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ON THE MAGNUS–SMELKIN EMBEDDING

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1. Introduction

The generalization of the Magnus embedding [7] proved by Smelkin [9] may be stated as follows. Let L be a free group freely generated by the set $x_i (i \in I)$, and let R be a normal subgroup of L with $G = L/R$. If V is any variety of groups and Π is the V-free group with free generating set the symbols $[g, x_i]$ ($g \in G$, $i \in I$), then $L/V(R)$ is embedded in the semidirect product $\Pi \rtimes G$ (where the action of G on Π is given by $h \cdot [g, x_i] = [hg, x_i]$, for $h, g \in G$).

In addition to the considerable number of direct applications of the Magnus–Smelkin embedding that are now known, the Magnus embedding itself may be used to obtain representations of certain subgroups of the automorphism group of a free group, for example the Burau representation of Artin's Braid Group (see e.g. [1]). In the present paper we give an extension of the Magnus–Smelkin embedding to the situation where the free group L is replaced by an arbitrary group F. The embedding result is stated in Section 2. We then show, in Section 4, that this generalised embedding also gives rise to representations of automorphism groups, in the same way as does the Magnus representation.

Our approach to these results may be summarized as follows. Let F be a group with normal subgroup R, and let $G = F/R$, where π is the natural map $F \to G$ with kernel R. Then there is a natural homomorphism π_1 from the free product $F * G$ to G (where π_1 is induced by the identity map on G and by π on F). Let Π be the kernel of π_1. Since $\Pi \cap G = 1$, and $f\pi(f)^{-1}$ belongs to Π for each $f \in F$, it is clear that $F * G$ can be described as the semidirect product $\Pi \rtimes G$ (where G acts on Π by conjugation). The embedding result arises by factoring out suitable G-invariant normal subgroups of Π from this product (the basic properties of Π required for this are given in Theorem 1 below). The results on representations of automorphisms are then obtained from the following observation: let H be the subgroup of $\operatorname{Aut} F$ consisting of those automorphisms α of F such that $\alpha(R) = R$. Each such α induces an automorphism α_1 of G, and hence an automorphism of $F * G$ (namely α on F, α_1 on G). The subgroup Π of $F * G$ is easily seen to be generated by all $f\pi(f)^{-1}$ ($f \in F$) and it follows that each $\alpha \in H$ induces an automorphism of Π. We thus obtain a homomorphism $H \to \operatorname{Aut} \Pi$ and hence a homomorphism $H \to \operatorname{Aut}(\Pi/N)$, for N any characteristic subgroup of Π.

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We note that the idea of using $F*G$ to prove an embedding result for F goes back to Artin [**11**], and has been used by Dunwoody [**3**] to give a short proof of the Magnus–Smelkin embedding. Another short proof has been given by Smelkin [**10**]. Also, in the proof of Theorem 1 we use the idea of attaching a cone to a covering space; this idea is used in Comerford's work [**2**] on subgroups of small cancellation groups and, as he notes, also occurs (at least implicitly) in the work of Fox [**4**].

2. The embedding

We have the group F with normal subgroup R, quotient group $G=F/R$ and π the natural homomorphism from F to G. The group Π is the kernel of the homomorphism π_1 above from $F*G$ to G. We take F to have presentation $\langle x_i\ (i\in I);\ S_j\ (j\in J)\rangle$ and put $g_i=\pi(x_i)$. As above, L denotes the free group on the x_i.

We take P to be the free group on the symbols $[g,x_i]$ $(g\in G,\ i\in I)$. It will be convenient to denote $[g,x_i]^{-1}$ by $[gg_i,x_i^{-1}]$. Note that there is an obvious action of G on P. We define a map Ψ from L to P by

$$\Psi(x_{i_1}^{\eta_1}x_{i_2}^{\eta_2}\dots x_{i_r}^{\eta_r})=\prod_{k=1}^{r}[g_{i_1}^{\eta_1}\dots g_{i_{k-1}}^{\eta_{k-1}},x_{i_k}^{\eta_k}] \tag{2.1}$$

(where each η_i is ± 1). We then have

Theorem 1. *With the above notation,*

(1) *Π has presentation*

$$\langle [g,x_i]\quad (g\in G, i\in I);\qquad g\cdot\Psi(S_j)\quad (g\in G, j\in J)\rangle.$$

(2) *The identification of Π as given by (1) with Π as a subgroup of $F*G$ is induced by*

$$[g,x_i]=gx_i\pi(x_i)^{-1}g^{-1}$$

(so that, in particular, the G action on Π is $g\cdot u=gug^{-1}$).

(3) *The map Ψ above induces a map (again denoted by Ψ) from F to Π, satisfying*

$$\Psi(\alpha)=\alpha\pi(\alpha)^{-1}\quad \textit{and}\quad \Psi(\alpha\beta)=\Psi(\alpha)\{\pi(\alpha)\cdot\Psi(\beta)\}$$

for all $\alpha,\beta\in F$.

(4) *Let S be a coset representative system of R in F, with $1\in S$, and write $\bar{x}$ for the representative of $x\in F$. Let $t_{\bar{x}}=\bar{x}\pi(\bar{x})^{-1}$ $(=\Psi(\bar{x}))$ for $\bar{x}\in S$. Then*

$$\Pi=R*F_{G-1},$$

where F_{G-1} is the free group freely generated by the $t_{\bar{x}}$'s, for $\bar{x}\in S-1$.

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(5) *In terms of the description of* Π *given in* (4), *the action of* G *is given by*

$$g \cdot r = t_{\bar{x}}^{-1}(\bar{x} r \bar{x}^{-1}) t_{\bar{x}},$$

$$g \cdot t_{\bar{y}} = t_{\bar{x}}^{-1}(\bar{x}\bar{y}\overline{xy}^{-1}) t_{\overline{xy}},$$

where $\pi(\bar{x}) = g$, $\bar{x}, \bar{y} \in S$, $r \in R$.

The proof of Theorem 1 will be given in the next section. In order to derive the generalized embedding from this result, we need to recall some notation.

Let A be a G-group, so that we have an action $g \cdot a$ of elements of G on elements of A (via a homomorphism $G \to \operatorname{Aut} A$). A map θ from F to A is called a G-derivation if $\theta(\alpha\beta) = \theta(\alpha)\{\pi(\alpha) \cdot \theta(\beta)\}$ for all $\alpha, \beta \in F$. The kernel of a G-derivation θ is the set $\{\alpha \in F; \theta(\alpha) = 1\}$. We have the following easily checked result (see e.g. [5], p. 196).

Lemma 2. *Let* A *be a* G*-group and* $\theta: F \to A$ *a* G*-derivation. Then the map* $g: F \to A \rtimes G$ *given by*

$$\eta(\alpha) = (\theta(\alpha), \pi(\alpha))$$

is a homomorphism, with $\ker \eta = \ker \pi \cap \ker \theta$.

We can now state

Lemma 3. *Let* N *be a* G*-invariant normal subgroup of* Π, *and* p *the natural homomorphism from* Π *to* Π/N. *Then the map* $\eta: F \to \Pi/N \rtimes G$ *given by*

$$\eta(\alpha) = (p\Psi(\alpha), \pi(\alpha))$$

is a homomorphism, with $\ker \eta = R \cap \ker p\Psi$.

Proof. From (3) of Theorem 1 we know that Ψ is a G-derivation, and it is then clear that $p\Psi$ is also a G-derivation. The result now follows from Lemma 2.

We now use Lemma 3 to obtain the generalized form of the Magnus–Smelkin embedding. For convenience, we restate the notation required.

Theorem 4. *Let* F *be a group with presentation*

$$\langle x_i \ (i \in I); \quad S_j \ (j \in J) \rangle.$$

R *a normal subgroup of* F, $G = F/R$, π *the natural map* $F \to G$.

Let the group Π *have presentation*

$$\langle [g, x_i] \quad (g \in G, i \in I); \ g \cdot \Psi(S_j) \quad (g \in G, j \in J) \rangle$$

where Ψ *is defined as in* (2.1).

Then, for any variety V of groups, Ψ induces a map $\Psi_V: F \to \Pi/V(\Pi)$, and the map $\eta_V: F/V(R) \to \Pi/V(\Pi) \rtimes G$ defined by

$$\eta_V(\alpha) = (\Psi_V(\alpha), \pi(\alpha))$$

is an embedding.

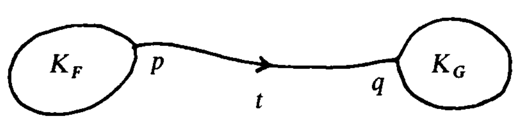
Proof. If V is the variety consisting of the trivial group we merely recover the natural homomorphism $\pi: F \to G$. Thus we assume that V is a non-trivial variety. We can apply Lemma 3 with $N = V(\Pi)$. We then have to establish that $\ker \Psi_V = V(R)$, where $\Psi_V: F \to \Pi/V(\Pi)$ is given by $\Psi_V = p\Psi$, p being the natural homomorphism from Π to $\Pi/V(\Pi)$. Now, using Theorem 1, we have

$$\ker p\Psi = \{x \in F; \Psi(x) \in V(\Pi)\} = \{x \in F; x\bar{x}^{-1} t_{\bar{x}} \in V(\Pi)\}.$$

Since $\Pi = R * F_{G-1}$, we note that $x\bar{x}^{-1} t_{\bar{x}} \in V(\Pi)$ implies that $x\bar{x}^{-1} \in V(R)$ and $t_{\bar{x}} \in V(F_{G-1})$. Since $t_{\bar{x}}$ belongs to a free generating set of F_{G-1} if $\bar{x} \neq 1$, the latter condition implies $\bar{x} = 1$, and then the former condition becomes $x \in V(R)$. Thus $\ker p\Psi = V(R)$ as required.

3. Proof of Theorem 1

We use the complex K, with fundamental group $F * G = \pi_1(K, q)$, given as



where K_F is the complex for F consisting of the single vertex p, a directed loop labelled x_i for each $i \in I$, and with attached two-cells corresponding to the relations S_j. K_G is the corresponding complex for G, consisting of a copy of K_F with two-cells r_u $(u \in U)$ added, where R is the normal closure in F of the set r_u $(u \in U)$.

The covering complex K_Π of K corresponding to the subgroup Π of $F * G$ may be realised as follows: the covering complex K_R of K_F for the subgroup R of F has vertices $g \in G$ and edges (g, x_i) $(g \in G, i \in I)$, where (g, x_i) has initial vertex g and terminal vertex gg_i. We denote the inverse of this edge by (gg_i, x_i^{-1}). For each vertex g of K_R we have a lift map Φ_g acting on paths in K_F, given by

$$\Phi_g(x_{i_1}^{\eta_1} \dots x_{i_r}^{\eta_r}) = \prod_{k=1}^{r} (g g_{i_1}^{\eta_1} \dots g_{i_{k-1}}^{\eta_{k-1}}, x_{i_n}^{\eta_k}),$$

(where $\eta_i = \pm 1$), and the two-cells of K_R are just all $\Phi_g(S_j)$. (We note, for future use, that

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Φ_g can be regarded as a map from F to equivalence classes of paths in K_R, or even as a map from the free group L on the x_i to the free group P on the (g, x_i).) Now take a copy of K_R and add two-cells corresponding to all $\Phi_g(r_t)$ $(r \in T)$. Call the resulting complex $\bar{K}_G$. Thus $\bar{K}_G$ is just the universal cover of the complex K_G. To distinguish vertices of $\bar{K}_G$ from those of K_R, let us agree that if $g \in G$ then $\bar{g}$ denotes the corresponding vertex of $\bar{K}_G$. Now join each vertex g of K_R to the corresponding vertex $\bar{g}$ of $\bar{K}_G$ by a (directed) edge labelled t_g. The resulting complex is K_Π.

The subcomplex $\bar{K}_G$ of K_Π is simply connected, and it follows easily, from the description of K_Π, that if we shrink this subcomplex to a point v to obtain a complex $\hat{K}_R$ which is just K_R "with a cone attached", then $\hat{K}_R$ will have fundamental group equal to Π. Formally, we define $\hat{K}_R$ by adding the new vertex v to K_R and joining each vertex g of K_R to v by a (directed) edge labelled t_g. There is a natural action of G on $\hat{K}_R$ (induced by that on K_Π), satisfying, for $g, h \in G$,

$$g \cdot h = gh \qquad g \cdot (h, x_i) = (gh, x_i)$$

$$g \cdot v = v \qquad g \cdot t_h = t_{gh}.$$

We note that the G action on K_R and the lift maps Φ_g are related by

$$\Phi_{gh}(\alpha) = g \cdot \Phi_h(\alpha)$$

and

$$\Phi_g(\alpha\beta) = \Phi_g(\alpha)\Phi_{g\pi(\alpha)}(\beta) = \{g \cdot \Phi(\alpha)\}\{(g\pi(\alpha)) \cdot \Phi(\beta)\},$$

where $g, h \in G$, $\alpha, \beta \in F$ and $\Phi = \Phi_1$.

We now modify the maps Φ_g so as to become maps from F to Π (induced by corresponding maps $L \to P$) by defining

$$\Psi_g(\alpha) = t_g^{-1}\Phi_g(\alpha)t_{g\pi(\alpha)}.$$

Regarding α as a loop in K_F, this merely adds tails to the corresponding path $\Phi_g(\alpha)$ in $\hat{K}_R$, making it into a loop at v. We then obtain immediately, from the corresponding properties of the Φ_g, that

$$\Psi_g(x_{i_1}^{\eta_1} \dots x_{i_r}^{\eta_r}) = \prod_{k=1}^{r} [gg_{i_1}^{\eta_1} \dots g_{i_{k-1}}^{\eta_{k-1}}, x_{i_k}^{\eta_k}],$$

where $[h, x_i]$ denotes the element $t_h^{-1}(h, x_i)t_{hg_i}$ of $\Pi = \pi_1(\hat{K}_R, v)$, and that

$$\Psi_{gh}(\alpha) = g \cdot \Psi_h(\alpha),$$

$$\Psi_g(\alpha\beta) = \{g \cdot \Psi(\alpha)\}\{(g\pi(\alpha)) \cdot \Psi(\beta)\},$$

where $g, h \in G$, $\alpha, \beta \in F$ and $\Psi = \Psi_1$.

Now $\hat{K}_R$ has two-cells the set of all $\Phi_g(S_j)$; this set may clearly be replaced by the set of all $\Psi_g(S_j)$ without altering the fundamental group. Choosing the set of all t_g as a maximal tree in $\hat{K}_R$, it then follows that Π has presentation

$$\langle [g, x_i] \quad (g \in G, i \in I); \qquad g \cdot \Psi(S_j) \quad (g \in G, j \in J) \rangle$$

(where Ψ here is regarded as a map from L to the free group on the symbols $[g, x_i]$.) This verifies (1) of Theorem 1.

We next note that if $x \in F$ and $\pi(x) = g$, then the loop $\Psi(x) = t_1^{-1}\Phi(x)t_g$ in $\hat{K}_R$ translates into the loop $t_1^{-1}\Phi(x)t_g\rho(x)^{-1}$ in K_Π, where $\rho(x)$ is the path in $\bar{K}_G$ from $\bar{1}$ to $\bar{g}$ corresponding to the path $\Phi(x)$ in K_R. Projecting this loop of K_Π to a loop in K clearly gives the element $x\pi(x)^{-1}$ of $F * G$. This verifies part (3) of the theorem. Part (2) may also be verified by translating into K_Π. Alternatively, if $\pi(x) = g$, we have

$$\begin{aligned} gx_i\pi(x_i)^{-1}g^{-1} &= gx^{-1}\{xx_i\pi(xx_i)^{-1}\} \\ &= \Psi(x)^{-1}\Psi(xx_i) = \Psi(x)^{-1}\Psi(x)\{g \cdot [1, x_i]\} \\ &= [g, x_i], \end{aligned}$$

as required.

Now let λ be the projection map from K_R to K_F. Choose a maximal tree in W in K_R. The choice of W gives a coset representative system S_1 for R in F as follows. If we regard the vertices of K_R as being the cosets Rx of R in F, and let $\tau(Rx)$ be the unique reduced path in W from R to Rx, then $\lambda\tau(Rx)$ is the element of S_1 representing Rx. We write $\tilde{x}$ for this element of S_1, and note that $\tilde{1} = 1$. Using the maximal tree W_1 of $\hat{K}_R$ obtained by adding the edge t_1 to W, we see that Π is the free product of R and the free group on the symbols $t_{\tilde{x}}$ ($\tilde{x} \in S_1 - 1$), and $t_{\tilde{x}}$ represents the loop $t_1^{-1}\tau(Rx)t_g$ of $\hat{K}_R$, where $\pi(x) = g$. Now $\tau(Rx) = \Phi(\tilde{x})$, so that $t_{\tilde{x}} = \Psi(\tilde{x}) = \tilde{x}\pi(\tilde{x})^{-1}$. This verifies (4) of the theorem for the particular representative system S_1. For an arbitrary representative system S, we have $t_{\bar{x}} = \Psi(\bar{x}) = \Psi(r_x\tilde{x})$ for some $r_x \in R$, so that

$$t_{\bar{x}} = \Psi(r_x)\Psi(\tilde{x}) = r_x t_{\tilde{x}}$$

(since $\Psi(r_x)$ is identified with r_x), and it follows that Π is the free product of R and the free group on the $t_{\bar{x}}$'s ($\bar{x} \neq 1$).

It remains to check part (5). Let $x, y \in F$, $r \in R$ and $g = \pi(x)$. Then

$$g \cdot r = grg^{-1} = g\bar{x}(\bar{x}r\bar{x}^{-1})\bar{x}g^{-1} = \Psi(\bar{x})^{-1}(\bar{x}r\bar{x}^{-1})\Psi(\bar{x}) = t_{\bar{x}}^{-1}(\bar{x}r\bar{x}^{-1})t_{\bar{x}}$$

and

$$g \cdot t_{\bar{y}} = g\bar{y}\pi(\bar{y})^{-1}g^{-1} = \pi(x)\bar{y}\pi(\bar{y})^{-1}\pi(\bar{x})^{-1} = \pi(x)\bar{x}^{-1}\bar{x}\bar{y}\overline{xy}^{-1}\overline{xy}\pi(xy)^{-1} = t_{\bar{x}}^{-1}(\bar{x}\bar{y}\overline{xy}^{-1})t_{\overline{xy}}$$

as required. This concludes the proof of the theorem.

In the particular case of Theorem 4 where **V** is an abelian variety, so that $\Pi' < V(\Pi)$, it is clear that $\Pi/V(\Pi)$ is the quotient of the direct sum $\bigoplus_{i\in I} ZGz_i$ of copies ZGz_i of the group ring ZG (where $z_i=[1,x_i]$). In the special case where F is free and $\mathbf{V}=\mathbf{A}$ is the variety of all abelian groups, $\Pi/A(\Pi)=\Pi/\Pi'$, is precisely $\bigoplus_{i\in I} ZGz_i$, and a comparison of (2.1) with the usual definition of the Fox derivatives $\partial x/\partial x_i$ shows immediately that $\Psi(x)$ is $\sum_{i\in I}(\partial x/\partial x_i)^\pi z_i$, where $(\partial x/\partial x_i)^\pi$ denotes the image of the element $\partial x/\partial x_i$ of ZF under the homomorphism from ZF to ZG induced by π. Thus we recover the Magnus embedding of F/R' into $(\bigoplus_{i\in I} ZGz_i) \rtimes G$ given by

$$\eta(x)=\left(\sum_{i\in I}\left(\frac{\partial x}{\partial x_i}\right)^\pi z_i, \pi(x)\right).$$

In the case where F has presentation $\langle x_i\ (i\in I);\ S_j\ (j\in J)\rangle$ and $\mathbf{V}=\mathbf{A}$, we see that Π/Π' is the quotient of $\bigoplus_{i\in I} ZGz_i$ by the submodule generated by all $\sum_{i\in I}(\partial S_j/\partial x_i)^\pi z_i$ $(j\in J)$. In this situation it follows immediately from Theorem 1 that the restriction of Ψ to R/R' is a G module embedding of R/R' into Π/Π', and that the quotient of Π/Π' by $\Psi(R/R')$ is isomorphic to the augmentation ideal IG of ZG (by the map $t_{\bar{x}}\to\pi(\bar{x})-1$). A straightforward computation shows that Π/Π' here is in fact $ZG\bigotimes_F IF$, and we have the well-known short exact sequence of G modules

$$R/R' \rightarrowtail Z\bigotimes_F IF \to IG$$

(see, e.g. [5]).

In the context of the situation of Theorem 4, Yabanzhi ([6], page 91, question 7.58) has asked (for the case F free on the x_i) if there is a criterion to determine those elements of $\Pi/V(\Pi)\rtimes G$ which belong to (the embedded image of) $F/V(R)$, and notes that such a criterion is well known in the case **V** is the variety **A** of all abelian groups. In other words, given $(w,g)\in\Pi/V(\Pi)\rtimes G$, where w is specified as a word on the $[g,x_i]$, can it be determined if $(w,g)\in F/V(R)$. Now $(w,g)\in F/V(R)$ iff $(w,g)=(\Psi_V(x),\pi(x))$ for some $x\in F$. Choosing $x'\in F$ so that $g=\pi(x')$, we then have

$$(w,g)\in F/V(R) \Leftrightarrow w=\Psi_v(rx')=\Psi_v(r)\Psi_v(x')$$

$$\Leftrightarrow w\Psi_v(x')^{-1}=\Psi_v(r)$$

for some $r\in R$. Thus the problem reduces to the question: given a word u (equal to $w\Psi_v(x')^{-1}$) in $\Pi/V(\Pi)$, is there a criterion to determine if $u\in R/V(R)$? Now if $\pi(x)=g$ then in Π we have

$$\Psi(xx_i)=\Psi(x)\{g\cdot\Psi(x_i)\}=\Psi(x)[g,x_i],$$

so that

$$[g,x_i]=\Psi(x)^{-1}\Psi(xx_i)=(x\bar{x}^{-1}t_{\bar{x}})^{-1}xx_i\overline{xx_i}^{-1}t_{\overline{xx_i}}=t_{\bar{x}}^{-1}\bar{x}x_i\overline{xx_i}^{-1}t_{\overline{xx_i}}$$

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and it is then easy to check that for $\eta = \pm 1$ we have

$$[g, x_i^\eta] = t_{\bar{x}}^{-1} \bar{x} x_i^\eta \overline{x x_i^\eta}^{-1} t_{\overline{x x_i^\eta}}. \tag{3.1}$$

Let $I_v G$ denote the free group of the variety **V** freely generated by the set s_g $(g \in G-1)$. From the structure of Π as $R * F_{G-1}$ it is clear that there is a homomorphism μ_v from $\Pi/V(\Pi)$ to $I_v G$ such that $\mu_v(t_{\bar{x}}) = s_g$ $(g = \pi(\bar{x}))$ and $\ker \mu_v$ is the normal closure of $R/V(R)$ in $\Pi/V(\Pi)$. From (3.1) it follows that $\mu_v[g, x_i] = s_g^{-1} s_{gg_i}$. Thus if $u = \prod_{i=1}^k [h_i, x_{\lambda_i}^{\eta_i}]$, then

$$u \in \ker \mu_v \Leftrightarrow \prod_{i=1}^{k} s_{h_i}^{-1} s_{h_i g_{\lambda_i}^{\eta_i}} = 1 \tag{3.2}$$

in $I_v G$. Hence (3.2) is a necessary condition that $u \in R/V(R)$; in case $\mathbf{V} = \mathbf{A}$ it is also sufficient, since then $\ker \mu_v = R/R'$, and translating into additive notation gives the known result in this case (see e.g. [**8**]). It does not seem likely that a general algebraic characterization of $u \in R/V(R)$ can be given, since such would seem to require knowledge of a normal form for elements of free **V** groups. However, under appropriate conditions, an algorithmic answer can be given. We have

Corollary 5. *Given F free and $G = F/R$. Then the generalised word problem is solvable for $F/V(R)$ and $R/V(R)$ in $\Pi/V(\Pi) \rtimes G$, provided that*

(1) *the word problem is solvable for G, and*

(2) *the word problem is solvable for the standard presentation of the free group of countably infinite rank of the variety* **V**.

Proof. We give an informal sketch. Condition (1) means that the index set I is at most countably infinite, and that R is a recursive subset of F. Thus we can construct (any required finite part of) a Schreier coset representative system S of R in F, and (any required finite part of) the corresponding free generating set Y of R. Using (3.1) it follows that given a word u on the $[g, x_i]$ generators of Π, we can express u as a reduced word on the free generating set $Y \cup \{t_{\bar{x}}; \bar{x} \in S-1\}$ of Π. Condition (2) then ensures that we can determine, in $\Pi/V(\Pi)$, whether or not $u \in R/V(R)$. The result then follows easily from the discussion above.

The special case of Corollary 5 when $\mathbf{V} = \mathbf{A}$ has been proved in [**8**].

4. Representations of automorphisms

Let F, G, R, **V** be as in Theorem 4, and let H be the subgroup of $\operatorname{Aut} F$ consisting of those automorphisms γ of F such that $\gamma(R) = R$. Each $\gamma \in H$ induces an automorphism $\bar{\gamma}$ of G, and hence an automorphism γ of $F * G$ (equal to the original γ on F, and to $\bar{\gamma}$ on G). Now, for $x \in F$,

$$\gamma\Psi(x) = \gamma(x\pi(x)^{-1}) = \gamma(x)\bar{\gamma}(\pi(x)^{-1}) = \gamma(x)\pi(\gamma(x))^{-1}$$
$$= \Psi\gamma(x).$$

Thus

$$\gamma\{g\cdot[1,x_i]\}=\gamma[g,x_i]=\gamma\{g\Psi(x_i)g^{-1}\}=\bar{\gamma}(g)\Psi(\gamma(x_i))\bar{\gamma}(g)^{-1}$$

$$=\bar{\gamma}(g)\cdot\Psi(\gamma(x_i))=\bar{\gamma}(g)\gamma\{[1,x_i]\}.$$

It follows that γ induces an automorphism of Π, and therefore we obtain a homomorphism $\mu':H\rightarrow \mathrm{Aut}(\Pi/V(\Pi)\rtimes G)$ and (by restriction) a homomorphism $\mu:H\rightarrow \mathrm{Aut}\,\Pi/V(\Pi)$, where $\mu(\gamma)[g,x_i]=\bar{\gamma}(g)\Psi_V(\gamma(x_i))$. Now $\Pi/V(\Pi)\rtimes G$ is generated by $F/V(R)$ and G, and it follows that $\ker\mu'$ consists of those $\gamma\in H$ which induce the identity on $F/V(R)$ (since such γ also induce the identity on G).

Now suppose $\gamma\in H$ induces the identity on $\Pi/V(\Pi)$. Then

$$\mu(\gamma)[1,x_i]=\Psi_V\gamma(x_i)=[1,x_i]$$

so that

$$\gamma(x_i)\overline{\gamma(x_i)}^{-1}t_{\overline{\gamma(x_i)}}=x_i\bar{x}_i^{-1}t_{\bar{x}_i}$$

in $\Pi/V(\Pi)$. From the free product structure of Π it now follows that $\overline{\gamma(x_i)}=\bar{x}_i$ for $i\in I$, and hence that γ induces the identity on G. Now for $x\in F/V(R)$, we have $x\pi(x)^{-1}\in\Pi/V(\Pi)$, and since γ fixes $x\pi(x)^{-1}$ and $\pi(x)$, it must also fix x. Hence $\ker\mu'=\ker\mu$. We summarise as

Theorem 6. *With the notation of Theorem* 4, *let H be the subgroup of* $\mathrm{Aut}\,F$ *consisting of those automorphisms γ of F such that $\gamma(R)=R$, and let K consist of the $\gamma\in H$ such that γ induces the identity on $F/V(R)$. Then the map $\mu:H\rightarrow\mathrm{Aut}\,\Pi/V(\Pi)$ given by*

$$\mu(\gamma)[g,x_i]=\bar{\gamma}(g)\cdot\Psi_V(\gamma(x_i))$$

is a homomorphism with $\ker\mu=K$.

Let H_1 be the subgroup of H consisting of those $\gamma\in H$ which induce the identity on G. In the case F is free on the x_i, the theorem yields a representation of H_1 (with kernel K) as a group of G-automorphisms of the free **V** group $\Pi/V(\Pi)$; the special case of this with $\mathbf{V}=\mathbf{A}$ yields the representations studied in chapter 3 of [**1**]. In this special case, as we noted previously, $\Pi/V(\Pi)$ is $\bigoplus_{i\in I}ZGz_i$ and thus we obtain matrix representations; in fact, since $z_i=[1,x_i]$, we have

$$\mu(\gamma)z_i=\Psi_A\gamma(x_i)=\sum_{j\in I}\left(\frac{\partial\gamma(x_i)}{\partial x_j}\right)^{\pi}z_j$$

so that if $M(\gamma)$ is defined to be the matrix over ZG whose jith entry is $(\partial\gamma(x_i)/\partial x_j)^{\pi}$, we have that the mapping $\gamma\rightarrow\Pi(\gamma)$ is a homomorphism with kernel K (note that we obtain the transpose of the representation of [1] since we apply our automorphism γ on the left).

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