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Étale Steenrod operations and the Artin–Tate pairing

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ABSTRACT

We prove a 1966 conjecture of Tate concerning the Artin–Tate pairing on the Brauer group of a surface over a finite field, which is the analog of the Cassels–Tate pairing. Tate asked if this pairing is always alternating and we find an affirmative answer, which is somewhat surprising in view of the work of Poonen–Stoll on the Cassels–Tate pairing. Our method is based on studying a connection between the Artin–Tate pairing and (generalizations of) Steenrod operations in étale cohomology. Inspired by an analogy to the algebraic topology of manifolds, we develop tools allowing us to calculate the relevant étale Steenrod operations in terms of characteristic classes.

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1. Introduction

1.1 Motivation

Let X be a smooth, projective, geometrically connected surface over \mathbf{F}_q , where $\text{char } \mathbf{F}_q = p$. For every prime $\ell \neq p$, Artin and Tate [Tat95] defined a pairing

$$\langle \cdot, \cdot \rangle_{\text{AT}}: \text{Br}(X)_{\text{nd}}[\ell^\infty] \times \text{Br}(X)_{\text{nd}}[\ell^\infty] \rightarrow \mathbf{Q}/\mathbf{Z}, \quad (1.1.1)$$

where $\text{Br}(X)_{\text{nd}}$ denotes the quotient of the Brauer group $\text{Br}(X)$ by its divisible part, and $\text{Br}(X)_{\text{nd}}[\ell^\infty]$ denotes its ℓ -power torsion subgroup. (Conjecturally the divisible part vanishes, implying that $\text{Br}(X)_{\text{nd}} = \text{Br}(X)$.) We will review the definition of (1.1.1) in §2.1; we henceforth call it the *Artin–Tate pairing*.

Artin and Tate’s investigation of $\text{Br}(X)$ was motivated by a dictionary relating the invariants of X to those appearing in the Birch and Swinnerton-Dyer conjecture for abelian varieties over

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function fields. In particular, under this dictionary $\text{Br}(X)$ corresponds to III, and the Artin–Tate pairing corresponds to the Cassels–Tate pairing.

It is not difficult to show that the pairing $\langle \cdot, \cdot \rangle_{\text{AT}}$ is skew-symmetric, but it is much less clear if it is *alternating*. For clarity, we recall that skew-symmetric means that

$$\langle x, y \rangle_{\text{AT}} + \langle y, x \rangle_{\text{AT}} = 0 \quad \text{for all } x, y \in \text{Br}(X)_{\text{nd}}[\ell^\infty],$$

while alternating means the stronger condition that

$$\langle x, x \rangle_{\text{AT}} = 0 \quad \text{for all } x \in \text{Br}(X)_{\text{nd}}[\ell^\infty].$$

Since the distinction between skew-symmetric and alternating disappears for $\ell \neq 2$, the difficulty lies entirely in the case $\ell = 2$. In Tate’s 1966 Bourbaki report on the Artin–Tate conjecture, he asks [Tat95, after Theorem 5.1] if the pairing (1.1.1) is alternating, conjecturing that the answer is ‘yes’.

CONJECTURE 1.1 (Tate, 1966). The Artin–Tate pairing is alternating.

Tate’s motivation for making Conjecture 1.1 was Cassels’ result [Cas65] that the analogous Cassels–Tate pairing is alternating for elliptic curves, which Tate had generalized in [Tat63] to abelian varieties with principal polarization ‘arising from a rational divisor’. Moreover, one can find at the end of [Tat95, § 1] the claim that the Cassels–Tate pairing is alternating for *all* Jacobians (with respect to their canonical principal polarizations). But this is rather ironic in hindsight, as Poonen and Stoll eventually demonstrated in [PS99] that the Cassels–Tate pairing actually need *not* be alternating in general for abelian varieties with principal polarization not satisfying the technical condition of ‘arising from a rational divisor’; in particular, it need not be alternating for Jacobians. See the introduction and § 1.2 of [She15] for a detailed explanation of these technical subtleties and amusing history.

The history of Conjecture 1.1 is perhaps even more tortuous than that of the analogous question for the Cassels–Tate pairing. Recall that any finite abelian group with a non-degenerate alternating pairing has order equal to a perfect square¹ [PS99, § 6], so Conjecture 1.1 implies that $\text{Br}(X)_{\text{nd}}[2^\infty]$ has square order. Manin computed examples [Man67, Man86] in which $\#\text{Br}(X)_{\text{nd}}[2^\infty]$ was purportedly $\mathbf{Z}/2\mathbf{Z}$, seemingly *disproving* Conjecture 1.1. However, in 1996 Urabe found mistakes in Manin’s calculations that invalidated the counterexamples (see the introduction to [Ura96]), and then proved that in characteristic $p \neq 2$, $\text{Br}(X)_{\text{nd}}[2^\infty]$ *always does* have square order!

There has been some other partial progress on Conjecture 1.1 besides Urabe’s theorem. We note in particular the following two results.

- Zarhin showed in [Zar89] that if X lifts to characteristic 0 and the Néron–Severi group of $X \times_{\mathbf{F}_q} \overline{\mathbf{F}}_q$ has vanishing 2-primary part, then $\langle \cdot, \cdot \rangle_{\text{AT}}$ is alternating for X .²
- Liu–Lorenzini–Raynaud [LLR05] proved that if $\text{Br}(X)$ is finite, implying that $\text{Br}(X) = \text{Br}(X)_{\text{nd}}$, then $\#\text{Br}(X)$ is a perfect square. (We emphasize that this applies even to the

¹ This should be thought of as analogous to the fact that a vector space with a non-degenerate alternating pairing has even dimension.

² We thank Yuri Zarhin for informing us about [Zar89], and for translating the statement of its main theorem into English. Strictly speaking, the result is for a pairing discovered independently in [Zar89], which should be the same as Artin–Tate’s, but no comparison with the Artin–Tate pairing is made in [Zar89].

p -primary part!)³ Amusingly, the argument of [LLR05] has nothing to do with the Artin–Tate pairing, but actually uses the work of Poonen–Stoll quantifying the failure of #III to be a perfect square.

1.2 Results

In this paper we answer Tate’s question in the *affirmative*, finally bringing closure to this eventful drama.

THEOREM 1.2. *Conjecture 1.1 is true.*

In fact, we deduce Theorem 1.2 from a more general result that we now describe. In [Jah15] Jahn defined a generalization of the Artin–Tate pairing to higher Brauer groups. Briefly, if X is a smooth projective variety of even dimension $2d$ over \mathbf{F}_q , then its *higher Brauer group* is

$$\mathrm{Br}^d(X) := H_L^{2d+1}(X; \mathbf{Z}(d)),$$

where H_L denotes Lichtenbaum cohomology [Jah15, §2]. The importance of this group lies in its relation to the (other) Tate conjecture concerning algebraic cycles in X . A completely analogous construction to Artin–Tate’s, which we will describe in §2.1, gives a non-degenerate skew-symmetric pairing for $\ell \neq p$

$$\langle \cdot, \cdot \rangle_{\mathrm{AT}}: \mathrm{Br}^d(X)_{\mathrm{nd}}[\ell^\infty] \times \mathrm{Br}^d(X)_{\mathrm{nd}}[\ell^\infty] \rightarrow \mathbf{Q}/\mathbf{Z}.$$

One wants to know if $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ is alternating; again the issue is for $\ell = 2$. In particular, this would imply that $\#\mathrm{Br}^d(X)_{\mathrm{nd}}[2^\infty]$ is a perfect square. Jahn generalized Urabe’s method to show that $\#\mathrm{Br}^d(X)_{\mathrm{nd}}[2^\infty]$ is indeed a perfect square if $\mathrm{char} \mathbf{F}_q = p > 2$ [Jah15, Theorem 1]. We prove the stronger statement that $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ is alternating for any such X , which of course recovers Theorem 1.2 when X is specialized to have dimension 2.

THEOREM 1.3. *Let X be a smooth, projective, geometrically connected variety of dimension $2d$ over \mathbf{F}_q with $\mathrm{char} \mathbf{F}_q = p \neq 2$. The pairing $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ on $\mathrm{Br}^d(X)_{\mathrm{nd}}[2^\infty]$ is alternating.*

In the course of proving Theorem 1.3, we establish several results, which may be of independent interest and utility, as our work involves developing algebro-geometric versions of techniques of fundamental importance in algebraic topology. Let us briefly summarize the idea, which will be elaborated upon in §1.3. The skew-symmetry of $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ implies that the assignment $x \mapsto \langle x, x \rangle_{\mathrm{AT}}$ is a *homomorphism*. Tautologically, $\langle \cdot, \cdot \rangle_{\mathrm{AT}}$ is alternating if and only if this homomorphism is 0. The strategy is to rewrite this homomorphism in terms of canonical cohomology operations called the *Steenrod squares*. Motivated by classical results on the algebraic topology of manifolds, we then develop a theory of ‘Stiefel–Whitney classes’ in étale cohomology of algebraic varieties, which facilitates the calculation of the relevant Steenrod squares.

Remark 1.4. Our approach is guided by an analogy between the Artin–Tate pairing and the *linking form* on a closed manifold of odd dimension. (See §2.2 for an explanation of this analogy.) Our method applies equally well to the topological situation, and it gives a necessary and sufficient criterion for the linking form to be alternating (see §7.3), which to our knowledge does not already exist in the topology literature.

³ Technically, the argument of [LLR05] uses an incorrect formula for $\#\mathrm{Br}(X)$, with the error stemming entirely from the false [Gor79, Lemma 4.2]. This formula is corrected in [Gei20]. The correction is by a factor that is a perfect square, and so preserves the conclusion that $\#\mathrm{Br}(X)[\ell^\infty]$ is a perfect square. Liu–Lorenzini–Raynaud have prepared a corrigendum [LLR18] that corrects or completes several more arguments in [Gor79], and amends the statements of [LLR05] correspondingly.

1.3 Overview of the proof

We now give a more detailed outline of our proof of Theorem 1.3.

Step 1: Reduction to an auxiliary pairing. In § 2 we explain that there is a surjection

$$H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \rightarrow \text{Br}(X)_{\text{nd}}[2^n],$$

so it suffices to prove that the pulled-back pairing on $H_{\text{ét}}^2(X; \mathbf{Z}/2^n\mathbf{Z}(d))$, which we denote $\langle \cdot, \cdot \rangle_n$, is alternating for all n . (Here $\mathbf{Z}/2^n\mathbf{Z}(d)$ is the constant sheaf $\mathbf{Z}/2^n\mathbf{Z}$ with an order d Tate twist.) One reason the pairing $\langle \cdot, \cdot \rangle_n$ is more tractable to study is that the coefficients $\mathbf{Z}/2^n\mathbf{Z}$ carry a ring structure, unlike $\mathbf{Q}_2/\mathbf{Z}_2$. As a consequence, the cohomology groups are enhanced with the structure of cohomology operations, which we exploit in the next step.

Step 2: Expression in terms of cohomology operations. We would like to understand the canonical linear functional $x \mapsto \langle x, x \rangle_n$ on $H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d))$. The key observation is that it can be expressed in terms of certain cohomology operations. To convey the spirit of this, we illustrate the flavor of the cohomology operations involved.

One is the Bockstein operation β , which is the boundary map

$$\beta: H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \rightarrow H_{\text{ét}}^{2d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(d))$$

induced by the short exact sequence of sheaves

$$0 \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \rightarrow \mathbf{Z}/2^{2n}\mathbf{Z}(d) \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \rightarrow 0.$$

The second operation is a Steenrod operation, which could be thought of informally as a kind of derived enhancement of the squaring operation. Concretely, the operation in question can be described in the following way (but see § 3 for a formal definition). Let $C_{\text{ét}}^*(X)$ be ‘the’ étale cochain complex computing $H_{\text{ét}}^*(X)$, which has the cup product

$$C_{\text{ét}}^*(X) \otimes C_{\text{ét}}^*(X) \xrightarrow{\sim} C_{\text{ét}}^*(X).$$

Since the cup product on $H_{\text{ét}}^*(X)$ is graded commutative, we can find a chain homotopy

$$\text{cup}_1: C_{\text{ét}}^*(X) \otimes C_{\text{ét}}^*(X) \rightarrow C_{\text{ét}}^{*-1}(X)$$

such that

$$d\text{cup}_1(u \otimes v) + \text{cup}_1(d(u \otimes v)) = u \smile v \mp v \smile u.$$

If $u \in Z_{\text{ét}}^{2d+1}(X)$, then $2^{n-1}\text{cup}_1(u \otimes u)$ defines a cohomology class in $H_{\text{ét}}^{4d+1}(X)$, if the coefficients are 2^n -torsion. For $n = 1$, the map $[u] \mapsto [2^{n-1}\text{cup}_1(u \otimes u)]$ is the Steenrod square Sq^{2d} . For larger n , it is a cohomology operation that we call $\widetilde{\text{Sq}}^{2d}$. These generalized Steenrod squares are carefully defined and studied in § 3.

The key identity that we have referred to is the following (the precise statement is Theorem 4.4).

THEOREM 1.5. *For all $x \in H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d))$ we have*

$$\langle x, x \rangle_n = \widetilde{\text{Sq}}^{2d}(\beta(x)).$$

The theorem is proved in § 4 using cohomology operations that we call ‘higher Bockstein operations’, which are those arising in the ‘Bockstein spectral sequence’ for $\mathbf{Z}/2^n\mathbf{Z}(2d)$. The argument is a little elaborate: we directly calculate the difference between the two sides as a differential in the spectral sequence. The game is then to deduce indirectly that this differential must vanish, by using Poincaré duality to infer information about the behavior of the E_∞ -page.

Step 3: Relation to characteristic classes. The previous step reduces the problem to that of understanding $\widetilde{\text{Sq}}^{2d}$ sufficiently well. A consequence of the structure of the $\widetilde{\text{Sq}}^{2d}$ discussed in §3 is that we only need to calculate the effect of the classical Steenrod square Sq^{2d} , which operates on cohomology with $\mathbf{Z}/2\mathbf{Z}$ -coefficients. For this purpose we draw inspiration from a theorem for smooth manifolds due originally to Wu (the precise version is explained in §6).

THEOREM 1.6 (Wu). *Let M be a closed smooth manifold of dimension d . For $x \in H^{d-i}(M; \mathbf{Z}/2\mathbf{Z})$, we have*

$$\text{Sq}^i x = P(w_1, w_2, \dots) \smile x,$$

where P is some explicit polynomial and the w_j are the Stiefel–Whitney classes of TM .

Our goal in this step is to establish a version of Wu’s theorem for the étale cohomology of smooth projective varieties over \mathbf{F}_q . The first task is to define an appropriate notion of Stiefel–Whitney classes, which is the subject of §5. Next, we establish an étale-cohomological analog of Wu’s theorem in §6. The overarching meta-strategy of the proofs is to attempt to imitate the theory as developed in algebraic topology. However there are a few possibly surprising subtleties, which result in this being the most technical part of the paper. For example, our argument employs the apparatus of *relative étale homotopy theory* developed by Harpaz–Schlank [HS13] and Barnea–Schlank [BS16], following in the tradition of Artin–Mazur and Friedlander. Hence our Theorem 1.2 is, in our humble opinion, a rather compelling example of how this abstract theory can be used to understand very concrete questions, which have no apparent grounding in homotopy theory.

Step 4: Calculation of characteristic classes. The upshot of the preceding steps is that we can express the obstruction for $\langle \cdot, \cdot \rangle_{\text{AT}}$ to be alternating explicitly in terms of our ‘étale Stiefel–Whitney classes’. We then need to show that this obstruction actually vanishes. After some elementary manipulations, it becomes clear that the key issue is whether or not a certain explicit polynomial in Stiefel–Whitney classes, which is a cohomology class with coefficients mod 2, lifts to an integral class. This calculation is carried out in §7. Motivated by an analogous fact for complex manifolds, we prove a formula expressing our Stiefel–Whitney classes in terms of Chern classes, and conclude that they lift because Chern classes do, which establishes Theorem 1.3.

1.4 Comparison with earlier arguments

The idea to use Steenrod squares on this problem goes back to Zarhin, who in [Zar89] studied the case when the surface admits a lift to characteristic 0. Zarhin’s argument was pushed further by Urabe in [Ura96] to show that $\# \text{Br}(X)_{\text{nd}}[2^\infty]$ is a perfect square.

Our argument also uses Steenrod operations, although both the operations used and the manner of use are quite different. The Steenrod operation at the focus of [Zar89] and [Ura96] is actually just the squaring operation on an element of $H_{\text{ét}}^2(X_{\overline{\mathbf{F}}_q}; \mathbf{Z}/2\mathbf{Z})$. By contrast, our argument studies a deeper connection between Tate’s pairing with subtler cohomology operations, as articulated in Theorem 4.4.

Our strategy to ‘compute’ these operations is based on setting up an analog of Wu’s theorem, as in [Zar89] and [Ura96]. The content of §5, especially the idea to lift Stiefel–Whitney classes to Chern classes, was inspired by [Ura96]. However, the proof of such an analog is significantly more difficult, because we work with arithmetic (as opposed to geometric) étale cohomology. For example our arithmetic version of Wu’s theorem is no easier to prove in the case when X lifts to characteristic 0, whereas Urabe’s geometric version follows immediately from the Artin comparison theorem and the classical Wu theorem in topology.

2. Pairings for varieties over finite fields

2.1 The Artin–Tate pairing

We briefly summarize the definition of the generalized pairing $\langle \cdot, \cdot \rangle_{\text{AT}}$ from [Jah15, §§2 and 3]. Let X be a geometrically connected, smooth, projective variety of even dimension $2d$ over \mathbf{F}_q . Jahn defines the *higher Brauer group*

$$\text{Br}^d(X) := H_L^{2d+1}(X; \mathbf{Z}(d)),$$

where H_L denotes Lichtenbaum cohomology. By [Jah15, Lemma 1] we have the following interpretation of its non-divisible quotient for $\ell \neq p$:

$$\text{Br}^d(X)_{\text{nd}}[\ell^\infty] \cong H_{\text{ét}}^{2d+1}(X; \mathbf{Z}_\ell(d))_{\text{tors}}.$$

The pairing $\langle \cdot, \cdot \rangle_{\text{AT}}$ on $\text{Br}^d(X)_{\text{nd}}[\ell^\infty]$ is defined as follows. For any abelian group G , let G_{nd} denote its non-divisible quotient (i.e. the quotient by the maximal divisible subgroup). Let

$$\tilde{\delta}: H_{\text{ét}}^{2d}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(d))_{\text{nd}} \rightarrow H_{\text{ét}}^{2d+1}(X; \mathbf{Z}_\ell(d))_{\text{tors}}$$

be the boundary map induced by the short exact sequence

$$0 \rightarrow \mathbf{Z}_\ell(d) \rightarrow \mathbf{Q}_\ell(d) \rightarrow \mathbf{Q}_\ell/\mathbf{Z}_\ell(d) \rightarrow 0.$$

The map $\tilde{\delta}$ is an isomorphism, so it suffices to define a pairing on $H_{\text{ét}}^{2d}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(d))_{\text{nd}}$. Now the key point is that X has a Poincaré duality of dimension $4d + 1$, since $X_{\overline{\mathbf{F}}_q}$ has a Poincaré duality of dimension $4d$ and $\text{Spec } \mathbf{F}_q$ has a Poincaré duality of dimension 1. (This may be deduced directly from the Hochschild–Serre spectral sequence and the usual Poincaré duality for $X_{\overline{\mathbf{F}}_q}$.) In particular, there is a canonical isomorphism

$$\int: H_{\text{ét}}^{4d+1}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(2d)) \xrightarrow{\sim} \mathbf{Q}_\ell/\mathbf{Z}_\ell.$$

DEFINITION 2.1. For $x, y \in H_{\text{ét}}^{2d}(X; \mathbf{Q}_\ell/\mathbf{Z}_\ell(d))_{\text{nd}}$, we define

$$\langle x, y \rangle_{\text{AT}} := \int (x \smile \tilde{\delta}y).$$

From Poincaré duality and the fact that $\tilde{\delta}$ is an isomorphism, it is evident that this pairing is non-degenerate. It is also skew-symmetric: this is proved in [Jah15, §3], and it also follows from combining Propositions 2.3 and 2.5 below.

2.2 The analogy to the linking form

An analogous pairing exists on any orientable manifold M of odd dimension $4d + 1$, and is called the linking form. Our approach was inspired by ideas of Browder used to study a variant of the linking form in [Bro62].

Actually, to make the analogy sharper it is better to work in a slightly more general setup. We do not assume that M is orientable, but we do assume that the orientation sheaf of M is given by the tensor square of a \mathbf{Z}_ℓ -local system \mathcal{L} . Then there is a pairing on $H^{2d}(M; \mathbf{Q}_\ell/\mathbf{Z}_\ell \otimes \mathcal{L})_{\text{nd}} \cong H^{2d+1}(M; \mathbf{Z}_\ell \otimes \mathcal{L})_{\text{tors}}$ given by

$$\langle x, y \rangle := \int x \smile \tilde{\delta}y,$$

where $\tilde{\delta}$ is the analogous boundary map to that in § 2.1 and

$$\int : H^{4d+1}(M; \mathbf{Q}_\ell/\mathbf{Z}_\ell \otimes \mathcal{L}^{\otimes 2})_{\text{nd}} \xrightarrow{\sim} \mathbf{Q}_\ell/\mathbf{Z}_\ell$$

is the isomorphism furnished by Poincaré duality. The same argument shows that this pairing is skew-symmetric (this is why we require the dimension to be 1 mod 4). This pairing is called the *linking form*.⁴

We were informed by an anonymous referee that the linking form on an orientable smooth 5-manifold should be alternating if and only if the manifold admits a spin^c -structure. We were not previously aware of this fact, nor have we been able to locate a proof in the literature, but after hearing it we realized that our method yields a necessary and sufficient criterion for the linking form on any odd-dimensional topological manifold (with orientation sheaf of the above form) to be alternating, which recovers the aforementioned result for orientable smooth 5-manifolds. This will be explained in § 7.3. Although our paper is phrased for étale cohomology, the reader can check that every one of the results has a corresponding statement for the singular cohomology of manifolds, which is either easier to prove or already a known theorem.

2.3 An auxiliary pairing

We define an auxiliary pairing on the group $H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d))$. As in § 2.1 there is a Poincaré duality for $H_{\text{ét}}^*(X; \mathbf{Z}/2^n\mathbf{Z}(*))$, which means in particular that there is a fundamental class inducing an isomorphism

$$\int : H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d)) \xrightarrow{\sim} \mathbf{Z}/2^n\mathbf{Z}.$$

DEFINITION 2.2. We have the short exact sequence of sheaves on X :

$$0 \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \rightarrow \mathbf{Z}/2^{2n}\mathbf{Z}(d) \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \rightarrow 0$$

inducing a boundary map

$$\beta : H_{\text{ét}}^i(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \rightarrow H_{\text{ét}}^{i+1}(X; \mathbf{Z}/2^n\mathbf{Z}(d)).$$

We define the pairing

$$\langle \cdot, \cdot \rangle_n : H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \times H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \rightarrow \mathbf{Z}/2^n\mathbf{Z}$$

by

$$\langle x, y \rangle_n := \int x \smile \beta y.$$

PROPOSITION 2.3. *The pairing $\langle \cdot, \cdot \rangle_n$ is skew-symmetric.*

Proof. The assertion is equivalent to

$$x \smile \beta y + y \smile \beta x = 0.$$

Since β is a derivation, we have $x \smile \beta y + y \smile \beta x = \beta(x \smile y)$. Then the result follows from the next lemma. □

⁴ Strictly speaking, this is the ℓ -primary part of the usual linking form.

LEMMA 2.4. *The boundary map $\beta: H_{\text{ét}}^{4d}(X; \mathbf{Z}/2^n\mathbf{Z}(2d)) \rightarrow H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d))$ is 0.*

Proof. By the obvious long exact sequence, the image is the kernel of

$$[2^n]: H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d)) \rightarrow H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^{2n}\mathbf{Z}(2d)),$$

which is identified with the inclusion $2^n\mathbf{Z}/2^{2n}\mathbf{Z} \hookrightarrow \mathbf{Z}/2^{2n}\mathbf{Z}$ by Poincaré duality. □

PROPOSITION 2.5. *The boundary map $H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \rightarrow H_{\text{ét}}^{2d+1}(X; \mathbf{Z}_2(d))$ induced by the short exact sequence*

$$0 \rightarrow \mathbf{Z}_2(d) \xrightarrow{2^n} \mathbf{Z}_2(d) \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \rightarrow 0$$

surjects onto $H_{\text{ét}}^{2d+1}(X, \mathbf{Z}_2(d))[2^n]$. Moreover, it is compatible for the pairings $\langle \cdot, \cdot \rangle_n$ and $\langle \cdot, \cdot \rangle_{\text{AT}}$ in the sense that the following diagram commutes.

$$\begin{array}{ccccc} H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) & \times & H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) & \xrightarrow{\langle \cdot, \cdot \rangle_n} & H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(d)) \\ \downarrow & & \downarrow & & \sim \downarrow \\ H_{\text{ét}}^{2d+1}(X, \mathbf{Z}_2(d))[2^n] & \times & H_{\text{ét}}^{2d+1}(X; \mathbf{Z}_2(d))[2^n] & \xrightarrow{\langle \cdot, \cdot \rangle_{\text{AT}}} & H_{\text{ét}}^{4d+1}(X; \mathbf{Q}_2/\mathbf{Z}_2(d))[2^n] \end{array}$$

Proof. The first claim is immediate from the long exact sequence. For the second claim, we will apply the following observation, which is an immediate consequence of naturality for the cup product: given a map of short exact sequences of sheaves

$$\begin{array}{ccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C \longrightarrow 0 \\ & & \downarrow f & & \downarrow g & & \downarrow h \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' \longrightarrow 0 \end{array}$$

and multiplications fitting into a commutative diagram

$$\begin{array}{ccc} A \otimes C & \longrightarrow & C \\ \downarrow f \otimes h & & \downarrow h \\ A' \otimes C' & \longrightarrow & C' \end{array}$$

then for $a \in H^*(X; A), c \in H^*(X; C)$, we have $h(a \smile c) = f(a) \smile h(c)$.

We apply this observation to each of the maps of short exact sequences in the following commutative diagram of sheaves.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) & \xrightarrow{2^n} & \mathbf{Z}/2^{2n}\mathbf{Z}(d) & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) \longrightarrow 0 \\ & & \uparrow & & \uparrow & & \parallel \\ 0 & \longrightarrow & \mathbf{Z}_2(d) & \xrightarrow{2^n} & \mathbf{Z}_2(d) & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \frac{1}{2^n} \\ 0 & \longrightarrow & \mathbf{Z}_2(d) & \longrightarrow & \mathbf{Q}_2(d) & \longrightarrow & \mathbf{Q}_2/\mathbf{Z}_2(d) \longrightarrow 0 \end{array}$$

Denote by $\tilde{\beta}$ the boundary map in cohomology corresponding to the middle horizontal sequence, recalling that β and $\tilde{\delta}$ denote the boundary maps for the top and bottom horizontal

sequences, respectively. The observation applied to the upper map of sequences shows that for $x, y \in H_{\text{ét}}^*(X; \mathbf{Z}/2^n\mathbf{Z}(d))$ we have

$$x \smile \beta(y) = x \smile \tilde{\beta}(y).$$

The observation applied to the lower map of sequences shows that

$$x \smile \tilde{\beta}(y) \mapsto \left[\frac{1}{2^n} \right]_* (x) \smile \tilde{\delta}(y).$$

Combining these equations yields the desired conclusion. □

Proposition 2.5 immediately implies the following.

COROLLARY 2.6. *If the pairing $\langle \cdot, \cdot \rangle_n$ on $H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d))$ is alternating then so is the pairing $\langle \cdot, \cdot \rangle_{\text{AT}}$ on $\text{Br}^d(X)_{\text{nd}}[2^n]$.*

Hence to prove Theorem 1.3 we are reduced to proving the following.

THEOREM 2.7. *The pairing $\langle \cdot, \cdot \rangle_n$ is alternating for all n .*

The proof of Theorem 2.7 will be the focus of the rest of the paper.

3. Steenrod squares

In this section we define the (generalized) Steenrod squares in étale cohomology and establish the key facts about them. The perspective we adopt here is that we can define our cohomology operations on topological spaces, and then transport them to étale cohomology via étale homotopy theory.

Let us emphasize that our construction is certainly not original to this paper. (We do introduce some generalized operations $\widetilde{\text{Sq}}^i$ that we have not seen defined elsewhere, but they are minor variants of the Steenrod squares.) The earliest construction of Steenrod squares, which was general enough to apply to étale cohomology occurs in work of Epstein [Eps66]. Our definition is perhaps closer to (a special case of) Jardine’s construction in [Jar89].

3.1 The Steenrod algebra in topology

We begin with a motivational pitch about Steenrod operations in algebraic topology. An old and fundamental observation in algebraic topology is that the singular cohomology of any space with $\mathbf{Z}/2\mathbf{Z}$ coefficients carries a natural module structure over a ring called the *Steenrod algebra*, and that it is fruitful to understand this additional structure. The Steenrod algebra may be characterized abstractly as the algebra of stable cohomology operations on $H^*(-; \mathbf{Z}/2\mathbf{Z})$, i.e. all natural transformations $H^j(-; \mathbf{Z}/2\mathbf{Z}) \rightarrow H^k(-; \mathbf{Z}/2\mathbf{Z})$ commuting with the suspension isomorphisms.

More concretely, one can exhibit a set of cohomology operations Sq^i , which generate the Steenrod algebra and which admit an explicit description in terms of homotopies defined on the cochain complex of a topological space, whose existence has to do with the failure of the cup product to be commutative at the level of cochains. This will be explained in § 3.3.

A key point in this paper is that we can and should ask about the analogous structure for $H^*(-; \mathbf{Z}/2^n\mathbf{Z})$ for every n . In particular, we need analogs of the Sq^i for $\mathbf{Z}/2^n\mathbf{Z}$ -coefficients. This leads to a construction of operations that we call $\widetilde{\text{Sq}}^i$. These turn out to all be induced by the Sq^i , so they are not fundamentally new operations. However, they do come up very directly in our calculations, so it will be useful to spell them out explicitly.

3.2 Étale homotopy theory

Using étale homotopy theory, we will be able to transport our definition of (singular) cohomology operations on topological spaces to étale cohomology of algebraic varieties. Here we just summarize the facts that we need.

To any locally noetherian scheme X there is attached a pro-object in simplicial sets, which is called its *étale topological type*,⁵ and which we denote $\acute{E}t(X)$. We refer to [Fri82, Definition 4.4] for the definition of $\acute{E}t(X)$. Given the awkwardness of the expression ‘pro-(simplicial set)’, we will henceforth use the phrase ‘pro-space’ to denote a pro-object in simplicial sets (however, it will be important at certain points that our ‘spaces’ are really simplicial sets).

DEFINITION 3.1. We define the category of *local coefficient systems* on a pro-space $\{T^i : i \in I\}$ as follows.

- An object is a local coefficient system on some T^j .
- A map between local coefficient systems, defined by \mathcal{L}_1 on T^i and \mathcal{L}_2 on T^j , is a map between the pullbacks of \mathcal{L}_1 and \mathcal{L}_2 to T^k for some $k > i, j$.

Remark 3.2. In Friedlander’s original definition [Fri82, §5, p. 48], a ‘local coefficient system’ is an isomorphism class of objects in our definition.

PROPOSITION 3.3 [Fri82, Corollary 5.8]. *There is an equivalence of categories between locally constant sheaves on the étale site of X , and local coefficient systems on the pro-space $\acute{E}t(X)$.*

DEFINITION 3.4. We define the *cochain complex* of a pro-space $\{T^i\}$ with coefficients in a local coefficient system \mathcal{F} to be the direct limit of the levelwise cochain complexes:

$$C^*(\{T^i\}; \mathcal{F}) := \varinjlim_i C^*(T^i; \mathcal{F}).$$

By the exactness of filtered colimits, we have

$$H^*(C^*(\{T^i\}; \mathcal{F})) \cong \varinjlim_i H^*(C^*(T^i; \mathcal{F})),$$

so this recovers the definition of the *cohomology* of a pro-space $\{T^i\}$ in [Fri82, Definition 5.1] as the direct limit of the levelwise cohomology.

In particular, if $\acute{E}t(X) = \{U^i : i \in I\}$ then

$$C^*(\acute{E}t(X); \mathcal{F}) := \varinjlim_i C^*(U^i; \mathcal{F}) \quad \text{and} \quad H^*(\acute{E}t(X); \mathcal{F}) := \varinjlim_i H^*(U^i; \mathcal{F}).$$

PROPOSITION 3.5 [Fri82, Proposition 5.9]. *If \mathcal{F} is a locally constant sheaf on X and $\acute{E}t(\mathcal{F})$ is the corresponding local coefficient system on $\acute{E}t(X)$ under the equivalence of categories in Proposition 3.3, then there is a natural isomorphism*

$$H^*_{\acute{e}t}(X; \mathcal{F}) \cong H^*(\acute{E}t(X); \acute{E}t(\mathcal{F})).$$

⁵ For the construction of $\acute{E}t(X)$ one also makes a choice of sufficiently many separably closed fields so that every residue field of a point of X is contained in one of them, but we can ignore this technicality.

3.3 Steenrod’s cup- i product

Let X be a topological space. Let R be a local coefficient system in commutative rings, and $C^*(X; R)$ the singular cochain complex. Steenrod defined sequence of maps

$$\begin{aligned} \text{cup}_i: C^r(X; R) \otimes C^s(X; R) &\rightarrow C^{r+s-i}(X; R) \\ u \otimes v &\mapsto u \smile_i v \end{aligned}$$

called the ‘cup- i products’. We will give a high-level exposition; a reference for this standard (in topology) material is [MT68, ch. 2].

The cup product for X is induced at the level of chain complexes by the composition of the Alexander–Whitney map and the restriction to the diagonal:

$$C^*(X; R) \leftarrow C^*(X \times X; R) \xleftarrow{\sim} C^*(X; R) \otimes C^*(X; R). \tag{3.3.1}$$

This composition is not S_2 -equivariant because the Alexander–Whitney map is not S_2 -equivariant; it is only S_2 -equivariant up to homotopy. However, there is a way to rectify it to be an S_2 -equivariant quasi-isomorphism, which we now describe.

Let ES_2 be a contractible space with a free S_2 -action; in fact, let us take the explicit model $ES_2 = S^\infty$. We view $C_*(ES_2; \underline{R})$ as a cochain complex in non-positive degrees, which provides a free resolution as S_2 -modules of the constant sheaf \underline{R} in degree 0. Then there is an S_2 -equivariant quasi-isomorphism

$$C^*(X \times X; R) \simeq C^*(X; R) \otimes C^*(X; R) \otimes C_*(ES_2; \underline{R}), \tag{3.3.2}$$

where the S_2 action on the right-hand side is diagonal for the ‘swap’ action on $C^*(X; R) \otimes C^*(X; R)$ and the tautological action on ES_2 . Tensoring (3.3.2) with $C^*(ES_2; \underline{R})$ and applying the evaluation pairing yields an S_2 -equivariant cochain map

$$C^*(X; R) \otimes C^*(ES_2; \underline{R}) \leftarrow C^*(X; R) \otimes C^*(X; R), \tag{3.3.3}$$

where the S_2 -action is via ‘swap’ on the right-hand side, and the tautological action on $C^*(ES_2; \underline{R})$ on the left-hand side.

Now we use the presentation of S^∞ as a simplicial complex with two cells d_i and Td_i in every dimension, which are interchanged under the S_2 -action. In the chain complex $C_*(ES_2; \underline{R})$ we then have two corresponding generators $e_i \otimes 1$ and $Te_i \otimes 1 \in C_i(ES_2; \underline{R})$. Contracting (3.3.3) with $e_i \otimes 1$ gives the cup- i product

$$C^{r+s-i}(X; R) \leftarrow C^r(X; R) \otimes C^s(X; R): \text{cup}_i.$$

We will also use the notation

$$u \smile_i v := \text{cup}_i(u \otimes v).$$

We have the coboundary formula [MT68, ch. 2, p. 16]

$$d(u \smile_i v) = (-1)^i du \smile_i v + (-1)^{i+r} u \smile_i dv - (-1)^i u \smile_{i-1} v - (-1)^{rs} v \smile_{i-1} u, \tag{3.3.4}$$

where $|u| = r, |v| = s$. We can rewrite (3.3.4) as

$$d(u \smile_i v) - (-1)^i \text{cup}_i(d(u \otimes v)) = (-1)^{i-1} u \smile_{i-1} v - (-1)^{rs} v \smile_{i-1} u. \tag{3.3.5}$$

It is the case $i = 1$ in (3.3.5) that will be most important for us. For concreteness, let us spell out the informal meaning of (3.3.5). The cup-0 product is just the multiplication on cochains. The cup-1 product furnishes a chain homotopy between $u \smile_0 v$ and $\pm v \smile_0 u$ ‘witnessing’ the graded commutativity of the cup product. The cup-2 product furnishes a chain homotopy between $u \smile_1 v$ and $\pm v \smile_1 u$, etc.

Remark 3.6. In order to bootstrap the cup_i -product from simplicial sets to étale topological type as in § 3.2, we take a model for the cup_i -product, which is *functorial* in maps of simplicial sets, whose existence is guaranteed by [Smi15, Appendix B]. (Although the cup_i -product in [Smi15] is phrased with integral coefficients, it exists for any local coefficient system of commutative rings, by base change.)

We now turn to the task of extracting cohomology operations out of the cup_i product. The cup_i product does not preserve cocycles, except in characteristic 2, so that is the simplest case in which we get cohomology operations, and we discuss it first.

3.4 Classical Steenrod squares

If $2 = 0$ in R , then it is easily checked from (3.3.4) that the operation

$$u \mapsto u \smile_i u$$

sends cocycles to cocycles and coboundaries to coboundaries, and hence descends to a cohomology operation

$$\text{Sq}_i : H^r(X; R) \rightarrow H^{2r-i}(X; R).$$

We then define the Steenrod square

$$\text{Sq}^i := \text{Sq}_{r-i} : H^r(X; R) \rightarrow H^{r+i}(X; R).$$

For $R = \mathbf{Z}/2\mathbf{Z}$, which is the case studied in [MT68, § 2], this construction recovers the classical Steenrod squares.

Properties of the Steenrod squares. We now recall the formal properties of these classical Steenrod squares. (Proofs can be found in [MT68, § 2,3].)

(i) (NATURALITY) For any $f : X' \rightarrow X$, we have

$$f^* \text{Sq}^i = \text{Sq}^i f^*.$$

(ii) (CARTAN FORMULA) We have

$$\text{Sq}^i(x \smile y) = \sum_{j=0}^i \text{Sq}^j(x) \smile \text{Sq}^{i-j}(y).$$

If we define the *total Steenrod operation* $\text{Sq} := \sum_i \text{Sq}^i$, then the Cartan formula can be neatly packaged as

$$\text{Sq}(x \smile y) = \text{Sq}(x) \smile \text{Sq}(y).$$

(iii) (ADEM RELATIONS) If $0 < i < 2j$ then

$$\text{Sq}^i \text{Sq}^j = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2k} \text{Sq}^{i+j-k} \text{Sq}^k.$$

(iv) (SPECIAL CASES) For $x \in H^j(X; \mathbf{Z}/2\mathbf{Z})$ we have

- $\text{Sq}^0(x) = x$;
- $\text{Sq}^1(x) = \beta(x)$ for β the connecting homomorphism $H^j(X; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\beta} H^{j+1}(X; \mathbf{Z}/2\mathbf{Z})$ induced by the short exact sequence $0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/4\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 0$;

- $Sq^j(x) = x \smile x$;
- for $i > j$, we have $Sq^i(x) = 0$.

(v) (STABILITY) The Steenrod operations commute with the suspension isomorphisms

$$H^i(X; \mathbf{Z}/2) \cong H^{i+1}(\Sigma X; \mathbf{Z}/2).$$

3.5 Generalized Steenrod squares

We now drop our assumption that $2 = 0$ in R . If u is a cocycle, we see from (3.3.4) that

$$d(u \smile_i u) = [(-1)^i - (-1)^{r^2}]u \smile_{i-1} u. \tag{3.5.1}$$

Suppose $2^n = 0$ in R . If $r - i$ is even, then (3.5.1) implies that $2^{n-1}u \smile_i u$ is a cocycle. Furthermore, one can check that the operation $u \mapsto 2^{n-1}u \smile_i u$ also takes coboundaries to coboundaries, and therefore descends to a cohomology operation

$$\widetilde{Sq}_i: H^r(X; R) \rightarrow H^{2r-i}(X; R).$$

DEFINITION 3.7. If i is even, we define

$$\widetilde{Sq}^i := \widetilde{Sq}_{r-i}: H^r(X; R) \rightarrow H^{r+i}(X; R).$$

LEMMA 3.8. Continue to assume that $2^n = 0$ in R . Let $\text{red}_2: H^*(X; R) \rightarrow H^*(X; R/2R)$ be the reduction mod 2, and let $[2^{n-1}]: H^*(X; R/2R) \rightarrow H^*(X; R)$ be the map induced by $R/2R \xrightarrow{2^{n-1}} R$. If i is even, then we have

$$\widetilde{Sq}^i = [2^{n-1}] \circ Sq^i \circ \text{red}_2.$$

Proof. This is immediate upon unwinding the definitions. □

Next suppose that $r - i$ is odd. In this case we do not assume a priori that $2^n = 0$ in R . (Although we do not need the operations where $r - i$ is odd in this paper, we construct them for the sake of completeness.) From (3.5.1) we see that if $r - i$ is odd, then $u \smile_i u$ is a cocycle if u is a cocycle. Similarly one checks that $u \mapsto u \smile_i u$ sends coboundaries to coboundaries, hence descends to a cohomology operation

$$\widetilde{Sq}_i: H^r(X; R) \rightarrow H^{2r-i}(X; R).$$

DEFINITION 3.9. If i is odd, we define

$$\widetilde{Sq}^i := \widetilde{Sq}_{r-i}: H^r(X; R) \rightarrow H^{r+i}(X; R).$$

Let us elucidate the relationship between the Steenrod squares constructed in the two cases. If u is cocycle, then by (3.3.4) we have

$$u \smile_i u = (-1)^i 2d(u \smile_{i+1} u).$$

Then the analog of Lemma 3.8 is as follows.

LEMMA 3.10. Suppose that the complex

$$0 \rightarrow R/2^n R \xrightarrow{2} R/2^{n+1} R \rightarrow R/2R \rightarrow 0$$

is short exact. Let $\beta_{2,2^n}: H^*(X; R/2R) \rightarrow H^{*+1}(X; R/2^n R)$ be the induced boundary map, and let red_2 be as in Lemma 3.8. If i is odd, then we have

$$\widetilde{Sq}^i = \beta_{2,2^n} \circ Sq^{i-1} \circ \text{red}_2.$$

Proof. This is immediate upon unwinding the definitions. □

3.6 Application to étale cohomology

Let $R := \bigoplus_{j \in \mathbf{Z}} \mathbf{Z}/2^n \mathbf{Z}(j)$, viewed as a locally constant sheaf on X (where ‘ (j) ’ denotes the Tate twist) valued in rings, with multiplicative structure given by the isomorphisms

$$\mathbf{Z}/2^n \mathbf{Z}(j) \otimes \mathbf{Z}/2^n \mathbf{Z}(j') \xrightarrow{\sim} \mathbf{Z}/2^n \mathbf{Z}(j + j').$$

Applying § 3.2 and the construction of § 3.5, we obtain operations

$$\widetilde{\text{Sq}}^i : H_{\text{ét}}^r(X; \mathbf{Z}/2^n \mathbf{Z}(j)) \rightarrow H_{\text{ét}}^{r+i}(X; \mathbf{Z}/2^n \mathbf{Z}(2j)).$$

For convenience of the reader, we summarize all the facts that we shall need about the $\widetilde{\text{Sq}}^i$ below.

Example 3.11. If $x \in H_{\text{ét}}^{i+1}(X; \mathbf{Z}/2^n \mathbf{Z}(j))$ then $\widetilde{\text{Sq}}^i(x)$ has the following description. Let $C_{\text{ét}}^*(X; \mathbf{Z}/2^n \mathbf{Z}(j))$ be the étale cochain complex for X , defined as in § 3.2. There is a chain homotopy

$$\begin{aligned} \text{cup}_1 : C_{\text{ét}}^r(X; \mathbf{Z}/2^n \mathbf{Z}(j)) \otimes C_{\text{ét}}^s(X; \mathbf{Z}/2^n \mathbf{Z}(j)) &\rightarrow C_{\text{ét}}^{r+s-1}(X; \mathbf{Z}/2^n \mathbf{Z}(2j)) \\ u \otimes v &\mapsto u \smile_1 v \end{aligned}$$

such that (by the $i = 1$ case of (3.3.5))

$$d(\text{cup}_1(u \otimes v)) + \text{cup}_1(d(u \otimes v)) = u \smile v - (-1)^{rs} v \smile u.$$

Let $u \in C_{\text{ét}}^{i+1}(X; \mathbf{Z}/2^n \mathbf{Z}(j))$ be a representative for x . If i is even, then we have

$$\widetilde{\text{Sq}}^i(x) = [2^{n-1}u \smile_1 u].$$

LEMMA 3.12. Let $[2^{n-1}] : H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z}(j)) \rightarrow H_{\text{ét}}^*(X; \mathbf{Z}/2^n \mathbf{Z}(j))$ be the map induced by the inclusion of sheaves $\mathbf{Z}/2\mathbf{Z}(j) \xrightarrow{2^{n-1}} \mathbf{Z}/2^n \mathbf{Z}(j)$. Let red_2 be the reduction mod 2. If i is even, then

$$\widetilde{\text{Sq}}^i = [2^{n-1}] \circ \text{Sq}^i \circ \text{red}_2.$$

Proof. This follows from combining Lemma 3.8, which implies the formula for all simplicial complexes in particular, and § 3.2, which transports the result to étale cohomology. \square

4. Bockstein operations

The goal of this section is to prove Theorem 4.4, which expresses the pairing of Definition 2.2 in terms of cohomology operations. Our argument originated from studying a generalized version of the Bockstein spectral sequence, and was motivated by a calculation in [Bro61]. However, we have found it cleaner for expository purposes to present a proof with the language of spectral sequences stripped out.

4.1 Higher Bockstein operations

The key technical ingredient in the proof is the study of ‘higher Bockstein operations’. These form a family of cohomology operations $\{\beta_r\}$ ‘growing off’ of the Bockstein β in the following sense. We have

$$\beta_1 := \beta : H_{\text{ét}}^*(X; \mathbf{Z}/2^n \mathbf{Z}(j)) \rightarrow H_{\text{ét}}^{*+1}(X; \mathbf{Z}/2^n \mathbf{Z}(j)).$$

The operation β_r is only defined on the kernel of $\beta_1, \dots, \beta_{r-1}$, and its image is only defined modulo the image of $\beta_1, \dots, \beta_{r-1}$. (These operations β_r arise as differentials in a spectral sequence, which explains this structure.) In fact we only need β_1 and β_2 for our purposes.

Remark 4.1. We must now confront the technical subtlety that \mathbf{Z}_ℓ -étale cohomology is not, as defined classically, the cohomology of a cochain complex with \mathbf{Z}_ℓ -coefficients, while the state of affairs is much more naturally reasoned about and phrased in terms of ‘integral cochains’. It is straightforward to translate all our statements into ones about compatible systems of ℓ -adic sheaves; for example the short exact sequence

$$0 \rightarrow \mathbf{Z}_\ell(j) \xrightarrow{\ell^n} \mathbf{Z}_\ell(j) \rightarrow \mathbf{Z}/\ell^n\mathbf{Z}(j) \rightarrow 0$$

should be replaced by the system of sequences

$$0 \rightarrow \mathbf{Z}/\ell^N\mathbf{Z}(j) \xrightarrow{\ell^n} \mathbf{Z}/\ell^{N+n}\mathbf{Z}(j) \rightarrow \mathbf{Z}/\ell^n\mathbf{Z}(j) \rightarrow 0$$

for all $N \gg 0$. We leave this translation to the reader so as not to complicate our exposition. (It should also be possible to deal with this problem more uniformly using the pro-étale topology of Bhatt–Scholze [BS15].)

Recall that the Bockstein β is induced by the short exact sequence of sheaves

$$0 \rightarrow \mathbf{Z}/\ell^n\mathbf{Z}(j) \xrightarrow{\ell^n} \mathbf{Z}/\ell^{2n}\mathbf{Z}(j) \rightarrow \mathbf{Z}/\ell^n\mathbf{Z}(j) \rightarrow 0.$$

Thanks to the commutative diagram

$$\begin{array}{ccccc} C^*(X; \mathbf{Z}_\ell(j)) & \xrightarrow{\ell^n} & C^*(X; \mathbf{Z}_\ell(j)) & \longrightarrow & C^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j)) \\ \downarrow & & \downarrow & & \parallel \\ C^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j)) & \longrightarrow & C^*(X; \mathbf{Z}/\ell^{2n}\mathbf{Z}(j)) & \longrightarrow & C^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j)) \end{array}$$

it admits the following alternative description. For $x \in H_{\text{ét}}^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))$ we let $\tilde{x} \in C_{\text{ét}}^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))$ be a representative for x , and a a lift of \tilde{x} in $C_{\text{ét}}^*(X; \mathbf{Z}_\ell(j))$. Since \tilde{x} is a cocycle, da is divisible by ℓ^n within $C_{\text{ét}}^{*+1}(X; \mathbf{Z}_\ell(j))$, so we may define

$$\tilde{\beta}(x) := \left[\frac{1}{\ell^n} da \right] \in H_{\text{ét}}^{*+1}(X; \mathbf{Z}_\ell(j)).$$

Then we define

$$\beta(x) := \overline{\tilde{\beta}(x)} \in H_{\text{ét}}^{*+1}(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))$$

to be the reduction of $\tilde{\beta}(x)$ mod ℓ^n . Note that $\tilde{\beta}$ is the boundary map for the short exact sequence of sheaves

$$0 \rightarrow \mathbf{Z}_\ell(j) \xrightarrow{\ell^n} \mathbf{Z}_\ell(j) \rightarrow \mathbf{Z}/\ell^n\mathbf{Z}(j) \rightarrow 0.$$

DEFINITION 4.2. We define operations

$$\begin{aligned} \tilde{\beta}_2 &: (\ker \beta_1 \subset H_{\text{ét}}^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))) \rightarrow H_{\text{ét}}^{*+1}(X; \mathbf{Z}_\ell(j))/\text{Im } \tilde{\beta}_1, \\ \beta_2 = \overline{\tilde{\beta}_2} &: (\ker \beta_1 \subset H_{\text{ét}}^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))) \rightarrow H_{\text{ét}}^{*+1}(X; \mathbf{Z}/\ell^n\mathbf{Z}(j))/\text{Im } \beta_1 \end{aligned}$$

as follows. If $x \in \ker \beta_1$, then (keeping the notation of the preceding paragraph) we have

$$\overline{\frac{1}{\ell^n} da} = d\tilde{y} \quad \text{for some } \tilde{y} \in C_{\text{ét}}^*(X; \mathbf{Z}/\ell^n\mathbf{Z}(j)),$$

where the overline means reduction mod ℓ^n . Hence we may and do choose a lift $b \in C_{\text{ét}}^*(X; \mathbf{Z}_\ell(j))$ of \tilde{y} such that $(1/\ell^n)da \equiv db \pmod{\ell^n}$, or in other words

$$da \equiv d(\ell^n b) \pmod{\ell^{2n}}.$$

Then we can form the cochain $(1/\ell^{2n})d(a - \ell^n b) \in C_{\text{ét}}^*(X; \mathbf{Z}_\ell(j))$, which is evidently a cocycle. Finally, we define

$$\begin{aligned} \tilde{\beta}_2(x) &:= \left[\frac{1}{\ell^{2n}} d(a - \ell^n b) \right] \in H_{\text{ét}}^{*+1}(X; \mathbf{Z}_\ell(j)) / \text{Im } \tilde{\beta}_1, \\ \beta_2(x) &:= \overline{\tilde{\beta}_2(x)} \in H_{\text{ét}}^{*+1}(X; \mathbf{Z}/\ell^n \mathbf{Z}(j)) / \text{Im } \beta_1. \end{aligned}$$

We leave it to the reader to check that this is indeed well-defined.

It is straightforward to define β_r in a similar way for all r . Since we only need β_1 and β_2 , we do not spell out the explicit construction.

In what follows, we will focus on the operations introduced in Definition 4.2 for $\ell = 2$.

PROPOSITION 4.3. *Let $\ell = 2$ in Definition 4.2. For any $x \in H_{\text{ét}}^{2k}(X; \mathbf{Z}/2^n \mathbf{Z}(j))$, we have the following identity:*

$$\beta_2(2^{n-1}x^2) = [x \cdot \beta(x) - \widetilde{\text{Sq}}^{2k}(\beta(x))] \in H_{\text{ét}}^{2k+1}(X; \mathbf{Z}/2^n \mathbf{Z}(2j)) / \text{Im } \beta.$$

Proof. Note that since β is a derivation, it indeed kills $2^{n-1}x^2$, hence $2^{n-1}x^2$ indeed lives in $\ker \beta$ so that $\beta_2(2^{n-1}x^2)$ is defined.

Let a be any integral cochain in $C_{\text{ét}}^{2k}(X; \mathbf{Z}_2(j))$ lifting a representative for x . Let $y := \beta(x) \in H_{\text{ét}}^{2k+1}(X; \mathbf{Z}/2^n \mathbf{Z}(j))$. Then by the definition of β , we have $da = 2^n b$, where $b \in C_{\text{ét}}^{2k+1}(X; \mathbf{Z}_2(j))$ lifts a representative for y .

According to the discussion in Definition 4.2, $\beta_2(2^{n-1}x^2)$ is calculated by finding an integral cochain lifting a representative for $2^{n-1}x^2$, whose boundary is divisible by 2^{2n} . We check that $2^{n-1}a^2 + 2^{2n-1}(a \smile_1 b)$ is such an integral cochain, using Example 3.11:

$$\begin{aligned} d(2^{n-1}a^2 + 2^{2n-1}(a \smile_1 b)) &= 2^{n-1}(a \cdot 2^n b + 2^n b \cdot a - 2^{2n}(b \smile_1 b)) + 2^n(ab - ba) \\ &= 2^{n-1}(2^{n+1}ab - 2^{2n}(b \smile_1 b)) \\ &= 2^{2n}(ab - 2^{n-1}(b \smile_1 b)). \end{aligned}$$

Hence by Definition 4.2 we have

$$\begin{aligned} \beta_2(2^{n-1}x^2) &= \left[\frac{1}{2^{2n}} d(2^{n-1}a^2 - 2^{2n-1}(a \smile_1 b)) \right] \\ &= [ab - 2^{n-1}(b \smile_1 b)] \\ &= [xy - [2^{n-1}(b \smile_1 b)]]. \end{aligned}$$

We then conclude by using Example 3.11 again to identify $[2^{n-1}(b \smile_1 b)] = \widetilde{\text{Sq}}^{2k} y$. □

THEOREM 4.4. *Let X be a smooth projective variety over \mathbf{F}_q of dimension $2d$. For $x \in H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n \mathbf{Z}(d))$, we have*

$$x \smile \beta(x) = \widetilde{\text{Sq}}^{2d}(\beta(x)) \in H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n \mathbf{Z}(2d)).$$

Proof. By Proposition 4.3, we have the identity

$$\beta_2(2^{n-1}x^2) = [x \cdot \beta(x) - \widetilde{\text{Sq}}^{2k}(\beta(x))] \in H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d))/\text{Im } \beta.$$

Therefore we will be done if we can show that the images of β and β_2 in $H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d))$ are both 0. Since β and β_2 are the reductions of $\tilde{\beta}$ and $\tilde{\beta}_2$, it suffices to prove the stronger statement that $\tilde{\beta}$ and $\tilde{\beta}_2$ vanish in the appropriate degree, which is what we shall do.

Note that the image of $\tilde{\beta}$ is automatically 2^n -torsion. Similarly, from the definition of $\tilde{\beta}_2$ it is immediate that its image is 2^{2n} -torsion. Indeed, referring to Definition 4.2 we see that $\ell^{2n}\tilde{\beta}_2(x) = [d(a - \ell^n b)]$ is manifestly a coboundary. (In general, $\text{Im}(\tilde{\beta}_r)$ is 2^{rn} -torsion.) But by Poincaré duality we have

$$H_{\text{ét}}^{4d+1}(X; \mathbf{Z}_2^n(2d)) \cong \mathbf{Z}_2$$

is torsion-free, so the images of $\tilde{\beta}$ and $\tilde{\beta}_2$ in $H_{\text{ét}}^{4d+1}(X; \mathbf{Z}/2^n\mathbf{Z}(2d))$ are necessarily 0. □

5. Stiefel–Whitney classes in étale cohomology

Theorem 4.4 recasts the pairing $\langle \cdot, \cdot \rangle_n$ in terms of the (generalized) Steenrod squares. But in order for this formula to be useful, we need some way to explicitly calculate the relevant Steenrod operations. In the classical theory of smooth manifolds there is a formula, due originally to Wu, relating the action of certain Steenrod operations as cupping with Stiefel–Whitney classes. This section and the next are concerned with establishing an analog of this formula in absolute étale cohomology for smooth proper varieties over finite fields. The first task, which we take up in this section, is to define an appropriate notion of Stiefel–Whitney classes.

Much of the material of this section was influenced by [Ura96]. The definition of the classes w_i already appears in [Ura96], though it is phrased in less generality there. The idea to find lifts of the w_i in terms of Chern classes was also inspired by [Ura96], as is the proof of Theorem 5.10.

5.1 Cohomology with supports

Let $i: X \hookrightarrow Y$ be a closed subscheme. We recall the definition of the ‘cohomology of Y with supports in X ’ [FK88, §I.10]. Let $j: U \hookrightarrow Y$ denote the inclusion of the (open) complement of X in Y . Then $H_X^k(Y; \mathcal{F})$ is defined to be the k th right derived functor of

$$\mathcal{F} \mapsto \ker(\mathcal{F}(Y) \mapsto j_*j^*\mathcal{F}(Y)).$$

We want to define Steenrod operations on $H_X^k(Y; \mathcal{F})$. Since we have been in the habit of defining Steenrod operations via étale homotopy theory, we need to realize the cohomology with supports in terms of étale homotopy theory, as the cohomology of a certain pro-space. This is explained in [Fri82, § 14]. The key features of this construction are summarized below.

DEFINITION 5.1. We define $\acute{E}t_X(Y)$ to be the mapping cylinder of $\acute{E}t(U) \rightarrow \acute{E}t(Y)$ in the sense of [Fri82, p. 140]. For any locally constant sheaf \mathcal{F} on the étale site of Y , we denote by $\acute{E}t(\mathcal{F})$ the corresponding local coefficient system on $\acute{E}t(Y)$ as in § 3.2, and also for its pullback to $\acute{E}t_X(Y)$ in the sense of [Fri82, p. 140]. Then we have a canonical identification $H^*(\acute{E}t_X(Y); \acute{E}t(\mathcal{F})) \xrightarrow{\sim} H_X^*(Y; \mathcal{F})$ [Fri82, Propositions 14.3 and 14.6, Corollary 14.5], and we define the Steenrod operations on $H_X^*(Y; \mathcal{F})$ via this identification as in § 3.

5.2 Construction of étale Stiefel–Whitney classes

Let k be a field of characteristic not equal to 2. Let $i: X \hookrightarrow Y$ be a codimension r closed embedding of smooth varieties over k . Then we have a *cycle class* $s_{X/Y} \in H_X^{2r}(Y; \mathbf{Z}_\ell(r))$, which can be described as the image of 1 under the *Gysin isomorphism*

$$\phi: H^j(X; \mathcal{F}) \xrightarrow{\sim} H_X^{j+2r}(Y; \mathcal{F}(r)), \tag{5.2.1}$$

which holds for any locally constant constructible sheaf \mathcal{F} on Y [FK88, §I.10].

We are going to apply this with Y being the total space of a vector bundle E over X , and $i: X \hookrightarrow E$ being the zero section.

DEFINITION 5.2. Let E be a vector bundle over X . We define the *j th Stiefel–Whitney class* of E by

$$w_j(E) = \phi^{-1}(\text{Sq}^j s_{X/E}). \tag{5.2.2}$$

Define the *total Stiefel–Whitney class* to be $w(E) := \sum w_i(E)$. If no vector bundle is mentioned, then by default we set $w_i := w_i(TX)$ and $w := \sum w_i$.

There is a possibly more intuitive way to phrase (5.2.2), which we will use later. The Gysin isomorphism (5.2.1) says that $H_X^*(E; \mathbf{Z}/2\mathbf{Z})$ is a free rank-one module over $H_{\text{ét}}^*(E; \mathbf{Z}/2\mathbf{Z})$, which can be identified with $H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z})$ via π^* since E is a vector bundle over X . Under this identification (5.2.2) is equivalent to

$$\text{Sq}^i(s_{X/E}) = w_i \cdot s_{X/E}. \tag{5.2.3}$$

Remark 5.3. The reason that we call these ‘Stiefel–Whitney classes’ is that Thom observed in [Tho52] that an exactly analogous construction for manifolds produces the usual Stiefel–Whitney classes.⁶ The construction goes as follows (a reference is [MS74, §8]). Let M be a topological manifold and E be a vector bundle of rank r over M . Let $i: M \hookrightarrow E$ denote the inclusion of M as the zero section of E . Let $E_0 = E - i(M)$. We have a Thom isomorphism

$$\phi: H^i(M; \mathbf{Z}/2\mathbf{Z}) \cong H^{i+2r}(E, E_0; \mathbf{Z}/2\mathbf{Z}),$$

and $w_i(E) = \phi^{-1}(\text{Sq}^i \phi(1))$.

5.3 Steenrod squares of Stiefel–Whitney classes

The following technical lemma is needed later in §7.1. The reader may safely skip this subsection for now and refer back to it when necessary.

LEMMA 5.4. For any i and j , $\text{Sq}^i(w_j)$ can be expressed as a polynomial in the Stiefel–Whitney classes $\{w_l\}$.

Remark 5.5. The analog of Lemma 5.4 for singular cohomology is immediate from the fact that the ring $H^*(\text{BO}(\mathbf{R}); \mathbf{Z}/2\mathbf{Z})$ is generated by Stiefel–Whitney classes. But because of the way that we have defined the classes w_i in étale cohomology, Lemma 5.4 is not quite obvious.

⁶ Unfortunately, this notation clashes with the established tradition of using the term ‘Stiefel–Whitney classes’ to denote the characteristic classes of quadratic bundles. The two definitions coincide for smooth manifolds, but in general there is no relation between them.

Proof of Lemma 5.4. We will use the identities of Steenrod squares from §3.4. Note that we may assume that $i < j$, since for $i > j$ we have $Sq^i(w_j) = 0$ and for $i = j$ we have $Sq^i(w_j) = w_j^2$.

We induct on j , and then (for fixed j) on i , with the base case $j = 0$ being trivial, and the base cases $i = 0$ being trivial for any j since $Sq^0 = Id$. Consider the expression

$$Sq^i Sq^j(s_{X/TX}).$$

On the one hand, we have by (5.2.3) that

$$Sq^i Sq^j(s_{X/TX}) = Sq^i(w_j \cdot s_{X/TX}).$$

By the Cartan formula,

$$\begin{aligned} Sq^i(w_j \cdot s_{X/TX}) &= \sum_{k+\ell=i} Sq^k(w_j) Sq^\ell(s_{X/TX}) \\ &= \left(Sq^i(w_j) + \sum_{k<i} Sq^k(w_j) w_{i-k} \right) s_{X/TX}. \end{aligned}$$

By the induction hypothesis, $Sq^k(w_j)$ is a polynomial in the $\{w_l\}$ for each $k < i$, so the upshot is that

$$Sq^i Sq^j(s_{X/TX}) = (Sq^i(w_j) + Poly(\{w_l\}))s_{X/TX}. \tag{5.3.1}$$

From (5.3.1) it is clearly sufficient to show that $Sq^i Sq^j(s_{X/TX})$ is a polynomial in the $\{w_l\}$ times $s_{X/TX}$. For this we use the Adem relations: for $0 < i < 2j$ we have

$$Sq^i Sq^j = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2k} Sq^{i+j-k} Sq^k,$$

hence

$$Sq^i Sq^j(s_{X/TX}) = \sum_{k=0}^{[i/2]} \binom{j-1-k}{i-2k} Sq^{i+j-k}(w_k \cdot s_{X/TX}).$$

Every index k in this sum is strictly less than j since we assumed $i < j$, so every summand is a polynomial in the $\{w_l\}$ times $s_{X/TX}$ by the induction hypothesis, which is what we wanted. \square

Remark 5.6. Since the preceding argument could have been carried out equally well in singular cohomology, the proof makes it clear that our $Sq^i w_j$ is given by the same formula as in algebraic topology.

5.4 Properties of the Stiefel–Whitney classes

We now record that the Stiefel–Whitney classes, as constructed in §5.2, enjoy the usual properties of topological Stiefel–Whitney classes.

- (i) We have $w_i(E) \in H^i(X; \mathbf{Z}/2\mathbf{Z})$, with $w_0 = 1$ and $w_i = 0$ for $i > 2 \text{rank } E$.
- (ii) (NATURALITY) If $f: X' \rightarrow X$ then

$$f^* w_i(E) = w_i(f^* E).$$

(iii) (WHITNEY PRODUCT FORMULA) We have

$$w_i(E \oplus E') = \sum_{k=0}^i w_k(E) \smile w_{i-k}(E').$$

If we set $w = \sum w_i$ to be the total Stiefel–Whitney class, then this can be written more succinctly as

$$w(E \oplus E') = w(E) \cdot w(E').$$

Proofs. It is well-known in the topological setting (cf. [MS74, §8]) that the characteristic properties of Stiefel–Whitney classes can be formally derived from those of the Steenrod squares. Since our étale Stiefel–Whitney classes are also based on Steenrod operations, essentially the same proofs go through. Nonetheless, we spell them out because they will be used in the proof of Theorem 5.10 below.

(i) Immediate from the fact that $Sq^0 = \text{Id}$ and Sq^i vanishes on H^j if $i > j$.

(ii) Immediate from the naturality of the Gysin map and Steenrod squares.

(iii) We begin by considering a general setup. Suppose we have two closed embeddings of smooth proper varieties

$$\begin{aligned} i: X &\hookrightarrow Y && \text{codimension } r, \\ i': X' &\hookrightarrow Y && \text{codimension } r'. \end{aligned}$$

We consider the two corresponding Gysin maps obtained:

$$\begin{aligned} \phi: H^*(X; \mathbf{Z}/2\mathbf{Z}) &\cong H_X^{*+r}(Y; \mathbf{Z}/2\mathbf{Z}), \\ \phi': H^*(X'; \mathbf{Z}/2\mathbf{Z}) &\cong H_{X'}^{*+r'}(Y'; \mathbf{Z}/2\mathbf{Z}). \end{aligned}$$

These send $\phi(1) = s_{X/Y}$ and $\phi'(1) = s_{X'/Y'}$. By the compatibility of the Gysin map for products, we have that for the closed embedding $X \times X' \hookrightarrow Y \times Y'$, the Gysin isomorphism

$$\phi \smile \phi': H^*(X \times X'; \mathbf{Z}/2\mathbf{Z}) \cong H_{X \times X'}^{*+r+r'}(Y \times Y'; \mathbf{Z}/2\mathbf{Z})$$

sends $1 \mapsto s_{X/Y} \smile s_{X'/Y'}$. Now taking Y and Y' to be the total spaces of E and E' , and applying the Cartan formula of Sq and the definition of Stiefel–Whitney classes, we obtain

$$\begin{aligned} w_{X \times X'} \smile (s_{X/Y} \smile s_{X'/Y'}) &= Sq(s_{X/Y} \smile s_{X'/Y'}) \\ &= Sq(s_{X/Y}) \smile Sq(s_{X'/Y'}) \\ &= (w_X \smile s_{X/Y}) \smile (w_{X'} \smile s_{X'/Y'}) \\ &= (w_X \smile w_{X'}) s_{X/Y} \smile s_{X'/Y'}. \end{aligned}$$

Finally, pulling back via the diagonal $\Delta: X \hookrightarrow X \times X$ and using naturality gives the result.

It is formal that the Whitney product formula for direct sums implies it for extensions.

LEMMA 5.7. *If*

$$0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$$

is a short exact sequence of vector bundles on X , then

$$w(E) = w(E') \smile w(E'').$$

Proof. The proof is the same as for [Ura96, Lemma 2.7]. □

5.5 Lifting Stiefel–Whitney classes to integral cohomology

We shall see in § 7 that it is crucial to know whether our Stiefel–Whitney classes lift to integral cohomology. The goal of this subsection is to prove Theorem 5.10, which answers this question.

Our first task is to address a technical subtlety that will come up in the proof of Theorem 5.10. There are the two short exact sequences

$$0 \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow \mathbf{Z}/4\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 0 \tag{5.5.1}$$

and

$$0 \rightarrow \mu_2 \rightarrow \mu_4 \rightarrow \mu_2 \rightarrow 0. \tag{5.5.2}$$

Since μ_2 is canonically identified with $\mathbf{Z}/2\mathbf{Z}$, as we are not in characteristic 2, both sequences induce Bockstein operations $H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z}) \rightarrow H_{\text{ét}}^{*+1}(X; \mathbf{Z}/2\mathbf{Z})$, but they are *not* necessarily the same. In § 3.4 we noted that the Bockstein operation for (5.5.1) is Sq^1 . Let us denote by $\beta^{(1)}$ the Bockstein operation for (5.5.2). We need to quantify the difference between these two operations. For this discussion, it will help to maintain a psychological distinction between μ_2 and $\mathbf{Z}/2\mathbf{Z}$.

LEMMA 5.8. *Let α be the image of $1 \in H_{\text{ét}}^0(X; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} H_{\text{ét}}^0(X; \mu_2)$ under the boundary map $\beta^{(1)}$. Then for all $c \in H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\sim} H_{\text{ét}}^*(X; \mu_2)$ we have*

$$\beta^{(1)}(c) = \text{Sq}^1(c) + \alpha \smile c.$$

Proof. Since μ_4 is a module over $\mathbf{Z}/4\mathbf{Z}$, the cohomology $H_{\text{ét}}^*(X; \mu_4)$ is a module over $H_{\text{ét}}^*(X; \mathbf{Z}/4\mathbf{Z})$. We similarly view $H_{\text{ét}}^*(X; \mu_2)$ as a module over $H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z})$.

The reduction map $\mu_4 \rightarrow \mu_2$, viewed as part of the short exact sequence (5.5.2), is compatible with the reduction map $\mathbf{Z}/4\mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z}$, viewed as part of (5.5.1), for the respective module structures. Hence the induced maps on cohomology satisfy the same compatibility: the reduction map

$$H_{\text{ét}}^*(X; \mu_4) \rightarrow H_{\text{ét}}^*(X; \mu_2)$$

is compatible as a map of modules with respect to the map of rings

$$H_{\text{ét}}^*(X; \mathbf{Z}/4\mathbf{Z}) \rightarrow H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z}).$$

More precisely, let $x \in H_{\text{ét}}^i(X; \mu_2)$ and $r \in H_{\text{ét}}^j(X; \mathbf{Z}/2\mathbf{Z})$, so that x is viewed as a module element and r is viewed as a ring element. Then $rx \in H_{\text{ét}}^{i+j}(X; \mu_2)$, and we are saying that

$$\beta^{(1)}(rx) = (\text{Sq}^1 r)x + r\beta^{(1)}(x). \tag{5.5.3}$$

This is seen immediately upon going back to the definition of the boundary map, using that the coboundary map on cochains is a derivation.

The lemma then follows from taking $r = c$ and $x = 1 \in H^0(X; \mu_2)$ in (5.5.3). □

Remark 5.9. The element $\alpha \in H_{\text{ét}}^1(X; \mu_2)$ is actually the pullback of a universal $\alpha \in H_{\text{ét}}^1(\text{Spec } \mathbf{F}_q; \mu_2)$, which vanishes if and only if $q \equiv 1 \pmod{4}$. Indeed, (5.5.1) and (5.5.2) are obviously the same for $q \equiv 1 \pmod{4}$. We also note for later use that α lifts to $H_{\text{ét}}^1(X; \mathbf{Z}_2(1))$, because $\beta^{(1)}$ is the reduction of the Bockstein for

$$0 \rightarrow \mathbf{Z}_2(1) \xrightarrow{2} \mathbf{Z}_2(1) \rightarrow \mu_2 \rightarrow 0.$$

THEOREM 5.10. *Let X be a smooth variety over a finite field \mathbf{F}_q of characteristic not 2 and E a vector bundle on X of rank r . Let α be as in Lemma 5.8. Then we have*

$$w(E) := 1 + w_1 + w_2 + \cdots + w_{2r} = \begin{cases} (1 + \alpha)\bar{c}_{\text{even}} + \bar{c}_{\text{odd}} & r \text{ odd,} \\ \bar{c}_{\text{even}} + (1 + \alpha)\bar{c}_{\text{odd}} & r \text{ even,} \end{cases}$$

where

$$c_{\text{even}} = 1 + c_2 + c_4 + \cdots \in H_{\text{ét}}^* \left(X; \bigoplus_{i \in \mathbf{Z}} \mathbf{Z}_2(i) \right),$$

$$c_{\text{odd}} = c_1 + c_3 + \cdots \in H_{\text{ét}}^* \left(X; \bigoplus_{i \in \mathbf{Z}} \mathbf{Z}_2(i) \right),$$

and \bar{c} means the reduction of c modulo 2.

Proof. Grothendieck showed [Gro58] that the definition of all characteristic classes can be obtained from the axioms in §5.4 plus the definition of the characteristic classes for arbitrary line bundles. Therefore, it suffices to check that the formula above satisfies the properties in §5.4 and is correct for all line bundles.

The fact that it satisfies axiom (i) of §5.4 is evident from the definition. The fact that it satisfies (ii) is immediate from the observation that the Chern classes satisfy the Whitney sum and naturality property. The fact that it satisfies (iii) also follows from the analogous property of Chern classes plus a case analysis of the formula. For example, when summing two bundles \mathcal{E} and \mathcal{E}' of odd rank with Chern classes c and c' , the product of the classes claimed in the formula is

$$\begin{aligned} ((1 + \alpha)\bar{c}_{\text{even}} + \bar{c}_{\text{odd}})((1 + \alpha)\bar{c}'_{\text{even}} + \bar{c}'_{\text{odd}}) &= (\bar{c}_{\text{even}}\bar{c}'_{\text{even}} + \bar{c}_{\text{odd}}\bar{c}'_{\text{odd}}) \\ &\quad + (1 + \alpha)(\bar{c}_{\text{even}}\bar{c}'_{\text{odd}} + \bar{c}_{\text{odd}}\bar{c}'_{\text{even}}) \end{aligned}$$

because $(1 + \alpha)^2 = 1$, and the Whitney sum formula for Chern classes implies that the right-hand side is indeed $c_{\text{even}}(\mathcal{E} \oplus \mathcal{E}') + (1 + \alpha)c_{\text{odd}}(\mathcal{E} \oplus \mathcal{E}')$.

Finally, we must check the formula for line bundles. What makes this possible is that we only have to verify the formula for w_1 and w_2 , since the higher Stiefel–Whitney classes vanish for degree reasons. Thus we only need to compute Sq^1 and Sq^2 , and we have ‘explicit’ descriptions of the Steenrod operations on degree 2 elements in these cases (§3.4).

Let Y be the total space of a line bundle \mathcal{L} on X . We view X as embedded in Y via the zero section, and identify their étale cohomology groups via pullback for the projection map $\pi: Y \rightarrow X$.

Calculation of w_1 . Recall from (5.2.3) that w_1 is defined by

$$\text{Sq}^1 s_{X/Y} = w_1 \smile s_{X/Y}.$$

But the cycle class $s_{X/Y}$ lifts compatibly to $H_X^2(Y; \mu_{2^j})$ for all j , and hence even to $H_Y^2(X; \mathbf{Z}_2(1))$ (cf. [FK88, §II.2]). In particular, $s_{X/Y}$ lies in the image of the reduction map $H_X^2(Y; \mu_4) \rightarrow H_X^2(Y; \mu_2)$. The long exact sequence for (5.5.2) then shows that $\beta^{(1)}(s_{X/Y}) = 0$. We are really interested in the *other* boundary map Sq^1 , but Lemma 5.8 tells us the difference between them:

$$\text{Sq}^1(s_{X/Y}) = \beta^{(1)}(s_{X/Y}) + \alpha \smile s_{X/Y} = \alpha \smile s_{X/Y}.$$

Hence $\alpha = w_1$, as required.

Calculation of w_2 . The argument is essentially the same as in the proof of [Ura96, Lemma 2.6]. Again, (5.2.3) tells us that

$$\text{Sq}^2 s_{X/Y} = w_2 \smile s_{X/Y} \in H^4_X(Y; \mathbf{Z}/2\mathbf{Z}).$$

Since $s_{X/Y} \in H^2_X(Y; \mathbf{Z}/2\mathbf{Z})$ we have that $\text{Sq}^2 s_{X/Y} = s_{X/Y} \smile s_{X/Y}$ (using one of the explicit ‘special cases’ from §3.4).

We now need to recall a property of the cycle class, which is a special case of a more general discussion to come in §6.3.2. If $X \hookrightarrow Y$ is a codimension 1 closed embedding of smooth varieties, then we have a cycle class $\text{cl}_Y(X) \in H^2(Y; \mu_2)$, which is the image of the line bundle $\mathcal{O}_Y(X)$ under the Chern class map $H^1(Y; \mathbf{G}_m) \rightarrow H^2(Y; \mu_2)$. This class $\text{cl}_Y(X)$ also coincides with the image of $s_{X/Y}$ under the natural map $H^*_X(Y) \rightarrow H^*_{\text{ét}}(Y)$. (A reference is [FK88, Propositions II.2.2 and II.2.6].⁷)

Consider the following commutative diagram.

$$\begin{CD} \text{Pic}(X) = H^1_{\text{ét}}(X; \mathbf{G}_m) @>>> H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z}) \\ @V \pi^* VV @VV \pi^* V \\ \text{Pic}(Y) = H^1_{\text{ét}}(Y; \mathbf{G}_m) @>>> H^2_{\text{ét}}(Y; \mathbf{Z}/2\mathbf{Z}) \end{CD}$$

An elementary calculation shows that the line bundle \mathcal{L} on X whose total space is Y pulls back to $\mathcal{O}_Y(X)$ on Y , i.e. the line bundle associated to the divisor of the zero-section in Y . The upshot is that in $H^4_X(Y; \mathbf{Z}/2\mathbf{Z})$, we have

$$\overline{c}_1(\mathcal{O}_Y(X)) \smile s_{X/Y} = \text{cl}_Y(X) \smile s_{X/Y} = s_{X/Y} \smile s_{X/Y} = \text{Sq}^2(s_{X/Y}),$$

which shows that $w_2 = \overline{c}_1(\mathcal{O}_Y(X)) \in H^2_{\text{ét}}(Y; \mathbf{Z}/2\mathbf{Z})$. Since we have already established that \mathcal{L} pulls back to $\mathcal{O}_Y(X)$ under the projection map $\pi: Y \rightarrow X$, naturality for Chern classes and the fact that π^* induces an isomorphism on cohomology shows that $w_2 = \overline{c}_1(\mathcal{L}) \in H^2_{\text{ét}}(X; \mathbf{Z}/2\mathbf{Z})$. \square

6. A Wu theorem for étale cohomology

Now we relate the Stiefel–Whitney classes just constructed in §5 with Steenrod operations. *In this section it is understood that all cohomology is with $\mathbf{Z}/2\mathbf{Z}$ -coefficients*, so we may suppress it from our notation.

6.1 Wu’s theorem for smooth manifolds

We first explain the classical version of Wu’s theorem. Let M be a closed smooth manifold of dimension n , so that the cup product induces a perfect duality on $H^*(M)$. Then for a cohomology class $x \in H^{n-i}(M; \mathbf{Z}/2\mathbf{Z})$ the map $x \mapsto \text{Sq}^i x \in H^n(M; \mathbf{Z}/2\mathbf{Z})$ must, by Poincaré duality, be represented by a class $v_i \in H^i(M; \mathbf{Z}/2\mathbf{Z})$, i.e.

$$v_i \smile x = \text{Sq}^i x \quad \text{for all } x \in H^i(M; \mathbf{Z}/2\mathbf{Z}).$$

This v_i is called the *i th Wu class*.

Let $v := \sum_i v_i$ be the total Wu class and $w := \sum w_i$ be the total Stiefel–Whitney class of TM . Then Wu’s formula relates the two in the following way.

⁷ The book [FK88] makes a blanket assumption that the ground field is separably closed, but the proofs of these particular facts do not require this assumption.

THEOREM 6.1 (Wu). *We have $w = \text{Sq} v$.*

Remark 6.2. Note that Sq is invertible, so Wu’s theorem completely describes v in terms of w .

Example 6.3. We use Wu’s theorem to calculate a few small examples. Equating terms of degree 1, we deduce that

$$v_1 = w_1.$$

Equating terms of degree 2, we deduce that $v_2 + \text{Sq}^1 v_1 = w_2$, which we can rewrite as

$$v_2 = w_2 + w_1^2.$$

6.2 A Wu theorem for varieties over finite fields

The aim of this section is to prove a version of Wu’s theorem in the setting of étale cohomology. For varieties over *separably closed* fields, this is done in [Ura96, Theorem 0.5]. In that case one can more or less transpose the usual proof for manifolds, essentially because the ℓ -adic cohomology of smooth varieties over separably closed fields behaves very similarly to the singular cohomology of complex manifolds. In particular, for a surface that lifts to characteristic 0, the classical version of Wu’s theorem implies the version for geometric ℓ -adic cohomology. The main result of this section (Theorem 6.5) is that the same formula also holds for *absolute* étale cohomology over *finite* fields, with our definitions of the w_i from §5. Because the ground field is not separably closed there are some significant new difficulties; one indication of this is that the proof requires étale homotopy theory.

Remark 6.4. The author has come to think about this philosophically as follows. A major defect in the analogy between varieties over \mathbf{F}_q and topological spaces fibered over S^1 is that in the latter situation one can forget the fibration and consider the bare topological space, while there is no corresponding move for varieties over \mathbf{F}_q . Thus any operation performed in the category of varieties over \mathbf{F}_q is really a ‘relative’ operation: the product of varieties over \mathbf{F}_q corresponds to the *fibred* product of manifolds over S^1 , the tangent bundle of a variety over \mathbf{F}_q corresponds to the *relative* tangent bundle over S^1 , etc. Because of this, there are some steps in the proof of Wu’s theorem that have no analog in the category of varieties over \mathbf{F}_q . However, passing to étale homotopy type allows one to disassociate a variety from this fibration, and thus acquire some of the additional flexibility enjoyed by topological spaces.

THEOREM 6.5. *Let X be a smooth, proper, geometrically connected variety over \mathbf{F}_q . Define the Wu class $v \in H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z})$ to be the unique cohomology class such that*

$$\int \text{Sq} x = \int v \smile x \quad \text{for all } x \in H_{\text{ét}}^*(X; \mathbf{Z}/2\mathbf{Z}).$$

Then we have $w = \text{Sq} v$.

The reader is recommended to skip the proof on the first pass through the paper, as it is quite lengthy and nothing but the statement of Theorem 6.5 will be used in the following.

6.3 Proof of Theorem 6.5

Our proof of Theorem 6.5 proceeds in four steps. Steps 2 and 3 are essentially a translation of the usual (topological) proof to algebraic geometry. Step 1 performs a technical reduction to the case where the topological argument begins, and is necessary because of the lack of ‘tubular neighborhoods’ in algebraic geometry. Finally, step 4 bridges a new technical difficulty, the spirit of which is discussed in Remark 6.4, that arises here because our ground field is not separably closed.

6.3.1 *Step 1.* Recall from (5.2.3) that we defined the Stiefel–Whitney classes w_i by

$$\mathrm{Sq}^i s_{X/TX} = \pi^*(w_i) \smile s_{X/TX},$$

where $\pi: TX \rightarrow X$ is the projection. Recall also that the normal bundle of X in $X \times X$ is isomorphic to TX . The purpose of this step is to prove the following lemma, which is motivated by the preceding considerations.

LEMMA 6.6. *Let $s_{X/X \times X} \in H_X^{*+2n}(X \times X)$ be the image of $1 \in H_{\text{ét}}^0(X)$ under the Gysin isomorphism*

$$H_{\text{ét}}^*(X) \xrightarrow{\sim} H_X^{*+2n}(X \times X).$$

Then we have

$$\mathrm{Sq}^i s_{X/X \times X} = \mathrm{pr}_1^*(w_i) \smile s_{X/X \times X}, \tag{6.3.1}$$

where $\mathrm{pr}_1: X \times X \rightarrow X$ denotes projection to the first factor.

From the definitions Lemma 6.6 is an immediate consequence of the following lemma.

LEMMA 6.7. *Let $X \hookrightarrow Y$ be a codimension n closed embedding of smooth varieties (over any field) and let*

$$\phi_1: H^*(X) \xrightarrow{\sim} H_X^{*+2n}(N_{X/Y}) \phi_2: H^*(X) \xrightarrow{\sim} H_X^{*+2n}(Y)$$

be the two Gysin isomorphisms. Then

$$w(N_{X/Y}) := \phi_1^{-1}(\mathrm{Sq} s_{X/N_{X/Y}}) = \phi_2^{-1}(\mathrm{Sq} s_{X/Y}).$$

Remark 6.8. If X were a smooth manifold, we could argue directly since we have an isomorphism

$$H_X^{*+2n}(N_{X/Y}) \cong H^{*+2n}(U, U \setminus X),$$

where U is a tubular neighborhood of the zero-section in Y , and we have

$$H^{*+2n}(U, U \setminus X) \cong H_X^{*+2n}(Y) := H^{*+2n}(Y, Y \setminus X)$$

by excision. Since these isomorphisms are pullbacks induced by maps of spaces, they commute with Steenrod squares. A referee has suggested that an analog of this argument may be carried out in our setting using the Morel–Voevodsky purity theorem.

Proof. The key fact is that if $X \hookrightarrow Y$ is a closed embedding, then there is a flat family deforming the inclusion $X \hookrightarrow Y$ into the zero-section $X \hookrightarrow N_{X/Y}$ (‘deformation to the normal cone’). This allows us to carry out the idea of Remark 6.8.

More precisely, there is a flat family $\mathcal{Y} \rightarrow \mathbf{A}^1$, which restricts to the trivial family away from the origin, $\mathcal{Y}|_{\mathbf{A}^1 - 0} \cong Y \times (\mathbf{A}^1 - 0)$, but such that $\mathcal{Y}|_0 \cong N_{X/Y}$. Furthermore, there is a closed embedding $X \times \mathbf{A}^1 \hookrightarrow \mathcal{Y}$, which restricts to the given embedding $X \hookrightarrow Y$ away from 0, and $X \hookrightarrow N_{X/Y}$ at 0. For the construction and proofs of the properties, see [Ful98, § 5]. The situation is depicted in the diagram below.

$$\begin{array}{ccccc} X & \hookrightarrow & X \times \mathbf{A}^1 & \hookleftarrow & X \\ \downarrow & & \downarrow & & \downarrow \\ N_{X/Y} & \hookrightarrow & \mathcal{Y} & \hookleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \hookrightarrow & \mathbf{A}^1 & \hookleftarrow & t \end{array}$$

Applying the Gysin morphism to $X \times \mathbf{A}^1 \hookrightarrow \mathcal{Y}$, we have an isomorphism

$$H_{\text{ét}}^*(X \times \mathbf{A}^1) \xrightarrow{\sim} H_{X \times \mathbf{A}^1}^{*+2n}(\mathcal{Y})$$

sending $1 \mapsto s_{X \times \mathbf{A}^1/\mathcal{Y}}$. Note that $X \times \mathbf{A}^1$ and Y (viewed as the fiber over t) intersect transversely in \mathcal{Y} , and similarly $X \times \mathbf{A}^1$ and $N_{X/Y}$. Hence from the diagram above we obtain a diagram of maps in cohomology (where the vertical maps are the respective Gysin isomorphisms).

$$\begin{array}{ccccc} H_{\text{ét}}^*(X) & \longleftarrow & H_{\text{ét}}^*(X \times \mathbf{A}^1) & \longrightarrow & H_{\text{ét}}^*(X) \\ \sim \downarrow & & \sim \downarrow & & \sim \downarrow \\ H_X^{*+2n}(N_{X/Y}) & \longleftarrow & H_{X \times \mathbf{A}^1}^{*+2n}(\mathcal{Y}) & \longrightarrow & H_X^{*+2n}(Y) \end{array}$$

Under this diagram the Thom classes are mapped as follows, by compatibility with base change (cf. §2 of Deligne’s exposé ‘Cycle’ in [SGA4½]).

$$\begin{array}{ccccc} 1 & \longleftarrow & 1 & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow \\ s_{X/N_{X/Y}} & \longleftarrow & s_{X \times \mathbf{A}^1/\mathcal{Y}} & \longrightarrow & s_{X/Y} \end{array}$$

Since the horizontal maps in the bottom row are pullbacks they are compatible with Sq, and hence send

$$\text{Sq}^i(s_{X/N_{X/Y}}) \longleftarrow \text{Sq}^i(s_{X \times \mathbf{A}^1/\mathcal{Y}}) \longrightarrow \text{Sq}^i(s_{X/Y}).$$

By definition $\text{Sq}^i(s_{X/N_{X/Y}}) = \pi^*(w_i) \smile s_{X/N_{X/Y}}$, and since the maps

$$H_X^{*+2n}(N_{X/Y}) \longleftarrow H_{X \times \mathbf{A}^1}^{*+2n}(\mathcal{Y}) \longrightarrow H_X^{*+2n}(Y)$$

are isomorphisms of $H_{\text{ét}}^*(X) \cong H_{\text{ét}}^*(X \times \mathbf{A}^1)$ -modules, they send

$$\begin{array}{ccccc} \text{Sq}^i(s_{X/N_{X/Y}}) & \longleftarrow & \text{Sq}^i(s_{X \times \mathbf{A}^1/\mathcal{Y}}) & \longrightarrow & \text{Sq}^i(s_{X/Y}) \\ \parallel & & \parallel & & \parallel \\ \text{pr}_1^*(w_i) \smile s_{X/N_{X/Y}} & \longleftarrow & \text{pr}_1^*(w_i) \smile s_{X \times \mathbf{A}^1/\mathcal{Y}} & \longrightarrow & \pi^*(w_i) \smile s_{X/Y} \end{array}$$

as desired. □

6.3.2 *Step 2.* For a regular embedding $X \hookrightarrow Y$, there is an attached *cycle class* $\text{cl}_Y(X) \in H^*(Y)$, which in topology would be the ‘Poincaré dual to the fundamental class of X in homology’. The goal of this step is to prove the following lemma.

LEMMA 6.9. *Let $\Delta := \text{cl}_{X \times X}(X) \in H_{\text{ét}}^*(X \times X)$ be the cycle class for the diagonal embedding of X . Then we have $w = (\text{pr}_1)_*(\text{Sq} \Delta) \in H_{\text{ét}}^*(X)$.*

We first review the definition of the pushforward in cohomology for a map of smooth proper varieties, and then the definition of the cycle class.

DEFINITION 6.10 (Pushforwards in cohomology). If $f: X \rightarrow Y$ is a map of smooth proper varieties over \mathbf{F}_q of dimensions m and n , respectively, then the pullback map

$$f^*: H_{\text{ét}}^*(Y) \rightarrow H_{\text{ét}}^*(X)$$

induces an adjoint map on the $\mathbf{Z}/2\mathbf{Z}$ -dual spaces:

$$(f^*)^\vee: H_{\text{ét}}^*(X)^\vee \rightarrow H_{\text{ét}}^*(Y)^\vee. \tag{6.3.2}$$

We can identify $H_{\text{ét}}^*(X)^\vee \cong H_{\text{ét}}^{2m+1-*}(X)$ by Poincaré duality, obtaining from (6.3.2) a map

$$f_*: H_{\text{ét}}^*(X) \rightarrow H_{\text{ét}}^{*+2n-2m}(Y).$$

In particular, we define the *cycle class of X in Y* to be $f_*(1) =: \text{cl}_Y(X)$. Unwrapping the definition, the class $\text{cl}_Y(X)$ is characterized by the identity

$$\int_X f^* \gamma = \int_Y \text{cl}_Y(X) \smile \gamma \quad \text{for all } \gamma \in H^*(Y).$$

We recall some basic properties of this pushforward. The proofs are all immediate from the definition except the last, which is [FK88, Proposition 2.7].

- It is functorial.
- We have the *product formula*

$$f_*(\alpha \smile f^* \beta) = (f_* \alpha) \smile \beta. \tag{6.3.3}$$

- If $X \hookrightarrow Y$ is a closed embedding, then the map $H_X^*(Y) \rightarrow H_{\text{ét}}^*(Y)$ sends $s_{X/Y} \mapsto \text{cl}_Y(X)$.

Proof of Lemma 6.9. We now apply the preceding discussion to the case $Y = X \times X$, with f being the diagonal embedding. By Lemma 6.6 we have

$$\text{Sq}^i s_{X/X \times X} = \text{pr}_1^*(w_i) \smile s_{X/X \times X} \in H_X^{2n+i}(X \times X).$$

Since the map $H_X^*(X \times X) \rightarrow H_{\text{ét}}^*(X \times X)$ is induced by a pullback (§ 5.1) it is automatically compatible with Steenrod operations, so it sends as follows.

$$\begin{array}{ccc} \text{Sq}^i s_{X/X \times X} & \longrightarrow & \text{Sq}^i \text{cl}_{X \times X}(X) \\ \parallel & & \parallel \\ \text{pr}_1^*(w_i) \smile s_{X/X \times X} & \longrightarrow & \text{pr}_1^*(w_i) \smile \text{cl}_{X \times X}(X) \end{array}$$

Hence by Definition 6.10 and (6.3.3) we have

$$(\text{pr}_1)_*(\text{pr}_1^*(w_i) \smile \text{cl}_{X \times X}(X)) = w_i \smile (\text{pr}_1)_* f_* 1 = w_i. \quad \square$$

6.3.3 *Step 3.* At this point, the classical proof of Wu’s theorem proceeds by computing $(\text{pr}_1)_* \text{cl}_{X \times X}(X)$ in a second way, which is predicated upon the Künneth formula

$$H_{\text{ét}}^*(X \times X) \cong H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X),$$

which unfortunately breaks down in our situation. To explain how to repair the argument, we need to make some observations. Note that $H_{\text{ét}}^*(X \times X)$ acts by correspondences on $H_{\text{ét}}^*(X)$, inducing the map

$$H_{\text{ét}}^*(X \times X) \rightarrow \text{End}(H_{\text{ét}}^*(X)) \tag{6.3.4}$$

given explicitly by sending $x \in H_{\text{ét}}^*(X \times X)$ to the endomorphism

$$\gamma \mapsto (\text{pr}_1)_*(x \smile \text{pr}_2^* \gamma). \tag{6.3.5}$$

LEMMA 6.11. Let $\Delta := \text{cl}_{X \times X}(X) \in H_{\text{ét}}^{2\dim X}(X \times X)$. Then the map (6.3.4) sends $\Delta \mapsto \text{Id}$.

Proof. Let $f: X \hookrightarrow X \times X$ denote the inclusion of the diagonal. Applying (6.3.5) to $x = \Delta$ yields the endomorphism

$$\gamma \mapsto (\text{pr}_1)_*(f_*(1) \smile p_2^*\gamma) = (\text{pr}_1)_*f_*(1 \smile f^*\text{pr}_2^*\gamma).$$

But since $\text{pr}_1 \circ f = \text{pr}_2 \circ f = \text{Id}$, this last expression is just γ again. □

The map (6.3.4) can be interpreted as a ‘pushforward’ in the following way. The projection morphisms $\text{pr}_1, \text{pr}_2: X \times X \rightarrow X$ induce maps $\text{pr}_1^*, \text{pr}_2^*: H_{\text{ét}}^*(X) \rightarrow H_{\text{ét}}^*(X \times X)$. From this we get a pullback map

$$H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) \xrightarrow{\text{pr}_1^* \smile \text{pr}_2^*} H_{\text{ét}}^*(X \times X).$$

Therefore, we get a dual map in the opposite direction

$$H_{\text{ét}}^*(X \times X)^\vee \rightarrow H_{\text{ét}}^*(X)^\vee \otimes H_{\text{ét}}^*(X)^\vee.$$

Each of these groups is canonically self-dual via Poincaré duality, so we can identify this with a map

$$\varphi_*: H_{\text{ét}}^*(X \times X) \rightarrow H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X). \tag{6.3.6}$$

Note that this map *increases the total degree* by 1. It is a straightforward exercise in unraveling the definitions to see that this map is the same as (6.3.4), once one makes the appropriate identifications.

Let $(p'_1)_*$ and $(p'_2)_*$ denote the ‘pushforward’ maps

$$\begin{array}{ccc} & H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) & \\ (p'_1)_* \swarrow & & \searrow (p'_2)_* \\ H_{\text{ét}}^*(X) & & H_{\text{ét}}^*(X) \end{array}$$

which are dual to the obvious ‘pullbacks’

$$\begin{array}{ccc} & H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) & \\ (p'_1)^* \nearrow & & \nwarrow (p'_2)^* \\ H_{\text{ét}}^*(X) & & H_{\text{ét}}^*(X) \end{array}$$

Remark 6.12. The maps $(p'_i)^*$ and $(p'_i)_*$ are *not* induced by maps of varieties; indeed $H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X)$ is not the cohomology of a variety over \mathbf{F}_q . However, $H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X)$ *is* the cohomology of the pro-space $\acute{\text{E}}\text{t}(X) \times \acute{\text{E}}\text{t}(X)$. This means, for instance, that it is equipped with a natural cup product, which is just the tensor product of the cup products on $H_{\text{ét}}^*(X)$. Now, $(p'_i)^*$ and $(p'_i)_*$ are induced by maps of pro-spaces, namely the obvious projection maps

$$p'_i: \acute{\text{E}}\text{t}(X) \times \acute{\text{E}}\text{t}(X) \rightarrow \acute{\text{E}}\text{t}(X).$$

This implies that $(p'_i)^*$ and $(p'_i)_*$ share the nice formal properties that are enjoyed by all pullbacks and pushforwards: for example, we will use that they satisfy the projection formula, and that $(p'_i)^*$ commutes with Steenrod operations.

However, it is not really necessary to use étale homotopy theory to see all this. We can just *formally define* the cup product on $H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X)$ to be the tensor product of the cup products on $H_{\text{ét}}^*(X)$, and *formally define* Sq on $H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X)$ to be the tensor product of $\text{Sq} \otimes \text{Sq}$. It is an exercise in elementary algebra to check that this induces a well-defined action of the Steenrod algebra, satisfying all the axioms of § 3.4. Similarly, the projection formula for $(p_i)_*$ boils down to a tautology.

LEMMA 6.13. *Let X be a smooth proper variety over a finite field. Let $\{e_i\}$ be a basis for $H_{\text{ét}}^*(X)$ and $\{f_i\}$ the dual basis under Poincaré duality. Then, letting Δ be as in Lemma 6.11, we have*

$$\varphi_* \Delta = \sum_i e_i \otimes f_i, \tag{6.3.7}$$

where φ_* is as in (6.3.6).

Proof. Lemma 6.11 says that the action of Δ induced on $H^*(X)$ by (6.3.5) is just the identity map. Therefore, it suffices to show that the right-hand side of (6.3.7) acts as the identity on $H^*(X)$, but this is just a straightforward linear algebra exercise about dual bases. \square

Since the pullback $H_{\text{ét}}^*(X) \xrightarrow{\text{pr}_1^*} H_{\text{ét}}^*(X \times X)$ obviously factors through

$$H_{\text{ét}}^*(X) \xrightarrow{(p'_1)^*} H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) \xrightarrow{\text{pr}_1^* \sim \text{pr}_2^*} H_{\text{ét}}^*(X \times X)$$

(morally, ‘ $\text{pr}_1 = \varphi \circ p'_1$ ’) we have

$$(p_1)_* = (p'_1)_* \varphi_*. \tag{6.3.8}$$

Now, let us summarize where we are. Combining Lemma 6.9 and (6.3.8), we know that

$$w = (p_1)_* \text{Sq} \Delta = (p'_1)_* \varphi_* \text{Sq} \Delta. \tag{6.3.9}$$

Lemma 6.13 gives us an expression for $\varphi_* \Delta$, and hence also $\text{Sq} \varphi_* \Delta$. If we could commute φ_* and Sq , then this would give us a formula for $\varphi_* \text{Sq} \Delta$. But although Steenrod squares commute with pullbacks, they do *not* in general commute with pushforwards. This is the key problem (note that the whole issue disappears when one has the Künneth formula, as in classical algebraic topology or in algebraic geometry over separably closed fields). To address this issue, in the last step of the proof, we will establish the following.

PROPOSITION 6.14. *Let X be a smooth proper variety over a finite field, and φ_* be as in (6.3.6). Then we have $\text{Sq} \varphi_* = \varphi_* \text{Sq}$.*

Assuming Proposition 6.14 for now (it will be shown in § 6.3.4), we complete the rest of the proof of Theorem 6.5. Let $\{e_i\}$ be a basis for $H_{\text{ét}}^*(X)$ and $\{f_i\}$ the dual basis under Poincaré duality, as above. By (6.3.9), Proposition 6.14, and Lemma 6.13 we have

$$w = (p'_1)_* \text{Sq} \left(\sum_i (p'_1)^* e_i \smile (p'_2)^* f_i \right). \tag{6.3.10}$$

By the Cartan formula and the projection formula (which hold by Remark 6.12), we have

$$(p'_1)_* \text{Sq} \left(\sum_i (p'_1)^* e_i \smile (p'_2)^* f_i \right) = \sum_i (p'_1)_* ((p'_1)^* \text{Sq} e_i \smile (p'_2)^* \text{Sq} f_i)$$

$$\begin{aligned}
 &= \sum_i \text{Sq } e_i \smile (p'_1)_*(p'_2)^* \text{Sq } f_i \\
 &= \sum_i \text{Sq } e_i \otimes (p'_1)_*(p'_2)^* \text{Sq } f_i.
 \end{aligned}
 \tag{6.3.11}$$

Now, unraveling the definitions shows that

$$(p'_1)_*(p'_2)^* \gamma = \int \gamma \quad \text{for all } \gamma \in H_{\text{ét}}^*(X),$$

where the right-hand side is viewed in $\mathbf{Z}/2\mathbf{Z} \cong H_{\text{ét}}^0(X)$. (It is also easy to see directly that this must be the case for degree reasons, since the left side can only be non-zero for γ in top degree.) Combining this with (6.3.10) and (6.3.11), we find that

$$w = \sum_i \text{Sq } e_i \cdot \int \text{Sq } f_i = \sum_i \text{Sq } e_i \cdot \int v \smile f_i = \text{Sq} \left(\sum_i e_i \cdot \int v \smile f_i \right) = \text{Sq } v,$$

with the last equality using that $\{e_i\}$ and $\{f_i\}$ are dual bases. □

6.3.4 *Step 4.* This step is devoted to the proof of Proposition 6.14. As foreshadowed in Remark 6.12, the difficulty stems from the inability to realize $H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X)$ as the cohomology of an actual variety over \mathbf{F}_q . For this reason it is useful to pass to étale topological type, where we can interpret

$$H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) = H_{\text{ét}}^*(\text{Ét}(X) \times \text{Ét}(X)).$$

The idea of the argument. The basic geometric idea is that the map

$$\varphi_*: H_{\text{ét}}^*(X \times X) \rightarrow H_{\text{ét}}^*(X) \otimes H_{\text{ét}}^*(X) \tag{6.3.12}$$

looks like a pushforward map on cohomology induced by a ‘homotopy quotient by $\widehat{\mathbf{Z}}$ ’ at the level of geometric objects. Proposition 6.14 is then motivated by the well-known fact (which we prove below in Proposition 6.23) that Steenrod operations commute with pushforward through a homotopy quotient by \mathbf{Z} , and that cohomologically (with finite coefficients) homotopy quotients by \mathbf{Z} and by $\widehat{\mathbf{Z}}$ look the same.

To see why the ‘basic geometric idea’ should be true, our heuristic is that for any field k and $G_k := \text{Gal}(\bar{k}/k)$, we should have

$$\text{‘Ét}(X) = \text{Ét}(X_{\bar{k}})_{hG_k}\text{’}. \tag{6.3.13}$$

Here if a group G acts on a space Y then we write $Y_{hG} := (Y \times EG)/G$ for the ‘homotopy quotient of Y by G ’, where EG is some contractible space with a free G action, and the quotient is for the (free) diagonal action. The heuristic (6.3.13) then suggests that

$$\text{Ét}(X \times_k X) \cong \text{Ét}((X \times_k X)_{\bar{k}})_{hG_k} \cong \text{Ét}(X_{\bar{k}} \times_{\bar{k}} X_{\bar{k}})_{hG_k} \cong (\text{Ét}(X_{\bar{k}}) \times \text{Ét}(X_{\bar{k}}))_{hG_k},$$

where the quotient is for the action of the diagonal G_k (leaving a residual action of G_k), while

$$\text{Ét}(X) \times \text{Ét}(X) \cong \text{Ét}(X_{\bar{k}})_{hG_k} \times \text{Ét}(X_{\bar{k}})_{hG_k} \cong (\text{Ét}(X_{\bar{k}}) \times \text{Ét}(X_{\bar{k}}))_{h(G_k \times G_k)}.$$

Hence we would have a homotopy quotient

$$\text{Ét}(X \times_k X) \rightarrow \text{Ét}(X \times_k X)_{hG_k} \cong \text{Ét}(X) \times \text{Ét}(X)$$

whose induced pushforward on cohomology agrees recovers (6.3.12).

In a previous version of this paper, we used some complicated gymnastics in cohomology as a substitute for the fact that we did not know how to rigorously formulate (6.3.13). We are very grateful to an anonymous referee for informing us that there already exists a framework to handle this sort of issue, namely the ‘relative étale homotopy theory’ developed in [HS13, BS16]. This formalism makes the argument much more efficient and transparent, so we review it next.

Relative étale homotopy theory. If X is a variety over a field k and L/k is a finite extension, then $\mathcal{E}t(X_L)$ is a pro-object in simplicial sets equipped with an action of $\text{Gal}(L/k)$ as a pro-object. However, it would be better to work with an object that enjoys a level-wise action, rather than an action as a pro-object. In [HS13] Harpaz–Schlank defined a refined variant of the étale homotopy type denoted $\mathcal{E}t_{/k}(X)$, called the *relative étale homotopy type* [HS13, §9.2.3]⁸ of X , which is a pro-object in the homotopy category of G_k -spaces (which means by definition that every simplex has an open stabilizer).⁹ The point is that $\mathcal{E}t_{/k}(X)$ is equipped with a *level-wise* action of G_k . In [BS16] Barnea–Schlank lifted this construction to the pro-category of G_k -spaces, and it is this refinement that we will use in this paper. The improvement is analogous to Friedlander’s refinement of the Artin–Mazur étale homotopy type, which is a pro-object in the homotopy category of spaces, to the étale topological type discussed in §3.2, which is a pro-object in spaces, although the methods of [BS16] are very different.

Here is a very brief summary of the difference between $\mathcal{E}t_{/k}(X)$ and $\mathcal{E}t(X)$; see [HS13, §9.2.3] and [BS16, §8] for the details. The basic idea is that the usual definition of $\mathcal{E}t(X)$ attaches to each hypercovering U of X the simplicial set $\pi_0(U)$ of its connected components. On the other hand, $\mathcal{E}t_{/k}(X)$ assigns to U the simplicial set $\pi_0(U \times_k \bar{k})$, which is equipped with an obvious G_k -action.

DEFINITION 6.15 [HS13, §9.6.2]. Given a G_k -space Y , we define the *homotopy quotient* Y_{hG_k} to be the pro-space

$$Y_{hG_k} := \{(Y \times E(G_k/H))/G_k\}_H,$$

where the index set runs over i and open normal subgroups $H \triangleleft G_k$.

Given a pro- G_k -space $\{Y_i\}_i$, we define the *homotopy quotient* $(\{Y_i\}_i)_{hG_k}$ to be the pro-space

$$(\{Y_i\}_i)_{hG_k} := \{(Y_i \times E(G_k/H))/G_k\}_{i,H},$$

where the index set runs over open normal subgroup $H \triangleleft G_k$.

Remark 6.16. Note that $(Y \times E(G_k/H))/G_k = (Y/H)_{h(G_k/H)}$. In particular, if H acts trivially on Y then $(Y \times E(G_k/H))/G_k = Y_{h(G_k/H)}$.

We next begin discussing the key properties of $\mathcal{E}t_{/k}(X)$. Actually, we replace $\mathcal{E}t_{/k}(X)$ by its Postnikov truncation denoted $\mathcal{E}t_{/k}(X)^\sharp$ in [HS13], which does not alter the cohomology. This is a technical device to guarantee certain finiteness conditions levelwise; for simplicity of notation we will omit the \sharp .

(i) By [HS13, Proposition 9.82], we have a homotopy equivalence

$$\mathcal{E}t_{/k}(X)_{hG_k} \xrightarrow{\sim} \mathcal{E}t(X).$$

⁸ Note that the arXiv version of [HS13] is numbered differently from the published version.

⁹ As in §3.2, we will refer to simplicial sets as ‘spaces’ to make the exposition smoother.

(ii) By the cofinality of the diagonal in the product of a left filtered index category with itself, [HS13, Proposition 9.82] also implies that we have a homotopy equivalence

$$(\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{hG_k \times hG_k} \xrightarrow{\sim} \acute{E}t(X) \times \acute{E}t(X).$$

(iii) Finally, [HS13, Proposition 9.19] asserts that the underlying pro-space of $\acute{E}t_{/k}(X)$ (obtained by forgetting the G_k -action) is homotopy equivalent to $\acute{E}t(X_{\bar{k}})$. Denote this forgetful functor by Oblv .

Now, $\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X)$ is naturally a pro- $G_k \times G_k$ -space. Taking the homotopy quotient for the diagonal G_k -action leaves a residual G_k -action, making $(\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{hG_k}$ a pro- G_k -space, so that we can take the homotopy quotient again. Putting the facts (i)–(iii) together, we realize the homotopy quotient map (6.3.13) as the vertical map between pro-spaces in the diagram below.

$$\begin{CD} \text{Oblv}((\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{hG_k}) @>\sim>> (\acute{E}t_{/k}(X \times_k X))_{hG_k} @>\sim>> \acute{E}t(X \times_k X) \\ @VV/hG_k V @. @. \\ (\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{h(G_k \times G_k)} @>\sim>>> \acute{E}t(X) \times \acute{E}t(X) \end{CD} \tag{6.3.14}$$

Comparing homotopy quotients by \mathbf{Z} and $\widehat{\mathbf{Z}}$. Let $Y := \{Y_i\}_i$ be any pro- $\widehat{\mathbf{Z}}$ -space. We can restrict the $\widehat{\mathbf{Z}}$ -action to a \mathbf{Z} -action, and then form $Y_{h\mathbf{Z}}$. By Definition 6.15 and Remark 6.16, we have $Y_{h\mathbf{Z}} = \{(Y_i)_{h\mathbf{Z}}\}_i$ while $Y_{h\widehat{\mathbf{Z}}} = \{(Y_i/n\mathbf{Z})_{h(\mathbf{Z}/n\mathbf{Z})}\}_{i,n}$. Thus there is a canonical map of pro- \mathbf{G}_k -spaces:

$$Y_{h\mathbf{Z}} \rightarrow Y_{h\widehat{\mathbf{Z}}}. \tag{6.3.15}$$

Our next goal is to show that if $Y = \acute{E}t_{/k}(X)$ arises as the relative homotopy type of a variety X/\mathbf{F}_q , then (6.3.15) induces an isomorphism on cohomology.

LEMMA 6.17. *Let X be a variety over \mathbf{F}_q . Then the natural map*

$$H^*(\acute{E}t_{/k}(X)_{h\mathbf{Z}}; \mathbf{Z}/2\mathbf{Z}) \leftarrow H^*(\acute{E}t_{/k}(X)_{h\widehat{\mathbf{Z}}}; \mathbf{Z}/2\mathbf{Z})$$

is an isomorphism.

Proof. We consider the pro G_k -space $Y = (Y_i) := \acute{E}t_{/k}(X)$. The map on cohomology induced by (6.3.15) is

$$\varinjlim_i H^*((Y_i)_{h\mathbf{Z}}) \leftarrow \varinjlim_i \varinjlim_n H^*((Y_i)_{h(\mathbf{Z}/n\mathbf{Z})}).$$

Therefore, it will certainly suffice to prove that

$$H^*((Y_i)_{h\mathbf{Z}}) \xleftarrow{\sim} \varinjlim_n H^*((Y_i/n\mathbf{Z})_{h(\mathbf{Z}/n\mathbf{Z})}) \tag{6.3.16}$$

individually for each i . Moreover, since we assume that $Y = \acute{E}t_{/k}(X)$ we can take Y_i to be *excellent*, i.e. we can assume that the $\widehat{\mathbf{Z}}$ -action on Y_i already factors through a finite quotient [HS13, §9.2.3, pp. 296–297]. Hence, by restricting to a cofinal subcategory of the indexing category, and renaming Y_i to Y , we may assume that Y is a $\widehat{\mathbf{Z}}$ -space (as opposed to pro-space) on which the action factors through $\mathbf{Z}/n\mathbf{Z}$. Then, obviously, for any $n \mid N$ we have

$$(Y/N\mathbf{Z})_{h(\mathbf{Z}/N\mathbf{Z})} = Y_{h(\mathbf{Z}/N\mathbf{Z})}.$$

The identity map $Y \rightarrow Y$ is obviously equivariant for the group homomorphism $\mathbf{Z} \rightarrow \mathbf{Z}/N\mathbf{Z}$, and induces the following map of spectral sequences.

$$\begin{array}{ccc} H^r(\mathbf{Z}; H^s(Y)) & \Longrightarrow & H^{r+s}(Y_{h\mathbf{Z}}) \\ \uparrow & & \uparrow \\ H^r(\mathbf{Z}/N\mathbf{Z}; H^s(Y)) & \Longrightarrow & H^{r+s}(Y_{h(\mathbf{Z}/N\mathbf{Z})}) \end{array} \tag{6.3.17}$$

The following map of spectral sequences is not necessarily an isomorphism for any fixed N , but after taking the direct limit as N runs over all positive multiples of n , it becomes an isomorphism by the well-known comparison of group cohomology for $\widehat{\mathbf{Z}}$ and \mathbf{Z} with finite coefficients.

$$\begin{array}{ccc} H^r(\mathbf{Z}; H^s(Y)) & \Longrightarrow & H^r(\mathbf{Z}; H^s(Y)) & \Longrightarrow & H^{r+s}(Y_{h\mathbf{Z}}) \\ \sim \uparrow & & \sim \uparrow & & \uparrow \\ H^r(\widehat{\mathbf{Z}}; H^s(Y)) & \xlongequal{\quad} & \varinjlim_{n|N} H^r(\mathbf{Z}/N\mathbf{Z}; H^s(Y)) & \Longrightarrow & H^{r+s}(Y_{h\widehat{\mathbf{Z}}}) \end{array} \tag{6.3.18}$$

Now (6.3.18) implies that the natural map in (6.3.16) induces an isomorphism on associated gradeds, and hence an isomorphism. \square

More on pushforwards in cohomology. We now turn towards studying the pushforward on cohomology. We begin by formalizing the construction of the pushforward in § 6.3.2. To help distinguish varieties from pro-spaces, we will use boldface letters to denote pro-spaces in this part.

DEFINITION 6.18. We say that a pro-space \mathbf{Y} has *Poincaré duality* if there exists an n and an isomorphism $\int: H^n(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \cong \mathbf{Z}/2\mathbf{Z}$ such that the cup product induces a perfect pairing

$$H^i(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \times H^{n-i}(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\smile} H^n(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\int} \mathbf{Z}/2\mathbf{Z}.$$

(This uniquely determines n .) We denote the non-zero element in $H^n(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z})$ by $\mu_{\mathbf{Y}}$, and call it the *fundamental class of \mathbf{Y}* . From \int we obtain a functional

$$H^*(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\text{project}} H^n(\mathbf{Y}; \mathbf{Z}/2\mathbf{Z}) \xrightarrow{\int} \mathbf{Z}/2\mathbf{Z},$$

which we will also denote by \int .

DEFINITION 6.19. Let $f: \mathbf{W} \rightarrow \mathbf{Y}$ be a map of pro-spaces with Poincaré duality, with fundamental classes $\mu_{\mathbf{W}} \in H^m(\mathbf{W})$ and $\mu_{\mathbf{Y}} \in H^n(\mathbf{Y})$. Then we have a pullback map on cohomology

$$f^*: H^*(\mathbf{Y}) \rightarrow H^*(\mathbf{W}),$$

which induces a dual map

$$(f^*)^\vee: H^*(\mathbf{W})^\vee \rightarrow H^*(\mathbf{Y})^\vee.$$

We define the *pushforward on cohomology*

$$f_*: H^i(\mathbf{W}) \rightarrow H^{i+n-m}(\mathbf{Y})$$

to be the map $(f^*)^\vee$, where the identifications

$$H^i(\mathbf{W}) = H^{m-i}(\mathbf{W})^\vee \quad \text{and} \quad H^{i+n-m}(\mathbf{Y}) = H^{m-i}(\mathbf{Y})^\vee$$

are induced by Poincaré duality. It is easily checked that f_* takes $\mu_{\mathbf{W}} \mapsto \mu_{\mathbf{Y}}$.

In particular, if $\mathbf{W} = \acute{E}t(X)$ for some smooth proper variety X over \mathbf{F}_q or k^{sep} , then \mathbf{W} inherits Poincaré duality from X . If the map $f: \mathbf{W} \rightarrow \mathbf{Y}$ arises from a map of varieties $X \rightarrow V$, then the pushforward f_* tautologically agrees, under the identifications $H^*(\mathbf{W}) = H_{\acute{e}t}^*(X)$ and $H^*(\mathbf{Y}) = H_{\acute{e}t}^*(V)$, with the pushforward we defined in § 6.3.2. In addition, $\acute{E}t(X) \times \acute{E}t(X)$ has Poincaré duality by the Künneth theorem. Hence, by the horizontal homotopy equivalences in (6.3.14) we find that $(\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{hG_k}$ and $(\acute{E}t_{/k}(X) \times \acute{E}t_{/k}(X))_{h(G_k \times G_k)}$ have Poincaré duality. It is tautological that the diagram

$$\begin{CD} H^*(\acute{E}t(X \times_{\mathbf{F}_q} X)) @<<< H^*(\acute{E}t(X) \times \acute{E}t(X)) \\ @| @| \\ H_{\acute{e}t}^*(X \times_{\mathbf{F}_q} X) @<<< H_{\acute{e}t}^*(X) \otimes H_{\acute{e}t}^*(X) \end{CD}$$

commutes, and dualizing it shows the following lemma.

LEMMA 6.20. *Let X be smooth and proper over \mathbf{F}_q . Then the map*

$$\varphi_*: H_{\acute{e}t}^*(X \otimes_{\mathbf{F}_q} X) \rightarrow H_{\acute{e}t}^*(X) \otimes H_{\acute{e}t}^*(X)$$

defined in § 6.3.2 coincides with the pushforward in cohomology (as defined in Definition 6.19) induced by the vertical map in (6.3.14)

$$H^*(\text{Oblv}(\acute{E}t_{/\mathbf{F}_q}(X) \times \acute{E}t_{/\mathbf{F}_q}(X))_{h\widehat{\mathbf{Z}}}) \rightarrow H^*((\acute{E}t_{/\mathbf{F}_q}(X) \times \acute{E}t_{/\mathbf{F}_q}(X))_{h(\widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}})})$$

under the identifications obtained by § 3.2 and the horizontal equivalences in (6.3.14):

$$\begin{aligned} H^*(\text{Oblv}(\acute{E}t_{/\mathbf{F}_q}(X) \times \acute{E}t_{/\mathbf{F}_q}(X))_{h\widehat{\mathbf{Z}}}) &= H_{\acute{e}t}^*(X \times_{\mathbf{F}_q} X), \\ H^*((\acute{E}t_{/\mathbf{F}_q}(X) \times \acute{E}t_{/\mathbf{F}_q}(X))_{h(\widehat{\mathbf{Z}} \times \widehat{\mathbf{Z}})}) &= H_{\acute{e}t}^*(X) \otimes H_{\acute{e}t}^*(X). \end{aligned}$$

There are other constructions of the pushforward in cohomology, which are more useful for studying the interaction with Steenrod operations, so we prove a general criterion for recognizing when a map on cohomology coincides with the pushforward as we have defined it.

LEMMA 6.21. *Suppose $f: \mathbf{W} \rightarrow \mathbf{Y}$ is a map of (pro-)spaces with Poincaré duality so that f_* is defined as in Definition 6.19. Keeping the notation of Definition 6.19, suppose $f'_*: H^*(\mathbf{W}) \rightarrow H^{*+n-m}(\mathbf{Y})$ is another map satisfying the following:*

- (i) $f'_*(\mu_{\mathbf{W}}) = \mu_{\mathbf{Y}}$;
- (ii) $f'_*(x \smile f^*\gamma) = f'_*x \smile \gamma$ for all $x \in H^*(\mathbf{W})$ and $\gamma \in H^*(\mathbf{Y})$.

Then $f'_* = f_*$.

Remark 6.22. In other words, condition (ii) is saying that f'_* is a module homomorphism for $H^*(\mathbf{Y})$, for its natural action on $H^*(\mathbf{Y})$ and its action on $H^*(\mathbf{W})$ via f^* .

Proof. By Poincaré duality on \mathbf{Y} , it suffices to show that

$$\int_{\mathbf{Y}} f'_*x \smile \gamma = \int_{\mathbf{Y}} f_*x \smile \gamma \quad \text{for all } x \in H^*(\mathbf{W}), \gamma \in H^*(\mathbf{Y}). \tag{6.3.19}$$

By definition (Definition 6.19), we have

$$\int_{\mathbf{Y}} f_* x \smile \gamma = \int_{\mathbf{W}} x \smile f^* \gamma. \tag{6.3.20}$$

Substituting (6.3.20) into (6.3.19) and using condition (ii), we see that we need to show that

$$\int_{\mathbf{Y}} f'_*(x \smile f^* \gamma) = \int_{\mathbf{W}} x \smile f^* \gamma. \tag{6.3.21}$$

By the definition of the fundamental class, we can write

$$x \smile f^* \gamma = \left(\int_{\mathbf{W}} x \smile f^* \gamma \right) \mu_{\mathbf{W}} + (\text{lower degree terms}),$$

hence condition (i) of the lemma implies that

$$f'_*(x \smile f^* \gamma) = \left(\int_{\mathbf{W}} x \smile f^* \gamma \right) \mu_{\mathbf{Y}} + (\text{lower degree terms}). \tag{6.3.22}$$

Substituting (6.3.22) into (6.3.21), we have

$$\int_{\mathbf{Y}} f'_*(x \smile f^* \gamma) = \left(\int_{\mathbf{W}} x \smile f^* \gamma \right) \cdot \left(\int_{\mathbf{Y}} f'_* \mu_{\mathbf{W}} \right) = \int_{\mathbf{W}} x \smile f^* \gamma,$$

where the last equality follows from condition (i) that $f'_* \mu_{\mathbf{W}} = \mu_{\mathbf{Y}}$ and the fact that the lower degree terms ‘integrate’ to 0 by definition. □

We use this discussion to study the pushforward in cohomology induced by a homotopy quotient of spaces $M \rightarrow M_{h\mathbf{Z}}$. The following proposition is presumably well-known, but we have included a proof since we could not find a reference.

PROPOSITION 6.23. *Let M be any simplicial set with \mathbf{Z} -action. If*

$$f: M \rightarrow M_{h\mathbf{Z}}$$

denotes the homotopy quotient map, then we can define a pushforward map

$$f_*: H^i(M) \rightarrow H^{i+1}(M_{h\mathbf{Z}}).$$

If M has Poincaré duality, then so does $M_{h\mathbf{Z}}$ and f_ agrees with the pushforward defined in Definition 6.19.*

Moreover, for all $x \in H^(M)$ we have*

$$\text{Sq} \circ f_*(x) = f_* \circ \text{Sq}(x).$$

Proof. We can take $\mathbf{R} = E\mathbf{Z}$ as a model for $E\mathbf{Z}$, so that a model for $M_{h\mathbf{Z}}$ is $(M \times \mathbf{R})/\mathbf{Z}$. (The map $M \rightarrow (M \times \mathbf{R})/\mathbf{Z}$ is what might classically be called the ‘inclusion of a fiber into the mapping torus’.) With this model there is an evident homeomorphism

$$((M \times \mathbf{R})/\mathbf{Z}, M) \cong (S^1 \wedge (M_+), \text{pt}),$$

where the left-hand side means the pointed space obtained from $(M \times \mathbf{R})/\mathbf{Z}$ by collapsing $M \times \{0\}$ to a point.

We always have a pushforward map $H^*(M) \rightarrow H^{*+1}(M_{h\mathbf{Z}})$ defined as the following composition.

$$\begin{array}{ccc}
 H^*(M) & \dashrightarrow & H^{*+1}(M_{h\mathbf{Z}}) \\
 \searrow a \sim & & \nearrow b \\
 & H^{*+1}(S^1 \wedge (M_+)) &
 \end{array}
 \tag{6.3.23}$$

If M has Poincaré duality then so does $M_{h\mathbf{Z}}$ by the same reasoning as in § 2.1, so we have fundamental classes μ_M and $\mu_{M_{h\mathbf{Z}}}$ and Definition 6.19 supplies another notion of pushforward $H^*(M) \rightarrow H^{*+1}(M_{h\mathbf{Z}})$. In this case, the alternate pushforward (6.3.23) evidently takes $\mu_M \mapsto \mu_{M_{h\mathbf{Z}}}$, and is an $H^*(M_{h\mathbf{Z}})$ -module homomorphism since b is induced by a map of spaces $M_{h\mathbf{Z}} \rightarrow S^1 \wedge (M_+)$. Therefore, Lemma 6.21 shows that the two notions of pushforward coincide.

Finally, the homomorphism b commutes with Steenrod operations because it is induced by the map of spaces, and the map a commutes with Steenrod operations because it is a suspension isomorphism, and Steenrod operations always commute with suspension (§ 3.4). \square

The punchline. We now finally complete the proof of Proposition 6.14. We transfer the Poincaré duality structure

$$\text{from } H^*(\text{Ét}(X \times_{\mathbf{F}_q} X)) \text{ to } H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h\mathbf{Z}})$$

and

$$\text{from } H^*(\text{Ét}(X) \times \text{Ét}(X)) \text{ to } H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h(\mathbf{Z} \times \mathbf{Z})})$$

using the horizontal isomorphisms in (6.3.14) plus Lemma 6.17. Then Definition 6.19 furnishes a notion of pushforward

$$H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h\mathbf{Z}}) \rightarrow H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h(\mathbf{Z} \times \mathbf{Z})}),
 \tag{6.3.24}$$

which by Lemma 6.20 is compatible the map in Proposition 6.14 in the sense that the following diagram commutes.

$$\begin{array}{ccc}
 H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h\mathbf{Z}}) & \longrightarrow & H^*((\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h(\mathbf{Z} \times \mathbf{Z})}) \\
 \parallel & & \parallel \\
 H^*_{\text{ét}}(X \times_{\mathbf{F}_q} X) & \xrightarrow{\varphi_*} & H^*_{\text{ét}}(X) \otimes H^*_{\text{ét}}(X)
 \end{array}$$

On the other hand, we may write $(\text{Ét}_{/k}(X) \times \text{Ét}_{/k}(X))_{h\mathbf{Z}} = \{U_i\}_i$, with each U_i being a \mathbf{Z} -space. Then by Lemma 6.21 the map (6.3.24) agrees with the colimit of the levelwise pushforwards

$$H^*(U_i) \rightarrow H^*((U_i)_{h\mathbf{Z}}),$$

each of which commutes with Steenrod operations by Lemma 6.23. \square

7. The obstruction to being alternating

7.1 Lifting Wu classes to integral cohomology

The goal of this subsection is to prove that Wu classes lift to integral cohomology, which will turn out to be important later.

LEMMA 7.1. Every Wu class v_s can be expressed as a polynomial in the Stiefel–Whitney classes $\{w_k\}$.

Proof. We induct on s . The base case is $v_0 = w_0 = 1$. Consider the equation

$$\text{Sq } v = w$$

from Theorem 6.5. Equating terms in cohomological degree s , we have

$$v_s + \text{Sq}^1 v_{s-1} + \cdots = w_s.$$

By the induction hypothesis each term v_{s-i} is a polynomial in the Stiefel–Whitney classes, so by the Cartan formula for Steenrod squares (§ 3.4) and Lemma 5.4, each $\text{Sq}^i v_{s-i}$ is a polynomial in the Stiefel–Whitney classes. Then solving for v_s completes the induction. \square

COROLLARY 7.2. The Wu class $v_s \in H_{\text{ét}}^s(X; \mathbf{Z}/2\mathbf{Z})$ is the reduction of a class in $H_{\text{ét}}^s(X; \mathbf{Z}_2(\lceil s/2 \rceil))$.

Proof. By Lemma 7.1 we can express v_s as a polynomial $P_s(\{w_i\})$ in just the Stiefel–Whitney classes. Using Theorem 5.10, rewrite $P_s(\{w_i\})$ as a polynomial in the (reductions of) Chern classes and α .

Note that the Chern classes all live in even cohomological degree while α has degree 1 and $\alpha^2 = 0$. Therefore, if $s = 2k$ is even then this polynomial can be written without α , while if $s = 2k + 1$ is odd then it can be written in the form $\alpha P'_s(\{c_i\})$. In any case, Theorem 5.10 tells us that the Chern classes lift to integral cohomology, and so does α by Remark 5.9. \square

7.2 Proof of the main theorem

We now combine the preceding ingredients to prove Theorem 2.7, which as already noted implies Theorem 1.3. We wish to show that

$$x \smile \beta(x) = 0 \quad \text{for all } x \in H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^n\mathbf{Z}(d)).$$

Theorem 4.4 tells us that

$$x \smile \beta(x) = \widetilde{\text{Sq}}^{2d}(\beta(x)). \tag{7.2.1}$$

Then Lemma 3.12 implies that

$$\widetilde{\text{Sq}}^{2d}(\beta(x)) = [2^{n-1}] \circ \text{Sq}^{2d}(\overline{\beta(x)}), \tag{7.2.2}$$

where $\overline{\beta(x)}$ denotes the reduction of $\beta(x) \pmod{2}$, and the notation $[2^{n-1}]$ is as in Lemma 3.12.

By the definition of the Wu classes in Theorem 6.5, we have

$$\widetilde{\text{Sq}}^{2d}(\beta(x)) = [2^{n-1}](v_{2d} \smile \overline{\beta(x)}). \tag{7.2.3}$$

It is immediate from the definition of $[2^{n-1}]$ that

$$[2^{n-1}](v_{2d} \smile \overline{\beta(x)}) = ([2^{n-1}]v_{2d}) \smile \beta(x). \tag{7.2.4}$$

Next, since β is a derivation we have

$$([2^{n-1}]v_{2d}) \smile \beta(x) = \beta([2^{n-1}]v_{2d} \smile x) - \beta([2^{n-1}]v_{2d}) \smile x. \tag{7.2.5}$$

Stringing together (7.2.1), (7.2.2), (7.2.3), and (7.2.4), it suffices to show that (7.2.5) is 0 for all x .

LEMMA 7.3. We have $\beta([2^{n-1}]v_{2d}) = \beta_{2,2^n}(v_{2d})$ where $\beta_{2,2^n}$ is the boundary map for

$$0 \rightarrow \mathbf{Z}/2^n\mathbf{Z}(d) \xrightarrow{2} \mathbf{Z}/2^{n+1}\mathbf{Z}(d) \rightarrow \mathbf{Z}/2\mathbf{Z}(d) \rightarrow 0.$$

Proof. This follows immediately from the below commutative diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) & \xrightarrow{2} & \mathbf{Z}/2^{n+1}\mathbf{Z}(d), & \longrightarrow & \mathbf{Z}/2\mathbf{Z}(d) \longrightarrow 0 \\ & & \parallel & & \downarrow 2^{n-1} & & \downarrow 2^{n-1} \\ 0 & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) & \xrightarrow{2^n} & \mathbf{Z}/2^{2n}\mathbf{Z}(d) & \longrightarrow & \mathbf{Z}/2^n\mathbf{Z}(d) \longrightarrow 0 \end{array}$$

Referring to (7.2.5), we have $\beta([2^{n-1}]v_{2d} \smile x) = 0$ for all x by Lemma 2.4. But we also have $\beta_{2,2^n}(v_{2d}) = 0$ as this is the obstruction to lifting v_{2d} to $H_{\text{ét}}^{2d}(X; \mathbf{Z}/2^{n+1}\mathbf{Z}(d))$, and v even lifts to $v_{2d} \in H_{\text{ét}}^{2d}(X; \mathbf{Z}_2(d))$ by Corollary 7.2. By Lemma 7.3 we also get $\beta([2^{n-1}]v_{2d}) = 0$, so the expression in (7.2.5) vanishes for all x , which completes the proof. \square

7.3 A criterion for the alternation of the linking form

The argument of §7.2, and the ingredients going into it, can be adapted in a straightforward manner to study linking form of an odd-dimensional manifold (§2.2). It yields the following conclusion, which we state in the orientable case for simplicity.

THEOREM 7.4. Let M be an orientable closed manifold of dimension $4d + 1$. Then the linking form on M is alternating if and only if v_{2d} lifts to $H^{2d}(M; \mathbf{Z})$, i.e. if and only if $\tilde{\beta}(v_{2d}) = 0$ where $\tilde{\beta}$ is the boundary map for the short exact sequence

$$0 \rightarrow \mathbf{Z} \xrightarrow{2} \mathbf{Z} \rightarrow \mathbf{Z}/2\mathbf{Z} \rightarrow 0.$$

Proof. The argument in §7.2, with the appropriate adaptations of its ingredients to singular cohomology, shows that the linking form is alternating if and only if $\beta_{2,2^n}(v_{2d}) = 0$ for all n . Since $H^{2d+1}(M; \mathbf{Z})$ is finitely generated, the reduction maps induce an injection

$$H^{2d+1}(M; \mathbf{Z}) \hookrightarrow \varprojlim_n H^{2d+1}(M; \mathbf{Z}/2^n\mathbf{Z}),$$

which implies that $\ker \tilde{\beta} = \bigcap_n \ker \beta_{2,2^n}$. \square

Remark 7.5. The statement of Theorem 7.4 is still true for a non-orientable manifold as in §2.2, if the short exact sequence is replaced with its twist by \mathcal{L} . Moreover, our theory shows that the condition can be reformulated in terms of (twisted) Stiefel–Whitney classes defined analogously to those in §5.2.

Example 7.6. Let us specialize Theorem 7.4 to recover the known criterion, which was stated in §2.2, for the linking form on an orientable 5-manifold to be alternating. Assume that M is a smooth, orientable 5-manifold. Then Theorem 7.4 tells us that the linking form is alternating if and only if v_2 lifts to integral cohomology. By Example 6.3, noting that $w_1 = 0$ since M is orientable, we have $v_2 = w_2(M) \in H^2(M; \mathbf{Z}/2\mathbf{Z})$. Finally, it is well-known (see for example [LM89, Theorem D.2]) that M is spin^c if and only if $w_2(M)$ lifts to integral cohomology.

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