific bequest goal. This correspondence is similar to the one found by Bayraktar and Young (2007), in which they relate the optimal strategies for maximizing the individual's expected utility of lifetime consumption and for minimizing her probability of lifetime ruin.* □

In the next corollary, we show how the solution in Theorem 3.2 varies with the ambiguity-aversion parameters α_1 and α_2. From the definition of Φ, we know that Φ decreases with respect to α_1 and α_2; however, we do not know *a priori* how the optimal controls change with α_1 and α_2. Recall that φ^* is the change of measure that reflects ambiguity with respect to the drift of the risky asset; ϕ^*, the hazard rate of mortality. See Appendix B for a proof of this corollary.

Corollary 3.2. *As the ambiguity-aversion parameter α_1 increases, $\pi^*(w)$ decreases for all $w \in (0, b)$. If $\alpha_1 > 1$, then $\pi^*(w)$ decreases with respect to α_2, and if $0 < \alpha_1 < 1$, then $\pi^*(w)$ increases with respect to α_2. If $q < 1$, then φ^* increases with respect to α_1, but ϕ^* is independent of α_1 for all values of $q > 0$. Both φ^* and ϕ^* increase with respect to α_2.* □

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Remark 3.2. *As the individual becomes more ambiguity averse about the drift of the risky asset, Corollary 3.2 shows that she becomes more cautious in her investment strategy, an intuitively pleasing result. As for how the measure φ^* changes with α_1 when $q \geq 1$, numerical work indicates that if $\lambda\delta$ is large relative to r, then φ^* first increases and then decreases with increasing α_1. On the other hand, if $\lambda\delta \leq r$, then $q < 1$ for all values of α_1, and Corollary 3.2 implies that φ^* monotonically increases with respect to α_1.* □

Remark 3.3. *It makes sense that the measure modifying the hazard rate for mortality, namely, ϕ^*, increases with respect to α_2. The more uncertain the individual is concerning her rate of dying, the greater her adjusted hazard rate $\lambda(1+\phi^*)$ will be. Note that this result is closely linked with the individual's goal of reaching a particular bequest. Indeed, if bequest is one's goal, then dying sooner is worse than dying later. By contrast, in Young and Zhang (2016), under the goal of minimizing the probability of lifetime ruin, the adjusted hazard rate decreases with increasing ambiguity because dying later is worse than dying sooner when one is concerned about running out of money before dying.*

It also makes sense that ϕ^* is independent of α_1. Indeed, the randomness in the stock price process and the randomness of the time of death are independent, so uncertainty about the drift of the risky asset does not affect ϕ^*. That said, because reaching the bequest goal involves investing in the risky asset, uncertainty about the hazard rate of mortality does *affect φ^*, the measure modifying the drift of the risky asset. Corollary 3.2 shows that φ^* increases with respect to α_2, which means that the adjusted drift $\mu - \sigma\varphi^*$ decreases with respect to α_2. If $\alpha_1 < 1$, then the individual will compensate for a decreased drift and larger hazard rate due to an increase in α_2 by investing more in the risky asset.* □

In the next corollary, we examine the limiting cases as $\alpha_i \to 0+$ and $\alpha_i \to \infty$ for $i = 1, 2$. The statements concerning $\alpha_1 \to 0+$ and $\alpha_2 \to 0+$ follow readily from Theorem 3.2 above and from Theorem 3.1 in Bayraktar and Young (2016), and the statements concerning $\alpha_i \to \infty$ follow from Theorem 3.2 above.

Corollary 3.3. *As $\alpha_1 \to 0+$ and $\alpha_2 \to 0+$, the parameter q in (3.10) approaches*

$$q\big|_{\alpha_1=0} = \frac{1}{2r}\left[(r+m+\lambda\delta) - \sqrt{(r+m+\lambda\delta)^2 - 4r\lambda\delta}\,\right] \in (0, 1),$$

and

$$q\big|_{\alpha_2=0} = \begin{cases} \dfrac{1}{2r(1-\alpha_1)}\left[\big(r+m+\lambda(1-\alpha_1)\big) - \sqrt{\big(r+m+\lambda(1-\alpha_1)\big)^2 - 4\lambda r(1-\alpha_1)}\,\right], & \text{if } \alpha_1 \neq 1, \\ \dfrac{\lambda}{r+m}, & \text{if } \alpha_1 = 1. \end{cases}$$

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If $\alpha_1 \to 0$ *and* $\alpha_2 \to 0$ *simultaneously, then the penalized probability* Φ *approaches the non-penalized maximum probability of reaching the bequest goal under the reference model, and the optimal investment strategy approaches the corresponding optimal investment strategy under the reference model.*

As $\alpha_1 \to \infty$*, the parameter q approaches*

$$q\big|_{\alpha_1=\infty} = \frac{\lambda\delta}{r},$$

the penalized probability Φ *approaches the worst-case probability of reaching the bequest goal with hazard rate* $\lambda\delta$*, namely,* $\Phi(w)\big|_{\alpha_1=\infty} = (w/b)^{\lambda\delta/r}$*, and the optimal investment strategy is to invest nothing in the risky asset.*

As $\alpha_2 \to \infty$*, the parameter q approaches*

$$q\big|_{\alpha_2=\infty} = \begin{cases} \dfrac{1}{1-\alpha_1}, & \text{if } \alpha_1 < 1, \\ \infty, & \text{if } \alpha_1 \geq 1, \end{cases}$$

the penalized probability Φ *approaches*

$$\Phi(w)\big|_{\alpha_2=\infty} = \begin{cases} \left(\dfrac{w}{b}\right)^{1/(1-\alpha_1)}, & \text{if } \alpha_1 < 1, \\ 0, & \text{if } \alpha_1 \geq 1, \end{cases}$$

and the optimal investment strategy increases to ∞ *if* $\alpha_1 < 1$ *and decreases to* 0 *if* $\alpha_1 > 1$. □

Remark 3.4. *As* $\alpha_1 \to \infty$*, from the discussion on page* 64 *of Bayraktar and Zhang* (2015)*, if the individual takes a worst-case approach concerning the risky asset, then she will invest nothing in the risky asset because the drift can be arbitrarily negative if one buys the stock and arbitrarily positive if one shorts the stock. Thus, wealth at time t equals* $W_t = we^{rt}$*, and the probability* W_t *reaches b before dying equals* $(w/b)^{\lambda\delta/r}$.

As $\alpha_2 \to \infty$*, the modified hazard rate* $\lambda\delta \to \infty$*, which means the individual dies immediately. Moreover, if* $\alpha_1 \to \infty$*, then the optimal investment strategy is to invest nothing in the risky asset, and the corresponding penalized probability equals* 0 *for wealth less than b.* □

Of interest is how the probability of reaching the bequest goal changes with increasing α_1 and α_2. We know that the *penalized* probability of reaching the bequest goal decreases with increasing α_i, and the next corollary considers the *un-penalized* probability of reaching the bequest goal under measure $\mathbb{Q}$ defined by φ^* and ϕ^* in (3.13) and (3.14), respectively, if the individual follows the investment strategy in (3.12). If we find an increasing classical solution of the following BVP, then that solution equals the probability of reaching the bequest goal:

$$\begin{cases} \lambda(\phi^* + 1)v = \big(rw + (\mu - \sigma\varphi^* - r)\pi^*(w)\big)v_w + \frac{1}{2}\sigma^2\big(\pi^*(w)\big)^2 v_{ww}, & 0 < w < b, \\ v(0) = 0, \qquad v(b) = 1. \end{cases}$$

The following corollary gives us the solution of this BVP.

Corollary 3.4. *Under measure* $\mathbb{Q}$ *defined by* (3.13) *and* (3.14)*, the probability of reaching the bequest goal when the individual follows the investment strategy in* (3.12) *equals*

$$\zeta(w) = \left(\frac{w}{b}\right)^{q'}, \qquad 0 \le w \le b, \tag{3.17}$$

in which $q' > 0$ *is given by*

$$q' = \frac{1}{2m}\Bigg[\Big(2mq - m - r\big(1 - (1-\alpha_1)q\big)^2\Big) + \sqrt{\Big(2mq - m - r\big(1-(1-\alpha_1)q\big)^2\Big)^2 + 4m\lambda e^{\alpha_2}\big(1-(1-\alpha_1)q\big)^2}\Bigg]. \tag{3.18}$$

Moreover, $q' > q$ *if* $\alpha_1 > 0$ *and* $\alpha_2 > 0$*,*

$$q'\big|_{\alpha_1=0,\alpha_2=0} = q\big|_{\alpha_1=0,\alpha_2=0},$$

as $\alpha_1 \to \infty$*,*

$$q'\big|_{\alpha_1=\infty} = \frac{\lambda e^{\alpha_2}}{r},$$

and as $\alpha_2 \to \infty$*,*

$$q'\big|_{\alpha_2=\infty} = \begin{cases} \dfrac{1+\alpha_1}{1-\alpha_1}, & \text{if } \alpha_1 < 1, \\ \infty, & \text{if } \alpha_1 \ge 1. \end{cases}$$

□

Numerical work indicates that if we fix α_2, q' first increases from $q\big|_{\alpha_1=0}$ and then decreases towards $\lambda e^{\alpha_2}/r$ as α_1 increases from 0 to ∞ if λ is large enough. If λ is small enough, then numerical work indicates that q' strictly increases from $q\big|_{\alpha_1=0}$ to $\lambda e^{\alpha_2}/r$ as α_1 increases from 0 to ∞. If we fix α_1 and if $\alpha_1 < 1$, then numerical work shows that q' first increases from $q'\big|_{\alpha_2=0}$ and then decreases towards $(1+\alpha_1)/(1-\alpha_1)$ as α_2 increases from 0 to ∞. If $\alpha_1 \ge 1$, then numerical work shows that q' increases from $q'\big|_{\alpha_2=0}$ to ∞ as α_2 increases from 0 to ∞.

4. Reaching a Bequest Goal With Life Insurance When $q < 1$

In this section, we add life insurance to the market and show that computing the penalized probability Φ^i is related to solving an Abel equation of the second kind; see Section 1.3 of Polyanin and Zaitsev (2003). We restrict our attention to the case for which $q < 1$, with q given in (3.10), because in that case, the bequest-goal problem without life insurance has a value, as shown in Corollary 3.1. Inequality $q < 1$ is equivalent to $m > \alpha_1(\lambda\delta - r)$, and we assume these inequalities hold throughout the remainder of this section, and we remind the reader regularly of this assumption.

If wealth is large enough, greater than or equal to the so-called *safe level* w_s (s for **s**afe), the individual can invest all her wealth in the riskless asset with the interest income sufficient to cover the insurance premium for a death benefit of $b - w_s$. That is, the safe level w_s generates interest of $rw_s = h(b - w_s)$, or equivalently,[1]

$$w_s = \frac{hb}{r + h} \, . \tag{4.1}$$

Thus, if $w \geq w_s$, $\Phi^i(w) = 1$. Also, $\Phi^i(0) = 0$ because ruin has occurred.

For completeness, we state the verification theorem for this problem, using the following differential operator $\mathcal{D}^{\pi,I,\varphi,\phi}$ for any quadruple $(\pi, I, \varphi, \phi) \in \mathbb{R}^4$ with $I, \phi \geq 0$:

$$\mathcal{D}^{\pi,I,\varphi,\phi} f = \big(rw + (\mu - \sigma\varphi - r)\pi - hI\big) f_w + \frac{1}{2}\sigma^2\pi^2 f_{ww} - \lambda(\phi + 1)\left(f - \mathbf{1}_{\{w+I \geq b\}}\right) + \frac{\varphi^2}{2\alpha_1} f + \frac{\lambda\big[(\phi + 1)\ln(\phi + 1) - \phi\big]}{\alpha_2} f.$$

Theorem 4.1 (Verification theorem). *Suppose* $v : [0, w_s] \to [0, 1]$*,* $\pi^* : (0, w_s) \to \mathbb{R}$*,* $I^* : (0, w_s) \to \mathbb{R}^+$*,* $\varphi^* : \mathbb{R} \times \mathbb{R}^+ \times (0, w_s) \to \mathbb{R}$*, and* $\phi^* : \mathbb{R} \times \mathbb{R}^+ \times (0, w_s) \to \mathbb{R}^+$ *are measurable functions satisfying the following conditions:*

(i) $v \in \mathcal{C}^2\big([0, w_s]\big)$ *is increasing.*
(ii) v *is a classical solution of*

$$\begin{cases} \sup\limits_{\pi \in \mathbb{R}, I \geq 0} \; \inf\limits_{\varphi \in \mathbb{R}, \phi \geq 0} \mathcal{D}^{\pi,I,\varphi,\phi} \, v(w) = 0, & 0 < w < w_s, \\ v(0) = 0, \qquad v(w_s) = 1. \end{cases} \tag{4.2}$$

(iii) $\pi^*(w)$ *and* $I^*(w)$ *attain the supremum in (ii) for each* $w \in (0, w_s)$*, and* $\varphi^*(\pi, I, w)$ *and* $\phi^*(\pi, I, w)$ *attain the infimum in (ii) for each* $\pi \in \mathbb{R}$*,* $I \geq 0$*, and* $w \in (0, w_s)$*.*
(iv) π^**,* I^**,* φ^**, and* ϕ^* *are bounded, and the first two are Lipschitz continuous.*

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Then,

$$\Phi^i = v$$

on $[0, w_s]$*, and* $\pi^*(\cdot)$*,* $I^*(\cdot)$*,* $\varphi^*(\pi^*(\cdot), I^*(\cdot), \cdot)$*, and* $\phi^*(\pi^*(\cdot), I^*(\cdot), \cdot)$ *are optimal feedback controls.* □

As in Section 3 of Bayraktar *et al.* (2016), which corresponds to the case for which $\alpha_1 = 0 = \alpha_2$, we hypothesize that there exists a so-called *buy level* $w_b \in (0, w_s)$ such that when wealth w is less than w_b, it is optimal not to buy life insurance, and when $w \geq w_b$, it is optimal to buy life insurance of $b - w$. Let ξ denote the penalized probability under this life-insurance strategy. Via a verification theorem similar to Theorem 4.1, if we find a classical solution of the following BVP, with bounded (Lipschitz continuous as necessary) feedback controls, then that solution equals ξ; therefore, we use ξ when stating the BVP:

$$\begin{aligned} 0 = &\big(rw - h(b-w)\mathbf{1}_{\{w \geq w_b\}}\big)\xi_w \\ &+ \sup_\pi \inf_{\varphi, \phi} \left\{ (\mu - \sigma\varphi - r)\pi\xi_w + \frac{1}{2}\sigma^2\pi^2\xi_{ww} - \lambda(\phi+1)\big(\xi - \mathbf{1}_{\{w \geq w_b\}}\big) + \frac{\varphi^2}{2\alpha_1}\xi \right. \\ &\left. + \frac{\lambda\big[(\phi+1)\ln(\phi+1) - \phi\big]}{\alpha_2}\xi \right\}, \end{aligned} \tag{4.3}$$

with boundary conditions $\xi(0) = 0$ and $\xi(w_s) = 1$. After we impose the first-order necessary conditions as in Section 3, we obtain

$$\varphi^*(\pi, w) = \alpha_1 \sigma \pi \, \frac{v_w}{v}, \tag{4.4}$$

and

$$\phi^*(\pi, w) = \begin{cases} e^{\alpha_2} - 1 > 0, & 0 < w < w_b, \\ 0, & w_b \leq w < w_s. \end{cases} \tag{4.5}$$

Because α_1 and α_2 are positive, the expression in the curly brackets in (4.3) is convex with respect to φ and ϕ; thus, φ^* and ϕ^* in (4.4) and (4.5), respectively, yield the global minimum.

For $0 < w < w_b$, after substituting for φ^* and ϕ^*, the HJB equation in (4.3) becomes

$$\lambda\delta\xi = rw\xi_w + \sup_\pi \left\{ (\mu - r)\pi\xi_w + \frac{1}{2}\sigma^2\pi^2 \left(\xi_{ww} - \alpha_1 \frac{\xi_w^2}{\xi} \right) \right\}, \tag{4.6}$$

in which δ is given in (3.6). The first-order necessary condition for π in (4.6) yields

$$\pi^*(w) = -\frac{\mu - r}{\sigma^2} \frac{\xi_w}{\xi_{ww} - \alpha_1 \xi_w^2/\xi}, \tag{4.7}$$

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as in (3.7). This expression for π^* gives the global maximum if $\xi\xi_{ww} - \alpha_1\xi_w^2 < 0$. By substituting π^* in (4.7) into (4.6), we obtain the following nonlinear differential equation:

$$\lambda\delta\xi = rw\xi_w - m\,\frac{\xi_w^2}{\xi_{ww} - \alpha_1\xi_w^2/\xi}\,,$$

which is identical to (3.8). Thus, we hypothesize

$$\xi(w) = \kappa\left(\frac{w}{w_b}\right)^q, \tag{4.8}$$

for $0 \le w < w_b$, for some positive constant κ, with q given in (3.10). Because $q < 1$, ξ is concave and satisfies $\xi\xi_{ww} - \alpha_1\xi_w^2 < 0$.

For $w_b < w < w_s$, after substituting for φ^* and ϕ^* and obtaining the same expression for π^* as in (4.7), the HJB equation in (4.3) becomes

$$\lambda\xi = \lambda + \big((r+h)w - hb\big)\xi_w - m\,\frac{\xi_w^2}{\xi_{ww} - \alpha_1\xi_w^2/\xi}\,, \tag{4.9}$$

with boundary conditions $\xi(w_s) = 1$, $\xi(w_b) = \kappa$, and $\xi_w(w_b) = \kappa q/w_b$. The buy level w_b solves

$$\lambda\big(1 + (\delta - 1)\xi(w_b)\big) = h(b - w_b)\xi_w(w_b), \tag{4.10}$$

which ensures that ξ is twice continuously differentiable if ξ and ξ_w are continuous. After we substitute $\xi(w_b) = \kappa$ and $\xi_w(w_b) = \kappa q/w_b$ into (4.10), we obtain the following relationship between the unknown proportionality constant κ and the unknown buy level w_b:

$$\lambda\big(1 + (\delta - 1)\kappa\big)w_b = h(b - w_b)\kappa q. \tag{4.11}$$

Define the variable x and the function ν by $x = \ln(w_s - w)$ and $\nu(x) = \xi\big(w_s - e^x\big)$, respectively; then, the differential equation (4.9) becomes

$$\lambda\nu = \lambda + (r+h)\nu_x - m\,\frac{\nu_x^2}{\nu_{xx} - \nu_x - \alpha_1\nu_x^2/\nu}\,, \tag{4.12}$$

with $\lim\limits_{x\to-\infty} \nu(x) = 1$. Note that x only appears in (4.12) via derivatives with respect to x.

Finally, define the function z on the range of ν, namely, $[\kappa, 1]$, by $z(\nu) = \nu_x$; in defining z, we assume that ν is strictly decreasing with respect to x. By differentiating $z(\nu) = \nu_x$ with respect to x, we obtain $z_\nu\nu_x = \nu_{xx}$ or $zz_\nu = \nu_{xx}$, which implies that the differential equation (4.12) becomes

$$\lambda\nu = \lambda + (r+h)z - m\,\frac{z}{z_\nu - 1 - \alpha_1 z/\nu}\,, \tag{4.13}$$

or equivalently

$$z_\nu = 1 + \frac{\alpha_1 z}{\nu} + \frac{mz}{(r+h)z + \lambda(1-\nu)}, \tag{4.14}$$

an Abel equation of the second kind for $z = z(\nu)$.

To obtain a boundary condition for z, we hypothesize that ξ_w approaches 0 as wealth approaches the safe level; this hypothesis is inspired by the corresponding result in Bayraktar *et al.* (2016) for the $\alpha_1 = 0 = \alpha_2$ case. Now, $\lim_{w \to w_s} \xi_w(w) = 0$ implies that $z(1) = 0$ because $\lim_{x \to -\infty} \nu(x) = 1$; thus, $z(1) = 0$ gives us a boundary condition at $\nu = 1$. On the other hand, this boundary condition implies that the differential equation (4.14) has a singularity at the boundary point $(\nu, z) = (1, 0)$, which can be difficult to work with numerically, but does not present a problem analytically, as we show in the following proposition. See Appendix C for its proof.

Proposition 4.1. *Fix $\varepsilon \in (0, 1)$. Then, there exists $\beta \in \left(\frac{\lambda}{r+h}, p\right)$ such that there is a continuously differentiable solution $z = z(\nu)$ of (4.14) on $[\varepsilon, 1]$ with $z(1) = 0$ and*

$$-p(1-\nu) < z(\nu) < -\beta(1-\nu), \tag{4.15}$$

for all $\nu \in [\varepsilon, 1)$, in which p is given by

$$p = \frac{1}{2(r+h)}\left[\left(r+h+m+\lambda\right) + \sqrt{\left(r+h+m+\lambda\right)^2 - 4(r+h)\lambda}\,\right] > 1. \tag{4.16}$$

Moreover, if ε is small enough and if $q < 1$, then we can choose β so that

$$\beta \ge \frac{\lambda - \left(rq - \lambda(\delta-1)\right)\varepsilon}{(r+h)(1-\varepsilon)}. \tag{4.17}$$

Consider the free boundary, that is, $\nu = \kappa$. If we define x_b by $x_b = \ln(w_s - w_b)$, then we have $\nu(x_b) = \kappa$ and

$$\nu_x(x_b) = -\frac{e^{x_b}\kappa q}{w_s - e^{x_b}}.$$

These expressions, together with the relationship in (4.11), give us a boundary condition for z at $\nu = \kappa$ in terms of the unknown buy level w_b or unknown proportionality constant κ, namely,

$$z(\kappa) = -\frac{(w_s - w_b)\kappa q}{w_b} = -\frac{\lambda - \left(rq - \lambda(\delta-1)\right)\kappa}{r+h}. \tag{4.18}$$

Thus, if we find the point of intersection between $z = z(\nu)$ and the line $y = -\frac{\lambda - (rq - \lambda(\delta-1))\nu}{r+h}$ for $0 < \nu < 1$, then the abscissa of that point gives us κ. In the following proposition, we prove that z intersects that line at a *unique* point if $q < 1$. See Appendix D for the proof.

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Proposition 4.2. *If $q < 1$, then the solution z of (4.14) for $0 < \nu \le 1$, with boundary condition $z(1) = 0$, intersects the line*

$$y(\nu) = -\frac{\lambda - \left(rq - \lambda(\delta - 1)\right)\nu}{r + h} \tag{4.19}$$

for a unique value of $\nu \in (0, 1)$. Denote that value by κ. □

The bounds on z in Proposition 4.1 lead to easily computable bounds on κ, the abscissa of intersection of $z = z(\nu)$ and the line $y = -\frac{\lambda - (rq - \lambda(\delta - 1))\nu}{r+h}$; see Proposition 4.2. Thus, we have the following corollary.

Corollary 4.1. *If β in Proposition 4.1 satisfies inequality (4.17), then we have the following bounds on κ:*

$$\frac{(r+h)\beta - \lambda}{(r+h)\beta - \lambda + (\lambda\delta - rq)} < \kappa < \frac{(r+h)p - \lambda}{(r+h)p - \lambda + (\lambda\delta - rq)}. \tag{4.20}$$

Moreover, the upper bound in (4.20) holds regardless of whether β satisfies inequality (4.17). □

Because $(r + h)p > \lambda$ and $\lambda\delta > rq$, the upper bound in (4.20) is less than 1, so it is a meaningful bound.

Inequality (4.17) ensures that $\frac{(r+h)\beta - \lambda}{(r+h)\beta - (rq - \lambda(\delta - 1))} \ge \varepsilon$, that is, the lower bound is within the domain of comparison between $z = z(\nu)$ and $y = -\beta(1 - \nu)$. To get tight bounds, it is best to obtain the largest value of $\beta = \beta(\varepsilon)$ possible, that is, $\beta = \beta_0(\varepsilon)$, and choose ε such that $\beta_0(\varepsilon)$ satisfies (4.17) with equality. Specifically, choose

$$\beta = \frac{1}{2(r+h)}\Bigg[\left(r + h + m + \lambda - \alpha_1(\lambda\delta - rq)\right) + \sqrt{\left(r + h + m + \lambda - \alpha_1(\lambda\delta - rq)\right)^2 - 4(r+h)\lambda}\,\Bigg], \tag{4.21}$$

and

$$\varepsilon = \frac{(r+h)\beta - \lambda}{(r+h)\beta - \left(rq - \lambda(\delta - 1)\right)}. \tag{4.22}$$

If $q < 1$, then $m - \alpha_1(\lambda\delta - rq) > 0$, which implies that β in (4.21) is real.

Remark 4.1. *Now that we have $z = z(\nu)$ for $\kappa \le \nu \le 1$, then we get x from z via the integral expression*

$$x(\nu) = x_b + \int_\kappa^\nu \frac{d\tilde{\nu}}{z(\tilde{\nu})}, \tag{4.23}$$

in which $x_b = \ln(w_s - w_b)$, with w_b obtained from κ via (4.11).

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In the solution of $z = z(\nu)$ in Proposition 4.1, we see that $z(\nu) < 0$ for $\nu < 1$; therefore, $x = x(\nu)$ in (4.23) is strictly decreasing with respect to ν on $[\kappa, 1)$. It follows that we can invert $x = x(\nu)$ to get $\nu = \nu(x)$, from which we obtain $\xi(w) = \nu(x)$, with $w = w_s - e^x$ for $w_b \le w \le w_s$. □

In the following theorem, we show that $\xi = \xi(w)$ obtained as described in Remark 4.1 for $w_b \le w \le w_s$ and as hypothesized in (4.8) for $0 \le w < w_b$ satisfies the conditions of Theorem 4.1; hence, it equals the penalized probability Φ^i. See Appendix E for the proof of this theorem.

Theorem 4.2. *If $q < 1$, then the penalized probability Φ^i equals ξ on $[0, w_s]$, in which ξ is given by*

$$\xi(w) = \begin{cases} \kappa \left(\dfrac{w}{w_b} \right)^q, & 0 \le w < w_b, \\ \nu\big(\ln(w_s - w) \big), & w_b \le w \le w_s, \end{cases} \tag{4.24}$$

with κ given in Proposition 4.2, w_b given by

$$w_b = \frac{hq\kappa}{\lambda + \big(hq + \lambda(\delta - 1)\big)\kappa} \, b, \tag{4.25}$$

and $\nu = \nu(x)$ equal to the functional inverse of $x = x(\nu)$ given in (4.23).

If wealth equals $w \in (0, w_s)$, then the optimal amount invested in the risky asset equals

$$\pi^*(w) = \begin{cases} \dfrac{\mu - r}{\sigma^2} \, \dfrac{w}{1 - (1 - \alpha_1) q}, & 0 < w < w_b, \\ \dfrac{2(w_s - w)}{\mu - r} \left[(r + h) + \lambda \, \dfrac{1 - \nu(x)}{\nu_x(x)} \right]_{x = \ln(w_s - w)}, & w_b \le w < w_s, \end{cases} \tag{4.26}$$

the optimal amount of life insurance equals

$$I^*(w) = \begin{cases} 0, & 0 < w < w_b, \\ b - w, & w_b \le w < w_s, \end{cases} \tag{4.27}$$

and the minimizing measures equal

$$\varphi^* = \begin{cases} \dfrac{\mu - r}{\sigma} \, \dfrac{\alpha_1 q}{1 - (1 - \alpha_1) q}, & 0 < w < w_b, \\ -\dfrac{2\alpha_1 \sigma}{\mu - r} \left[(r + h) \, \dfrac{\nu_x(x)}{\nu(x)} + \lambda \, \dfrac{1 - \nu(x)}{\nu(x)} \right]_{x = \ln(w_s - w)}, & w_b \le w < w_s, \end{cases} \tag{4.28}$$

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and

$$\phi^* = \begin{cases} e^{\alpha_2} - 1, & 0 < w < w_b, \\ 0, & w_b \le w < w_s. \end{cases} \tag{4.29}$$

Remark 4.2. *When it is optimal not to buy life insurance, that is, when $0 < w < w_b$, the optimal investment strategy π^* is* identical *to the strategy found in Section 3 when life insurance is not available in the market.* (*Recall that this investment strategy is independent of the specific goal, as observed in the discussion following Theorem 3.2. Thus, when wealth is less than the buy level w_b, the individual invests in order to reach w_b so that she can buy life insurance and, thereby, reach her bequest goal.*) *We see a similar phenomenon for the optimal measure φ^*. Other goal-seeking problems demonstrate similar myopia; see the extended discussion in Angoshtari et al.* (2015). □

In the first corollary of Theorem 4.2, parallel to Corollary 3.1, we deduce that Φ^i's zero-sum, differential game has an equilibrium value because Φ^i is concave, that is, if we switch sup and inf in (4.2), the penalized probability Φ^i does not change. The proof of the corollary follows that of Corollary 3.1, so we omit it.

Corollary 4.2. *If $q < 1$, then the game embodied in* (4.2) *has a value, that is, Φ^i also equals*

$$\Phi^i(w) = \inf_{\mathbb{Q} \in \mathcal{Q}} \sup_{(\pi, I) \in \mathcal{A}^{(\pi, I)}} \left\{ \mathbb{Q}_w(W_{\tau_d \wedge \tau_0} \ge b) + \frac{1}{\alpha_1} \mathbb{E}^{\mathbb{Q}}_w \int_0^{\tau_d \wedge \tau_0} \frac{\varphi_s^2}{2} \Phi^i(W_s)\, ds \right.$$
$$\left. + \frac{1}{\alpha_2} \mathbb{E}^{\mathbb{Q}}_w \int_0^{\tau_d \wedge \tau_0} \lambda \big[(\phi_s + 1) \ln(\phi_s + 1) - \phi_s \big] \Phi^i(W_s)\, ds \right\}.$$

Moreover, the optimal controls are given in feedback form as in Theorem 4.2. □

In the next corollary, parallel to Corollary 3.2, we show how the solution in Theorem 4.2 varies with the ambiguity-aversion parameters α_1 and α_2. From the definition of Φ^i, we know that Φ^i decreases with respect to α_1 and α_2; thus, in this corollary, we show how π^* changes with respect to α_1 and α_2. See Appendix F for the proof of this corollary.

Corollary 4.3. *As the ambiguity-aversion parameter α_1 increases, $\pi^*(w)$ decreases for all $w \in (0, w_s)$. When $w \in (w_b, w_s)$, $\pi^*(w)$ is independent of α_2. When $w \in (0, w_b)$, if $\alpha_1 > 1$, then $\pi^*(w)$ decreases with respect to α_2, and if $0 < \alpha_1 < 1$, then $\pi^*(w)$ increases with respect to α_2.* □

Remark 4.3. *Corollaries 3.2 and 4.3 show that the optimal amount invested in the risky asset decreases with increasing ambiguity towards the drift of the risky asset both when life insurance is available and when it is not. Uncertainty about*

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the financial market leads to less investment in that market, regardless of whether the individual buys life insurance. □

In the final corollary, parallel to Corollary 3.3, we examine limiting cases of the ambiguity parameter α_1, specifically, $\alpha_1 \to 0+$ and $\alpha_1 \to \infty$. In both cases, we have explicit expressions for the solution. The case for which $\alpha_1 \to 0+$ with $\alpha_2 > 0$ is identical in form to the solution in Section 3 of Bayraktar *et al.* (2014) when $\alpha_1 = 0 = \alpha_2$.

Corollary 4.4. *As $\alpha_1 \to 0+$, the penalized probability Φ^i is given by*

$$\Phi^i(w) = \begin{cases} \dfrac{p(1-q)}{p-q}\left(\dfrac{w}{w_b}\right)^q, & 0 \le w < w_b, \\[2ex] 1 - \dfrac{q(p-1)}{p-q}\left(\dfrac{w_s - w}{w_s - w_b}\right)^p, & w_b \le w \le w_s, \end{cases}$$

in which $q = q\big|_{\alpha_1 = 0}$ and p is given in (4.16). The buy level w_b equals

$$w_b = \frac{1-q}{p-q}\, w_s,$$

and the optimal investment strategy equals

$$\pi^*(w) = \begin{cases} \dfrac{\mu - r}{\sigma^2}\,\dfrac{w}{1-q}, & 0 \le w < w_b, \\[2ex] \dfrac{\mu - r}{\sigma^2}\,\dfrac{w_s - w}{p-1}, & w_b \le w \le w_s. \end{cases}$$

Because q increases with respect to α_2, it is easy to see that w_b decreases with respect to α_2.

If $\lambda\delta \le r$, then as $\alpha_1 \to \infty$, the parameter q approaches $\lambda\delta/r$, the penalized probability Φ^i approaches the worst-case probability of reaching the bequest goal, namely, $(w/w_s)^{\lambda\delta/r}$, the optimal investment strategy is to invest nothing in the risky asset, and the optimal strategy for purchasing life insurance is not to buy life insurance until wealth reaches the safe level.[2] □

Analytical results concerning how π^* changes with respect to wealth and how w_b changes with respect to the ambiguity parameters α_1 and α_2 are difficult to obtain. However, we present numerical examples for which π^* decreases with respect to wealth when the individual buys life insurance, that is, when $w_b < w < w_s$, which matches the case for which $\alpha_1 = 0 = \alpha_2$; see Figures 1 through 5. In other numerical work, not shown here, we found that π^* increases with respect to wealth for wealth near $w = w_b$ when $\lambda > r + h$ and q is close to 1. In all cases, when $q < 1$, eventually π^* decreases to 0 as wealth approaches the safe level w_s.

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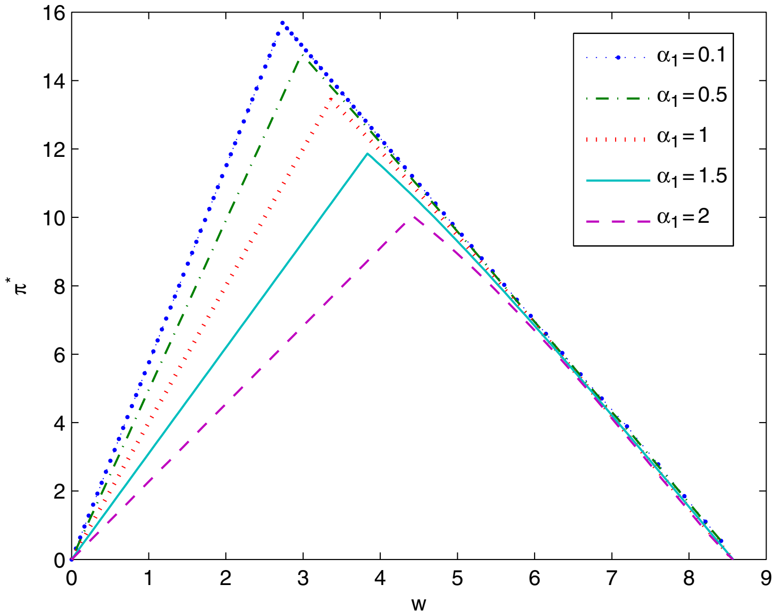

FIGURE 1: Graph of the optimal investment strategy for various levels of ambiguity of the drift of the risky asset. For this figure, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.10$, $\lambda = 0.04$, $h = 0.06$, $b = 10$, and $\alpha_2 = 0.1$.

We interpret these results concerning how π^* changes with wealth as follows. When the individual optimally does not purchase life insurance, that is, when her wealth is less than w_b, she invests more with increasing wealth in order to reach the buy level w_b so that she can "afford" life insurance and, thereby, cover her bequest goal. If her ambiguity towards the drift of the risky asset increases, then she invests less in the risky asset when her wealth is below w_b (although that amount still increases with increasing wealth), and she anticipates a lower return in the market. The modified drift is $\mu - \sigma\varphi^*$ which decreases with increasing α_1 for $0 < w < w_b$. Because her anticipated drift decreases and because she invests less in the risky asset, she requires greater wealth in order to afford life insurance, that is, the buy level increases with α_1.

As noted above, when wealth is greater than w_b, in this example, the individual invests less in the risky asset as her wealth increases towards the safe level w_s. In Figures 1 through 5, we see that the decrease is nearly linear; when $\alpha_1 = 0 = \alpha_2$, then the decrease is exactly linear, as found in Bayraktar *et al.* (2016). Young (2004) observes a similar linearly decreasing investment strategy when minimizing the probability of lifetime ruin under a constant rate of consumption. It is as if, when purchasing life insurance, the individual invests conservatively in order to avoid hitting the buy level w_b, so that she can continue to be able to purchase life insurance.

Figure 1 confirms that π^* decreases with increasing ambiguity towards the risky asset, which we proved in Corollary 4.3. Also, Figure 1 indicates that

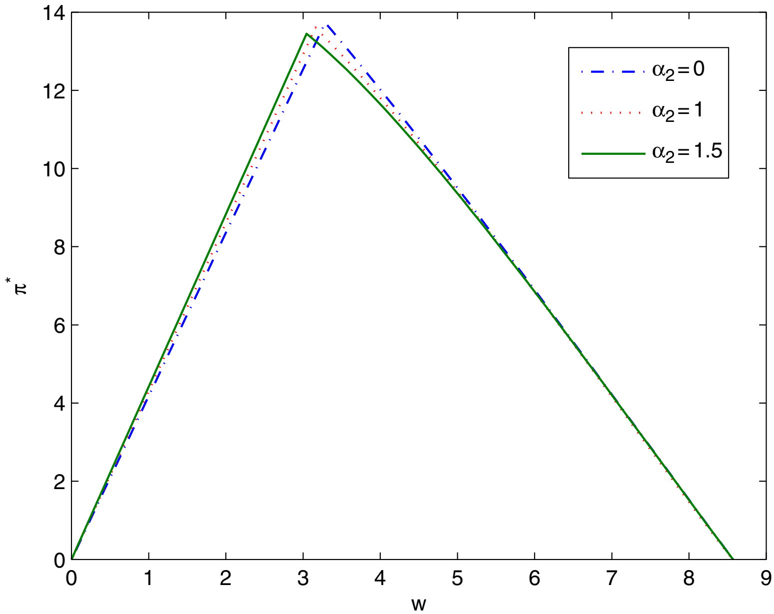

FIGURE 2: Graph of the optimal investment strategy for various levels of ambiguity of the hazard rate for mortality. For this figure, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.10$, $\lambda = 0.04$, $h = 0.06$, $b = 10$, and $\alpha_1 = 0.9$.

w_b increases with increasing ambiguity towards the drift of the risky asset; intuitively, as α_1 increases, the individual needs more wealth in order to justify purchasing life insurance because her investment returns are expected to be smaller.

Figures 2 and 3 indicate that w_b decreases with increasing ambiguity towards the hazard rate for mortality when $\alpha_1 < 1$. This monotonicity in the buy level w_b makes sense because, for a fixed premium h, life insurance becomes more attractive as the individual becomes more uncertain about her future expected lifetime. We also know that when $0 < \alpha_1 < 1$ (which is true for Figures 2 and 3), π^* increases with increasing α_2 when the individual is not purchasing life insurance, that is, when wealth is less than w_b, and we also see this in Figures 2 and 3. Intuitively, if the individual is more uncertain about the value of her hazard rate, then, when she is not purchasing life insurance, she wants to get her wealth greater than w_b sooner and the only way to do that is to invest more in the risky asset.

Figures 4 and 5 indicate that w_b increases with increasing ambiguity towards the hazard rate for mortality when $\alpha_1 > 1$. We also know that when $\alpha_1 > 1$ (which is true for Figures 4 and 5), π^* decreases with increasing α_2 when the individual is not purchasing life insurance, that is, when wealth is less than w_b, and we also see this in Figures 4 and 5.

In Figure 6, we graph the penalized probabilities Φ and Φ^i to demonstrate the gain in (robust) probability due to the presence of life insurance in the market.

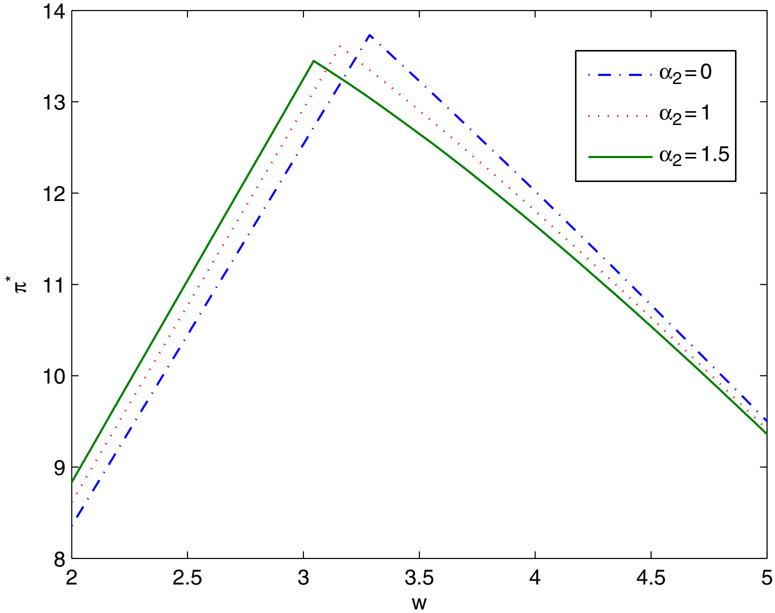

FIGURE 3: Enlargement of Figure 2 in order to see that w_b decreases with increasing α_2 when $\alpha_1 = 0.9$.

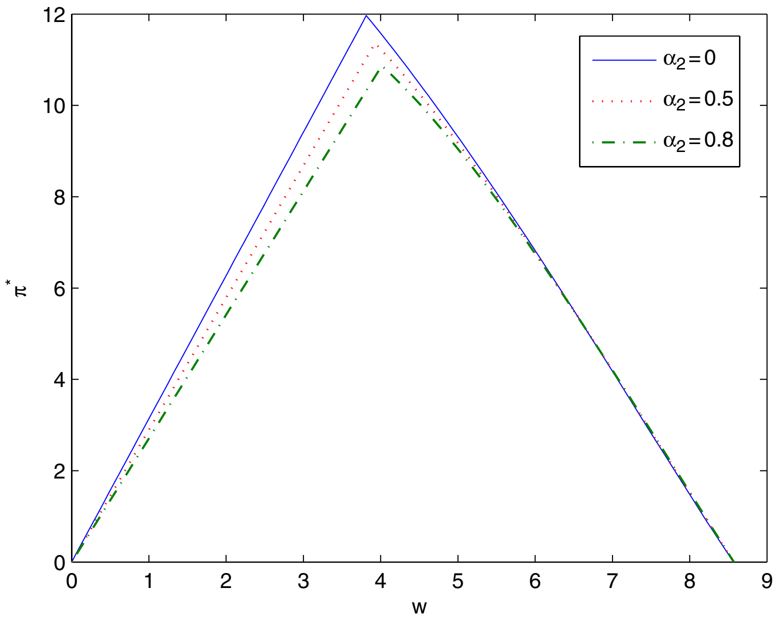

FIGURE 4: Graph of the optimal investment strategy for various levels of ambiguity of the hazard rate for mortality. For this figure, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.10$, $\lambda = 0.04$, $h = 0.06$, $b = 10$, and $\alpha_1 = 1.5$.

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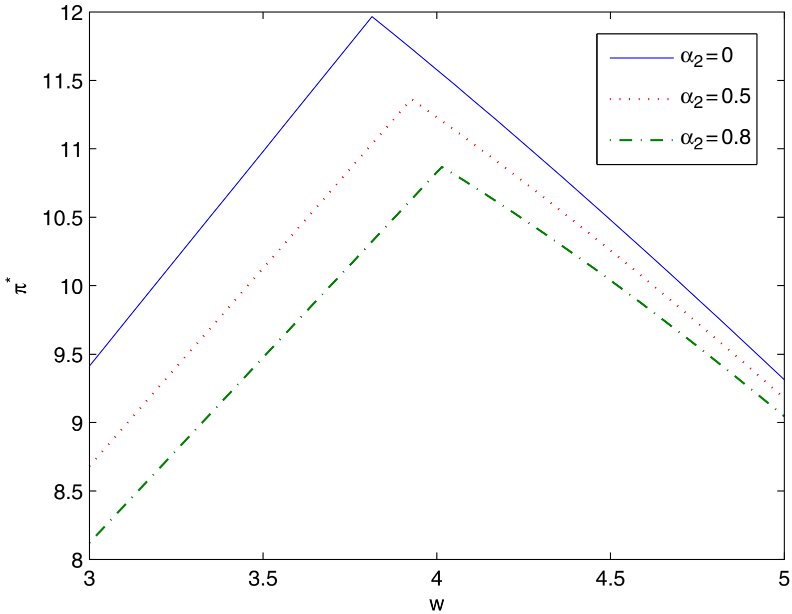

FIGURE 5: Enlargement of Figure 4 in order to see that w_b increases with increasing α_2 when $\alpha_1 = 1.5$.

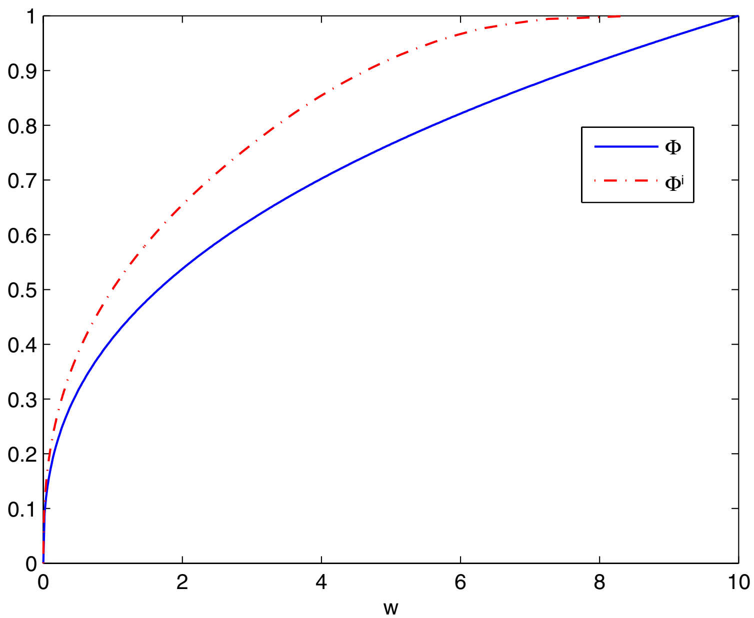

FIGURE 6: Graph of Φ versus Φ^i. For this figure, $r = 0.01$, $\mu = 0.05$, $\sigma = 0.10$, $\lambda = 0.04$, $h = 0.06$, $b = 10$, $\alpha_1 = 0.5$, and $\alpha_2 = 0.1$.

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NOTES

1. Recall that h is the premium rate per unit of death benefit for term life insurance.
2. See Proposition 3.2 of Bayraktar *et al.* (2014).
3. Note that this integration is proper because $F(\nu, z)$ is bounded on $[\varepsilon, 1]$. Indeed, from (4.15), the problematic term in $F(\nu, z)$ is bounded as follows:

$$0 < \frac{m}{r+h-\dfrac{\lambda}{p}} \leq \frac{m}{r+h+\lambda\,\dfrac{1-\nu}{z}} \leq \frac{m}{r+h-\dfrac{\lambda}{\beta}} < \infty.$$

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APPENDIX A. PROOF OF COROLLARY 3.1

The differential equation in (3.15) equals

$$\inf_{\varphi,\phi}\sup_{\pi}\left\{(\mu-\sigma\varphi-r)\pi v_w+\frac{1}{2}\sigma^2\pi^2 v_{ww}-\lambda(\phi+1)v+\frac{\varphi^2}{2\alpha_1}v+\frac{\lambda\big[(\phi+1)\ln(\phi+1)-\phi\big]}{\alpha_2}v\right\}$$
$$+rwv_w=0.$$

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The first-order necessary condition for π yields

$$\pi^*(\varphi, w) = -\frac{\mu - \sigma\varphi - r}{\sigma^2}\,\frac{v_w}{v_{ww}},$$

which gives the global maximum if $v_{ww} < 0$ (which holds for $v = \Phi$ if $q < 1$). When we substitute this expression for π^* into the differential equation, we obtain

$$\inf_{\varphi,\phi}\left\{-\frac{1}{2}\left(\frac{\mu - \sigma\varphi - r}{\sigma}\right)^2 \frac{v_w^2}{v_{ww}} + \frac{\varphi^2}{2\alpha_1} v - \lambda(\phi+1)v + \frac{\lambda\big[(\phi+1)\ln(\phi+1) - \phi\big]}{\alpha_2} v\right\} + rwv_w = 0,$$

and the first-order necessary conditions for φ and ϕ yield, respectively,

$$\varphi^*(w) = \frac{\mu - r}{\sigma}\,\frac{\alpha_1 v_w^2/v}{\alpha_1 v_w^2/v - v_{ww}}, \qquad \text{and} \qquad \phi^*(w) = e^{\alpha_2} - 1,$$

which gives the global minimum if $\frac{vv_{ww} - \alpha_1 v_w^2}{v_{ww}} > 0$ (which holds for $v = \Phi$ if $q < 1$). When we substitute these expressions for φ^* and ϕ^*, we obtain the differential equation

$$\lambda\delta v = rwv_w - m\,\frac{v_w^2}{v_{ww} - \alpha_1 v_w^2/v},$$

which is identical to (3.8), and Φ given in Theorem 3.2 solves this equation, with the boundary conditions. Moreover, after we substitute for $v = \Phi$, π^*, φ^*, and ϕ^* above equal the corresponding optimal controls in Theorem 3.2.

APPENDIX B. PROOF OF COROLLARY 3.2

We begin this proof by determining how q in (3.10) changes with respect to α_1 and α_2. q is the smaller root of the quadratic equation

$$r(1-\alpha_1)q^2 - \big(r + m + \lambda\delta(1-\alpha_1)\big)q + \lambda\delta = 0; \tag{B.1}$$

thus, by differentiating this quadratic equation and by simplifying the result via (3.10), we obtain

$$\frac{\partial q}{\partial \alpha_1} = \frac{(\lambda\delta - rq)q}{\sqrt{\big(r + m + \lambda\delta(1-\alpha_1)\big)^2 - 4\lambda\delta r(1-\alpha_1)}}, \tag{B.2}$$

which is positive if and only if $\lambda\delta - rq > 0$, which is straightforward to demonstrate. Thus, q increases with respect to α_1. As an aside, note that this implies Φ decreases with increasing α_1, as expected.

From the definition of δ in (3.6), we have

$$\frac{\partial \delta}{\partial \alpha_2} = \frac{1}{\alpha_2^2}\left\{(\alpha_2 - 1)e^{\alpha_2} + 1\right\} > 0. \tag{B.3}$$

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Again, by differentiating (B.1) with respect to δ, by simplifying the result via (3.10), and by (B.3), we obtain

$$\frac{\partial q}{\partial \alpha_2} = \frac{\partial q}{\partial \delta}\,\frac{\partial \delta}{\partial \alpha_2} = \frac{\lambda\big(1-(1-\alpha_1)q\big)}{\sqrt{\big(r+m+\lambda\delta(1-\alpha_1)\big)^2 - 4\lambda\delta r(1-\alpha_1)}}\,\frac{\partial \delta}{\partial \alpha_2} > 0.$$

Thus, q also increases with respect to α_2. Again, as an aside, note that this implies Φ decreases with increasing α_2, as expected.

Also, from (3.12) and (B.2), we deduce

$$\begin{aligned}\frac{\partial \pi^*(w)}{\partial \alpha_1} &\propto \frac{\partial}{\partial \alpha_1}\big((1-\alpha_1)q\big) = -q + (1-\alpha_1)\frac{\partial q}{\partial \alpha_1}\\ &\propto -\sqrt{\big(r+m+\lambda\delta(1-\alpha_1)\big)^2 - 4\lambda\delta r(1-\alpha_1)} + (1-\alpha_1)(\lambda\delta - rq),\end{aligned}$$

which is automatically negative for $\alpha_1 \geq 1$ because $\lambda\delta - rq > 0$. Thus, suppose $0 < \alpha_1 < 1$; then, after substituting for the square-root term via (3.10), we see that the above inequality is equivalent to

$$r(1-\alpha_1)q < r+m,$$

which is straightforward to demonstrate. Thus, we have shown that $\pi^*(w)$ decreases with respect to α_1 for all $0 < w < b$.

Moreover, by taking the derivative of $\pi^*(w)$ with respect to α_2, we have

$$\frac{\partial \pi^*(w)}{\partial \alpha_2} \propto (1-\alpha_1)\frac{\partial q}{\partial \alpha_2} \propto 1-\alpha_1.$$

Hence, if $\alpha_1 > 1$, $\pi^*(w)$ decreases with respect to α_2; if $0 < \alpha_1 < 1$, $\pi^*(w)$ increases with respect to α_2.

Next,

$$\begin{aligned}\frac{\partial \varphi^*}{\partial \alpha_1} &\propto \big(1-(1-\alpha_1)q\big)\,\frac{\partial(\alpha_1 q)}{\partial \alpha_1} - \alpha_1 q\left(-\frac{\partial q}{\partial \alpha_1} + \frac{\partial(\alpha_1 q)}{\partial \alpha_1}\right)\\ &= q(1-q) + \alpha_1\,\frac{\partial q}{\partial \alpha_1},\end{aligned}$$

which is positive if $q < 1$, and

$$\frac{\partial \varphi^*}{\partial \alpha_2} = \frac{\partial \varphi^*}{\partial q}\,\frac{\partial q}{\partial \alpha_2} \propto \big(1-(1-\alpha_1)q\big) + (1-\alpha_1)q = 1 > 0.$$

Finally, it is clear that $\phi^* = e^{\alpha_2} - 1$ is independent of α_1 and increases with α_2.

APPENDIX C. PROOF OF PROPOSITION 4.1

To prove this proposition, we rely on the following comparison lemma.

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Lemma C.1 (Comparison lemma on $[\varepsilon, 1)$**).** *Define the operator* F *by*

$$F(\nu, f) = 1 + \frac{\alpha_1 f}{\nu} + \frac{mf}{(r+h)f + \lambda(1-\nu)}. \tag{C.1}$$

Fix $\varepsilon \in (0, 1)$*. If* f *and* g *are differentiable functions on* $[\varepsilon, 1)$*, continuous on* $[\varepsilon, 1]$*, such that* $f(1) \le g(1)$ *and* $f_\nu - F(\nu, f) > g_\nu - F(\nu, g)$ *on* $[\varepsilon, 1)$*, then*

$$f < g$$

on $[\varepsilon, 1)$*.*

Proof. First, if the maximum of $f - g$ occurs at $\nu = 1$, but not at any point in $[\varepsilon, 1)$, then $f < g$ on $[\varepsilon, 1)$ because $f(1) - g(1) \le 0$. Second, if $f - g$ attains a strictly negative maximum on $[\varepsilon, 1)$, then we also have $f < g$ on $[\varepsilon, 1)$.

Third, if $f - g$ attains a nonnegative maximum at $\nu_0 \in [\varepsilon, 1)$, then $f(\nu_0) - g(\nu_0) \ge 0$ and $f_\nu(\nu_0) - g_\nu(\nu_0) \le 0$. Thus, $f_\nu - F(\nu, f) > g_\nu - F(\nu, g)$ on $[\varepsilon, 1)$ implies that

$$0 \ge f_\nu(\nu_0) - g_\nu(\nu_0) > F\big(\nu_0, f(\nu_0)\big) - F\big(\nu_0, g(\nu_0)\big) \ge 0,$$

a contradiction. The last inequality follows because F increases with respect to its second argument. We have shown that $f < g$ on $[\varepsilon, 1)$. □

Proof of Proposition 4.1. Observe that

$$\frac{d}{d\nu}\big(-p(1-\nu)\big) - F\big(\nu, -p(1-\nu)\big) = \frac{\alpha_1 p(1-\nu)}{\nu},$$

which is positive for $\nu \in [\varepsilon, 1)$. Thus, $-p(1-\nu)$ is a sub-solution of $f_\nu = F(\nu, f)$ and, by Lemma C.1, is less than a possible solution of $f_\nu = F(\nu, f)$. To show that there exists $\beta \in \left(\frac{\lambda}{r+h}, p\right)$, such that $-\beta(1-\nu)$ is a super-solution of $f_\nu = F(\nu, f)$ on $[\varepsilon, 1]$, it is enough to find β such that $\frac{d}{d\nu}\big(-\beta(1-\nu)\big) - F\big(\nu, -\beta(1-\nu)\big) < 0$ for all $\nu \in [\varepsilon, 1)$, which is equivalent to

$$\beta - 1 + \alpha_1 \beta \left(\frac{1}{\varepsilon} - 1\right) < \frac{m\beta}{(r+h)\beta - \lambda}. \tag{C.2}$$

Define the function f by

$$f(\chi) = \frac{m\chi}{(r+h)\chi - \lambda} - \chi + 1 - \alpha_1 \chi \left(\frac{1}{\varepsilon} - 1\right). \tag{C.3}$$

It is straightforward to show that f decreases from positive infinity to $-\alpha_1 p \left(\frac{1}{\varepsilon} - 1\right) < 0$ as χ increases from $\frac{\lambda}{r+h}$ to p. Thus, there exists a unique value of β_0 between $\frac{\lambda}{r+h}$ and p such that $f(\beta_0) = 0$. If we choose any value of β between $\frac{\lambda}{r+h}$ and β_0, then inequality (C.2) holds.

Let $\mathscr{Z}$ denote the set of upper-semi-continuous sub-solutions of $f_\nu = F(\nu, f)$. Note that $-p(1-\nu) \in \mathscr{Z}$. Define the function z by

$$z(\nu) = \sup\left\{\hat{z}(\nu) \,\Big|\, -p(1-\nu) \le \hat{z}(\nu) \le -\beta(1-\nu) \text{ and } \hat{z} \in \mathscr{Z}\right\}. \tag{C.4}$$

By following the proof of Proposition 4.2 in Bayraktar and Zhang (2015), one can show that z defined in (C.4) is a continuous viscosity solution of $f_\nu = F(\nu, f)$ with $z(1) = 0$. Also, as in Bayraktar and Zhang (2015), one can extend the comparison lemma, Lemma C.1, to viscosity sub- and super-solutions; therefore, we conclude that z defined in (C.4) is the *unique* continuous viscosity solution.

Next, to show that z is continuously differentiable on $[\varepsilon, 1)$, note that we can integrate $F(\nu, z)$ to get a continuously differentiable function ζ, specifically, $\zeta(\nu) = -\int_\nu^1 F(y, z(y))dy$ for $\nu \in [\varepsilon, 1)$.[3] Because z is the unique viscosity solution of $f_\nu = F(\nu, f)$ with $z(1) = 0$, it follows that $z = \zeta$, and z is continuously differentiable. Moreover, we can repeat this argument to prove that z is infinitely continuously differentiable on $[\varepsilon, 1)$.

To be able to choose β such that inequality (4.17) holds, we need β_0 to satisfy that inequality. Because f in (C.3) strictly decreases with χ, β_0 satisfies inequality (4.17) if and only if f evaluated at $\chi = \frac{\lambda - (rq - \lambda(\delta-1))\varepsilon}{(r+h)(1-\varepsilon)}$ is nonnegative, or equivalently $g(\varepsilon) \geq 0$, in which g is given by

$$g(\varepsilon) = \frac{\lambda - \big(rq - \lambda(\delta - 1)\big)\varepsilon}{\varepsilon(\lambda\delta - rq)}\big(m - \alpha_1(\lambda\delta - rq)\big) - \frac{\lambda - \big(rq - \lambda(\delta - 1)\big)\varepsilon}{1 - \varepsilon} + (r + h). \tag{C.5}$$

We assert that g strictly decreases from $+\infty$ to $-\infty$ as ε increases from 0 to 1. Thus, we can choose ε small enough so that β_0 satisfies inequality (4.17). The assertion concerning g's behavior with respect to ε relies on $m - \alpha_1(\lambda\delta - rq) > 0$, which holds if and only if $q < 1$. Indeed, rewrite q's quadratic equation in (B.1) as

$$\lambda\delta - rq = \frac{mq}{1 - (1 - \alpha_1)q};$$

thus, $m > \alpha_1(\lambda\delta - rq)$ if and only if

$$m > \frac{\alpha_1 mq}{1 - (1 - \alpha_1)q},$$

or, because $1 - (1 - \alpha_1)q > 0$ generally, if and only if $q < 1$.

APPENDIX D. PROOF OF PROPOSITION 4.2

From Proposition 4.1, we can choose ε and β such that inequality (4.17) holds, which implies that $z(\varepsilon) < -\beta(1 - \varepsilon) \leq y(\varepsilon)$. Also, $z(1) = 0 > -\frac{\lambda\delta - rq}{r+h} = y(1)$. Thus, $z(\nu) = y(\nu)$ for some $\nu \in (0, 1)$.

Let κ denote any value in $(0, 1)$ such that $z(\kappa) = y(\kappa)$; κ is unique if we show that $z_\nu(\kappa) > y_\nu(\kappa) = \frac{rq - \lambda(\delta-1)}{r+h}$. By setting $\nu = \kappa$ in the differential equation for z in (4.14), we obtain

$$\begin{aligned}
z_\nu(\kappa) &= 1 + \frac{\alpha_1 z(\kappa)}{\kappa} + \frac{mz(\kappa)}{(r+h)z(\kappa) + \lambda(1-\kappa)} \\
&= 1 + \frac{\alpha_1 y(\kappa)}{\kappa} + \frac{my(\kappa)}{(r+h)y(\kappa) + \lambda(1-\kappa)} \\
&= \frac{1}{\kappa(r+h)}\left\{\kappa(r+h) + \left(\frac{m}{\lambda\delta - rq} - \alpha_1\right)\Big(\lambda - \big(rq - \lambda(\delta-1)\big)\kappa\Big)\right\}.
\end{aligned}$$

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This expression is greater than $\frac{rq-\lambda(\delta-1)}{r+h}$ if and only if

$$\kappa\big(h+\lambda(\delta-1)\big)+\lambda\big(1+(\delta-1)\kappa\big)\left(\frac{m}{\lambda\delta-rq}-\alpha_1\right)>0.$$

From the proof of Proposition 4.1, we know that $m-\alpha_1(\lambda\delta-rq)>0$ if $q<1$ (recall that $\lambda\delta-rq>0$ and $\delta>1$ generally); thus, $z_\nu(\kappa)>y_\nu(\kappa)$. It follows that z and y intersect at a single point.

APPENDIX E. PROOF OF THEOREM 4.2

By construction, ξ in (4.24) is a continuously twice differentiable, increasing function that solves the differential equation (4.3) with boundary conditions $\xi(0)=0$ and $\xi(w_s)=1$. The optimal investment strategy π^* in (4.26) comes from (4.7). This expression for π^* gives the global maximum if $\xi\xi_{ww}-\alpha_1\xi_w^2<0$, which is clear for $w<w_b$ and which is equivalent to $z_\nu-1-\alpha_1 z/\nu>0$ for $w\geq w_b$. The latter inequality is equivalent to $\frac{mz}{(r+h)z+\lambda(1-\nu)}>0$, which is true because $z<0$ and $(r+h)z+\lambda(1-\nu)<0$ for $\kappa\leq\nu<1$.

To show that ξ satisfies the HJB equation in (4.2), it remains to show that

$$\lambda\big(1+(\delta-1)\xi\big)-h(b-w)\xi_w(w)\geq 0$$

if and only if $w\geq w_b$. If ξ is concave, then we are done because this inequality holds with equality at $w=w_b$, and the left side will be increasing with w. Clearly, ξ is concave for $w<w_b$ because $q<1$, so consider ξ for $w\geq w_b$: $\xi_{ww}<0$ for $w\geq w_b$ if and only if $\nu_{xx}-\nu_x<0$ for $x\leq x_b$, which is equivalent to $z_\nu>1$ for $\kappa\leq\nu\leq 1$, or

$$\frac{\alpha_1}{\nu}+\frac{m}{(r+h)z+\lambda(1-\nu)}<0,$$

or $z(\nu)>\ell(\nu)$ in which ℓ is the line defined by

$$\ell(\nu)=-\frac{\lambda}{r+h}(1-\nu)-\frac{m\nu}{\alpha_1(r+h)}\,. \tag{E.1}$$

We know that $z(\nu)\geq-\frac{\lambda-(rq-\lambda(\delta-1))\nu}{r+h}$ for $\kappa\leq\nu\leq 1$ (strictly unless $\nu=\kappa$); thus, if we show that

$$\ell(\nu)<-\frac{\lambda-\big(rq-\lambda(\delta-1)\big)\nu}{r+h},$$

for $\kappa\leq\nu\leq 1$, then we are done. This inequality holds if and only if

$$m-\alpha_1(\lambda\delta-rq)>0,$$

which is true from the proof of Proposition 4.1. Thus, Conditions (i), (ii), and (iii) of Theorem 4.1 hold.

Clearly, I^*, φ^*, and ϕ^* are bounded, and I^* is Lipschitz continuous. From the bounds for z in (4.15), we deduce that π^* is bounded because

$$\frac{1-\nu}{\nu_x}=\frac{1-\nu}{z}\in\left[-\frac{1}{\beta},\,-\frac{1}{p}\right], \tag{E.2}$$

for $\kappa \le \nu \le 1$ and for some $\beta \in \left(\frac{\lambda}{r+h}, p\right)$. Also, clearly π^* is Lipschitz continuous for $w < w_b$. To show that π^* is Lipschitz continuous for $w \ge w_b$, it is enough to show that π^* has a bounded first derivative on $[w_b, w_s]$. To that end, differentiate π^* to obtain

$$\frac{d\pi^*(w)}{dw} = \frac{2}{\mu - r}\left\{\lambda - r - h + \frac{\alpha_1\lambda(1-\nu)}{\nu} + \frac{m\lambda(1-\nu)}{(r+h)z + \lambda(1-\nu)}\right\},$$

and from (E.2), we deduce that this expression is bounded on $[\kappa, 1]$.

APPENDIX F. PROOF OF COROLLARY 4.3

For $0 < w < w_b$, $\pi^*(w)$ in (4.26) equals $\pi^*(w)$ in (3.12), which (from Corollary 3.2) we know decreases with respect to α_1. Thus, consider $\pi^*(w)$ for $w_b \le w < w_s$, namely,

$$\pi^*(w) = \frac{2(w_s - w)}{\mu - r}\left[(r+h) + \lambda\,\frac{1-\nu(x)}{\nu_x(x)}\right]_{x=\ln(w_s - w)}.$$

This expression decreases with respect to α_1 if and only if

$$\frac{1-\nu}{z(\nu)} = \frac{1-\Phi^i(w)}{z(\nu)}, \tag{F.1}$$

decreases with respect to α_1. We know that the numerator of (F.1) increases with respect to α_1, but $z(\nu)$ is negative, so

$$\frac{1}{z(\nu)}\,\frac{\partial\left(1-\Phi^i(w)\right)}{\partial\alpha_1} < 0.$$

Because the numerator of (F.1) is positive, to show that (F.1) decreases with respect to α_1 it is enough to show that the denominator z increases with α_1. To that end, we use the comparison lemma, Lemma C.1, to show that z increases with α_1.

Let $0 < a_1 < a_2$, and write z_i to denote the solution z of (4.14) corresponding to $\alpha_1 = a_i$ for $i = 1, 2$. Similarly, write F_i for F in (C.1) and κ_i for κ in Proposition 4.2, in which we set $\alpha_1 = a_i$ for $i = 1, 2$. Note that $z_1(1) = z_2(1) = 0$, and $z_1'(\nu) - F_1(\nu, z_1) = 0$; thus, from Lemma C.1, to show that $z_1(\nu) < z_2(\nu)$ for all $\varepsilon \le \nu < 1$, in which $\varepsilon \in \left(0, \min[\kappa_1, \kappa_2]\right]$, it is enough to show that $z_2'(\nu) - F_1(\nu, z_2) < 0$ for $\varepsilon \le \nu < 1$:

$$z_2'(\nu) - F_1(\nu, z_2) = \left(z_2'(\nu) - F_2(\nu, z_2)\right) + \left(F_2(\nu, z_2) - F_1(\nu, z_2)\right) = \frac{(a_2 - a_1)z_2(\nu)}{\nu} < 0,$$

in which the inequality holds because $z_2(\nu) < 0$ and $a_1 < a_2$. Thus, we have shown that $\pi^*(w)$ decreases with respect to α_1 for $w_b \le w < w_s$.

The statements concerning how $\pi^*(w)$ changes with α_2 follow readily from the expression of $\pi^*(w)$ in (4.26) and from Corollary 3.2.

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