Periodic points of post-critically algebraic holomorphic endomorphisms

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Abstract. A holomorphic endomorphism of $\mathbb{CP}^n$ is post-critically algebraic if its critical hypersurfaces are periodic or preperiodic. This notion generalizes the notion of post-critically finite rational maps in dimension one. We will study the eigenvalues of the differential of such a map along a periodic cycle. When $n = 1$, a well-known fact is that the eigenvalue along a periodic cycle of a post-critically finite rational map is either superattracting or repelling. We prove that, when $n = 2$, the eigenvalues are still either superattracting or repelling. This is an improvement of a result by Mattias Jonsson [Some properties of 2-critically finite holomorphic maps of P2. *Ergod. Th. & Dynam. Sys.* **18**(1) (1998), 171–187]. When $n \geq 2$ and the cycle is outside the post-critical set, we prove that the eigenvalues are repelling. This result improves one obtained by Fornæss and Sibony [Complex dynamics in higher dimension. II. *Modern Methods in Complex Analysis (Princeton, NJ, 1992) (Annals of Mathematics Studies, 137)*. Ed. T. Bloom, D. W. Catlin, J. P. D'Angelo and Y.-T. Siu, Princeton University Press, 1995, pp. 135–182] under a hyperbolicity assumption on the complement of the post-critical set.

Key words: holomorphic dynamics, periodic points, eigenvalues, post-critically finite, post-critically algebraic
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Contents

1. *Introduction*

In this article, we will work with holomorphic endomorphisms of $\mathbb{CP}^n$. Without any further indication, every endomorphism considered in this article is holomorphic. Let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be an endomorphism. The critical locus C_f is the set of points where the differential $D_z f : T_z \mathbb{CP}^n \to T_{f(z)} \mathbb{CP}^n$ is not surjective. The endomorphism f is called *post-critically algebraic (PCA)* if the post-critical set of f

$$PC(f) = \bigcup_{j \geq 1} f^{\circ j}(C_f)$$

is an algebraic set of codimension one. If $n = 1$, such an endomorphism is called *post-critically finite (PCF)* since proper algebraic sets in $\mathbb{CP}^1$ are finite sets. A point $z \in \mathbb{CP}^n$ is called a periodic point of f of period m if $f^{\circ m}(z) = z$ and m is the smallest positive integer satisfying such a property. We define an eigenvalue of f along the cycle of z as an eigenvalue of $D_z f^{\circ m}$. We will study eigenvalues along periodic cycles of PCA endomorphisms of $\mathbb{CP}^n$ of degree $d \geq 2$.

When $n = 1$, we have the following fundamental result.

THEOREM 1.1. *Let $f : \mathbb{CP}^1 \to \mathbb{CP}^1$ be a PCF endomorphism of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

This theorem relies on the following relation of critical orbits and periodic points of endomorphisms of $\mathbb{CP}^1$. More precisely, without loss of generality, let λ be the eigenvalue of an endomorphisms f of $\mathbb{CP}^1$ (not necessary PCF) at a fixed point z. Then:

- if $0 < |\lambda| < 1$ or if λ is a root of unity, then z is the limit of the infinite orbit of some critical point;
- if $|\lambda| = 1$ and λ is not a root of unity, then either:
  - z is accumulated by the infinite orbit of some critical point; or
  - z is contained in a Siegel disc whose boundary is accumulated by the infinite orbit of some critical point.

We refer the reader to [**Mil11**] for further reading on this topic. In this article, we study how this result may be generalized to dynamics in dimension $n \geq 2$. We study the following question.

Question A. Let f be a PCA endomorphism of $\mathbb{CP}^n$, $n \geq 2$, of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle. Can we conclude that either $\lambda = 0$ or $|\lambda| > 1$?

In this article, we give an affirmative answer to this question when $n = 2$.

THEOREM 1.2. *Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

We note that this question has been studied by several authors and some partial conclusions have been achieved. We refer the reader to [**Ast20**, **FS92**, **FS94**, **GHK**, **GV19**, **Ji20**, **Jon98**, **PI19**, **Ron08**, **Ued98**] for the study of PCA endomorphisms in higher dimensions. Concerning eigenvalues along periodic cycles, following Fornæss–Sibony

[**FS94**, Theorem 6.1], one can deduce that, for a PCA endomorphism f of $\mathbb{CP}^n$, $n \geq 2$, such that $\mathbb{CP}^n \setminus PC(f)$ is Kobayashi hyperbolic and hyperbolically embedded, an eigenvalue λ of f along a periodic cycle outside $PC(f)$ has modulus at least or equal to one. Following Ueda [**Ued98**], one can show that the differential of a PCA endomorphism f of $\mathbb{CP}^n$, $n \geq 2$ along a periodic cycle which is not critical has modulus at least one (see Corollary 2.5 in this article). When $n = 2$, Jonsson [**Jon98**] considered PCA endomorphisms of $\mathbb{CP}^2$ whose critical locus does not have a periodic irreducible component. He proved that, for such class of maps, every eigenvalue along a periodic cycle outside the critical set has modulus strictly bigger than one. Recently, in [**Ast20**], Astorg studied Question A under a mild transversality assumption on irreducible components of $PC(f)$.

Our approach to proving Theorem 1.2 is subdivided into two main cases: the cycle is either outside, or inside the post-critical set.

When the cycle is outside the post-critical set, we improve the method of [**FS94**] to remove the Kobayashi hyperbolic assumption and exclude the possibility of eigenvalues of modulus one. We obtain the following general result.

THEOREM 1.3. *Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle outside the post-critical set. Then $|\lambda| > 1$.*

When the cycle is inside the post-critical set, we restrict our study to dimension $n = 2$. Let $z \in PC(f)$ be a periodic point of period m. Counting multiplicities, $D_z f^{\circ m}$ has two eigenvalues λ_1 and λ_2. We consider two subcases: either z is a regular point of $PC(f)$ or z is a singular point of $PC(f)$.

If the periodic point z is a regular point of $PC(f)$, the tangent space $T_z PC(f)$ is invariant by $D_z f^{\circ m}$. Then $D_z f^{\circ m}$ admits an eigenvalue λ_1 with associated eigenvectors in $T_z PC(f)$. The other eigenvalue λ_2 arises as the eigenvalue of the linear endomorphism $\overline{D_z f^{\circ m}} : T_z \mathbb{CP}^2 / T_z PC(f) \to T_z \mathbb{CP}^2 / T_z PC(f)$ induced by $D_z f^{\circ m}$. By using the normalization of irreducible algebraic curves and Theorem 1.1, we prove that the eigenvalue λ_1 has modulus strictly bigger than one. Regarding the eigenvalue λ_2, following the idea used to prove Theorem 1.3, we also deduce that either $\lambda_2 = 0$ or $|\lambda_2| > 1$.

If the periodic point z is a singular point of $PC(f)$, in most cases, there exists a relationship between λ_1 and λ_2. Then, by using Theorem 1.1, we deduce that, for $i = 1, 2$, either $\lambda_j = 0$ or $|\lambda_j| > 1$. This has been already observed in [**Jon98**] but, for the sake of completeness, we will recall the detailed statements and include the proof.

Structure of the article. In §2, we recall the results of Ueda and prove that when a fixed point is not a critical point, then every eigenvalue has modulus at least one. In §3, we present the strategy and the proof of Theorem 1.3. In §4, since the idea is the same as the proof of Theorem 1.3, we prove that, for an eigenvalue λ of a PCA map along a periodic cycle which is a regular point of the post-critical set and the associated eigenvectors which are not tangential to the post-critical set, then λ is either zero or of modulus strictly bigger than one. In §5, we study the dynamics of PCA endomorphisms of $\mathbb{CP}^2$ restricting on invariant curves and then prove Theorem 1.2.

Notation.

- $\mathbb{D}(0, r) = \{x \in \mathbb{C} | \|x\| < r\}$: the ball of radius r in $\mathbb{C}$ (or simply $\mathbb{D}$ when $r = 1$).
- $\mathbb{D}(0, R)^*$ or $\mathbb{D}^*$: the punctured disc $\mathbb{D}(0, R) \setminus \{0\}$.
- For two paths $\gamma, \eta : [0, 1] \to X$ in a topological space X such that $\gamma(1) = \eta(0)$, the concatenation path $\gamma * \eta : [0, 1] \to X$ is defined as

$$\gamma * \eta(t) = \begin{cases} \gamma(2t) & \text{if } t \in [0, \frac{1}{2}], \\ \eta(2t - 1) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$$

- $\text{Spec}(L)$: the set of eigenvalues of a linear endomorphism L of a vector space V.
- For an algebraic set (analytic set) X in a complex manifold, we denote by $\text{Sing } X$ the set of singular points of X and by $\text{Reg } X$ the set of regular points of X.

2. *Periodic cycles outside the critical set*

In this section, we prove that the eigenvalues of a PCA endomorphism of $\mathbb{CP}^n$ at a fixed point, which is not a critical point, have modulus at least or equal to one. The proof relies on the existence of an open subset on which we can find a family of inverse branches and the fact that the family of inverse branches of endomorphisms of $\mathbb{CP}^n$ is normal. These results are due to Ueda [**Ued98**].

We recall the definition of finite branched covering.

Definition 2.1. A proper, surjective continuous map $f : Y \to X$ of complex manifolds of the same dimension is called a *finite branched covering* (or *finite ramified covering*) if there exists an analytic set D of codimension one in X such that the map

$$f : Y \setminus f^{-1}(D) \to X \setminus D$$

is a covering. We say that f is *ramified over* D or a *D-branched covering*. The set D is called the *ramification locus*.

We refer the reader to [**Gun90**] for more information about the theory of finite branched coverings.

Recall that endomorphisms f of $\mathbb{CP}^n$ of degree $d \geq 2$ are finite branched coverings ramifying over $f(C_f)$. If f is PCA, then, for every $j \geq 1$, $f^{\circ j}$ is ramified over $PC(f)$.

Let $z \notin C_f$ be a fixed point of f. Then $f^{\circ j}$ is locally invertible in a neighborhood of z. The following result, which is due to Ueda, ensures that we can find a common open neighborhood on which inverse branches of $f^{\circ j}$ fixing z are well defined for every $j \geq 1$.

LEMMA 2.2. [**Ued98**, Lemma 3.8] *Let X be a complex manifold and let D be an analytic subset of X of codimension one. For every point $x \in X$, if W is a simply connected open neighborhood of x such that $(W, W \cap D)$ is homeomorphic to a cone over $(\partial W, \partial W \cap D)$ with vertex at x, then, for every branched covering $\eta : Y \to W$ ramifying over $D \cap W$, the set $\eta^{-1}(x)$ consists of only one point.*

Given a topological space T, the cone over a set $K \subset T$ with vertex at a point $x \in T$ is the quotient space $Cone(K) = K \times [0, 1]/K \times \{0\}$, where x is identified with the equivalent class of $K \times \{0\}$, which is a point in $Cone(K)$.

PROPOSITION 2.3. *Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$ and let $z \notin C_f$ be a fixed point of f. Let W be a simply connected open neighborhood of z such that $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with vertex at z. Then there exists a family of holomorphic inverse branches $h_j : W \to \mathbb{CP}^n$ of $f^{\circ j}$ fixing z, that is,*

$$h_j(z) = z, \ f^{\circ j} \circ h_j = \mathrm{Id}|_W.$$

Note that, for a fixed point z of a PCA endomorphism f of $\mathbb{CP}^n$, since $PC(f)$ is an algebraic set, there always exists a simply connected neighborhood W of z such that $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with vertex at z. Indeed, if $z \notin PC(f)$, then we can take any simply connected neighborhood W of z in $\mathbb{CP}^n \setminus PC(f)$. If $z \in PC(f)$, then it follows from [**Sea19**, Theorem 3.2] that such a neighborhood always exists. We refer the reader to [**Mil68**] for an approach when z is an isolated singularity of $PC(f)$ (see also [**Sea06**, Remark 2.3]).

Proof of Proposition 2.3. For every $j \geq 1$, denote by W_j the connected component of $f^{-j}(W)$ containing z. Since $f^{\circ j}$ are branched coverings ramifying over $PC(f)$, $f^{\circ j}$ induces a branched covering

$$f_j := f^{\circ j}|_{W_j} : W_j \to W$$

ramifying over $W \cap PC(f)$. By Lemma 2.2, we deduce that $f_j^{-1}(z)$ consists of only one point, which is, in fact z. Since W is simply connected and hence connected, the order of the branched covering f_j coincides with the branching order of $f^{\circ j}$ at z. Note that z is not a critical point of $f^{\circ j}$ and therefore the branching order of $f^{\circ j}$ at z is one. This means that f_j is a branched covering of order one of complex manifolds, and thus f_j is a homeomorphism and hence a biholomorphism (see [**Gun90**, Corollary 11Q]). The holomorphic map $h_j : W \to W_j$, defined as the inverse of f_j, is the map we are looking for.   □

Once we obtain a family of inverse branches, the following theorem, which is due to Ueda, implies that this family is normal.

THEOREM 2.4. [**Ued98**, Theorem 2.1] *Let f be an endomorphism of $\mathbb{CP}^n$. Let X be a complex manifold with a holomorphic map $\pi : X \to \mathbb{CP}^n$. Let $\{h_j : X \to \mathbb{CP}^n\}_j$ be a family of holomorphic lifts of $f^{\circ j}$ by π, that is, $f^{\circ j} \circ h_j = \pi$. Then $\{h_j\}_j$ is a normal family.*

Thus, for a fixed point z of a PCA endomorphism f, if z is not a critical point (or, equivalently, if $D_z f$ is invertible), then we can obtain an open neighborhood W of z in $\mathbb{CP}^n$ and a normal family of holomorphic maps $\{h_j : W \to \mathbb{CP}^n\}_j$ such that

$$f^{\circ j} \circ h_j = \mathrm{Id}_W, \ h_j(z) = z.$$

The normality of $\{h_j\}_j$ implies that $\{D_z h_j\}_j$ is a uniformly bounded sequence (with respect to a fixed norm $\|\cdot\|$ on $T_z \mathbb{CP}^n$). Since $D_z h_j = (D_z f)^{-j}$, we can deduce that every eigenvalue of $D_z f$ has modulus at least one. Consequently, we can find a $D_z f$-invariant

decomposition of $T_z\mathbb{CP}^n$ as the direct sum

$$T_z\mathbb{CP}^n = \Big(\bigoplus_{\lambda \in \text{Spec } D_z f, |\lambda|=1} E_\lambda \Big) \oplus \Big(\bigoplus_{\lambda \in \text{Spec } D_z f, |\lambda|>1} E_\lambda \Big) = E_n \oplus E_r,$$

where E_λ is the generalized eigenspace of the eigenvalue λ. We call E_n the neutral eigenspace and E_r the repelling eigenspace of $D_z f$ (see also 3.2). If E_λ is not generated by eigenvectors (or, equivalently, if $D_z f|_{E_\lambda}$ is not diagonalizable), we can find at least two generalized eigenvectors e_1, e_2 of $D_z f$ corresponding to λ such that

$$D_z f(e_1) = \lambda e_1, \ D_z f(e_2) = \lambda e_2 + e_1.$$

Then $D_z h_j(e_2) = \lambda^{-j} e_2 - j\lambda^{-(j+1)} e_1$. If $|\lambda| = 1$, then $\|D_z h_j(e_2)\|$ tends to infinity as j tends to infinity. This contradicts the uniform boundedness of $\{D_z h_j\}_j$. Hence $D_z f|_{E_\lambda}$ is diagonalizable. Thus, we have proved the following corollary.

COROLLARY 2.5. *Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$ and let λ be an eigenvalue of f at a fixed point $z \notin C_f$. Then $|\lambda| \geq 1$. Moreover, if $|\lambda| = 1$, then $D_z f|_{E_\lambda}$ is diagonalizable.*

3. *Periodic cycles outside the post-critical set*
In this section, we prove Theorem 1.3.

THEOREM 3.1. *Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle outside the post-critical set. Then $|\lambda| > 1$.*

Observe that a periodic point of f is simply a fixed point of some iterate of f. Moreover, any iterate of a PCA maps is still PCA. Thus, it is enough to proof Theorem 1.3 when λ is an eigenvalue of f at a fixed point $z \notin PC(f)$. We will consider an equivalent statement and then prove it.

3.1. *Equivalent problem in the affine case.* Recall that, for an endomorphism $f : \mathbb{CP}^n \to \mathbb{CP}^n$ of degree d, there exists a polynomial endomorphism

$$F = (P_1, \dots, P_{n+1}) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1},$$

where P_i are homogeneous polynomials of the same degree $d \geq 1$ and $F^{-1}(0) = \{0\}$ such that

$$f \circ \pi = \pi \circ F,$$

where $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is the canonical projection. The integer d is called the *algebraic degree* (or *degree*) of f. Such a map F is called a *lift* of f to $\mathbb{C}^{n+1}$. Further details about holomorphic endomorphisms of $\mathbb{CP}^n$ and their dynamics can be found in [FS94, FS95, Gun90, Sib99, DS10].

Lifts to $\mathbb{C}^{n+1}$ of an endomorphism of $\mathbb{CP}^n$ belong to a class of *non-degenerate homogeneous polynomial endomorphisms* of $\mathbb{C}^{n+1}$. More precisely, a non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^n$ of algebraic degree d is a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ such that $F(\lambda z) = \lambda^d z$ for every $z \in \mathbb{C}^n, \lambda \in \mathbb{C}$ and $F^{-1}(0) = \{0\}$.

Conversely, such a map induces an endomorphism of $\mathbb{CP}^n$. This kind of map has been studied extensively in [**HP94**]. If we consider $\mathbb{C}^{n+1}$ as a dense open set of $\mathbb{CP}^{n+1}$ by the inclusion $(\zeta_0, \zeta_1, \ldots, \zeta_n) \mapsto [\zeta_0 : \zeta_1 : \cdots : \zeta_n : 1]$, then F can be extended to an endomorphism of $\mathbb{CP}^{n+1}$. Moreover, this extension fixes the hypersurface at infinity $\mathbb{CP}^{n+1} \setminus \mathbb{C}^{n+1} \cong \mathbb{CP}^n$ and the restriction to this hypersurface is the endomorphism of $\mathbb{CP}^n$ induced by F.

Thus, if f is a PCA endomorphism of $\mathbb{CP}^n$, every lift F of f to $\mathbb{C}^{n+1}$ is the restriction of a PCA endomorphism of $\mathbb{CP}^{n+1}$ to $\mathbb{C}^{n+1}$. PCA non-degenerate homogeneous polynomial endomorphisms of $\mathbb{C}^{n+1}$ have similar properties, which are proved in §2, as PCA endomorphisms of $\mathbb{CP}^n$. More precisely, we can summarize this in the following proposition.

PROPOSITION 3.2. *Let F be a PCA non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^{n+1}$ of degree $d \geq 2$ and let $z \notin C_F$ be a fixed point of F. Then we have the following assertions.*

(a) *Let X be a complex manifold and let $\pi : X \to \mathbb{C}^{n+1}$ be a holomorphic map. Then every family of holomorphic maps $\{h_j : X \to \mathbb{C}^{n+1}\}_j$, which satisfies that $F^{\circ j} \circ h_j = \pi$ for every $j \geq 1$, is normal.*

(b) *There exist a simply connected open neighborhood W of z in $\mathbb{C}^{n+1}$ and a family $\{h_j : W \to \mathbb{C}^{n+1}\}_j$ of inverse branches of iterates of F, that is, $F^{\circ j} \circ h_j = \mathrm{Id}_W$, fixing z.*

(c) *Every eigenvalue λ of $\mathrm{Spec}(D_z F)$ has modulus at least one. The tangent space $T_z \mathbb{C}^{n+1}$ admits a $D_z F$-invariant decomposition $T_z \mathbb{C}^{n+1} = E_n \oplus E_r$, where the neutral eigenspace E_n is the sum of generalized eigenspaces corresponding to eigenvalues of modulus one and the repelling E_r is the sum of generalized eigenspaces corresponding to eigenvalues of modulus strictly bigger than one.*

(d) *If $|\lambda| = 1$, then $D_z f|_{E_\lambda}$ is diagonalizable.*

Remark 3.3. Studying eigenvalues at fixed points of PCA endomorphisms of $\mathbb{CP}^n$ is equivalent to studying eigenvalues at fixed points of a PCA non-degenerate homogeneous polynomial self-map of $\mathbb{C}^{n+1}$. More precisely, let $f : \mathbb{CP}^n \to \mathbb{CP}^n$ be an endomorphism of degree $d \geq 2$ and let F be a lift of f. Assume that z is a fixed point of f. Then, the complex line L containing $\pi^{-1}(z)$ is invariant under F and the map induced by F on L is conjugate to $x \mapsto x^d$. In particular, there exists a fixed point $w \in L \setminus \{0\}$ of F such that $\pi(w) = z$ and $D_w F$ preserves $T_w L \subset T_w \mathbb{C}^{n+1}$ with an eigenvalue d. Then, $D_w F$ descends to a linear endomorphism of the quotient space $T_w \mathbb{C}^{n+1} / T_w(\mathbb{C}w)$ which is conjugate to $D_z f : T_z \mathbb{CP}^n \to T_z \mathbb{CP}^n$. Hence a value λ is an eigenvalue of $D_w F$ if and only if either λ is an eigenvalue of $D_z f$ or $\lambda = d$.

Conversely, if w is a fixed point of F, then either $w = 0 \in \mathbb{C}^{n+1}$ (and the eigenvalues of $D_0 F$ are all equal to zero) or w induces a fixed point $\pi(w)$ of f. Since we consider only PCA endomorphisms of degree $d \geq 2$, Question A is equivalent to the following question.

Question B. Let f be a non-degenerate homogeneous polynomial PCA endomorphism of $\mathbb{C}^n$ and let λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.

The advantage of this observation is that we can make use of some nice properties of the affine space $\mathbb{C}^n$. More precisely, we will use the fact that the tangent bundle of $\mathbb{C}^n$ is trivial.

3.2. *Strategy of the proof of Theorem 1.3.* Recall that an eigenvalues λ of $D_z f : T_z\mathbb{C}^n \to T_z\mathbb{C}^n$ is called:

- superattracting if $\lambda = 0$;
- attracting if $0 < |\lambda| < 1$;
- neutral if $|\lambda| = 1$:
  - parabolic or rational if λ is a root of unity;
  - elliptic or irrational if λ is not a root of unity;
- repelling if $|\lambda| > 1$.

By Remark 3.3, in order to prove Theorem 1.3, it is enough to prove the following result.

THEOREM 3.4. *Let f be a PCA non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^n$ of degree $d \geq 2$ and let λ be an eigenvalue of f at a fixed point $z \notin PC(f)$. Then λ is repelling.*

The strategy of the proof is as follows.

Step 1. Set $X = \mathbb{C}^n \setminus PC(f)$ and let $\pi : \widetilde{X} \to X$† be its universal covering. We construct a holomorphic map $g : \widetilde{X} \to \widetilde{X}$ such that

$$f \circ \pi \circ g = \pi$$

and g fixes a point $[z]$ such that $\pi([z]) = z$.

Step 2. We prove that the family $\{g^{\circ m}\}_m$ is normal. Then there exists a closed complex submanifold M of $\widetilde{X}$ passing through $[z]$ such that $g|_M$ is an automorphism and $\dim M$ is the number of eigenvalues of $D_z f$ of modulus one, that is, neutral eigenvalues, counted with multiplicities. Due to Corollary 2.5, it is enough to prove that $\dim M = 0$.

Step 3. In order to prove that $\dim M = 0$, we proceed by contradiction. Assume that $\dim M > 0$. We then construct a holomorphic mapping $\Phi : M \to T_{[z]}M$ such that $\Phi([z]) = 0$, $D_{[z]}\Phi = \mathrm{Id}$ and

$$\Phi \circ g = D_{[z]}g \circ \Phi.$$

We deduce that $D_z f$ has no parabolic eigenvalue.

Step 4. Assume that λ is a neutral irrational eigenvalue and that v is an associated eigenvector. We prove that the irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ containing $[z]$ is smooth and that $\Phi|_\Gamma$ maps Γ biholomorphically onto a disc $\mathbb{D}(0, R)$ with $0 < R < +\infty$.

Step 5. Denote by $\kappa : \mathbb{D}(0, R) \to M$ the inverse of $\Phi|_\Gamma$. We prove that $\pi \circ \kappa$ has radial limits almost everywhere on $\partial\mathbb{D}(0, R)$ and that these radial limits land on ∂X. This results in a contradiction and Theorem 3.4 is proved.

† Since we are now only work on $\mathbb{C}^n$, we will not use π as the canonical projection from $\mathbb{C}^{n+1}$ to $\mathbb{CP}^n$.

3.3. *Lifting the backward dynamics via the universal covering.* Denote by $X = \mathbb{C}^n \setminus PC(f)$ the complement of $PC(f)$ in $\mathbb{C}^n$. Since $PC(f)$ is an algebraic set, the set X is a connected open subset of $\mathbb{C}^n$ and then the universal covering of X is well defined. Denote by $\pi : \widetilde{X} \to X$ the universal covering of X defined by

$$\widetilde{X} = \big\{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } z_0 \big\} \quad \text{and} \quad \pi\big([\gamma]\big) = \gamma(1),$$

where $[\gamma]$ denotes the homotopy class of γ in X, fixing the endpoints $\gamma(0)$ and $\gamma(1)$. Denote by $[z]$ the element in $\widetilde{X}$ representing the homotopy class of the constant path at z. We endow $\widetilde{X}$ with a complex structure such that $\pi : \widetilde{X} \to X$ is a holomorphic covering map.

Set $Y = f^{-1}(X) \subset X$ and $\widetilde{Y} = \pi^{-1}(Y) \subset \widetilde{X}$. Since $f : Y \to X$ is a covering map, every path γ in X starting at z_0 lifts to a path $f^*\gamma \subset Y$ starting at z_0. In addition, if γ_1 and γ_2 are homotopic in X, then $f^*\gamma_1$ and $f^*\gamma_2$ are homotopic in $Y \subset X$; in particular, in X. Thus, this pull-back map f^* induces a map $g : \widetilde{X} \to \widetilde{X}$ such that the following diagram commutes.

$$
\begin{array}{ccc}
\widetilde{X} & \xleftarrow{\ \ g\ \ } & \widetilde{X} \\
{\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\
X & \xrightarrow[\ \ f\ \ ]{} & X
\end{array}
$$

Note that $g([z]) = [z]$. In addition, g is holomorphic since, in local charts given by π, it coincides with inverse branches of f.

3.4. *Normality of maps on the universal covering.* We prove that the family $\{g^{\circ j}\}_j$ is a normal family. For every integer $j \geq 1$, define $k_j = \pi \circ g^{\circ j}$ so that $f^{\circ j} \circ k_j = \pi$.

LEMMA 3.5. *The family $\{k_j : \widetilde{X} \to X\}_j$ is normal and any limit takes values in X.*

Proof. By Proposition 3.2, the family $\{k_j : \widetilde{X} \to \mathbb{C}^n\}_j$ is normal. Denote by $Q : \mathbb{C}^n \to \mathbb{C}$ a polynomial such that $PC(f)$ is the zeros locus of Q. Consider the family

$$Q_j = Q \circ k_j : \widetilde{X} \to \mathbb{C}.$$

Since $k_j(\widetilde{X}) \subset X$, the family $\{Q_j\}$ is a normal family of non-vanishing functions. Then by Hurwitz's theorem, every limit map is either a non-vanishing function or a constant function. But $Q_j([z]) = Q(z) \neq 0$ and hence every limit map is a non-vanishing function, that is, every limit map of $\{k_j\}$ is valued in X. Thus $\{k_j : \widetilde{X} \to X\}$ is normal. $\square$

We can deduce the normality of $\{g^{\circ j}\}_j$.

PROPOSITION 3.6. *The family $\{g^{\circ j}\}_j$ is normal.*

Proof. Let $\{g^{\circ j_s}\}_s$ be a sequence of iterates of g. Extracting subsequences, if necessary, we can assume that k_{j_s} converges to a holomorphic map $k : \widetilde{X} \to X$.

Since $\widetilde{X}$ is simply connected and $\pi : \widetilde{X} \to X$ is a holomorphic covering map, there exists a holomorphic map $g_0 : \widetilde{X} \to \widetilde{X}$ such that $\pi \circ g_0 = k$ and $g_0([z]) = [z]$. Note that, for every $j \geq 1$, $g^{\circ j}([z]) = [z]$, and thus the sequence $\big\{g^{\circ j}([z])\big\}_j$ converges to $g_0([z])$.

According to [**AC03**, Theorem 4], the sequence $\big\{g^{\circ j_s}\big\}_s$ converges locally uniformly to g_0. This shows that $\big\{g^{\circ j}\big\}_j$ is normal. □

Remark 3.7. The proof of Lemma 3.5 relies on the PCA hypothesis. Without the PCA assumption, for a fixed point z which is not accumulated by the critical set, we can still consider the connected component U of $\mathbb{C}^n \setminus \overline{PC(f)}$ containing z, and the construction follows. Then we will need some control on the geometry of U to prove that the family $\{g^{\circ j}\}$ is normal. For example, if U is a pseudoconvex open subset of $\mathbb{C}^n$ or, in general, if U is a taut manifold, then $\{g^{\circ j}\}_j$ is normal.

3.5. *Consequences of normality.* The normality of the family of iterates of g implies much useful information. In particular, following Abate [**Aba89**, Corollaries 2.1.30 and 2.1.31], we derive the existence of a center manifold of g on $\tilde{X}$.

THEOREM 3.8. *Let X be a connected complex manifold and let g be an endomorphism of X. Assume that g has a fixed point z. If the family of iterates of g is normal, then:*

(1) *every eigenvalue of $D_z g$ is contained in the closed unit disc;*
(2) *the tangent space $T_z X$ admits a $D_z g$-invariant decomposition $T_z X = \widetilde{E}_n \oplus \widetilde{E}_a$ such that $D_z g|_{\widetilde{E}_n}$ has only neutral eigenvalues and $D_z g|_{\widetilde{E}_a}$ has only attracting or superattracting eigenvalues;*
(3) *the linear map $D_z g|_{\widetilde{E}_n}$ is diagonalizable;*
(4) *there exists a limit map ρ of iterates of g such that $\rho \circ \rho = \rho$;*
(5) *the set of fixed points of ρ, which is $\rho(X)$, is a closed submanifold of X; set $M = \rho(X)$;*
(6) *the submanifold M is invariant by g; in fact, $g|_M$ is an automorphism; and*
(7) *the submanifold M contains z and $T_z M = \widetilde{E}_n$.*

Applying this theorem to $\widetilde{X}$ and $g : \widetilde{X} \to \widetilde{X}$ fixing $[z]$, we deduce that $D_{[z]}g$ has only eigenvalues of modulus at most one and $T_{[z]}\widetilde{X}$ admits a $D_{[z]}g$-invariant decomposition as $T_{[z]}\widetilde{X} = \widetilde{E}_n \oplus \widetilde{E}_a$. Differentiating both sides of $f \circ \pi \circ g = \pi$ at $[z]$,

$$D_z f \cdot D_{[z]}\pi \cdot D_{[z]}g = D_{[z]}\pi.$$

Hence λ is a neutral eigenvalue of $D_z f$ if and only if λ^{-1} is a neutral eigenvalue of $D_{[z]}g$. Consequently, $D_{[z]}\pi$ maps $\widetilde{E}_n$ to the neutral eigenspace E_n of $D_z f$, $\widetilde{E}_a$ onto the repelling eigenspace E_r of $D_z f$ (see Proposition 3.2).

We also obtain a closed center manifold M of g at $[z]$, that is, if λ is a neutral eigenvalue of $D_{[z]}g$ of eigenvector v, then $v \in T_{[z]}M$. So, in order to prove Theorem 3.4, it is enough to prove that $\dim M = 0$. The first remarkable property of M is that $\pi(M)$ is a bounded set in $\mathbb{C}^n$.

PROPOSITION 3.9. *The image $\pi(M)$ is bounded.*

Proof. For every $[\gamma] \in \widetilde{X}$, note that $\{\pi(g^{\circ j}([\gamma]))\}_j$ is, in fact, a sequence of backward iterations of $\gamma(1)$ by f and that the ω-limit set of backward images of $\mathbb{C}^n$ by f is bounded. More precisely, since f is a homogeneous polynomial endomorphism of $\mathbb{C}^n$ of degree

$d \geq 2$, the origin 0 is superattracting and the basin of attraction $\mathcal{B}$ bounded with the boundary is $\partial \mathcal{B} = H_f^{-1}(0)$, where

$$H_f(w) = \lim_{j \to \infty} \frac{1}{d^j} \log \| f^{\circ j}(w) \|$$

for $w \in \mathbb{C}^n \setminus \{0\}$. The function H_f is called the *potential function* of f (see [**HP94**]). It is straightforward by computation to show that $H_f(\pi(m)) = 0$ for every $m \in M$. Hence $\pi(M) \subset \partial \mathcal{B}$ is bounded. $\qquad \square$

3.6. *Semi-conjugacy on the center manifold.* Assume that $\dim M > 0$. Denote by Λ the restriction of $D_{[z]} g$ on $T_{[z]} M$. The following proposition ensures that we can semi-conjugate $g|_M$ to Λ.

PROPOSITION 3.10. *Let M be a complex manifold and let g be an endomorphism of M such that the family of iterates of g is normal. Assume that g has a fixed point z such that $D_z g$ is diagonalizable with only neutral eigenvalues and that there exists a holomorphic map $\varphi : M \to T_z M$ such that $\varphi(z) = 0$, $D_z \varphi = \mathrm{Id}$. Then there exists a holomorphic map $\Phi : M \to T_z M$ such that $\Phi(z) = 0$, $D_z \Phi = \mathrm{Id}$ and*

$$D_z g \circ \Phi = \Phi \circ g.$$

Proof. Consider the family $\{(D_z g)^{-n} \circ \varphi \circ g^{\circ n} : M \to T_{[z]} M\}_n$. We know that $\{g^{\circ n}\}_n$ is normal, and thus $\{\varphi \circ g^{\circ j}\}_j$ is locally uniformly bounded. The linear map $D_z g$ is diagonalizable with neutral eigenvalues, so $\{D_z g^{-n}\}_n$ is uniformly bounded on any bounded set. Then $\{(D_z g)^{-n} \circ \varphi \circ g^{\circ n}\}_n$ is a normal family. Denote by Φ_N the Cesaro average of $\{(D_z g)^{-n} \circ \varphi \circ g^{\circ n}\}_n$, that is,

$$\Phi_N = \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ \varphi \circ g^{\circ n}.$$

The family $\{\Phi_N\}_N$ is also locally uniformly bounded and thus is normal. Observe that

$$\Phi_N \circ g = \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ \varphi \circ g^{\circ(n+1)}$$

$$= D_z g \Phi_N + D_z g \left(-\frac{1}{N} \varphi + \frac{1}{N} ((D_z g)^{-(N+1)} \circ \varphi \circ g^{\circ(N+1)}) \right).$$

For every subsequence $\{N_k\}$, the second term on the right-hand side converges locally uniformly to zero. So, for every limit map Φ of $\{\Phi_N\}_N$, Φ satisfies that

$$\Phi \circ g = D_z g \circ \Phi.$$

Since g fixes z, we have that, for every $N \geq 1$,

$$D_z \Phi_N = \frac{1}{N} \sum_{n=0}^{N-1} (D_z g)^{-n} \circ D_z \varphi \circ D_z g^{\circ n} = \mathrm{Id} .$$

So $D_z \Phi = \mathrm{Id}$ for every limit map Φ of $\{\Phi_N\}_N$. $\qquad \square$

Now we consider the complex manifold M obtained in Step 3.5 and the restriction of g on M which is an automorphism with a fixed point $[z]$. Since Λ has only neutral eigenvalues, in order to apply Proposition 3.10, we need to construct a holomorphic map $\varphi : M \to T_{[z]}M$ such that $D_{[z]}\varphi = \text{Id}$. The map φ is constructed as the composition

$$M \xrightarrow{\mathfrak{i}} \widetilde{X} \xrightarrow{\pi} X \xrightarrow{\delta} T_z X \xrightarrow{(D_{[z]}\pi)^{-1}} T_{[z]}\widetilde{X} \xrightarrow{\pi_{\widetilde{E_a}}} T_{[z]}M,$$

where $\delta : X \to T_z X$ is a holomorphic map tangential to identity, $\pi_{\widetilde{E_a}} : T_{[z]}\widetilde{X} \to T_{[z]}M$ is the projection parallel to $\widetilde{E_a}$, $\mathfrak{i} : M \to \widetilde{X}$ is the canonical inclusion and its derivative $D_{[z]}\mathfrak{i} : T_{[z]}M \to T_{[z]}\widetilde{X}$ is again the canonical inclusion. Then $D_{[z]}\varphi : T_{[z]}M \to T_{[z]}M$ is

$$\begin{aligned} D_{[z]}\varphi &= D_{[z]}(\pi_{\widetilde{E_a}} \circ (D_{[z]}\pi)^{-1} \circ \delta \circ \pi \circ \mathfrak{i}) \\ &= \pi_{\widetilde{E_a}} \circ (D_{[z]}\pi)^{-1} \circ D_z\delta \circ D_{[z]}\pi \circ D_{[z]}\mathfrak{i} = \text{Id}. \end{aligned}$$

Remark 3.11. The existence of a holomorphic map $\delta : X \to T_z X$ tangential to identity is one of the advantages we mentioned in Remark 3.3. It comes from the intrinsic nature of the tangent space of affine spaces. In this case, X is an open subset of $\mathbb{C}^n$ which is an affine space directed by $\mathbb{C}^n$.

COROLLARY 3.12. *Let z be a fixed point of a non-degenerate homogeneous polynomial PCA endomorphism f of $\mathbb{C}^n$. Assume that $z \notin PC(f)$. If λ is a neutral eigenvalue of $D_z f$, then λ is an irrational eigenvalue.*

Proof. It is equivalent to consider a neutral eigenvalue λ of $D_{[z]}g$ and assume that $\lambda = e^{2\pi i(p/q)}$. Hence $(D_{[z]}g)^q$ fixes pointwise the line $\mathbb{C}v$ in $T_{[z]}M$. This means that locally near $[z]$, $g^{\circ q}$ fixes $\Phi^{-1}(\mathbb{C}v)$ and hence $f^{\circ q}$ fixes $\pi(\Phi^{-1}(\mathbb{C}v))$ near z. Note that Φ is locally invertible near $[z]$ and hence $\Phi^{-1}(\mathbb{C}v)$ is a complex manifold of dimension one near $[z]$. Then $\pi(\Phi^{-1}(\mathbb{C}v))$ is a complex manifold near z because π is locally biholomorphic. In particular, $\pi(\Phi^{-1}(\mathbb{C}v))$ contains uncountably many fixed points of $f^{\circ q}$. This is a contradiction because $f^{\circ q}$ has only finitely many fixed points (see [**DS10**, Proposition 1.3]). Hence λ is an irrational eigenvalue. $\square$

3.7. *Linearization along the neutral direction.* We obtain a holomorphic map $\Phi : M \to T_{[z]}M$, $\Phi([z]) = 0$, $D_{[z]}\Phi = \text{Id}$ and

$$\Phi \circ g|_M = \Lambda \circ \Phi, \tag{3.1}$$

where $\Lambda = D_{[z]}g|_{T_{[z]}M}$. Let $\lambda = e^{2\pi i\theta}$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational eigenvalue of Λ and let $\mathbb{C}v$ be a complex line of direction v in $T_{[z]}M$. The line $\mathbb{C}v$ is invariant by Λ, that is, $\Lambda(\mathbb{C}v) = \mathbb{C}v$, and hence $\Sigma := \Phi^{-1}(\mathbb{C}v)$ is invariant by g. Denote by Γ the irreducible component of $\Phi^{-1}(\mathbb{C}v)$ containing $[z]$.

LEMMA 3.13. *Set $\Gamma_0 = \Gamma \setminus \text{Sing } \Sigma$. Then $g(\Gamma) = \Gamma$ and $g(\Gamma_0) = \Gamma_0$.*

Proof. On the one hand, since g is an automorphism, it maps irreducible analytic sets to irreducible analytic sets. On the other hand, since $D_{[z]}\Phi = \text{Id}$ then, by inverse function

theorem, $\Phi^{-1}(\mathbb{C}v)$ is smooth near $[z]$, and hence Γ is the only irreducible component of $\Phi^{-1}(\mathbb{C}v)$ near $[z]$. Then $g(\Gamma) = \Gamma$.

Concerning Γ_0, we observe that $\mathrm{Sing}\,\Sigma = \Sigma \cap C_\Phi$, where $C_\Phi = \{x \in M \mid \mathrm{rank}\,D_x \Phi < \dim M\}$ which is the set of critical points of Φ. By differentiating (3.1) at $x \in M$,

$$D_{g(x)}\Phi \circ D_x g|_M = \Lambda \circ D_x \Phi.$$

We deduce that $g(C_\Phi) = C_\Phi$. Moreover, $g(\Sigma) = \Sigma$. Thus $g(\mathrm{Sing}\,\Sigma) = \mathrm{Sing}\,\Sigma$ and hence $g(\Gamma_0) = \Gamma_0$. $\qquad \square$

Note that $\Gamma_0 \subset \mathrm{Reg}\,\Gamma$ is smooth since $\mathrm{Sing}\,\Gamma \subset \Gamma \cap \mathrm{Sing}\,\Sigma$ and, in fact, Γ_0 is a Riemann surface. In particular, $[z] \in \Gamma_0$.

We prove that Γ_0 is biholomorphic to a disc and that $\Phi|_{\Gamma_0}$ is conjugate to an irrational rotation. Then we deduce from this $\Gamma = \Gamma_0$ and Φ_Γ conjugates $g|_\Gamma$ to an irrational rotation. First, we recall an important theorem in the theory of dynamics in one complex dimension.
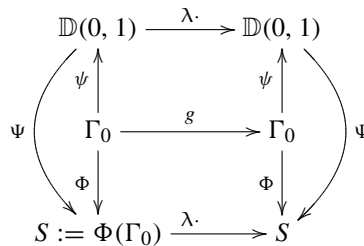
THEOREM 3.14. (See [Mil11, Theorem 5.2]) *Let S be a hyperbolic Riemann surface and let $g : S \to S$ be a holomorphic map with a fixed point z. If z is an irrational fixed point with multiplier λ, then S is biholomorphic to the unit disc and g is conjugate to the irrational rotation $\zeta \mapsto \lambda\zeta$.*

LEMMA 3.15. *The Riemann surface Γ_0 is hyperbolic, $\Phi(\Gamma_0) = \mathbb{D}(0, R)$ with $R \in (0, +\infty)$ and $\Phi|_{\Gamma_0} : \Gamma_0 \to \mathbb{D}(0, R)$ is a biholomorphism conjugating $g|_{\Gamma_0}$ to the irrational rotation $\zeta \mapsto \lambda\zeta$, that is,*

$$\Phi \circ g|_{\Gamma_0} = \lambda \cdot \Phi|_{\Gamma_0}.$$

Proof. Recall that $\pi(M)$ is bounded in $\mathbb{C}^n$. Thus π induces a non-constant bounded holomorphic function from Γ_0 to $\mathbb{C}^n$. Therefore Γ_0 is a hyperbolic Riemann surface. Note that $[z]$ is a fixed point of the holomorphic map $g|_{\Gamma_0}$ with the irrational multiplier λ. Then we can apply Theorem 3.14 to obtain a conjugacy $\psi : \Gamma_0 \to \mathbb{D}(0, 1)$ such that $\psi \circ g|_{\Gamma_0} = \lambda \cdot \psi$.

Denote $\Psi := \Phi \circ \psi^{-1} : \mathbb{D}(0, 1) \to S := \Phi(\Gamma_0)$ (see the diagram below). Then we have $\Psi(\lambda z) = \lambda\Psi(z)$ for every $z \in \mathbb{D}$.

$$
\begin{array}{ccc}
\mathbb{D}(0, 1) & \xrightarrow{\;\lambda\cdot\;} & \mathbb{D}(0, 1) \\
{\scriptstyle\psi}\big\uparrow & & \big\uparrow{\scriptstyle\psi} \\
\Gamma_0 & \xrightarrow{\;g\;} & \Gamma_0 \\
{\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\
S := \Phi(\Gamma_0) & \xrightarrow{\;\lambda\cdot\;} & S
\end{array}
$$

It follows that $\Psi(z) = \Psi'(0)z$ for every $z \in \mathbb{D}$. Therefore $\Phi|_{\Gamma_0} = \Psi'(0) \cdot \psi$ is a conjugacy conjugating $g|_{\Gamma_0}$ to $z \mapsto \lambda z$. In particular, $\Phi(\Gamma_0) = \mathbb{D}(0, R)$, $R = |\Psi'(0)| \in (0, +\infty)$ and $\Phi|_{\Gamma_0}$ is a biholomorphism. $\qquad \square$

PROPOSITION 3.16. *The analytic set Γ is smooth and and the map*

$$\Phi|_\Gamma : \Gamma \to \Phi(\Gamma) = \mathbb{D}(0, R)$$

is biholomorphic with $R \in (0, +\infty)$.

Proof. It is enough to prove that $\Gamma = \Gamma_0$. From Lemma 3.15, we deduce that Γ_0 is simply connected. Note that $\Gamma_0 \subset \text{Reg }\Gamma$ is the complement of a discrete set $\Gamma \cap \text{Sing }\Sigma$ in Γ. We denote by $\hat{\Gamma}$ the normalization of Γ, a Riemann surface (see [**Chi89**]) and by $\widetilde{\Gamma}$ the universal covering of $\hat{\Gamma}$. Since $\Gamma_0 \subset \text{Reg }\Gamma$, the preimage $\hat{\Gamma}_0$ of Γ_0 by the normalization, which is isomorphic to Γ_0, is simply connected. Hence the preimage of $\hat{\Gamma}_0$ by the universal covering in $\widetilde{\Gamma}$ is a simply connected open subset in $\widetilde{\Gamma}$ with discrete complement. Then either $\widetilde{\Gamma}$ is biholomorphic to the unit disc (or $\mathbb{C}$) and $\Gamma_0 = \Gamma$ or $\widetilde{\Gamma}$ is biholomorphic to $\mathbb{CP}^1$ and $\Gamma \setminus \Gamma_0$ is only one point. Since $\pi|_\Gamma$ is a non-constant bounded holomorphic function valued in $\mathbb{C}^n$, the only case possible is that $\widetilde{\Gamma}$ is biholomorphic to a disc and $\Gamma_0 = \Gamma$. □

Thus, we obtain a biholomorphic map $\kappa := (\Phi|_\Gamma)^{-1} : \mathbb{D}(0, R) \to \Gamma \hookrightarrow M, \kappa(0) = [z]$ with $R \in (0, +\infty)$.

3.8. *End of the proof.* Denote $\tau = \pi \circ \kappa$. Note that $\tau(0) = z$. Since $\tau(\mathbb{D}(0, R)) \subset \pi(M)$ is bounded, by Fatou–Riesz's theorem (see [**Mil11**, Theorem A.3]) the radial limit

$$\tau_\theta = \lim_{r \to R^-} \pi \circ \kappa(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi)$.

Remark 3.17. This is another advantage that we mentioned in Remark 3.3.

PROPOSITION 3.18. *If τ_θ exists, then $\tau_\theta \in PC(f)$.*

Proof. Consider θ such that τ_θ exists and $\tau_\theta \notin PC(f)$, that is, $\tau_\theta \in X$. Note that

$$\gamma_R : [0, 1] \to X,$$

where $\gamma_R(t) = \tau(tRe^{i\theta})$, $\gamma_R(1) = \tau_\theta$ is a well-defined path in X starting at z and hence it defines an element in $\widetilde{X}$. Moreover, in $\widetilde{X}$, the family of paths $\{[\gamma_r]\}_{0 \leq r \leq R}$

$$\gamma_r : [0, 1] \to X,$$

where $\gamma_r(t) = \tau(tre^{i\theta})$, converges to $[\gamma_R]$ as $r \to R^-$. A quick observation is that, in $\widetilde{X}$, we have $[\gamma_r] = \kappa(re^{i\theta}) \subset M$ for every $r \in [0, 1)$. Since M is a closed submanifold of $\widetilde{X}$, $[\gamma_R] \in M$ or, in fact, $[\gamma_R] \in \Gamma$. Recall that $\Phi : \Gamma \to \mathbb{D}(0, R)$ is a biholomorphic mapping and hence

$$re^{i\theta} = \Phi([\gamma_r]) \xrightarrow{r \to R^-} \Phi([\gamma_R]) \in \mathbb{D}(0, R).$$

But $re^{i\theta} \xrightarrow{r \to R^-} Re^{i\theta} \notin \mathbb{D}(0, R)$, which yields a contradiction. Thus $\tau_\theta \in PC(f)$. □

Now, we denote by Q a defining polynomial of $PC(f)$. Then

$$Q \circ \tau : \mathbb{D}(0, R) \to \mathbb{C}$$

has vanishing radial limit $\lim_{r \to R^-} Q \circ \tau(re^{i\theta})$ for almost every $\theta \in [0, 2\pi)$. Then, $Q \circ \tau$ vanishes identically on $\mathbb{D}(0, R)$ (see [**Mil11**, Theorem A.3]). In particular, $Q \circ \tau(0) = Q(z) = 0$ and hence $z \in PC(F)$. It is a contradiction and our proof of Proposition 3.18 and Theorem 3.4 is complete.

4. *Periodic cycles in the regular locus: the transversal eigenvalue*

Now we consider a periodic point z of period m in the post-critical set of a PCA endomorphism f of $\mathbb{CP}^n$. Note that $f^{\circ m}$ is also PCA and $PC(f^{\circ m})$ is exactly $PC(f)$. It is enough to assume that z is a fixed point.

If z is a regular point of $PC(f)$, then $T_z PC(f)$ is well defined and it is a $D_z f$-invariant subspace of $T_z \mathbb{CP}^n$. On the one hand, it is natural to expect that our method of the previous case can be extended to prove that $D_z f|_{T_z PC(f)}$ has only repelling eigenvalues (it cannot have superattracting eigenvalues; see Remark 4.5 below). Unfortunately, there are some difficulties due to the existence of singularities of codimension higher than one that we cannot overcome easily. On the other hand, we are able to adapt our method to prove that the transversal eigenvalue with respect to $T_z PC(f)$, that is, the eigenvalue of $\overline{D_z f} : T_z \mathbb{CP}^n / T_z PC(f) \to T_z \mathbb{CP}^n / T_z PC(f)$, is repelling. More precisely, we will prove the following proposition.

PROPOSITION 4.1. *Let f be a PCA endomorphism of $\mathbb{CP}^n$ of degree $d \geq 2$ and let $z \in \mathrm{Reg}\, PC(f)$ be a fixed point. Then the eigenvalue of the linear map $\overline{D_z f} : T_z \mathbb{CP}^n / T_z PC(f) \to T_z \mathbb{CP}^n / T_z PC(f)$ is either repelling or superattracting.*

By Remark 3.3, it is equivalent to prove the following proposition.

PROPOSITION 4.2. *Let f be a PCA non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^n$ of degree $d \geq 2$ and let $z \in \mathrm{Reg}\, PC(f)$ be a fixed point. Then the eigenvalue of the linear map $\overline{D_z f} : T_z \mathbb{C}^n / T_z PC(f) \to T_z \mathbb{C}^n / T_z PC(f)$ is either repelling or superattracting.*

Since $PC(f)$ has codimension one, $\overline{D_z f}$ has exactly one eigenvalue and we denote it by λ. The value λ is also an eigenvalue of $D_z f$. The proof of Proposition 4.2 will occupy the rest of this section.

4.1. *Strategy of the proof.* Denote by $X = \mathbb{C}^n \setminus PC(f)$.

Step 1. We first prove that if $\lambda \neq 0$, then $|\lambda| \geq 1$ and z is not a critical point. Then we prove that $|\lambda| = 1$ will lead to a contradiction. By assuming that $|\lambda| = 1$, following from the discussion in §2, there exists an eigenvector v of $D_z f$ corresponding to λ such that $v \notin T_z PC(f)$.

Our goal is to build a holomorphic map $\tau : \mathbb{D}(0, R)^* \to X$ such that τ can be extended holomorphically to $\mathbb{D}(0, R)$ so that $\tau(0) = z$, $\tau'(0) = v$. Then, we show that τ has radial

limit almost everywhere and these radial limits land on $PC(f)$ whenever they exist. The construction of τ occupies Steps 2–6 and the contradiction will be deduced in Step 7.

Step 2. We construct a connected complex manifold $\widetilde{X}$ of dimension n with two holomorphic maps $\pi : \widetilde{X} \to X$, $g : \widetilde{X} \to \widetilde{X}$ such that

$$f \circ \pi \circ g = \pi.$$

Step 3. We prove that $\{g^{\circ j}\}_j$ is a normal family. Then we extract a subsequence $\{g^{\circ j_k}\}_k$ converging to a retraction $\rho : \widetilde{X} \to \widetilde{X}$, that is, $\rho \circ \rho = \rho$.

Step 4. We will study $M = \rho(\widetilde{X})$. More precisely, we will prove that $\pi(M)$ can be extended to a center manifold of f at z.
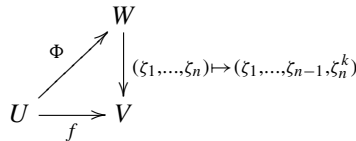
Step 5. We construct a holomorphic map $\Phi : M \to E_n$ which semi-conjugates g to the restriction of $(D_z f)^{-1}$ to the neutral eigenspace E_n (see Proposition 3.2.c).

Step 6. We prove that there exists an irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ which is smooth and biholomorphic to the punctured disc. More precisely, we prove that $\Phi(\Gamma) = \mathbb{D}(0, R)^*$ with $R \in (0, +\infty)$ and the map $\tau := \pi \circ (\Phi|_\Gamma)^{-1}$ extends to a holomorphic map from $\mathbb{D}(0, R)$ to $\mathbb{C}^n$ so that $\tau(0) = z$, $\tau'(0) = v$.

Step 7. We prove that the map τ has radial limit almost everywhere and that the limit belongs to $PC(f)$ if it exists. This implies that $\pi \circ \tau \subset PC(f)$, which contradicts the fact that $v \notin T_z PC(f)$. This means that the assumption $|\lambda| = 1$ is false and thus Proposition 4.2 is proved.

4.2. *Existence of the transversal eigenvector.* We recall the following result due to Grauert.

PROPOSITION 4.3. [**GR58**, Satz 10] *Let U, V be an open neighborhood of zero in $\mathbb{C}^n$ and let $f : U \to V$ be a holomorphic branched covering of order k ramifying over $V_f = \{\zeta_n = 0\} \cap V$. Then there exists a biholomorphism $\Phi : U \to W$ such that the following diagram commutes.*

$$
\begin{array}{ccc}
 & W & \\
{\scriptstyle \Phi}\nearrow & \Big\downarrow {\scriptstyle (\zeta_1,\ldots,\zeta_n) \mapsto (\zeta_1,\ldots,\zeta_{n-1},\zeta_n^k)} & \\
U \xrightarrow{\;\;f\;\;} & V &
\end{array}
$$

In particular, the branched locus $B_f = \Phi^{-1}(\{\zeta_n = 0\} \cap W)$ is smooth and $f|_{B_f} : B_f \to V_f$ is a biholomorphism.

This is, in fact, a local statement and we can apply it to a PCA non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^n$ to obtain the following proposition.

PROPOSITION 4.4. [**Ued98**, Lemma 3.5] *Let f be a PCA non-degenerate homogeneous polynomial endomorphism of $\mathbb{C}^n$ of degree $d \geq 2$. Then*

$$f^{-1}(\operatorname{Reg} PC(f)) \subset \operatorname{Reg} PC(f)$$

and

$$f : f^{-1}(\operatorname{Reg} PC(f)) \to \operatorname{Reg} PC(f)$$

is locally a biholomorphism.

Remark 4.5. In particular, Proposition 4.4 implies that if f has a fixed point $z \in$ Reg $PC(f)$, then $D_z f|_{T_z PC(f)}$ is invertible. Hence $D_z f|_{T_z PC(f)}$ does not have any superattracting eigenvalues.

If $\lambda \neq 0$, then z is not a critical point. By Proposition 3.2, the modulus of λ is at least one. Then we will prove Proposition 4.1 by contradiction by assuming that $|\lambda| = 1$. If $|\lambda| = 1$, there exists an associated eigenvector v of $D_z f$ such that $v \notin T_z PC(f)$. Indeed, note that

$$\mathrm{Spec}(D_z f) = \mathrm{Spec}(D_z f|_{T_z PC(f)}) \cup \mathrm{Spec}(\overline{D_z f}),$$

where $\mathrm{Spec}(\overline{D_z f})$ has only one eigenvalue λ of modulus one. Then the repelling eigenspace E_r is included in $T_z PC(f)$. The diagonalizability of $D_z f|_{E_n}$ implies that E_n is generated by a basis of eigenvectors. The vector v is such an eigenvector which is not in $T_z PC(f)$.

4.3. (X, z)-homotopy and related constructions.   Denote $X = \mathbb{C}^n \setminus PC(f)$.

4.3.1. *Construction of $\widetilde{X}$.*   We construct a complex manifold $\widetilde{X}$, a covering map $\pi : \widetilde{X} \to X$ and a holomorphic map $g : \widetilde{X} \to \widetilde{X}$ such that

$$f \circ \pi \circ g = \pi.$$

Denote by

$$\Xi = \{\gamma : [0, 1] \to \mathbb{C}^n \text{ continuous map such that } \gamma(0) = z, \gamma((0, 1]) \subset X\}$$

the space of paths starting at z and varying in X. Let $\gamma_0, \gamma_1 \in \Xi$. We say that γ_0 and γ_1 are (X, z)-*homotopic* if there exists a continuous map $H : [0, 1] \times [0, 1] \to \mathbb{C}^n$ such that

$$H(0, s) = z, \, H(1, s) = \gamma_0(1) = \gamma_1(1),$$

$$H(t, 0) = \gamma_0(t), \, H(t, 1) = \gamma_1(t),$$

$$H(t, s) \subset X \text{ for all } t \neq 0.$$

Denote $\gamma_0 \sim_X \gamma_1$ if γ_0 and γ_1 are (X, z)-homotopic (see Figure 1). In other words, γ_0 and γ_1 are homotopic by a homotopy of paths $\{\gamma_t, t \in [0, 1]\}$ such that $\gamma_t \in \Xi$ for every t. It is easy to see that (X, z)-homotopy is an equivalence relation on Ξ. Denote by $\widetilde{X}$ the quotient space of Ξ by this relation and by $[\gamma]$ the equivalent class of $\gamma \in \Xi$. Denote the projection by

$$\pi : \widetilde{X} \to X, \pi([\gamma]) = \gamma(1).$$

We endow $\widetilde{X}$ with a topology constructed in the same way as the topology of a universal covering. More precisely, let $\mathfrak{B}$ be the collection of simply connected open subsets of X. Note that $\mathfrak{B}$ is a basis for the usual topology of X. We consider the topology on $\widetilde{X}$ which is defined by a basis of open subsets $\{U_{[\gamma]}\}_{U \in \mathfrak{B}, [\gamma] \in X}$, where $\gamma(1) \in U$ and

$$U_{[\gamma]} = \{[\gamma * \alpha] | \alpha \text{ is a path in } U \text{ starting at } \gamma(1)\}.$$

We can transport the complex structure of X to $\widetilde{X}$ and this will make $\widetilde{X}$ a complex manifold of dimension n. Note that π is also a holomorphic covering map.
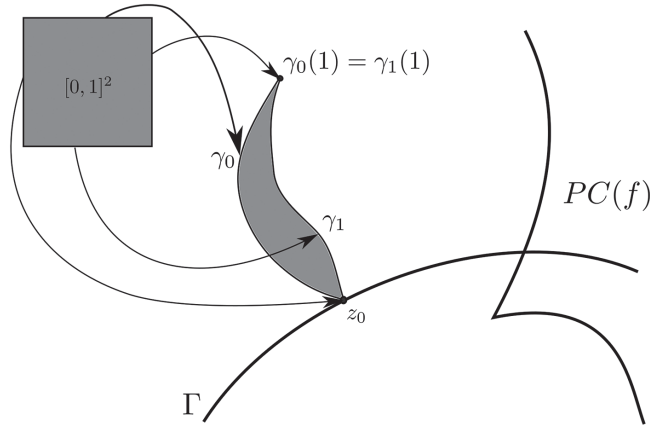
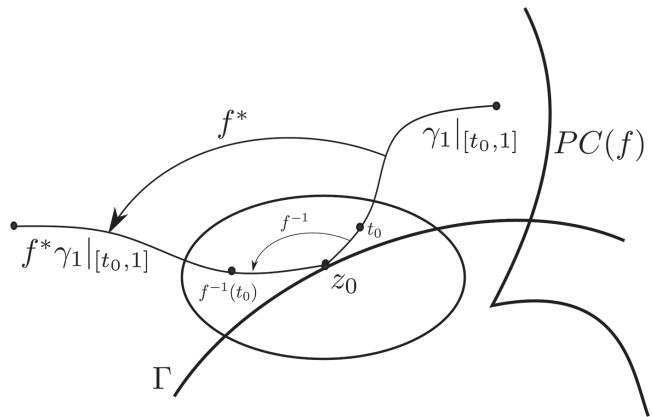FIGURE 1. Two paths γ_0, γ_1 which are (X, z)-homotopic.

FIGURE 2. Pulling back of an element in Ξ.

4.3.2. *Lifts of inverse branches of f.* We will construct a holomorphic mapping

$$g : \widetilde{X} \to \widetilde{X}$$

which is induced by the pull-back action of f on paths in Ξ.

LEMMA 4.6. *Let γ be a path in Ξ. Then there exists a unique path $f^*\gamma \in \Xi$ such that $f \circ f^*\gamma = \gamma$.*

Proof. Since f is locally invertible at z and since $f^{-1}(X) \subset X$, there exists $t_0 \in [0, 1]$ such that $\gamma|_{[0,t_0]} \in \Xi$ and $f^{-1} \circ \gamma|_{[0,t_0]}$ is a well-defined element in Ξ. Then the path $f^*\gamma$ is the concatenation of $f^{-1} \circ \gamma|_{[0,t_0]}$ with the lifting $f^*\gamma|_{[t_0,1]}$ of the path $\gamma|_{[t_0,1]}$ by the covering $f : f^{-1}(X) \to X$ (see Figure 2). This construction does not depend on the choice of t_0. $\qquad\square$

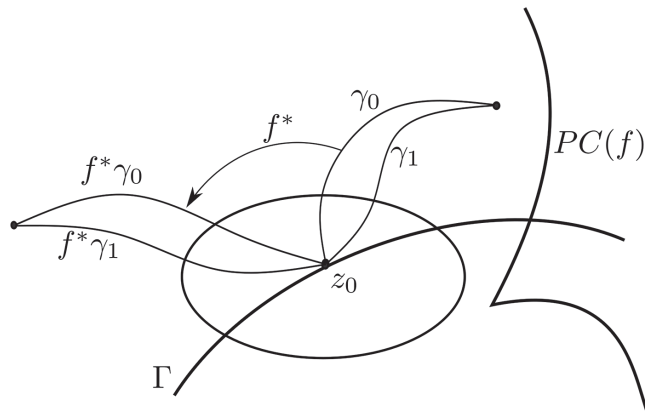LEMMA 4.7. *Let γ_0, $\gamma_1 \in \Xi$. If $[\gamma_0] = [\gamma_1]$, then $[f^*\gamma_0] = [f^*\gamma_1]$.*

FIGURE 3. Pull-back preserves the (X, z)-homotopic paths.

Proof. Lemma 4.6 implies that the pull-back of a homotopy of paths in Ξ between γ_0 and γ_1 is a homotopy of paths in Ξ between $f^*\gamma_0$ and $f^*\gamma_1$ (see also Figure 3). ☐

The two previous lemmas allow us to define a map $g : \widetilde{X} \to \widetilde{X}$ as

$$g([\gamma]) = [f^*\gamma].$$

Then $f \circ \pi \circ g = \pi$.

4.3.3. *The connectedness of $\widetilde{X}$.*

The connectedness of $\widetilde{X}$ is not obvious from the construction. We will introduce the notion of a *regular neighborhood*, which is not only useful for proving that $\widetilde{X}$ is connected but will also be very important later.

Definition 4.8. A bounded open subset W of $\mathbb{C}^n$ containing z is called a *regular neighborhood* of z if:
(1) $(W, W \cap PC(f))$ is homeomorphic to a cone over $(\partial W, \partial W \cap PC(f))$ with a vertex at z; and
(2) for every path $\gamma_0, \gamma_1 \in \Xi$ such that $\gamma_0([0, 1]), \gamma_1([0, 1]) \subset W$ and $\gamma_0(1) = \gamma_1(1)$. Then $\gamma_0 \sim_X \gamma_1$.

Let W be an open subset of $\mathbb{C}^n$ containing z. Set

$$\widetilde{W} = \{[\gamma] | \gamma \in \Xi, \gamma((0, 1]) \subset W\}.$$

LEMMA 4.9. *If W is a regular neighborhood of z, then $\pi : \widetilde{W} \to W \setminus PC(f)$ is a biholomorphism.*

Proof. We can observe that $\widetilde{W}$ is open. Indeed, for an element $[\gamma]$ in $\widetilde{W}$, let U be an open set in $W \setminus PC(f)$ containing $\gamma(1)$. Then $U_{[\gamma]} \subset \widetilde{W}$. The projection $\pi|_{\widetilde{W}} : \widetilde{W} \to W \setminus PC(f)$ is surjective since $W \setminus PC(f)$ is path-connected. So we need to prove that $\pi : \widetilde{W} \to W \setminus PC(f)$ is injective. Indeed, let z be a point in $W \setminus PC(f)$ and let $[\gamma_0], [\gamma_1]$ be two elements in $\widetilde{W}$ such that $\pi([\gamma_0]) = \pi([\gamma_1]) = z$, that is, $\gamma_0(1) = \gamma_1(1)$.

Since W is regular, we have $\gamma_0 \sim_X \gamma_1$ or $[\gamma_0] = [\gamma_1]$. So $\pi|_{\widetilde{W}}$ is injective and hence biholomorphic. $\qquad\square$

In particular, $\widetilde{W}$ is path-connected. If $\gamma : [0, 1] \to X$ is a path in Ξ, then the path $\gamma_s : [0, 1] \to \mathbb{C}^n$ defined by $\gamma_s(t) = \gamma(t(1 - s))$ also belongs to Ξ and $\gamma_s([0, 1]) = \gamma([0, 1 - s])$. It follows that every element in $\widetilde{X}$ can be joined by paths to an element in $\widetilde{W}$ and thus $\widetilde{X}$ is path-connected and hence connected.

Now we prove that we can indeed find a regular neighborhood when z is a regular point of $PC(f)$. Let $(\zeta_1, \ldots, \zeta_n)$ be local coordinates vanishing at z in which $PC(f)$ is given by $\{\zeta_1 = 0\}$. Let U be the unit polydisc centered at z.

PROPOSITION 4.10. *Any polydisc centered at z in U is a regular neighborhood.*

Proof. Let γ_0 and γ_1 be two elements of Ξ such that $\gamma_0([0, 1]), \gamma_1([0, 1]) \in U$ and $\gamma_0(1) = \gamma_1(1)$. Consider the loop $\eta = \gamma_0 * (-\gamma_1)$ and the continuous map $H : [0, 1] \times [0, 1] \to \mathbb{C}^n$ defined by

$$H(t, s) = s \cdot \eta(t).$$

The loop η bounds $H([0, 1] \times [0, 1] \setminus \{(0, 0)\}) \subset X$, which implies that $\gamma_0 \sim_X \gamma_1$. $\qquad\square$

Remark 4.11. The construction above also implies that z admits a basis of neighborhoods consisting of regular neighborhoods.

4.3.4. *Dynamics of g on regular neighborhoods.* Let W be a regular neighborhood of z and let

$$\sigma : W \setminus PC(f) \to \widetilde{W}$$

be the inverse of $\pi|_{\widetilde{W}}$. Note that if W is constructed as above, then, by Proposition 3.2(b), there exists a family of holomorphic maps $h_j : W \to \mathbb{C}^n$, $j \geq 1$ such that

$$h_j(z) = z, \ f^{\circ j} \circ h_j = \mathrm{Id}_W .$$

We can deduce from the definition of g that, for every $j \geq 1$, $\pi \circ g^{\circ j} \circ \sigma = h_j|_{W \setminus PC(f)}$. More precisely, let $[\gamma] \in \widetilde{W}$, that is, $\gamma((0, 1]) \subset W \setminus PC(f)$. Then, by definition, for every $j \geq 1$,

$$g^{\circ j}([\gamma]) = [h_j \circ \gamma]. \tag{4.1}$$

Recall that the family $\{h_j : W \to \mathbb{C}^n\}_j$ is normal. The assumption $|\lambda| = 1$ allows us to control the value taken by any limit maps of this family. Note that $f^{-1}(X) \subset X$ and hence $h_j(W \setminus PC(f)) \subset X$ for every $j \geq 1$.

LEMMA 4.12. *Let $h = \lim_{s \to \infty} h_{j_s}$ be a limit map of $\{h_j : W \to \mathbb{C}^n\}_j$. Then*

$$h(W \setminus PC(f)) \subset X.$$

Proof. Recall that $PC(f)$ is the zero locus of a polynomial $Q : \mathbb{C}^n \to \mathbb{C}$. Then

$$Q \circ h = \lim_{s \to +\infty} Q \circ h_{j_s}.$$

Since $h_{j_s}(W \setminus PC(f)) \subset X$, the map $Q \circ h|_{W \setminus PC(f)}$ is the limit of a sequence of non-vanishing holomorphic functions. By Hurwitz's theorem, $Q \circ h|_{W \setminus PC(f)}$ is either a non-vanishing function or identically zero, that is, either $h(W \setminus PC(f)) \subset X$ or $h(W \setminus PC(f)) \subset PC(f)$.

Let v be an eigenvector of $D_z f$ associated to the eigenvalue λ. Then $D_z h_j(v) = (1/\lambda^j)v$. Hence

$$D_z h(v) = \frac{1}{\lambda'}v$$

for some limit value λ' of $\{\lambda^j\}_j$. Since we assumed that $|\lambda| = 1$, we have $|\lambda'| = 1$. The fact that $v \notin T_z PC(f)$ implies that $D_z h(v) \notin T_z PC(f)$. Consequently,

$$h(W \setminus PC(f)) \cap X \neq \emptyset$$

and thus $h(W \setminus PC(f)) \subset X$. □

4.4. *Normality of the family of maps lifted via the relative homotopy.* We will prove that $\{g^{\circ j} : \widetilde{X} \to \widetilde{X}\}_j$ is normal. Following 3.4, it is enough to prove the following two lemmas.

LEMMA 4.13. *The family $\{k_j = \pi \circ g^{\circ j} : \widetilde{X} \to X\}_j$ is normal and any limit map can be lifted by π to a holomorphic endomorphism of $\widetilde{X}$.*

Proof. Note that $\{k_j : \widetilde{X} \to \mathbb{C}^n\}_j$ is locally uniformly bounded and hence is normal (see Proposition 3.9). Consider a limit map k of this family. By using Hurwitz's theorem, we deduce that either $k(\widetilde{X}) \subset X$ or $k(\widetilde{X}) \subset PC(f)$.

Let W be a regular neighborhood of z. Then there exists a family $\{h_j : W \to \mathbb{C}^n\}_j$ of $f^{\circ j}$ fixing z (see 4.3.4). We have

$$k_j|_{\widetilde{W}} \circ \sigma = h_j|_{W \cap X},$$

where $\sigma : W \setminus PC(f) \to \widetilde{W}$ is the section of $\pi|_{\widetilde{W}}$. Therefore $k|_{\widetilde{W}} \circ \sigma$ is a limit map of $\{h_j|_{W \setminus PC(f)}\}$. Lemma 4.12 implies that $k(\widetilde{W}) \subset X$, and hence, $k(\widetilde{X}) \subset X$. Thus $\{k_j : \widetilde{X} \to X\}_j$ is normal and any limit map takes values in X.

We now show that the map $k : \widetilde{X} \to X$ can be lifted to a map from $\widetilde{X}$ to $\widetilde{X}$. Set

$$h := k|_{\widetilde{W}} \circ \sigma.$$

Then h is a limit map of $\{h_j|_{W \setminus PC(f)}\}$. For each element $[\gamma] \in \widetilde{X}$, we denote by η the image of γ under the analytic continuation of h along γ. Note that h fixes z. Thus, Lemma 4.12 implies that $\eta \in \Xi$. This construction does not depends on the choice of γ in the equivalence class $[\gamma]$. Thus, the map

$$\begin{array}{cccc} \widetilde{k}: & \widetilde{X} & \to & \widetilde{X} \\ & [\gamma] & \mapsto & [\eta] \end{array}$$

is well defined. The map $\pi \circ \widetilde{k}$ coincides with k on an open set $\widetilde{W}$ in $\widetilde{X}$ and hence coincides with k on $\widetilde{X}$. In other words, $\widetilde{k}$ is a lifted map of k by π. □

Hence we deduce the following proposition.

PROPOSITION 4.14. *The family $\{g^{\circ j} : \widetilde{X} \to \widetilde{X}\}_j$ is normal.*

Proof. Let $\{g^{\circ j_s}\}_s$ be sequence of iterates of g. Extracting subsequences, if necessary, we can assume that $\{k_{j_s}\}_s$ converges locally uniformly to a holomorphic map $k : \widetilde{X} \to X$. By Lemma 4.13, there exists a holomorphic map $\widetilde{k} : \widetilde{X} \to \widetilde{X}$ so that $\pi \circ \widetilde{k} = k$. We prove that $\{g^{\circ j_s}\}_s$ converges locally uniformly to $\widetilde{k}$. Applying [AC03, Theorem 4], it is enough to prove that there exists an element $[\gamma] \in \widetilde{X}$ such that $g^{\circ j_s}([\gamma])$ converges to $\widetilde{k}([\gamma])$.

We consider a regular neighborhood W of z and the family $\{h_j : W \to \mathbb{C}^n\}_j$ of $f^{\circ j}$ fixing z (see 4.3.4). Let $[\gamma]$ be an element in $\widetilde{W}$† and an associated path be $\widetilde{\gamma} : [0, 1] \to \widetilde{X}, \widetilde{\gamma}(t) = [\gamma|_{[0,t]}]$ in $\widetilde{X}$. Since $[\gamma] \in \widetilde{W}$, $\widetilde{\gamma} = \sigma \circ \gamma$. Then

$$\widetilde{k}([\gamma]) = [k \circ \widetilde{\gamma}] = \lim_{s \to \infty} [k_{j_s} \circ \widetilde{\gamma}]$$
$$= \lim_{s \to \infty} [h_{j_s} \circ \gamma] = \lim_{s \to \infty} g^{\circ j_s}([\gamma]).$$

Thus we conclude the proof of the proposition. □

Following [Aba89, Corollary 2.1.29], the normality of $\{g^{\circ j}\}_j$ implies that:
- there exists a subsequence $\{g^{\circ j_k}\}_k$ converging to a holomorphic retraction

$$\rho : \widetilde{X} \to \widetilde{X}$$

  of $\widetilde{X}$, that is, $\rho \circ \rho = \rho$;
- by [Car86], the image $M = \rho(\widetilde{X})$ is a closed submanifold of $\widetilde{X}$; and
- by [Aba89, Corollary 2.1.31], M is invariant by g and $g|_M$ is an automorphism.

4.5. *Existence of the center manifold.* We will study the dynamics of g restricted on M. The difference between the construction of the universal covering used in the first case (the fixed point is outside $PC(f)$) and the construction of $\widetilde{X}$ in this case is that $\widetilde{X}$ does not contain a point representing z. Hence it is not straightforward that we can relate the dynamics of g on M with the dynamics of f near z.

We consider the objects introduced in §4.3.4. In particular, we consider a regular neighborhood W of z in $\mathbb{C}^n$ and the family $\{h_j : W \to \mathbb{C}^n\}_j$ of inverse branches fixing z of $f^{\circ j}$ on W. Recall that

$$\sigma : W \setminus PC(f) \to \tilde{W}$$

is the inverse of the biholomorphism $\pi : \tilde{W} \to W \setminus PC(f)$ and that $\lim_{k \to \infty} g^{\circ j_k} = \rho$ is a holomorphic retraction on $\widetilde{X}$.

Define a holomorphic map $\widetilde{H} : W \setminus PC(f) \to \mathbb{C}^n$ as

$$\widetilde{H} = \pi \circ \rho \circ \sigma = \lim_{k \to +\infty} \pi \circ g^{\circ j_k} \circ \sigma.$$

By (4.1), we have $\widetilde{H} = \lim_{k \to +\infty} h_{j_k}|_{W \setminus PC(f)}$. Since $\{h_j : W \to \mathbb{C}^n\}_j$ is normal, by passing to subsequences, we can extend $\widetilde{H}$ to a holomorphic map $H : W \to \mathbb{C}^n$ such that $H = \lim_{k \to +\infty} h_{j_k} : W \to \mathbb{C}^n$.

---

† In §3.4, we choose an element representing z. Such an element does not exist in this case but it is enough to consider an element representing a point near z.

Note that $h_j(z) = z$ for every $j \geq 1$. Then $H(z) = z$. By continuity of H, there exists an open neighborhood U of z in W such that $H(U) \subset W$. Note that we choose U to be a regular neighborhood of z (see Remark 4.11) and we can shrink U whenever we need to. Recall that, for every $[\gamma] \in \widetilde{W}$, we have $g^{\circ j}([\gamma]) = [h_j \circ \gamma]$. Then for $[\gamma] \in \widetilde{U} := \sigma(U \setminus PC(f))$,

$$\rho([\gamma]) = \lim_{k \to \infty} g^{\circ j_k}([\gamma]) = \lim_{k \to \infty} [h_{j_k} \circ \gamma] = [H \circ \gamma] \subset \widetilde{W}.$$

In other words, $\rho(\widetilde{U}) \subset \widetilde{W}$. Hence

$$H(U \setminus PC(f)) = \pi \circ \rho \circ \sigma(U \setminus PC(f)) \subset W \setminus PC(f). \qquad (4.2)$$

Moreover, since $\sigma \circ \pi|_{\widetilde{W}} = \mathrm{Id}_{\widetilde{W}}$, the composition

$$\widetilde{H} \circ \widetilde{H} = \pi \circ \rho \circ \sigma \circ \pi \circ \rho \circ \sigma$$

is well defined on $U \setminus PC(f)$ and is equal to $\widetilde{H}|_{U \setminus PC(f)}$. Since H is the extension of $\widetilde{H}$, we deduce that

$$H \circ H(U) = H(U).$$

PROPOSITION 4.15. *The set $H(U)$ is a submanifold of W containing z whose dimension is the number of neutral eigenvalues of $D_z f$ counted with multiplicities. Moreover, $T_z H(U) = E_n$ and $D_z f|_{T_z H(U)}$ is diagonalizable.*

Proof. The first assertion is due to [Car86] since $H \circ H = H$ on U. The rest are consequences of the fact that H is a limit map of the family $\{h_j : W \to \mathbb{C}^n\}_j$ of inverse branches fixing z of f (see Corollary 2.5). $\qquad \square$

LEMMA 4.16. $\dim M = \dim H(U)$.

Proof. Since $\widetilde{X}$ is connected, M is also a connected complex manifold. Thus $\dim M = \mathrm{rank}\, D_x \rho$ for every $x \in \widetilde{M}$. In particular, if we choose $x = \sigma(z)$ with $w \in U \setminus PC(f)$, then

$$\mathrm{rank}\, D_x \rho = \mathrm{rank}_w H = \dim H(U)$$

and thus $\dim M = \dim H(U)$. $\qquad \square$

In other words,

$$M_X := \pi(M) \cup H(U)$$

is a submanifold of $\mathbb{C}^n$ in a neighborhood of z. Moreover, following Proposition 3.9, we can deduce that $\pi(M)$ is a bounded set in $\mathbb{C}^n$.

4.6. *Semi-conjugacy on the center manifold.* Denote $\Lambda = (D_z f|_{E_n})^{-1}$. We will construct a holomorphic $\Phi : M \to E_n$ such that

$$\Phi \circ g|_M = \Lambda \circ \Phi.$$

The construction follows the idea in §3.6 and the connection established in Step 3 between g and inverse branches of f at z.

LEMMA 4.17. *There exists a holomorphic map $\Phi : M \to E_n$ such that $\Phi \circ g = \Lambda \circ \Phi$.*

Proof. We consider a holomorphic map $\varphi : M \to E_n$ constructed as the composition

$$M \hookrightarrow \widetilde{X} \xrightarrow{\pi} \mathbb{C}^n \xrightarrow{\delta} T_z \mathbb{C}^n \xrightarrow{\pi_{E_r}} E_n,$$

where $\delta : \mathbb{C}^n \to T_z \mathbb{C}^n$ is a holomorphic map such that $\delta(z) = 0$, $D_z \delta = \mathrm{Id}$, $\pi_{E_r} : T_z \mathbb{C}^n \to E_n$ is a projection on E_n parallel to E_r. Note that, since $\pi(M)$ is bounded, $\varphi(M)$ is also a bounded set in E_n.

We consider the family

$$\Lambda^{-j} \circ \varphi \circ g^{\circ j} : M \to E_n, \, j \geq 0.$$

Since Λ is diagonalizable with only neutral eigenvalues, this family is uniformly bounded and hence so is the family of its Cesàro averages $\{\Phi_N = (1/N) \sum_{j=0}^{N-1} \Lambda^{-j} \circ \varphi \circ g^{\circ j}\}_N$. Therefore, $\{\Phi_N\}_N$ is normal and every limit map Φ of $\{\Phi_N\}_N$ satisfies that

$$\Phi \circ g = \Lambda \circ g.$$

Note that $\Phi(M)$ is also a bounded set in E_n. $\qquad \square$

Let us fix such a limit map $\Phi = \lim_{k \to \infty} \Phi_{N_k}$. We prove that Φ restricted to $\widetilde{U} \cap M$ is a biholomorphism. In order to do so, we consider the holomorphic function

$$\Phi_1 := \Phi \circ \sigma|_{H(U)} : H(U \setminus PC(f)) \to \mathbb{C}^n.$$

Since $H(U \setminus PC(f)) \subset \pi(M)$ is a bounded set in $\mathbb{C}^n$, the map Φ_1 is bounded and hence we can extend it to a holomorphic function on $H(U)$. By an abuse of notation, we denote the extension by Φ_1. We prove that Φ_1 is invertible in a neighborhood of z in $H(U)$.

More precisely, on $H(U) \setminus PC(f)$,

$$\Phi_1 \quad = \quad \lim_{k \to \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi \circ g^{\circ j} \circ \sigma|_{h(U) \setminus PC(f)}.$$

Consider the map $\varphi_1 : \pi(M) \cup H(U) \to E_n$, $\varphi_1 = \pi_{E_r} \circ \delta|_{M_X}$. Then $\varphi_1(z) = 0$, $D_z \varphi_1 = \mathrm{Id}$ and $\varphi = \varphi_1|_{\pi(M)} \circ \pi$. It follows that

$$\Phi_1 \quad = \quad \lim_{k \to \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi_1|_{\pi(M)} \circ \pi \circ g^{\circ j} \circ \sigma|_{H(U) \setminus PC(f)}$$

$$= \quad \lim_{k \to \infty} \frac{1}{N_k} \sum_{j=0}^{N_k-1} \Lambda^{-j} \circ \varphi_1|_{\pi(M)} \circ h_j|_{H(U) \setminus PC(f)}.$$

Note that $D_z h_j|_{H(U)} = \Lambda^j$. Then we can deduce that $\Phi_1(z) = 0$ and $D_z \Phi_1 = \mathrm{Id}$. Therefore, there exists a regular neighborhood V of z in W such that Φ_1 is biholomorphic on $V \cap H(U)$. Consequently, since σ is a biholomorphism, the neighborhood V induces an open neighborhood $\widetilde{V} = \sigma(V \setminus PC(f))$ in $\widetilde{W}$ such that Φ is biholomorphic on $\widetilde{V} \cap \mathcal{M}$. By shrinking U, we can assume that $V = U$ and hence $\Phi|_{\widetilde{U} \cap M}$ is a biholomorphism.
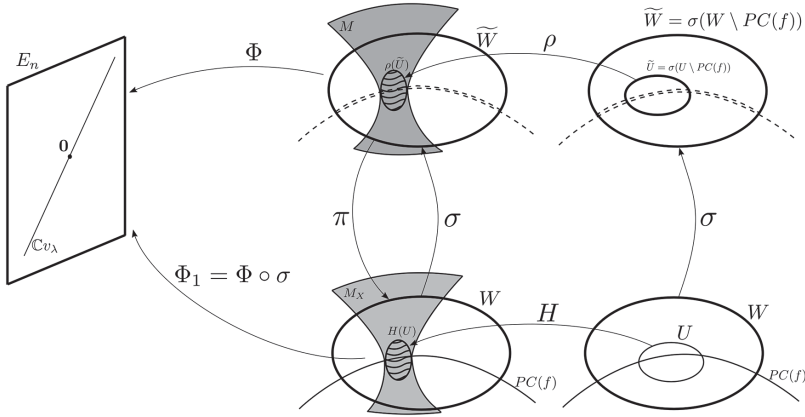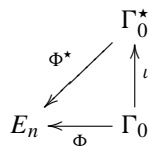
FIGURE 4. Constructions on M.

4.7. *Linearization along the neutral direction.* The map Φ_1 extends the image of Φ in the sense that $\Phi(M) \cup \Phi_1(H(U))$ contains a full neighborhood of zero in E_n. Let $v \in E_n$ be an eigenvector of $D_z f$ associated to λ. We will study $\Phi^{-1}(\mathbb{C}v)$ by studying $\Phi_1^{-1}(\mathbb{C}v)$.

Denote by Γ_1 the irreducible component of $\Phi_1^{-1}(\mathbb{C}v)$ containing z. Since $D_z\Phi_1 = \mathrm{Id}$, Γ_1 is a submanifold of dimension one of $H(U)$ near z and $T_z\Gamma_1 = \mathbb{C}v$. Note that $v \notin T_z PC(f)$. Then by shrinking U, if necessary, we can assume that $\Gamma_1 \cap PC(f) = \{z\}$. In other words, $\Gamma_1 \setminus \{z\}$ is a smooth component of $\Phi_1^{-1}(\mathbb{C}v)$ in $H(U) \setminus PC(f)$.

Since $\Phi_1 = \Phi \circ \sigma|_{H(U)\setminus PC(f)}$ and σ is a biholomorphism, there exists a unique irreducible component Γ of $\Phi^{-1}(\mathbb{C}v)$ such that Γ contains $\sigma(\Gamma_1 \setminus \{z\})$. Moreover, $\Phi(\Gamma)$ is a punctured neighborhood of zero in $\mathbb{C}v$. This means that $0 \notin \Phi(\Gamma)$ but $\Phi(\Gamma) \cup \{0\}$ contains an open neighborhood of zero in $\mathbb{C}v$. We will prove that Γ is, in fact, biholomorphic to a punctured disc and that $\Phi|_\Gamma$ is a biholomorphism conjugating $g|_\Gamma$ to the irrational rotation $\zeta \mapsto \lambda\zeta$ (see Figure 4).

Following §3.7, we consider $\Gamma_0 = \Gamma \setminus C_\Phi$, where C_Φ the set of critical points of Φ. Then Γ_0 is a hyperbolic Riemann surface which is invariant by g. The map g induces an automorphism $g|_{\Gamma_0}$ on Γ_0 such that $g|_{\Gamma_0}^{\circ j_k}$ converges to $\rho = \mathrm{Id}_M$, which is identity on Γ_0.

On the one hand, Γ_0 contains $\sigma(\Gamma_1 \setminus PC(f))$ and hence $\Phi(\Gamma_0)$ is also a punctured neighborhood of zero in $\mathbb{C}v$. On the other hand, g restricted on $\sigma(\Gamma_1 \setminus PC(f))$ is conjugate to h restricted on $\Gamma_1 \setminus PC(f)$. Note that h fixes $z = \Gamma_1 \cap PC(f)$. Hence we can consider an abstract Riemann surface $\Gamma_0^\star = \Gamma_0 \cup \{z\}$ and two holomorphic maps $\iota : \Gamma_0 \to \Gamma_0^\star$, $\Phi^\star : \Gamma_0^\star \to \mathbb{C}v \subset E_n$ so that ι is an injective holomorphic map, $\Gamma_0^\star \setminus \iota(\Gamma_0) = \{z\}$, $\Phi^\star(z) = 0$ and the following diagram commutes.

$$
\begin{array}{ccc}
 & & \Gamma_0^\star \\
 & {\scriptstyle\Phi^\star}\nearrow & \uparrow{\scriptstyle\iota} \\
E_n & \xleftarrow{\;\;\Phi\;\;} & \Gamma_0
\end{array}
$$

Moreover, $\Gamma_0^{\star}$ admits an automorphism $g^{\star}$ fixing the point z with multiplier λ and extends $g|_{\Gamma_0}$ in the sense that $g^{\star} \circ \iota = \iota \circ g$. Note that $\Phi^{\star}(\Gamma_0^{\star}) = \Phi(\Gamma_0) \cup \{0\}$ is bounded in E_n. Then, by arguing as in Lemma 3.15, we deduce the following Lemma 4.18.

LEMMA 4.18. *The Riemann surface $\Gamma_0^{\star}$ is biholomorphic to a disc $\mathbb{D}(0, R)$, $R \in (0, +\infty)$ and $\Phi^{\star} : \Gamma_0^{\star} \to \Phi^{\star}(\Gamma_0^{\star}) = \mathbb{D}(0, R)$, $\Phi^{\star}(z) = 0$ is a biholomorphism conjugating $g^{\star}$ to the irrational rotation $\zeta \to \lambda\zeta$.*

Consequently, Γ_0 is biholomorphic to $\mathbb{D}(0, R)$ and $\Phi|_{\Gamma}$ is a biholomorphism.

PROPOSITION 4.19. *The set Γ is smooth and the map*

$$\Phi|_{\Gamma} : \Gamma \to \Phi(\Gamma) = \mathbb{D}(0, R)^*$$

is a biholomorphism with $R \in (0, +\infty)$.

Proof. It is enough to prove that $\Gamma = \Gamma_0$. The idea is similar to the proof of Proposition 3.16

Note that Γ_0 is the complement of a discrete set $\Gamma \cap \mathrm{Sing}\, \Phi^{-1}(\mathbb{C}v)$ in Γ and Γ_0 is biholomorphic to a punctured disc $\mathbb{D}(0, R)^*$. Moreover, $\Phi(\Gamma_0) \subset \Phi(\Gamma)$ is also a punctured neighborhood of zero and $\Gamma_0 \subset \mathrm{Reg}\, \Gamma$ has discrete complement.

Then we can consider an abstract one-dimensional analytic space $\Gamma^{\star} = \Gamma \cup \{z\}$ such that $\Gamma_0^{\star} \subset \Gamma^{\star}$ and $\Gamma^{\star} \setminus \Gamma_0^{\star}$ is a discrete set containing singular points of $\Gamma^{\star}$ (which is exactly $\Gamma \setminus \Gamma_0$). Then, by an argument similar to that of Proposition 3.16, we can deduce that $\Gamma^{\star}$ is biholomorphic to $\mathbb{D}(0, R)$ and hence the proposition is proved. $\square$

4.8. *End of the proof.* Denote $\tau_1 := \pi \circ (\Phi|_{\Gamma})^{-1} : \mathbb{D}(0, R)^* \to \mathbb{C}^n$. The map τ_1 has a holomorphic extension to the map $\tau : \mathbb{D}(0, R) \to \mathbb{C}^n$ such that $\tau(0) = z$, $\tau'(0) = v$. The map τ takes values in $\pi(M)$, which is bounded, and hence the radial limit

$$\tau_{\theta} = \lim_{r \to R^-} \tau(re^{i\theta})$$

exists for almost every $\theta \in [0, 2\pi)$.

PROPOSITION 4.20. $\tau_{\theta} \in PC(f)$ *if it exists.*

Proof. Note that $\tau(\mathbb{D}(0, R)) \cap PC(f) = (\pi(\Gamma) \cup \Gamma_1) \cap PC(f) = \{z\}$. Hence $\tau(\mathbb{D}(0, R) \setminus \{0\}) \subset M$. This implies that $\tau(\mathbb{D}(0, R) \setminus \{0\}) \subset X$. Then, by an argument similar to that of Proposition 3.18, we deduce that the radial limit $\tau_{\theta} \in PC(f)$ if this limit exists. $\square$

Recall that Q is the defining polynomial of $PC(f)$. Then $Q \circ \tau$ has vanishing radial limit for almost every $\theta \in [0, 2\pi)$. This means that $Q \circ \tau$ is identically zero. Hence $\tau(\mathbb{D}(0, 1)) \subset PC(f)$. This is a contradiction since $\tau'(0) = v \notin T_z PC(f)$. The proof of Proposition 4.2 is complete.

5. *Periodic cycles of PCA endomorphisms of $\mathbb{CP}^2$*
This section is devoted to the proof of Theorem 1.2.

THEOREM 5.1. *Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$ and let λ be an eigenvalue of f along a periodic cycle. Then either $\lambda = 0$ or $|\lambda| > 1$.*

If the periodic cycle is not in $PC(f)$, then $\lambda \neq 0$ and the result follows from Theorem 1.3. Therefore, without loss of generality, we may assume that z is a fixed point of f in $PC(f)$. Note that if λ is an eigenvalue of f at a fixed point z, then λ^j is an eigenvalue of $f^{\circ j}$ at the fixed point z. If we can prove that λ^j is either superattracting or repelling, then so is λ. Thus, in order to prove Theorem 1.2, we can always consider f up to some iterates, if necessary.

After passing to an iterate, we may assume that the fixed point belongs to an invariant irreducible component Γ of $PC(f)$. The reason why we need to restrict to dimension $n = 2$ is that, in this case, Γ is an algebraic curve. There is a normalization $\mathfrak{n} : \hat{\Gamma} \to \Gamma$, where $\hat{\Gamma}$ is a smooth compact Riemann surface and $\mathfrak{n}$ is a biholomorphism outside a finite set (see [RS13], [Gun90] or [Chi89]) and there is a holomorphic endomorphism $\hat{f} : \hat{\Gamma} \to \hat{\Gamma}$ such that $\mathfrak{n} \circ \hat{f} = f \circ \mathfrak{n}$.

In §5.1, we analyze the dynamics of $\hat{f} : \hat{\Gamma} \to \hat{\Gamma}$ and, in particular, we show that when f is PCA, then $\hat{f}$ is PCF. In §5.2, we complete the proof in the case where the fixed point belongs to the regular part of $PC(f)$, and in §5.3, we complete the proof in the case where the fixed point belongs to the singular part of $PC(f)$.

5.1. *Dynamics on an invariant curve.* Assume that $f : \mathbb{CP}^2 \to \mathbb{CP}^2$ is an endomorphism of degree $d \geq 2$ (not necessarily PCA) and that $\Gamma \subset \mathbb{CP}^2$ is an irreducible algebraic curve such that $f(\Gamma) = \Gamma$. Let $\mathfrak{n} : \hat{\Gamma} \to \Gamma$ be a normalization of Γ and let $\hat{f} : \hat{\Gamma} \to \hat{\Gamma}$ be an endomorphism such that $\mathfrak{n} \circ \hat{f} = f \circ \mathfrak{n}$.

According to [FS94, Theorem 7.4], the endomorphism $\hat{f} : \hat{\Gamma} \to \hat{\Gamma}$ has degree $d \geq 2$. It follows from the Riemann–Hurwitz formula that the compact Riemann surface $\hat{\Gamma}$ has genus zero or one. In addition, if the genus is one, then $\hat{f}$ has no critical point and all fixed points of $\hat{f}$ are repelling with common repelling eigenvalue λ satisfying $|\lambda| = \sqrt{d'}$. If the genus is zero, then the following lemma implies that the post-critical sets of $\hat{f}$ and f are closely related.

LEMMA 5.2. *Denote by $V_{\hat{f}}$ and V_f the set of critical values of $\hat{f}$ and f, respectively. Then*

$$V_{\hat{f}} \subset \begin{cases} \mathfrak{n}^{-1}(V_f) & \text{if } \Gamma \not\subset V_f, \\ \mathfrak{n}^{-1}(\text{Sing } V_f) & \text{if } \Gamma \subset V_f. \end{cases}$$

Proof. The set of critical values of $\hat{f}$ is characterized by the property that $x \notin V_{\hat{f}}$ if and only if, for every $y \in \hat{f}^{-1}(x)$, $\hat{f}$ is injective near y. Note that $\mathfrak{n} : \hat{\Gamma} \to \Gamma$ induces a parametrization of the germ (Γ, x) such that, for every $x \in \Gamma$ and for every $y \in \mathfrak{n}^{-1}(x)$, $\mathfrak{n}$ is injective near y (see also [Wal04, §2.3]).
- If $\Gamma \not\subset V_f$, let $x \notin \mathfrak{n}^{-1}(V_f)$ and let $y \in \hat{f}^{-1}(x)$. Then $\mathfrak{n}(y) \in f^{-1}(\mathfrak{n}(x))$. Since $\mathfrak{n}(x) \notin V_f$, f is injective near $\mathfrak{n}(y)$. Combining this with the fact that $\mathfrak{n}$ is locally injective, we deduce that $\hat{f}$ is injective near y. Thus $x \notin V_{\hat{f}}$.

- If $\Gamma \subset V_f$, let $x \notin \mathfrak{n}^{-1}(\mathrm{Sing}\, V_f)$ and let $y \in \hat{f}^{-1}(x)$. Then $\mathfrak{n}(y) \in f^{-1}(\mathfrak{n}(x))$ and $\mathfrak{n}(x) \in \mathrm{Reg}\, V_f$. By Proposition 4.4, we can deduce that $\mathfrak{n}(y) \in f^{-1}(\mathrm{Reg}\, V_f)$ and that $f|_{f^{-1}(\mathrm{Reg}\, V_f)}$ is locally injective. This implies that $\hat{f}$ is also injective near y. Hence $x \notin V_{\hat{f}}$.

Thus we obtain the conclusion of the lemma. $\qquad\square$

PROPOSITION 5.3. *If $\hat{\Gamma}$ has genus zero and f is a PCA endomorphism, then $\hat{f}$ is a PCF endomorphism.*

Proof. We have that

$$PC(f) = \bigcup_{j \geq 1} V_{f \circ j} \quad \text{and} \quad PC(\hat{f}) = \bigcup_{j \geq 1} V_{\hat{f} \circ j}.$$

Since $\mathfrak{n} \circ \hat{f}^{\circ j} = f^{\circ j} \circ \mathfrak{n}$ for all $j \geq 1$, applying the previous lemma to $f^{\circ j}$ and $\hat{f}^{\circ j}$ yields

$$PC(\hat{f}) \subset \begin{cases} \mathfrak{n}^{-1}(PC(f)) & \text{if } \Gamma \not\subset PC(f), \\ \mathfrak{n}^{-1}(\mathrm{Sing}\, PC(f)) & \text{if } \Gamma \subset PC(f). \end{cases}$$

In both cases, $PC(\hat{f})$ is contained in the preimage by $\mathfrak{n}$ of a proper algebraic subset of Γ, which therefore is finite. Since $\mathfrak{n}$ is proper, $PC(\hat{f})$ is finite and so $\hat{f}$ is PCF. $\qquad\square$

Assume that f has a fixed point z which is a regular point of Γ (which is not necessarily an irreducible component of $PC(f)$). Since $\mathfrak{n}$ is a biholomorphism outside the preimage of singular points of Γ, the point $\mathfrak{n}^{-1}(z)$ is a fixed point of $\hat{f}$ and $\mathfrak{n}$ will conjugate $D_z f|_{T_z\Gamma}$ and $D_{\mathfrak{n}^{-1}(z)} \hat{f}$. Denote by λ the eigenvalue of $D_z f|_{T_z\Gamma}$. Then λ is also the eigenvalue of $D_{\mathfrak{n}^{-1}(z)} \hat{f}$. The previous discussion allows us to conclude that either $\lambda = 0$ or $|\lambda| > 1$. Thus we can deduce the following lemma.

LEMMA 5.4. *Let f be a PCA endomorphism of $\mathbb{CP}^2$ of degree $d \geq 2$, let $\Gamma \subset \mathbb{CP}^2$ be an invariant irreducible algebraic curve, let $z \in \mathrm{Reg}\,\Gamma$ be a fixed point of f and let λ be the eigenvalue of $D_z f|_{T_z\Gamma}$. Then either $\lambda = 0$ or $|\lambda| > 1$.*

5.2. *Periodic cycles in the regular locus of the post-critical set.*

Proof of Theorem 1.2—first part. Let f be a PCA endomorphism of $\mathbb{CP}^2$ with a fixed point z that is a regular point of $PC(f)$. Denote by Γ the irreducible component of $PC(f)$ containing z. Then Γ is invariant by f. Denote by $\overline{D_z f} : T_z\mathbb{CP}^2/T_z\Gamma \to T_z\mathbb{CP}^2/T_z\Gamma$ the linear endomorphism induced by $D_z f$. Note that

$$\mathrm{Spec}(D_z f) = \mathrm{Spec}(D_z f|_{T_z\Gamma}) \cup \mathrm{Spec}(\overline{D_z f}).$$

By Proposition 4.4, the eigenvalue of $D_z f|_{T_z\Gamma}$ is not zero and hence repelling by Lemma 5.4. By Proposition 4.1, the eigenvalue of $\overline{D_z f}$ is either superattracting or repelling. Thus Theorem 1.2 is proved when the fixed point is a regular point of the post-critical set. $\qquad\square$

5.3. *Periodic cycles in the singular locus of the post-critical set.* When the fixed point z is a singular point of $PC(f)$, by passing to some iterates of f, we can assume that f

induces a holomorphic germ at z that fixes a singular germ of the curve at z which is induced by some irreducible components of $PC(f)$. On the one hand, from the local point of view, there exists (in most cases) a relationship between the two eigenvalues of $D_z f$ as a holomorphic germ fixing a singular germ of the curve. On the other hand, from the global point of view, these eigenvalues can be identified with the eigenvalue of the germ at a fixed point of the lifts of f via the normalization of $PC(f)$. Then, by Proposition 5.3, we can conclude Theorem 1.2.

5.3.1. *Holomorphic germ of* $(\mathbb{C}^2, 0)$ *fixing a singular germ of the curve.* Let $(\Sigma, 0)$ be an irreducible germ of the curve at 0 in $(\mathbb{C}^2, 0)$ defined by a holomorphic germ $g : (\mathbb{C}^2, 0) \to (\mathbb{C}, 0)$. In local coordinates (x, y) of $\mathbb{C}^2$, if $g(0, y) \not\equiv 0$, that is, if g does not identically vanish on $\{x = 0\}$, it is well known that there exists an injective holomorphic germ $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ of the form

$$\gamma(t) = (t^m, \alpha t^n + O(t^{n+1}))$$

parameterizing Σ, that is, $\gamma((\mathbb{C}, 0)) = (\Sigma, 0)$ (see [**Wal04**, Theorem 2.2.6]). If Σ is singular, after a change of coordinates, α can be 1 and m and n satisfy that $1 < m < n$, $m \nmid n$. The germ γ is called a Puiseux parametrization. In fact, if Σ is a germ induced by an algebraic curve Γ in $\mathbb{CP}^2$, then γ coincides with the germ induced by the normalization morphism. When Σ is singular, the integers m and n are called the first two Puiseux characteristics of Γ and they are invariants of the equisingularity class of Γ. In particular, m and n do not depend on the choice of local coordinates. We refer the reader to [**Wal04**] for further discussion about Puiseux characteristics. We also refer to [**Z73**] and references therein for discussion about equisingular invariants.

Now consider a proper† holomorphic germ $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ and a singular germ of the curve $(\Sigma, 0)$. If Σ is invariant by g, that is, if $g(\Sigma) = \Sigma$, then g acts as a permutation on irreducible branches of Σ. Then, by passing to some iterates of g, we assume that there exists an invariant branch. The following propositions show that there exists a relationship between the two eigenvalues of $D_0 g$.

When g has an invariant singular branch, the following result was observed by Jonsson [**Jon98**].

PROPOSITION 5.5. *Let* Σ *be an irreducible singular germ of a curve parametrized by* $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ *of the form*

$$\gamma(t) = (t^m, t^n + O(t^{n+1})), \ 1 < m < n, m \nmid n.$$

Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ *and* $\hat{g} : (\mathbb{C}, 0) \to (\mathbb{C}, 0), \hat{g}(t) = \lambda t + O(t^2)$ *be holomorphic germs such that*

$$g \circ \gamma = \gamma \circ \hat{g}.$$

Then the eigenvalues of $D_0 g$ *are* λ^m *and* λ^n.

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† Proper germ means that $g^{-1}(0) = 0$. In particular, an endomorphism of $\mathbb{CP}^2$ induces a proper germ at its fixed points.

Proof. The germ g has an expansion of the form

$$g(x, y) = (ax + by + h_1(x, y), cx + dy + h_2(x, y)),$$

where $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|^2)$. Replacing those expansions in the equation $\gamma \circ \hat{g} = g \circ \gamma$ gives

$$\left(\lambda^m t^m + O(t^{m+1}), \lambda^n t^n + O(t^{n+1})\right) = \left(at^m + bt^n + h_1(t^m, t^n + O(t^{n+1})),\right.$$
$$\left. ct^m + dt^n + h_2(t^m, t^n + O(t^{n+1}))\right).$$

Comparing coefficients of the term t^m in each coordinate, we deduce that $a = \lambda^m$ and $c = 0$. Comparing coefficients of the term t^n in the second coordinate, since $m \nmid n$, the expansion of h_2 cannot contribute any term of order t^n, and hence $d = \lambda^n$. The linear part of g has the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ and hence a, d are eigenvalues of $D_0 g$. In other words, λ^m and λ^n are eigenvalues of $D_0 g$. $\qquad\square$

When g has an invariant smooth branch which is the image of another branch, g is not an injective germ and hence 0 is an eigenvalue of $D_0 g$. This case was not considered in [Jon98] since Jonsson assumed that there is no periodic critical point.

PROPOSITION 5.6. *Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be irreducible germs of curves at zero such that $\Sigma_1 \neq \Sigma_2, g(\Sigma_1) = \Sigma_2, g(\Sigma_2) = \Sigma_2$. If Σ_2 is smooth, then the eigenvalues of $D_z g$ are zero and λ, where λ is the eigenvalue of $D_0 g|_{T_0 \Sigma_2}$.*

Proof. Since Σ_2 is smooth, we choose a local coordinates (x, y) of $(\mathbb{C}^2, 0)$ such that $\Sigma_2 = \{x = 0\}$. Since Σ_1 and Σ_2 are distinct irreducible germs, the defining function of Σ_1 does not identically vanish on Σ_2. Then we can find a Puiseux parametrization of Σ_1 of the form

$$\gamma(t) = (t^m, \alpha t^n + O(t^{n+1})), \alpha \in \mathbb{C} \setminus \{0\},$$

where m, n are positive integers (see [Wal04, Theorem 2.2.6]). The germ g has an expansion of the form

$$g(x, y) = (ax + by + h_1(x, y), cx + dy + h_2(x, y)),$$

where $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|^2)$. The invariance of Σ_2 implies that $b = 0$ and that g has the form

$$g(x, y) = (x(a + h_3(x, y)), cx + dy + h_2(x, y)),$$

where $h_3(x, y) = O(\|(x, y)\|)$, $h_2(x, y) = O(\|(x, y)\|^2)$. Replacing γ and f in the equation $g(\Sigma_1) = \Sigma_2$ gives

$$t^m (a + h_3(t^m, \alpha t^n + O(t^{n+1}))) = 0$$

and hence $a = 0$. Then the linear part of $D_0 g$ is $(0, cx + dy)$ and hence zero and d are the eigenvalues of $D_0 g$, $\qquad\square$

Finally, if g has two invariant smooth branches which are tangential, then we have the following proposition (see also [Jon98]).

PROPOSITION 5.7. *Let $g : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ be a proper holomorphic germ and let Σ_1, Σ_2 be irreducible invariant germs of smooth curves at zero. If Σ_1 and Σ_2 intersect tangentially, that is, if $\Sigma_1 \neq \Sigma_2$ and $T_0\Sigma_1 = T_0\Sigma_2$, then there exists a positive integer m such that the eigenvalues of D_0g are λ and λ^m, where λ is the eigenvalue of $D_0g|_{T_0\Sigma_1}$.*

Proof. Since Σ_1 is smooth, we can choose local coordinates (x, y) such that $\Sigma_1 = \{y = 0\}$. The defining function of Σ_2 cannot identically vanish on $\{x = 0\}$ since, otherwise, Σ_2 and Σ_1 would not be tangential. Then Σ_2 has a parametrization of the form

$$\gamma(t) = (t, t^m + O(t^{m+1})).$$

Since $\Sigma_1 = \{y = 0\}$ is invariant, g has an expansion in the coordinates (x, y) of the form

$$g(x, y) = \big(\lambda x + by + h_1(x, y), y(d + h_2(x, y))\big),$$

where $\lambda, b, d \in \mathbb{C}$, $h_1(x, y) = O(\|(x, y)\|^2)$, $h_2(x, y) = O(\|(x, y)\|)$. The linear part of D_0g is $(\lambda x + by, dy)$ and thus λ, d are eigenvalues of $D_z f$. Letting $x = t, y = t^m + O(t^{m+1})$, we have

$$f(t, t^m + O(t^{m+1})) = (\lambda t + O(t^2), dt^m + O(t^{m+1})).$$

Since $f(\Sigma_2) = \Sigma_2$, we deduce that $d = \lambda^m$. Note that λ is the eigenvalue of $D_0g|_{T_0\Sigma_1}$. □

Using these observations, we can conclude the proof of Theorem 1.2.

Proof of Theorem 1.2—final. Let f be a PCA endomorphism of $\mathbb{CP}^2$ and let z be a fixed point such that z is a singular point of $PC(f)$. We look at the germ $(PC(f), z)$ induced by $PC(f)$ at z and prove Theorem 1.2 depending on how f acts on irreducible branches of $(PC(f), z)$. Passing to an iterate of f, if necessary, we assume that there exists a branch Σ of $(PC(f), z)$ such that $f(\Sigma) = \Sigma$. Denote by Γ the irreducible component of $PC(f)$ inducing Σ. Then Γ is also invariant by f. Denote by $\mathfrak{n} : \hat{\Gamma} \to \Gamma$ the normalization of Γ and by $\hat{f}$ the lifting of f by the normalization $\mathfrak{n} : \hat{\Gamma} \to \Gamma$.

If Σ is singular, then $\mathfrak{n}^{-1}(z)$ is a finite set and $\hat{f}(\mathfrak{n}^{-1}(z)) \subset \mathfrak{n}^{-1}(z)$ since z is a fixed point of f. Then, by passing up to some iterations, we can assume that $\hat{f}$ fixes a point $w_0 \in \mathfrak{n}^{-1}(z)$. By Proposition 5.5, the eigenvalues of $D_z f$ are λ^m, λ^n, where λ is the eigenvalue of $D_{w_0}\hat{f}$ and m and n are the first two Puiseux characteristics of Σ. By Proposition 5.3, λ is either superattracting or repelling. Hence so are λ^m and λ^n.

If Σ is smooth, then the tangent space $T_z\Sigma$ is well defined and invariant by $D_z f$. Denote by λ the eigenvalue of $D_z f|_{T_z\Sigma}$. Even if Γ can be singular (for example, z can be a self-intersection point of Γ), there exists a point $w \in \hat{\Gamma}, \mathfrak{n}(w) = z$ such that w is a fixed point of $\hat{f}$ and $\mathfrak{n}$ induces an invertible germ $\mathfrak{n} : (\hat{\Gamma}, w) \to (\Sigma, z)$. By an argument similar to that of Lemma 5.4, we can deduce that either $\lambda = 0$ or $|\lambda| > 1$. To deal with the other eigenvalue, since $(PC(f), z)$ is singular, we have one of the following cases.

(1) There exists a branch Σ_1 such that $f(\Sigma_1) = \Sigma$. By Proposition 5.6, the eigenvalues of $D_z f$ are zero and λ, where λ is the eigenvalue of $D_z f|_{T_z\Sigma}$. Hence the proof is complete by the previous discussion.

(2) There exists a smooth invariant branch Σ_1 such that Σ and Σ_1 intersect transversally. Denote by Γ_1 the irreducible component of $PC(f)$ containing Σ_1 (which can be Γ). Thus Γ_1 are invariant by f. Since Σ and Σ_1 are transversal, the eigenvalues of $D_z f$ are λ and λ_1, where λ_1 is the eigenvalue of $D_z f|_{T_z \Sigma_1}$. Arguing similarly to the case of λ, we deduce that λ_1 is also either superattracting or repelling.

(3) There exists a smooth invariant branch Σ_1 such that Σ and Σ_1 intersect tangentially. Denote by Γ_1 the irreducible component of $PC(f)$ inducing Σ_1 (which is possibly equal to Γ). Thus Γ_1 are invariant by f. By Proposition 5.7, the eigenvalues of $D_z f$ are λ and λ^m. Note that λ is either superattracting or repelling and hence so is λ^m. $\qquad\square$

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