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Steenrod operators, the Coulomb branch and the Frobenius twist

Gus Lonergan

For Joy

Abstract

We observe a fundamental relationship between Steenrod operations and the Artin–Schreier morphism. We use Steenrod's construction, together with some new geometry related to the affine Grassmannian, to prove that the quantum Coulomb branch is a Frobenius-constant quantization. We also demonstrate the corresponding result for the K-theoretic version of the quantum Coulomb branch. At the end of the paper, we investigate what our ideas produce on the categorical level. We find that they yield, after a little fiddling, a construction which corresponds, under the geometric Satake equivalence, to the Frobenius twist functor for representations of the Langlands dual group. We also describe the unfiddled answer, conditional on a conjectural 'modular derived Satake', and, though it is more complicated to state, it is in our opinion just as neat and even more compelling.

Contents

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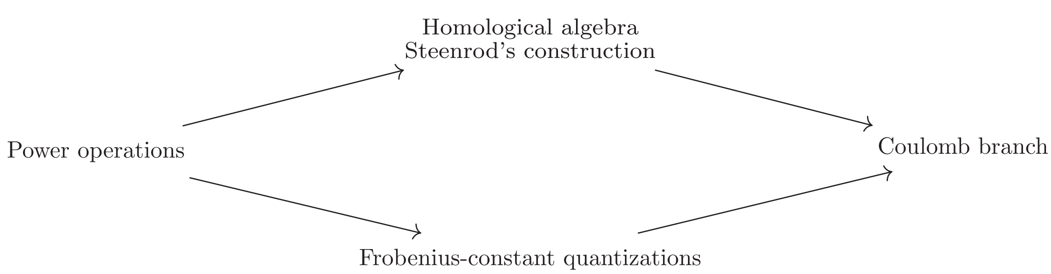
1. Introduction

1.1 This paper is about power operations.



1.2 Overview

A power operation is an enhanced version of a pth-power, or more generally nth-power, map. One of the most famous examples is Steenrod's operations [SE62], a cornerstone of algebraic topology. We will give an account of Steenrod's construction in the language of derived categories. To be precise, we produce a categorical version of the 'total external Steenrod operation'. When we look at what this construction does to the constant sheaf and compose with a diagonal restriction functor, we recover the Steenrod cohomology operations; when instead we focus attention on the dualizing complex of an E_2-group and compose with a pushforward along the multiplication, we recover Kudo–Araki–Dyer–Lashof operations [KA56, DL62]. However, the new context in which we put these old ideas allows us to make new observations about them, including a fundamental relationship between Steenrod operations and the Artin–Schreier morphism, and one between Kudo–Araki–Dyer–Lashof operations and the concept of Frobenius constancy, proving the Frobenius constancy of all quantum Coulomb branches [BFN16] in the process. We use a version of this construction adapted to perverse sheaves to provide a geometric description of the functor which corresponds under geometric Satake [MV07] to the Frobenius twist functor from

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the theory of algebraic groups, and give a conjectural description of the compatibility between these structures and the (unproven at the time of writing) 'derived geometric Satake' (see [BF08] for the characteristic 0 case), through the newly proposed 'Frobenius character twist'. A key geometric input is our extension of the theory of Beilinson–Drinfeld Grassmannians [BD91] which in particular yields a generalization of the 'global convolution diagram' to the case of the variety of triples [BFN16].

A reader who knows about equivariant constructible derived categories on complex algebraic varieties will be able to understand these constructions even if they do not know any homotopy theory. Perhaps this is an advantage.

1.3 In § 2, we will give an account of Steenrod's construction in the language of derived categories. In these terms, the construction itself is very simple, and it yields not only Steenrod's cohomology operations but also operations in Borel–Moore homology, which are related to the Kudo–Araki–Dyer–Lashof operations.

1.4 In § 3, we will introduce a different type of power operation, due to Bezrukavnikov and Kaledin [BK08], which is an important tool in non-commutative algebraic geometry. Such a power operation is known as a Frobenius-constant quantization. Essentially, a Frobenius-constant quantization of a commutative algebra A over $\mathbb{F}_{\mathrm{p}}$ is a one-parameter flat deformation $A_\hbar$ of A in associative algebras which has a large center; see § 3.1 for a precise definition which also justifies regarding such a thing as a power operation. The main example is the Weyl algebra

$$\mathbb{F}_{\mathrm{p}}[\hbar]\langle x, \partial\rangle/([\partial, x] = \hbar)$$

which contains x^p, ∂^p in its center.

1.5 We will then illustrate a general method to apply Steenrod's construction to produce Frobenius-constant quantizations.

Remark 1.1. The method is on the fundamental level purely homotopical, and is known to the author to apply to any S^1-framed E_3-ring spectrum over $\mathbb{F}_p$. For example, it can be used in conjunction with ideas of derived algebraic geometry to explain the famous result that the algebra of differential operators on a smooth variety of characteristic p is a Frobenius-constant quantization of the ring of functions on its cotangent bundle. This appears on the surface to be of a very different flavor than the main thrust of the present paper, which is primarily concerned with mod p homology rings on complex algebraic varieties. As previously mentioned, in this paper we take a point of view practically devoid of homotopy theory. I hope that homotopy theorists will be able to read between the lines!

The example which we use to illustrate the method is the *quantum Coulomb branch* of Braverman, Finkelberg and Nakajima [BFN16], or rather, its natural characteristic p version. That is, we prove the following theorem.

Theorem 1.2. *For any complex reductive algebraic group G, and finite-dimensional representation $\mathbf{N}$ of G, and any odd prime p, the corresponding mod p quantum Coulomb branch is a Frobenius-constant quantization.*

The Coulomb branch is the G-equivariant mod p Borel–Moore homology of a certain complex algebraic space $\mathcal{R}$ related to the affine Grassmannian Gr of G. The space $\mathcal{R}$ is equipped with a 'loop-rotation' $\mathbb{C}^*$-action, and the quantum Coulomb branch is obtained by switching on loop-rotation equivariance; the deformation parameter $\hbar$ enters as the $\mathbb{C}^*$-equivariant cohomology of a point. The key geometric insight behind Theorem 1.2 is that, following ideas of Beilinson and Drinfeld [BD91], one may deform the space $\mathcal{R}$ into to $\mathcal{R}^p$, in such a way that the action of C_p (the cyclic group of order p) on $\mathcal{R}$, which applies the loop rotation by pth roots of unity, is deformed to the C_p action on $\mathcal{R}^p$, which permutes the factors cyclically.

Example 1.3. Recall that the affine Grassmannian Gr of the complex reductive group G is an algebraization of the space $\Omega^{\mathrm{pol}}K$ of based polynomial loops in a fixed compact form K of G; this is homotopy equivalent to the space ΩK of all based loops in K. The S^1-action on ΩK given by rotating loops and then renormalizing (by dividing on the right, say, by the value at $1 \in K$) corresponds to the loop-rotation action of $\mathbb{C}^*$ on Gr. We give a heuristic explanation, in terms of ΩK and its 'renormalized loop-rotation' action, of the key geometric insight mentioned above, in the case $\mathcal{R} = Gr$.

Recall first that ΩK has a (strict) group structure given by pointwise multiplication of loops (using the group structure of K), and this is reflected in the existence of a convolution diagram for Gr. Next, recall that ΩK has a (homotopy) group structure given by concatenation of based loops, and this is reflected in the existence of the Beilinson–Drinfeld affine Grassmannian, which is a deformation of Gr into Gr^2. Moreover, these two group structures of ΩK commute (up to homotopy) and are part of an S^1-family of group structures. This is reflected in the existence of a 'global convolution diagram' linking the convolution diagram and the Beilinson–Drinfeld Grassmannian. Thus we have a kind of dictionary between some homotopy-theoretical structures and some algebraic structures. The algebraic structures are not strict algebraizations of their topological counterparts (at least in any obvious sense), but we shall say that they correspond under the 'algebraization paradigm' (for want of a better term).

Now, consider the map

$$(\Omega K)^p \to \Omega K$$

which speeds up each loop by a factor of p and then concatenates the results. Observe that the action of C_p on the left by cyclic permutation of the factors is compatible under this map by the 'renormalized loop-rotation' action of C_p as a subgroup of S^1 on the right. Our 'key geometric insight' is essentially what corresponds to this under the algebraization paradigm.

Remark 1.4. We will never again use the notation C_p, since every finite cyclic group considered will be given as a subgroup of $\mathbb{C}^*$. We will use the notation μ_p instead. Given an algebraic space X and a finite set S, we will write X^S for $\mathrm{Hom}(S, X)$, that is, the product of X with itself $|S|$ times with factors being labeled by the elements of S. Bringing these together, be warned that the expression X^{μ_p} denotes the p-fold product of X with itself labeled by the elements of μ_p, and does not denote the fixed points of some action of μ_p on X.

1.6 This is already quite a broad class of examples: for instance, partially spherical rational Cherednik algebras; see [BEF20, Web19]. It is expected that the same underlying geometry will lead to the discovery of large centers of related algebras. In fact, in § 4 we indicate how the same underlying geometry shows that the K-theoretic version of *integral* quantum Coulomb branch, which is itself a q-deformation of the K-theoretic version of the Coulomb branch, admits a large center when q is evaluated at any complex root of unity (not necessarily of prime order). Essentially the only difference with the homological case is to replace Steenrod's construction with a so-called 'Adams construction' which is to Adams operations as Steenrod's construction is to Steenrod's operations.

Question 1.5. Is it possible to develop the theory of equivariant *elliptic* Borel–Moore homology far enough to deduce the analogous 'large center' statements for the elliptic analogues of the Coulomb branch?

1.7 In the final section, we investigate the meaning of our constructions on the level of the constructible derived category. We are especially interested in understanding what our constructions correspond to in terms of representations of the Langlands dual group $G^\vee_{\mathbb{F}_p}$ over $\mathbb{F}_p$.

Conjecture 1.6. (i) For sufficiently large p, there are monoidal equivalences ('derived geometric Satake with coefficients modulo p'; see [BF08] for the version with complex coefficients):

$$D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \cong Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])),$$

$$D^b_{G(\mathcal{O})\rtimes\mathbb{C}^*}(Gr, \mathbb{F}_p) \cong Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(\mathcal{U}_\hbar(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])).$$

Here the left-hand categories denote the equivariant constructible derived categories with coefficients in $\mathbb{F}_p$ in the sense of Bernstein and Lunts [BL06]. As for the right-hand side, $\mathfrak{g}^\vee_{\mathbb{F}_p}[-2]$ denotes the Lie algebra of $G^\vee_{\mathbb{F}_p}$ placed in cohomological degree 2, $\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])$ denotes its symmetric algebra (considered as a commutative differential graded (dg) algebra with trivial differential) and $\mathcal{U}_\hbar(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])$ denotes its canonical (Poincaré–Birkhoff–Witt) one-parameter deformation to a non-commutative dg-algebra with $\hbar$ in cohomological degree 2 (also with trivial differential). The expression $Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(*)$ denotes the full triangulated subcategory of the (derived) category of $G^\vee_{\mathbb{F}_p}$-equivariant dg-modules generated by the objects which are free over the $G^\vee_{\mathbb{F}_p}$-equivariant dg-algebra $(*)$.

(ii) (Paraphrasing), our geometric construction, when reinterpreted on the level of equivariant constructible derived categories, yields a triangulated functor

$$D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to D^b_{G(\mathcal{O})\rtimes\mathbb{C}^*}(Gr, \mathbb{F}_p)[\hbar^{-1}]$$

which corresponds under the above equivalences to the functor

$$Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])) \to Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(\mathcal{U}_\hbar(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])[\hbar^{-1}])$$

which is the composition of the *external Steenrod/Tate power* functor

$$Coh^{fr}_{G^\vee_{\mathbb{F}_p}}(\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])) \to Coh^{fr}_{(G^\vee_{\mathbb{F}_p})^{(1)}}(\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])^{(1)}[\hbar^{\pm1}]),$$

followed by the restriction of equivariance along the Frobenius map

$$G^\vee_{\mathbb{F}_p} \to (G^\vee_{\mathbb{F}_p})^{(1)},$$

followed by the base change along the $G^\vee_{\mathbb{F}_p}$-equivariant central flat algebra extension

$$\mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])^{(1)}[\hbar^{\pm1}] \to \mathcal{U}_\hbar(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])[\hbar^{-1}].$$

In particular, for a representation V of $G^\vee_{\mathbb{F}_p}$, this functor sends the free object

$$V \otimes \mathrm{Sym}(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])$$

to the free object

$$V^{(1)} \otimes \mathcal{U}_\hbar(\mathfrak{g}^\vee_{\mathbb{F}_p}[-2])[\hbar^{-1}].$$

Here $V^{(1)}$ denotes the Frobenius twist of V, that is, the $G^\vee_{\mathbb{F}_p}$-representation obtained from V by pulling back along the Frobenius map. It is therefore reasonable to regard our functor as the correct 'Frobenius twist' functor in this context (indeed we do not believe there to be any alternative). We call this functor the 'Frobenius character twist', since its source and target are in some sense categories of character sheaves.

(iii) The Frobenius character twist is a central monoidal functor, that is, it admits a monoidal lift through the monoidal center of its target. This is not really conjectural: it is provably true of both our geometrically constructed functor and the Frobenius character twist, and the compatibility of the two lifts would surely follow from any proof of the earlier parts of the conjecture.

Although the above picture is conjectural (mainly because we cannot prove derived geometric Satake with coefficients modulo p), we can prove an underived version. Namely, we give a geometric description of the functor

$$Perv_{\mathrm{sph}}(Gr, \mathbb{F}_{\mathrm{p}}) \to Perv_{\mathrm{sph}}(Gr, \mathbb{F}_{\mathrm{p}})$$

which corresponds to the Frobenius twist functor under the underived geometric Satake equivalence [MV07]

$$S : Rep(G^{\vee}_{\mathbb{F}_{\mathrm{p}}}) \to Perv_{\mathrm{sph}}(Gr, \mathbb{F}_{\mathrm{p}}).$$

The construction is as follows. Consider the μ_p-equivariant deformation $Gr_{(p)}$ of Gr (with loop rotation) to Gr^p (with cyclic permutation). Given a spherical perverse sheaf $\mathcal{F} \in Perv_{\mathrm{sph}}(Gr, \mathbb{F}_{\mathrm{p}})$, one may view its pth external power as a spherical perverse sheaf on $Gr_{(p)}|_{\mathbb{A}^1 - \{0\}}$ and take the IC sheaf of the result:

$$IC(\mathcal{F}^{\boxtimes p}).$$

Let $i : Gr \to Gr_{(p)}$ denote the inclusion of the zero fiber of the deformation. Then we have a Thom map $i^*[-1] \to i^![1]$, and this gives us a map of perverse sheaves

$$i^* IC(\mathcal{F}^{\boxtimes p})[-1] \to i^! IC(\mathcal{F}^{\boxtimes p})[1].$$

Let $Fr(\mathcal{F})$ denote the image of the above Thom map.

Theorem 1.7. *Fr is the required functor. That is, if*

$$Fr : Rep(G^{\vee}_{\mathbb{F}_{\mathrm{p}}}) \to Rep(G^{\vee}_{\mathbb{F}_{\mathrm{p}}})$$

denotes the Frobenius twist functor, then we have an equivalence

$$Fr \circ S \cong S \circ Fr$$

where S is the geometric Satake equivalence.

This appears as Theorem 5.1 in the text.

1.8 Background required to read this paper.

(i) *Homotopy theory.* As previously mentioned, no homotopy theory (besides the basics of homology and cohomology) is required to read this paper, despite the fact that its arguments are actually quite relevant to homotopy theory. Familiarity with Steenrod operations may help, but on the other hand may not because our approach is somewhat non-traditional and can be confusing to homotopy theorists. In any case that aspect of this paper is self-contained.

(ii) *Algebraic geometry.* We will use heavily S. Raskin's theory of placid ind-schemes [Ras15]. This is a beautiful theory describing a certain class of infinite-dimensional algebro-geometric objects (placid ind-schemes) which can be handled, via some categorical bookkeeping, in essentially the same way as finite-type schemes. It does not rely on any algebraic geometry deeper than a basic understanding of the functorial approach based on the Yoneda embedding and fundamental concepts such as flatness and smoothness, and nor does the present paper. We will try to explain enough of the material of [Ras15] ourselves that the reader need not constantly refer to it (it is nonetheless highly recommended reading).

All the algebro-geometric objects we consider will be defined over $\mathbb{C}$, and although we will use the Yoneda embedding to define them and reason about them, we will always think of them as complex analytic spaces when considering their invariants (homology, sheaf categories etc.). No understanding of stacks is required, even though much of the geometry can be interpreted in terms of quotient stacks.

(iii) *Sheaf theory.* We use Bernstein and Lunts' theory of equivariant constructible derived categories [BL06] in order to build invariants out of our complex geometric objects. In particular, this involves forgetting about their algebro-geometric nature and replacing them by their sets of $\mathbb{C}$-points with the analytic topology. The primary alternative is to stay in the world of algebraic geometry and use the étale topology. Although using the étale topology has some well-known advantages (e.g. the theory of weights), we do not need them in the present work and so prefer to stick to the (in our opinion) conceptually easier analytic topology framework. The reader who is more comfortable with the étale topology most likely has no difficulty translating our results in those terms. Section 5 also uses the theory of perverse sheaves; cf. [DBB83].

The reader who is not familiar with either theory of constructible derived categories but knows a little more homotopy theory may still be able to understand everything apart from § 5 which is explicitly about Bernstein and Lunts' category. For, apart from in that section, we are only really using the constructible derived category as a mechanism for producing structures on Borel–Moore homology, and for this there are purely homotopical means. To be precise, although we describe operations on the level of the constructible derived category, ultimately we restrict attention to their effect on a single object, the dualizing complex ω; and even then we only really care about its global sections complex $\Gamma(X, \omega)$, which is (by definition) just the Borel–Moore complex of X, that is, the reduced singular complex of the one-point compactification of X (denoted $\mathbb{F}_{\mathrm{p}} \wedge \bar{X}$). If X is equipped with the action of a group G, then so is $\mathbb{F}_{\mathrm{p}} \wedge \bar{X}$, and the notation

$$\mathrm{Hom}_{D_G(X,\mathbb{F}_{\mathrm{p}})}(\mathbb{F}_{\mathrm{p}}, \Sigma^n \omega)$$

will refer to the $-n$th homotopy groups of the G-homotopy invariants of $\mathbb{F}_{\mathrm{p}} \wedge \bar{X}$. If the reader bears this in mind, they should be able to translate enough of what is phrased in terms of constructible derived categories to fully understand the main result about the (quantum) Coulomb branch.

(iv) *Representation theory.* The definition of the Coulomb branch (originally found in [BFN16]) takes as input a complex reductive algebraic group plus a finite-dimensional representation, but one does not really need to know to this theory well to understand the definition and the other arguments. The constructions of [MV07, BFN16, BFN17] are used throughout the paper but are explained in reasonable detail here so, while helpful, are not necessary reading, except for § 5 where Mirković and Vilonen's geometric Satake equivalence [MV07] is essential.

2. Steenrod's construction

2.1 Overview

Let p be an odd prime number, and let μ_p be the group of complex pth roots of unity. Let R be a commutative ring. Let k be a field of characteristic p, and let $F : k \to k$ be the Frobenius map.

Let X be a topological space and let $D^+(X,R)$ denote the bounded-below derived category of sheaves of k-modules on X. We write X^{μ_p} for $\mathrm{Map}(\mu_p, X)$. Following Steenrod [SE62], we construct a functor

$$St : D^+(X,R) \to D^+_{\mu_p}(X^{\mu_p}, R)$$

where $D^+_{\mu_p}(X^{\mu_p}, R)$ denotes the bounded-below μ_p-equivariant derived category of sheaves of R-modules on X^{μ_p}. This functor is not linear or triangulated, but nonetheless if we take $R = k$, compose with restriction to the diagonal and apply to morphisms between shifted constant sheaves we obtain linear maps

$$F^*H^n(X,k) \to H^{pn}_{\mu_p}(X,k) \cong \bigoplus_{i+j=pn} H^i(X,k) \otimes H^j(\mathrm{B}\mu_\mathrm{p}, k) \tag{2.1}$$

for each $n \geq 0$. Recall that $H^*(\mathrm{B}\mu_\mathrm{p}, k) = k[a, \hbar]$ is the super-polynomial algebra in one variable a of degree 1 and one variable $\hbar$ of degree 2. Here $\hbar$ is the first Chern class of the tautological complex line bundle on $\mathrm{B}\mu_\mathrm{p}$ arising from the embedding $\mu_p \subset \mathbb{C}^*$.

The direct sum of the maps of (2.1) is not in the most naive sense an algebra homomorphism. This fact led Steenrod to introduce certain correction factors which make it so; his famous cohomology operations are then defined to be the coefficients of the resulting algebra homomorphism in the monomial basis of $k[a, \hbar]$. However, the sum of maps of (2.1) *does* give a homomorphism of super-graded algebras

$$H^*(X,k)^{(1)} \to H^*_{\mu_p}(X,k) \cong H^*(X,k)[a,\hbar] \tag{2.2}$$

where $H^*(X,k)^{(1)}$ denotes the *Frobenius twist* of $H^*(X,k)$. Naively one might think that this is just the p-dilation of $F^*H^*(X,k)$. This is wrong: rather, the natural and correct definition of the Frobenius twist of an algebra A in any symmetric monoidal category over k is as the Tate cohomology

$$A^{(1)} := \hat{H}^0_{\mu_p}(A^{\otimes \mu_p})$$

where the symmetric monoidal structure endows $A^{\otimes \mu_p}$ with the structure of μ_p-equivariant algebra. In the case of the super-graded k-algebra $H^*(X,k)$, the underlying super-graded k-module of this construction is the same as the p-dilation of $F^*H^*(X,k)$, but the multiplication differs by a sign (at least when $p \equiv 3 \bmod 4$), removal of which is part of the purpose of Steenrod's correction factors.

We prefer therefore to use Steenrod's operations in their raw form, that is, without the correction factors and packaged as in (2.2). This has the advantage of revealing the following fundamental connection between Steenrod's operations and the Artin–Schreier map, which is obscured by the correction factors.

Fact 2.1. (i) Let $X = BT$ for some complex torus T. Then the Picard group of X is canonically isomorphic to the character lattice $\mathbb{X}^\bullet(T)$ of T, and the cohomology ring is the polynomial algebra

$$H^*(X,\mathbb{Z}) = \mathrm{Sym}_\mathbb{Z}\, \mathbb{X}^\bullet(T)$$

with $\mathbb{X}^\bullet(T)$ in degree 2. This is equal to the ring $\mathcal{O}(\mathfrak{t}_\mathbb{Z})$ of polynomial functions on the canonical $\mathbb{Z}$-form of the scheme $\mathfrak{t} = \mathrm{Lie}(T)$. Likewise we have

$$H^*(X,k) = \mathcal{O}(\mathfrak{t}_k)$$

where $\mathfrak{t}_k$ denotes the canonical k-form of $\mathfrak{t}$.

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(ii) Under this identification, the map of (2.2) factors as

$$\mathcal{O}(\mathfrak{t}_k)^{(1)} \xrightarrow{AS_\hbar} \mathcal{O}(\mathfrak{t}_k)[\hbar] \subset \mathcal{O}(\mathfrak{t}_k)[a, \hbar]$$

where $AS_\hbar$ corresponds, on the level of $\bar{k}$-points, to the $\bar{k}^\times$-equivariant family, parameterized by $\hbar \in \bar{k}$, of additive maps of free $\bar{k}$-modules

$$\begin{array}{ccc} \bar{k} \otimes \mathfrak{t}_k & \to & \bar{k}^{(1)} \otimes \mathfrak{t}_k \\ \sum_i x_i \otimes v_i & \mapsto & \sum_i (x_i^p - \hbar^{p-1} x_i) \otimes v_i \end{array}$$

for a basis $\{v_i\}$ of $\mathfrak{t}_{\mathbb{F}_p}$. This family interpolates between the usual Artin–Schreier map for $\hbar = 1$ and the Frobenius map for $\hbar = 0$.

Proof. (i) This is a standard fact which boils down to the calculation

$$H^*(\mathbb{CP}^\infty, \mathbb{Z}) = \mathbb{Z}[[\mathbb{CP}^1]^*],$$

the free algebra on the degree 2 cochain $v^* = [\mathbb{CP}^1]^*$ dual to the degree 2 chain represented by the embedded $\mathbb{CP}^1$. An isomorphism $T = (\mathbb{C}^*)^d$ gives an isomorphism

$$H^*(BT, R) \cong R[v_1^*, \ldots, v_d^*]$$

for any commutative ring R, and we will fix one for ease of exposition.

(ii) This is just saying that the map of (2.2), when applied to $X = BT$, sends each generator v_i^* to $(v_i^*)^p - \hbar^{p-1} v_i^*$. Since we have not actually defined the map (2.2) yet, we postpone the proof until § 2.8. However, notice that since v_i^* has degree 2 the expected image

$$(v_i^*)^p - \hbar^{p-1} v_i^*$$

is exactly the generating function of the Steenrod powers of v_i^* (with placeholder variable $\hbar$), up to some sign in the coefficients. Therefore the result will follow from a careful study of Steenrod's normalization factors (or rather, what happens if we don't use them!). □

Remark 2.2. The appearance of $AS_\hbar$ in the topological setting was the first indication that Steenrod's construction might be related to the theory of Frobenius-constant quantizations, where $AS_\hbar$ plays a central role; see Fact 3.2.

2.2 Steenrod's construction

Recall that p is an odd prime, μ_p is the group of complex pth roots of unity, R is a commutative ring, k is a field of characteristic p and X is a topological space. We denote by $C^+(X, R)$ (respectively, $D^+(X, R)$) the category of bounded-below cochain complexes of sheaves of R-modules on X (respectively, the corresponding bounded-below derived category). Recall that the latter is obtained from the former by inverting quasi-isomorphisms (this is important to us, and does not hold in the unbounded case). If Y is a topological space with an action of μ_p, we denote by $C^+_{\mu_p}(Y, R)$, $D^+_{\mu_p}(Y, R)$ the corresponding μ_p-equivariant categories. Since μ_p is a finite group, these are the same as the (bounded-below) cochain, derived categories of μ_p-equivariant sheaves of R-modules on Y.

Consider the functor of pth external tensor power

$$C^+(X, R) \xrightarrow{\boxtimes p} C^+(X^{\mu_p}, R).$$

It sends quasi-isomorphisms to quasi-isomorphisms, and so descends to a (non-triangulated) functor

$$D^+(X, R) \xrightarrow{\boxtimes p} D^+(X^{\mu_p}, R)$$

by the universal property of derived categories. Notice that the cochain-level functor factors as

$$\boxtimes p : C^+(X,R) \xrightarrow{St_C} C^+_{\mu_p}(X^{\mu_p},R) \to C^+(X^{\mu_p},R).$$

To make this explicit, we first choose an isomorphism

$$\mu_p \cong \mathbb{Z}/p \cong \{1,\dots,p\} =: [p];$$

the result will be independent of this choice. Write σ for the generator of μ_p corresponding to 1 under the isomorphism. Then, for a complex $A^\bullet$, we give the complex $(A^\bullet)^{\boxtimes p}$ with degree n term

$$\big((A^\bullet)^{\boxtimes p}\big)^n = \bigoplus_{i_1+\cdots+i_p=n} A^{i_1} \boxtimes \cdots \boxtimes A^{i_p}$$

the μ_p-equivariant structure by letting the generator σ act in degree n by the direct sum of the canonical isomorphisms of sheaves

$$A^{i_1} \boxtimes \cdots \boxtimes A^{i_p} \cong \sigma^*(A^{i_2} \boxtimes \cdots \boxtimes A^{i_p} \boxtimes A^{i_1})$$

each twisted by the sign $(-1)^{i_1(n-i_1)}$. The sign twist is the natural (Koszul) choice which makes the action of μ_p commute with the differential. Moreover, given a chain map $f : A^\bullet \to B^\bullet$, $f^{\boxtimes p}$ is automatically a μ_p-equivariant chain map. Since the functor $C^b_{\mu_p}(X^{\mu_p},R) \to C^b(X^{\mu_p},R)$ reflects quasi-isomorphisms, it follows immediately that St_C descends to a functor St_D as below:

$$\boxtimes p : D^+(X,R) \xrightarrow{St_D} D^+_{\mu_p}(X^{\mu_p},R) \to D^+(X^{\mu_p},R).$$

Writing Σ for the suspension functor, we have $St_D\Sigma \cong \Sigma^p St_D$. Also, St_D is not triangulated, nor additive, nor even linear. To control the failure of linearity, and for future reference, we now introduce the following special element of $R[\mu_p]$.

Definition 2.3. The *norm* element $N \in R[\mu_p]$ is given by

$$N = \sum_{x\in\mu_p} x.$$

Observe that, given two objects A, B of $D^+_{\mu_p}(X,R)$, we have an action of μ_p on

$$\mathrm{Hom}_{D^+(X,R)}(A,B)$$

where by definition, $x \in \mu_p$ sends the non-equivariant morphism f to xfx^{-1}. The following two propositions control the failure of linearity.

Proposition 2.4. *Suppose given two parallel morphisms $f, g : A^\bullet \to B^\bullet$ in $D^+(X,R)$. Then the morphism*

$$St_D(f+g) - St_D(f) - St_D(g) : St_D(A^\bullet) \to St_D(B^\bullet)$$

is an induced map. That is, there exists some non-equivariant map

$$h : (A^\bullet)^{\boxtimes \mu_p} \to (B^\bullet)^{\boxtimes \mu_p}$$

such that the equivariant map

$$N(h) = \sum_{x\in\mu_p} xhx^{-1} : St_D(A^\bullet) \to St_D(B^\bullet)$$

is equal to $St_D(f+g) - St_D(f) - St_D(g)$.

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Proof. The maps f, g, h appearing in the statement are morphisms of derived categories. Let us right away replace $A^\bullet, B^\bullet$ by quasi-isomorphic complexes so that f, g may be realized as genuine maps of complexes. Let $\underline{f}, \underline{g}$ denote the constant functions $\mu_p \to \{f, g\}$ with respective values f, g. Then μ_p acts freely on $\{f, g\}^{\mu_p} - \{\underline{f}, \underline{g}\}$; choose a set $\{h_1, \ldots, h_d\}$ of orbit representatives ($d = (2^p - 2)/p$). Then each h_i determines a non-equivariant map $(A^\bullet)^{\boxtimes \mu_p} \to (B^\bullet)^{\boxtimes \mu_p}$, hence so does their sum h. Then we have

$$(f+g)^{\boxtimes \mu_p} - f^{\boxtimes \mu_p} - g^{\boxtimes \mu_p} = \sum_{x \in \mu_p} x h x^{-1}$$

where, by definition, xhx^{-1} is the composition

$$xhx^{-1} : (A^\bullet)^{\boxtimes \mu_p} \cong x^*(A^\bullet)^{\boxtimes \mu_p} \xrightarrow{x^*(h)} x^*(B^\bullet)^{\boxtimes \mu_p} \cong (B^\bullet)^{\boxtimes \mu_p}$$

where the two isomorphisms are given by the equivariant structures. □

Let $Fr : R \to R$ denote the pth-power morphism of commutative monoids (using the multiplication on R and forgetting the addition).

Proposition 2.5. *St_D is Frobenius-multiplicative with respect to the action of the multiplicative monoid R on hom-sets. That is, St_D determines a functor*

$$St_D : \operatorname{Ind}_R^R D^+(X, R) \to D^+_{\mu_p}(X^{\mu_p}, R)$$

which respects multiplication by R. Here the category on the left is obtained from $D^b(X, R)$ by regarding each hom-set as a set with an action of the multiplicative monoid R and inducing along the pth-power map of monoids $R \to R$.

Proof. Clear. □

Note that $\operatorname{Ind}_R^R D^b(X, R)$ is not an additive category in general. However, suppose that $R = k$ and k is perfect. In that case, the Frobenius map of monoids is actually a map of rings F, and is, moreover, bijective. Write $M : k\text{-mod} \to k\text{-set}$ for the forgetful functor, where k-set denotes the category of sets with action of the multiplicative monoid k. We have the following lemma.

Lemma 2.6. *Suppose that k is a perfect field of characteristic p. Then we have*

$$M \circ F^* \cong \operatorname{Ind}_k^k \circ M.$$

Proof. Indeed, in that case both F^* and Ind_k^k are equivalent to the functor of restriction along the inverse of Frobenius. □

It follows that if k is a perfect field of characteristic p, we have produced a k-multiplicative functor

$$St_D : F^* D^+(X, k) \to D^+_{\mu_p}(X^{\mu_p}, k).$$

Since the source $F^* D^+(X, k)$ is triangulated, we find this statement somewhat nicer than the version for general R. However, the functor St_D itself is still not linear, triangulated etc.

2.3 Localization

The category $D^+_{\mu_p}(X^{\mu_p}, R)$ is enriched, in a triangulated sense, over $H^*_{\mu_p}(X^{\mu_p}, R)$. That is, the monoidal structure of $D^+_{\mu_p}(X^{\mu_p}, R)$ gives maps of R-modules

$$H^n_{\mu_p}(X^{\mu_p}, R) \cong \operatorname{Hom}_{D^+_{\mu_p}(X^{\mu_p}, R)}(R, \Sigma^n R) \to \operatorname{Hom}_{D^+_{\mu_p}(X^{\mu_p}, R)}(\mathrm{id}, \Sigma^n)$$

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for each $n \geq 0$, whose sum is a map of algebras. In particular, $D^+_{\mu_p}(X^{\mu_p}, R)$ is enriched in the same sense over

$$H^*_{\mu_p}(*, R) \cong H^*(\mathrm{B}\mu_\mathrm{p}, R) \cong R \otimes^L_{\mathbb{Z}} (\mathbb{Z}[\hbar]/p\hbar).$$

Here $\hbar$ is the first Chern class of the tautological line bundle on $\mathrm{B}\mu_\mathrm{p}$ corresponding to the embedding $\mu_p \to \mathbb{C}^*$. In particular, this super-commutative ring receives a map from $R \otimes^L_{\mathbb{Z}} \mathbb{Z}[\hbar] = R[\hbar]$ so that $D^+_{\mu_p}(X^{\mu_p}, R)$ is enriched over $R[\hbar]$. Thus we may consider the 2-periodic R-linear triangulated category $D^+_{\mu_p}(X^{\mu_p}, R)[\hbar^{-1}]$, which is enriched over

$$R \otimes^L_{\mathbb{Z}} (\mathbb{Z}[\hbar]/p\hbar)[\hbar^{-1}] \cong R \otimes^L_{\mathbb{Z}} \mathbb{F}_\mathrm{p}[\hbar^{\pm 1}].$$

The degree 0 component of this ring is R/p, and the natural map from R to here is the modular reduction map. In particular, there is a Frobenius map of rings

$$F : R \to R \otimes^L_{\mathbb{Z}} \mathbb{F}_\mathrm{p}[\hbar^{\pm 1}]$$

In this way, it makes sense to ask whether a functor from a triangulated category enriched over R to one enriched over $R \otimes^L_{\mathbb{Z}} \mathbb{F}_\mathrm{p}[\hbar^{\pm 1}]$ is Frobenius-linear.

Proposition 2.7. *The composition*

$$St'_D : D^+(X, R) \xrightarrow{St_D} D^+_{\mu_p}(X^{\mu_p}, R) \to D^+_{\mu_p}(X^{\mu_p}, R)[\hbar^{-1}]$$

is exact, Frobenius-linear and preserves direct sums.

Proof. First, we prove that St'_D preserves direct sums. Let $A^\bullet$, $B^\bullet$ be complexes in $D^+(X, R)$. We argue as in the proof of Proposition 2.4 that we have

$$St_D(A^\bullet \oplus B^\bullet) \cong St_D(A^\bullet) \oplus St_D(B^\bullet) \oplus \mathrm{Ind}_1^{\mu_p} C^\bullet$$

for some complex $C^\bullet$ in $D^b(X^{\mu_p}, R)$. Here $\mathrm{Ind}_1^{\mu_p}$ is the averaging functor

$$\mathrm{Ind}_1^{\mu_p} : D^+(X^{\mu_p}, R) \to D^+_{\mu_p}(X^{\mu_p}, R)$$

bi-adjoint to the restriction functor $\mathrm{Res}_1^{\mu_p}$. Therefore, it suffices to prove that the composition

$$D^+(X^{\mu_p}, R) \xrightarrow{\mathrm{Ind}_1^{\mu_p}} D^+_{\mu_p}(X^{\mu_p}, R) \to D^+_{\mu_p}(X^{\mu_p}, R)[\hbar^{-1}]$$

is isomorphic to 0. This follows by adjunction from the fact that $Res_1^{\mu_p}(\hbar) = 0$.

Next, we prove Frobenius-linearity. By Proposition 2.5, it suffices to prove that St'_D respects addition of parallel morphisms. By Proposition 2.4, it is enough to see that the image of an induced morphism in $D^+_{\mu_p}(X^{\mu_p}, R)$ in the localized category $D^+_{\mu_p}(X^{\mu_p}, R)[\hbar^{-1}]$ is 0. This we have shown in the previous paragraph.

Finally, we prove exactness. First we must specify an exact structure, that is, an isomorphism $e : \Sigma St'_D \cong St'_D \Sigma$, which makes the image under St'_D of any triangle a triangle. Note that since p is odd, there is a morphism in $D^+_{\mu_p}(X^{\mu_p}, R)$ of functors $\hbar^{(p-1)/2} : \Sigma \to \Sigma^p$ which becomes an isomorphism when $\hbar$ is inverted. Already on the level of complexes we have a canonical isomorphism $\Sigma^p St_C \cong St_C \Sigma$. The exact structure is taken to be the composition

$$e : \Sigma St'_D \xrightarrow{((p-1)/2)!\hbar^{(p-1)/2}} \Sigma^p St'_D \cong St'_D \Sigma.$$

This is indeed an isomorphism since the localized category is enriched over $\mathbb{F}_\mathrm{p}$. The reason for the factor $((p-1)/2)!$ will be explained shortly. Thus, given a triangle

$$B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} \Sigma A^\bullet \xrightarrow{-\Sigma f} \Sigma B^\bullet$$

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in $D^+(X,R)$, we have a triangle

$$\begin{array}{ccccccc} St'_D(B^\bullet) & \xrightarrow{St'_D(g)} & St'_D(C^\bullet) & \xrightarrow{St'_D(h)} & St'_D(\Sigma A^\bullet) & \xrightarrow{St'_D(-\Sigma f)=-St'_D(\Sigma f)} & St'_D(\Sigma B^\bullet) \\ &&&&&& \downarrow \wr\!\| \, e^{-1} \\ &&&&&& \Sigma St'_D(B^\bullet) \end{array}$$

in $D^+_{\mu_p}(X^{\mu_p},R)[\hbar^{-1}]$. We must show that this is exact whenever the original triangle is. Since any exact triangle in a derived category is isomorphic to a semi-split one, we may assume that f is a chain map and $C^\bullet$ is the usual mapping cone satisfying

$$C^n = A^{n+1} \oplus B^n$$

with differential

$$\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix}.$$

So we have

$$St_C(C^\bullet) = \Big(\bigoplus_{i_1+\cdots+i_p=\bullet} (A^{i_1+1} \oplus B^{i_1}) \boxtimes \cdots \boxtimes (A^{i_p+1} \oplus B^{i_p})\Big)^\bullet$$

with some differential (whose precise form is not important). This complex has a $(p+1)$-step equivariant increasing filtration

$$St_C(B^\bullet) = F_0 St_C(C^\bullet) \subset \cdots \subset F_p St_C(C^\bullet) = St_C(C^\bullet)$$

where $F_i St_C(C^\bullet)$ consists of the subcomplex of $St_C(C^\bullet)$ in which at most i summands $A^?$ are taken in the expansion of the external tensor product. The inclusion of the zeroth piece of this filtration is equal to $St_C(g)$, while the quotient map

$$St_C(C^\bullet) \twoheadrightarrow St_C(C^\bullet)/F_{p-1}St_C(C^\bullet) \cong St_C(\Sigma A^\bullet)$$

is equal to $St_C(h)$. Furthermore, arguing as in Proposition 2.4, we see that $F_i St_C(C^\bullet)/F_{i-1}St_C(C^\bullet)$ is an induced complex for each $1 \le i \le p-1$. Therefore, the map

$$St_C(C^\bullet)/St_C(B^\bullet) \twoheadrightarrow St_C(\Sigma A^\bullet)$$

becomes an isomorphism in $D^b_{\mu_p}(X^{\mu_p},R)[\hbar^{-1}]$. Consider now the commutative diagram

$$\begin{array}{ccccccc} St_C(B^\bullet) & \xrightarrow{St_C(g)} & St_C(C^\bullet) & \xrightarrow{\alpha} & St_C(C^\bullet)/St_C(B^\bullet) & \xrightarrow{\beta} & \Sigma St_C(B^\bullet) \\ \downarrow \| && \downarrow \| && \downarrow \gamma && \\ St_C(B^\bullet) & \xrightarrow{St_C(g)} & St_C(C^\bullet) & \xrightarrow{St_C(h)} & \Sigma^p St_C(A^\bullet) & \xrightarrow{-\Sigma^p St_C(f)} & \Sigma^p St_C(B^\bullet) \end{array}$$

of equivariant chain complexes. Here α, β are the usual chain maps which make the image of the top row in $D^+_{\mu_p}(X,R)$ an exact triangle, γ is the quotient map, whose image in $D^+_{\mu_p}(X,R)[\hbar^{-1}]$ is an isomorphism, and $St_C\Sigma$ has been identified with $\Sigma^p St_C$. Comparing with the definition of the triangle obtained by applying St'_D to $B^\bullet \xrightarrow{g} C^\bullet \xrightarrow{h} \Sigma A^\bullet \xrightarrow{-\Sigma f} \Sigma B^\bullet$, we see that it is enough to prove that the diagram

$$\begin{array}{ccc} St_D(C^\bullet)/St_D(B^\bullet) & \xrightarrow{\beta} & \Sigma St_D(B^\bullet) \\ \downarrow \gamma && \downarrow ((p-1)/2)!\hbar^{(p-1)/2} \\ \Sigma^p St_D(A^\bullet) & \xrightarrow{-\Sigma^p St_D(f)} & \Sigma^p St_D(B^\bullet) \end{array}$$

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commutes in $D^+_{\mu_p}(X,R)[\hbar^{-1}]$. Here we have written β,γ for their own images in $D^+_{\mu_p}(X,R)$. Let $D^\bullet$ be the cocone of $A^\bullet \xrightarrow{\mathrm{id}} A^\bullet$, so that f induces a map $D^\bullet \to C^\bullet$. We have a commutative diagram

$$\begin{array}{ccc} St_C(D^\bullet)/St_C(A^\bullet) & \xrightarrow{\epsilon} & \Sigma St_C(A^\bullet) \\ \Big\downarrow \delta & & \Big\downarrow \Sigma St_C(f) \\ St_C(C^\bullet)/St_C(B^\bullet) & \xrightarrow{\beta} & \Sigma St_C(B^\bullet) \end{array}$$

where ϵ is the standard boundary map, and the vertical arrows are induced by f (no signs necessary). Now $\gamma\delta$ becomes an isomorphism in $D^b_{\mu_p}(X^{\mu_p},R)$ for the same reason that γ does; therefore so does δ. So it is enough to show that the composition of these two diagrams is commutative. By functoriality of $\hbar$, the resulting composition equals

$$\begin{array}{ccc} St_D(D^\bullet)/St_D(A^\bullet) & \xrightarrow{\epsilon} & \Sigma St_D(A^\bullet) \\ \Big\downarrow \gamma\delta & & \Big\downarrow \Sigma^p St_D(f)\circ((p-1)/2)!\hbar^{(p-1)/2} \\ \Sigma^p St_D(A^\bullet) & \xrightarrow{-\Sigma^p St_D(f)} & \Sigma^p St_D(B^\bullet) \end{array}$$

Let

$$\mathfrak{p} = (R \to R[\mu_p] \xrightarrow{\sigma-1} R[\mu_p] \xrightarrow{N} \cdots \xrightarrow{\sigma-1} R[\mu_p])$$

be the equivariant resolution of the trivial $R[\mu_p]$-module supported in degrees $(1-p),\ldots,0$. Here N is the norm element as defined in Definition 2.3. We have the standard chain maps $\mathfrak{p}\to R$, which is an isomorphism in the equivariant derived category, and $\mathfrak{p}\to\Sigma^{p-1}R$, which equals $\hbar^{(p-1)/2}$ by definition. Now ϵ is a chain map, which is an isomorphism in the equivariant derived category but not in the equivariant complex category. However, there *is* a chain map

$$\mathfrak{p}\otimes\Sigma St_C(A^\bullet) \xrightarrow{\zeta} St_C(D^\bullet)/St_C(A^\bullet)$$

such that $\epsilon\zeta$ is induced by the standard chain map $\mathfrak{p}\to R$ (i.e. counit in degree 0). To see this, it is enough to do the case $A^\bullet = R$ and then tensor on the right with $St_C(A^\bullet)$. In that case, we are looking for an equivariant chain map from the complex

$$R \to R[\mu_p] \xrightarrow{\sigma-1} R[\mu_p] \xrightarrow{N} \cdots \xrightarrow{\sigma-1} R[\mu_p]$$

supported in degrees $-p,\ldots,-1$ to the complex $E^\bullet$ satisfying

$$E^i = \bigoplus_{\substack{S\subset[p]\\ |S|=-i}} R_S$$

where R_S is a copy of R, and with differential sending R_S to $\bigoplus_{s\in S} R_{S-\{s\}}$ by $(1,-1,1,\ldots)$. Let us write 1_S for the canonical generator of R_S. One example of such a map is the map which sends the element 1 in the degree $-(2i+1)$ copy of $R[\mu_p]$ to the term

$$-i! \sum_{\substack{T\subset\{2,\ldots,p\}\\ |T|=2i\\ \text{even block lengths}}} 1_{\{1\}\cup T}$$

and sends the element 1 in the degree $-(2i+2)$ copy of $R[\mu_p]$ to the term

$$-i! \sum_{\substack{T\subset\{3,\ldots,p\}\\ |T|=2i\\ \text{even block lengths}}} 1_{\{1\}\cup\{2\}\cup T}.$$

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The sign is chosen so that $\epsilon\zeta$ is induced by the standard chain map $\mathfrak{p} \to R$. We compute

$$\gamma\delta\zeta = -((p-1)/2)!\hbar^{(p-1)/2}.$$

Therefore, the two paths $St_D(D^\bullet)/St_D(A^\bullet) \to \Sigma^p St_D(B^\bullet)$ in this composed diagram are equalized by ζ. Since ζ is an isomorphism in $D^+_{\mu_p}(X^{\mu_p}, R)[\hbar^{-1}]$, they coincide in the localized category as required. □

Corollary 2.8. *Suppose $R = k$ is a field of characteristic p. Then we have a triangulated k-linear functor*

$$St'_D : F^*D^+(X, k) \to D^+_{\mu_p}(X^{\mu_p}, k)[\hbar^{-1}].$$

2.4 Six functors

We shall henceforth restrict ourselves to the geometric context of [BL06]. Thus from now on every topological space X will be the Borel quotient $EG\overset{\times}{_G}Y$ of a complex algebraic variety Y by the action of some affine algebraic group G, every coefficient ring R will be a Noetherian ring of finite homological dimension, and we will restrict attention to the full subcategory

$$D^+_G(Y, R) \subset D^+(X, R)$$

spanned by those objects which descend from Y.

For the rest of this paper, we will cut down our scope even further (perhaps unnecessarily) to the constructible equivariant derived category

$$D^b_C(c, G)(Y, R).$$

This is the full subcategory of $D^+_G(Y, R)$ spanned by those complexes $\mathcal{F}$ which have only finitely many non-zero cohomology sheaves, and all of them are algebraically constructible, that is, locally constant along every stratum of some algebraic stratification of Y (depending on $\mathcal{F}$).

The constructions of the previous section preserve constructibility, so we have a Steenrod construction

$$St_D : D^b_{c,G}(Y, R) \to D^b_{c,G^{\mu_p}\rtimes\mu_p}(Y^{\mu_p}, R).$$

Recall that we have the deep 'six-functor formalism' for constructible derived categories; see Bernstein and Lunts [BL06]. We assume that the reader is familiar with this material, but remind him/her of the standard notation: for a G-equivariant algebraic map $f : Y \to Y'$, we have the adjoint pairs of exact functors

$$f^*\colon D^b_{c,G}(Y', R) \rightleftarrows D^b_{c,G}(Y, R) : f_*,$$
$$f_!\colon D^b_{c,G}(Y, R) \rightleftarrows D^b_{c,G}(Y', R) : f^!$$

and also a pair of bi-exact bifunctors

$$(-) \otimes (-) : D^b_{c,G}(Y, R) \times D^b_{c,G}(Y, R) \to D^b_{c,G}(Y, R),$$
$$\mathcal{H}om(-, -) : D^b_{c,G}(Y, R)^{op} \times D^b_{c,G}(Y, R) \to D^b_{c,G}(Y, R)$$

related by a tensor-hom adjunction. There is also a Verdier duality functor $\mathbb{D}$, and an exceptional tensor product $\otimes^!$, which can be written in terms of the other functors, as can the external tensor product $\boxtimes$. We call the collection of all of these functors the *six plus functors*. Notice that $G^{\mu_p} \rtimes \mu_p$ is also an affine algebraic group, so the six-functor formalism exists for the target category of St_D. Also if $f : Y \to Y'$ is G-equivariant then $f^{\mu_p} : Y^{\mu_p} \to (Y')^{\mu_p}$ is $G^{\mu_p} \rtimes \mu_p$-equivariant. The following fact is essentially a consequence of the same fact for $^{\boxtimes p}$:

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Proposition 2.9. *Steenrod's construction is compatible with the six-functor formalism. That is, we have canonical isomorphisms*

$$(f^{\mu_p})^* St_D \cong St_D f^*,$$
$$(f^{\mu_p})_* St_D \cong St_D f_*,$$
$$(f^{\mu_p})_! St_D \cong St_D f_!,$$
$$(f^{\mu_p})^! St_D \cong St_D f^!,$$
$$St_D(-) \otimes St_D(-) \cong St_D(- \otimes -),$$
$$St_D(-) \otimes^! St_D(-) \cong St_D(- \otimes^! -),$$
$$St_D(-) \boxtimes St_D(-) \cong St_D(- \boxtimes -),$$
$$\mathcal{H}om(St_D(-), St_D(-)) \cong St_D \mathcal{H}om(-,-)$$

commuting with any and all adjunction morphisms of the six-functor formalism.

We have the same compatibilities with functors St'_D.

Remark 2.10. The four functors f^*, f_*, $f_!$, $f^!$ and the three tensor product functors may all be extended to the bounded-below equivariant derived category, and are also compatible with St_D there, in the same way. However, the internal hom (and its special case, Verdier duality) do not make sense there, so we prefer to confine ourselves to the constructible case going forward.

Proof. That the claim for St'_D follows from the claim for St_D, is a direct consequence of the $H^*_{\mu_p}(*)$-linearity of the six functors in the μ_p-equivariant context. So let us focus on St_D. It is known (cf. [BL06]) that the six functors intertwine with external tensor product, so it is just a matter of showing that the μ_p-equivariant structures are carried along. This is easy for f^*: it is defined on the cochain level, and strictly intertwines with St_C on that level (no homotopy involved). It follows by adjointness that we have a canonical map

$$St_D f_* \to (f^{\mu_p})_* St_D$$

which is an isomorphism upon forgetting equivariance, hence itself an isomorphism.

We may argue similarly for $f_!$, $f^!$. Indeed, the thrust of the argument for f^* was to lift both f^* and St_D to the concrete model of bounded-below cochain complexes and observe strict intertwining there. There is an alternative concrete model which works just as well for the purposes of defining St_D, namely the category of bounded-below cochain complexes of soft sheaves. But $f_!$ is defined on the level of the soft model, and the intertwining holds strictly there. We deduce the desired claim for $f^!$ in the same way we deduced it for f_* from f^*.

The canonical isomorphism

$$St_D(-) \boxtimes St_D(-) \cong St_D(- \boxtimes -)$$

is a tautological consequence of the cochain-level construction of St_D, and intertwining for the other two tensors follows by composing with $*$- and !- restriction along the diagonal. Finally, the canonical isomorphism

$$\mathcal{H}om(St_D(-), St_D(-)) \cong St_D \mathcal{H}om(-,-)$$

can be deduced using the hom-tensor adjunction. □

Corollary 2.11. *Let R denote the constant sheaf and ω_R denote the dualizing complex (see [CG97] for details and evidence of its ubiquity in geometric representation theory). We have*

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canonical isomorphisms

$$St_D R \cong R$$

and

$$St_D \omega_R \cong \omega_R.$$

Proof. The result is easy over the point $*$: we have $\omega_R \cong R$ and the construction of St_D is degenerate. For general Y, apply the intertwining relation between St_D and f^*, $f^!$ where f is the constant map $Y \to *$. $\square$

2.5 Tate's construction

Note that the object $St_D(\Sigma^n R)$ is canonically isomorphic to $\Sigma^{pn} R$ with its trivial μ_p-equivariant structure. Here R is the constant sheaf. Since a degree n cohomology class is just a morphism $R \to \Sigma^n R$ in $D^b(X, R)$, we thus obtain a Frobenius-multiplicative map of multiplicative R-sets:

$$St^H : H^n(X, R) \to H^{pn}_{\mu_p}(X^{\mu_p}, R).$$

This map is not Frobenius-linear, but as in Proposition 2.4, its deviation from additivity is by a class induced from $H^{pn}(X^{\mu_p}, R)$. To see how these maps interact with multiplication, we make the following definition.

Definition 2.12. Let $(\mathcal{C}, *, \mathbb{1}, e, a, s)$ be a symmetric monoidal abelian category enriched over some commutative ring R, that is, $\mathcal{C}$ is an abelian category, $*$ is a bi-exact R-linear functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $\mathbb{1}$ is an object of $\mathcal{C}$, e is a pair of equivalences $\mathbb{1} * \mathrm{id} \cong \mathrm{id} \cong \mathrm{id} * \mathbb{1}$, a is an associativity constraint for $*$ and s is a commutativity constraint for $*$, satisfying natural compatibilities. Let A be an object of $\mathcal{C}$. Then s determines an action of μ_p on $A^{*\mu_p}$, and we define

$$A^{(1)} := \hat{H}^0_{\mu_p}(A^{\mu_p}) := \ker_{A^{*\mu_p}}(1-\sigma)/\operatorname{im}_{A^{*\mu_p}}(N).$$

Here N is the norm element defined in Definition 2.3. This is the so-called *Tate construction*; see [CE99] for more details. For a morphism $f : A \to B$ the morphism $f^{*\mu_p} : A^{*\mu_p} \to B^{*\mu_p}$ is μ_p-equivariant and so induces a morphism $A^{(1)} \to B^{(1)}$, so that $(-)^{(1)}$ becomes a functor.

Lemma 2.13. *$(-)^{(1)}$ is additive over $\mathbb{Z}$.*

Proof. We essentially rehash the proof of Propositions 2.4 and 2.7. First we show that $(-)^{(1)}$ is linear over $\mathbb{Z}$. Suppose $f, g : A \to B$ are two parallel morphisms in $\mathcal{C}$. Let $\underline{f}, \underline{g}$ denote the constant functions $\mu_p \to \{f, g\}$ with respective values f, g. Then μ_p acts freely on $\{f, g\}^{\mu_p} - \{\underline{f}, \underline{g}\}$; choose a set $\{h_1, \ldots, h_n\}$ of orbit representatives ($n = (2^p - 2)/p$). Then each h_i determines a non-equivariant map $A^{*\mu_p} \to B^{*\mu_p}$, and we have

$$(f+g)^{*\mu_p} - f^{*\mu_p} - g^{*\mu_p} = \sum_{x \in \mu_p} \sum_{i=1}^{n} x h_i x^{-1}.$$

Restricting to $\ker_{A^{*\mu_p}}(1-\sigma)$, this becomes

$$((f+g)^{*\mu_p} - f^{*\mu_p} - g^{*\mu_p})_{\ker_{A^{*\mu_p}}(1-\sigma)} = \sum_{x \in \mu_p} \sum_{i=1}^{n} x h_i = N \sum_{i=1}^{n} h_i$$

which factors through $\operatorname{im}_{B^{*\mu_p}}(N)$ as required.

Next we show that $(-)^{(1)}$ preserves direct sums. Let $\underline{A}, \underline{B}$ denote the constant functions $\mu_p \to \{A, B\}$ with respective values A, B. Then μ_p acts freely on $\{A, B\}^{\mu_p} - \{\underline{A}, \underline{B}\}$; choose a set $\{C_1, \ldots, C_n\}$ of orbit representatives. Then each C_i determines an object of $\mathcal{C}$, and as a

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μ_p-module in $\mathcal{C}$ we have

$$(A \oplus B)^{*\mu_p} \cong A^{*\mu_p} \oplus B^{*\mu_p} \oplus \bigoplus_{i=1}^{n} C_i[\mu_p].$$

The result then follows from the fact that $\hat{H}^0_{\mu_p}(k[\mu_p]) = 0$ in R-mod. □

Let SVect_k denote the symmetric monoidal category of $\mathbb{Z}/2$-super-graded k-vector spaces.

Lemma 2.14. *Suppose that $R = k$ is a field of characteristic p and that $\mathcal{C}$ admits a super-fiber functor $\mathcal{C} \to \mathrm{SVect}_k$. Then $(-)^{(1)}$ is exact, monoidal and Frobenius-linear over k. In the case $\mathcal{C} = \mathrm{SVect}_k$, $(-)^{(1)}$ is equivalent to the functor $k \otimes_F (\text{-})$ which tensors the k-linear structure along the Frobenius map $F : k \to k$.*

Proof. Since Tate's construction commutes with the fiber functor, it is enough to take $\mathcal{C} = \mathrm{SVect}_k$, where it is a simple calculation using bases. □

Now suppose that A, B are objects of $\mathcal{C}$. We have a μ_p-equivariant isomorphism $(A * B)^{*\mu_p} \cong (A^{*\mu_p} * B^{*\mu_p})$. We have also the natural inclusions

$$\ker_{A^{*\mu_p}}(1-\sigma) * \ker_{B^{*\mu_p}}(1-\sigma) \to \ker_{A^{*\mu_p} * B^{*\mu_p}}(1-\sigma),$$
$$\mathrm{im}_{A^{*\mu_p}}(N) * \ker_{B^{*\mu_p}}(1-\sigma) \to \mathrm{im}_{A^{*\mu_p} * B^{*\mu_p}}(N),$$
$$\ker_{A^{*\mu_p}}(1-\sigma) * \mathrm{im}_{A^{*\mu_p}}(N) \to \mathrm{im}_{A^{*\mu_p} * B^{*\mu_p}}(N)$$

which induce a map $A^{(1)} * B^{(1)} \to (A * B)^{(1)}$. Suppose that $(A, \mathbb{1}_A, m_A)$ is a unital ring in $\mathcal{C}$. Then $A^{(1)}$ still has a multiplication

$$m_{A^{(1)}} : A^{(1)} * A^{(1)} \to (A * A)^{(1)} \xrightarrow{m_A^{(1)}} A^{(1)}.$$

Also, there is a canonical isomorphism $\ker_{\mathbb{1}^{*\mu_p}}(1-\sigma) \cong \mathbb{1}$, hence a canonical surjection $\mathbb{1} \to \mathbb{1}^{(1)}$ which determines a map

$$\mathbb{1}_{A^{(1)}} : \mathbb{1} \to \mathbb{1}^{(1)} \xrightarrow{\mathbb{1}_A^{(1)}} A^{(1)}.$$

One may check that this makes $A^{(1)}$ into a ring, and, moreover, that $A^{(1)}$ is associative or commutative if A is. The following lemma explains how this looks in the main example.

Lemma 2.15. *Suppose $A = \bigoplus_{i \in \mathbb{Z}/2} A_i$ is a unital ring in $\mathcal{C} = \mathrm{SVect}_k$. Then the ring structure on $A^{(1)}$ corresponds under the identification $A^{(1)} = k \otimes_F A$ to the ring structure with unit $1 \otimes_F \mathbb{1}_A$ and multiplication*

$$m_{A^{(1)}}(r \otimes a, r' \otimes a') = (-1)^{ij\binom{p}{2}} rr' \otimes m_A(a, a')$$

for $a \in A_i, a' \in A_j$ and $r, r' \in k$.

Proof. The isomorphism of $k \otimes_F A_i$ with $(A^{(1)})_i$ sends the element $r \otimes a$ to the class of $r.\underbrace{a \otimes \cdots \otimes a}_{p \text{ times}}$. The natural map

$$(A^{(1)})_i \otimes (A^{(1)})_j \cong \hat{H}^0_{\mu_p}((A_i)^{*\mu_p}) \otimes \hat{H}^0_{\mu_p}((A_j)^{*\mu_p}) \to \hat{H}^0_{\mu_p}((A_i \otimes A_j)^{*\mu_p})$$

sends the class of $\underbrace{a \otimes \cdots \otimes a}_{p \text{ times}} \otimes \underbrace{a' \otimes \cdots \otimes a'}_{p \text{ times}}$ to the class of $(-1)^{ij\binom{p}{2}} \underbrace{(a \otimes a') \otimes \cdots \otimes (a \otimes a')}_{p \text{ times}}$, since it entails permuting the xth copy of A_i with the yth copy of A_j for every $p \geq x > y \geq 1$. □

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Arguing the same way, we have the following proposition.

Proposition 2.16. *Let A be a Hopf algebra in SVect_k. Then $A^{(1)}$ is naturally a Hopf algebra in SVect_k. It has multiplication and unit given as in Lemma 2.15, comultiplication given by*

$$\Delta_{A^{(1)}}(r \otimes a) = r \otimes (-1)^{\binom{p}{2} \deg \otimes \deg} \Delta_A(a),$$

counit given by $\epsilon_{A^{(1)}}(r \otimes a) = r(\epsilon_A(a))^p$ and antipode given by $S_{A^{(1)}}(r \otimes a) = r \otimes S_A(a)$. Moreover, the functor $(-)^{(1)}$ on SVect_k upgrades to functor

$$(-)^{(1)} : A\text{-comod} \to A^{(1)}\text{-comod}.$$

For an A-comodule M, the $A^{(1)}$-comodule structure on $M^{(1)}$ is given by

$$\Delta_{M^{(1)}}(r \otimes m) = r \otimes (-1)^{\binom{p}{2} \deg \otimes \deg} \Delta_M(m).$$

Example 2.17. Suppose $\mathcal{O}$ is a commutative Hopf algebra in SVect_k. Then the monoidal category $\mathcal{C}$ of $\mathcal{O}$-comodules is symmetric. Taking pth powers gives a (Frobenius) map of Hopf algebras $F_{\mathcal{O}} : \mathcal{O}^{(1)} \to \mathcal{O}$. Then Tate's construction on $\mathcal{C}$ factors as

$$\mathcal{O}\text{-comod} \xrightarrow{(-)^{(1)}} \mathcal{O}^{(1)}\text{-comod} \xrightarrow{F_{\mathcal{O}}^*} \mathcal{O}\text{-comod}.$$

For instance, we could take $\mathcal{O}$ to be the ring of functions $\mathcal{O}(\mathbb{G}_m)$ on the multiplicative group $\mathbb{G}_m$ over k, concentrated in degree $0 \in \mathbb{Z}/2$. Then $\mathcal{O}(\mathbb{G}_m)$-comod contains as a full subcategory over SVect_k the category of $\mathbb{Z}$-super-graded vector spaces, and Tate's construction there is isomorphic to the functor which applies $k \otimes_F (-)$ and multiplies degrees by p.

Recall we have the Frobenius-multiplicative maps of multiplicative k-sets

$$St^H : H^n(X, k) \to H^{pn}_{\mu_p}(X^{\mu_p}, k).$$

We view cohomology rings as commutative ring objects of $\mathbb{Z}$-super-graded vector spaces; in particular, we can apply functor $(-)^{(1)}$ to them. By Lemma 2.6, if k is perfect then it gives a map of $\mathbb{Z}$-super-graded k-sets

$$St_{\mathrm{ex}} : H^*(X, k)^{(1)} \to H^*_{\mu_p}(X^{\mu_p}, k).$$

The following fact is immediate from the constructions.

Proposition 2.18. *St_{ex} respects the multiplicative k-monoidal structures.*

Remark 2.19. If k is not perfect, then the map $H^*(X, k) \to H^*_{\mu_p}(X^{\mu_p}, k)$ of $\mathbb{Z}/2$-super-graded sets respects the multiplicative monoidal structures up to the sign change of Lemma 2.15. There is presumably an appropriate nonlinear version of Tate's construction which would allow us to say that we really have a certain $\mathbb{Z}$-super-graded k-monoid $H^*(X, k)^{(1)}_{nl}$ and a map of monoids $H^*(X, k)^{(1)}_{nl} \to H^*_{\mu_p}(X^{\mu_p}, k)$, but we prefer for simplicity not to do so.

2.6 Borel–Moore homology

We return to the setting of § 2.4. By Corollary 2.11, we have a canonical isomorphism

$$St_D(\omega) \cong \omega.$$

Here the ω on the left-hand side denotes the G-equivariant dualizing complex on Y with coefficients in R, while the ω on the left-hand side denotes the $G^{\mu_p} \rtimes \mu_p$-equivariant dualizing complex on Y^{μ_p} with coefficients in R. By definition, the G-equivariant Borel–Moore homology

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of Y is

$$H_n^{\mathrm{BM},G}(Y,R) := \mathrm{Hom}_{D^b_G(Y,R)}(R,\Sigma^n\omega).$$

See § 3.10 for more about this. Altogether $H_n^{\mathrm{BM},G}(Y,R)$ form a $\mathbb{Z}$-super-graded $H^*_G(Y,R)$-module; in particular, it is a module for $H^*_G(*,R)$. By functoriality we have the nonlinear maps

$$St^{\mathrm{BM}} : H_n^{\mathrm{BM},G}(Y,R) \to H_{pn}^{\mathrm{BM},G^{\mu_p}\rtimes\mu_p}(Y^{\mu_p},R).$$

This is a map of St^H-monoids. Its discrepancy from additivity it averaged from $H_{pn}^{\mathrm{BM},G^{\mu_p}}(Y^{\mu_p},R)$. If R is a perfect field k of characteristic p, we can say that we have a nonlinear graded map of St_{ex}-monoids:

$$St^{\mathrm{BM}}_{\mathrm{ex}} : H_*^{\mathrm{BM},G}(Y,k)^{(1)} \to H_*^{\mathrm{BM},G^{\mu_p}\rtimes\mu_p}(Y^{\mu_p},k).$$

2.7 Steenrod operations

For simplicity let us assume k to be perfect from now on. Let us compose St_{ex} with the restriction map to the diagonal:

$$St_{\mathrm{in}} : H^*(X,k)^{(1)} \xrightarrow{St_{\mathrm{ex}}} H^*_{\mu_p}(X^{\mu_p},k) \xrightarrow{\Delta^*} H^*_{\mu_p}(X,k) \cong H^*(X,k)[a,\hbar].$$

This is again a map of multiplicative k-monoids. Tautologically we have $St_{\mathrm{in}}(x) = x^p \bmod (a,\hbar)$. Also St_{in} is compatible with pullback maps in cohomology in the natural way. Since induction commutes with restriction, the difference between $St_{\mathrm{in}}(x+y)$ and $St_{\mathrm{in}}(x)+St_{\mathrm{in}}(y)$ is induced from a cohomology class $z\in H^\bullet(X;k)$. Since μ_p acts trivially on X, that means that it is equal to $pz=0$, so St_{in} is linear. That is, we have a map of super-commutative k-algebras

$$St_{\mathrm{in}} : H^*(X,k)^{(1)} \to H^*(X,k)[a,\hbar].$$

Remark 2.20. The coefficients of $\hbar^m, a\hbar^m$ in St_{in} are *not* the Steenrod operations. More precisely, they are the Steenrod operations only up to some non-zero scalars. Even more precisely, let $x\in H^n(X,k)$ and let $p=2q+1$. Consider

$$(-1)^{qn(n-1)/2}(q!)^{-n}St_{\mathrm{in}}(x).$$

where x is viewed as a degree pn element of $H^*(X,k)^{(1)}$. The coefficient of $\hbar^m$ in this expression vanishes unless $m=\frac{1}{2}(p-1)(n-2s)$ for some s such that $2s\le n$, in which case that coefficient is equal to $(-1)^sP^s(x)$ where P^s is the sth Steenrod operation. Similarly, the coefficient of $a\hbar^m$ in that expression vanishes unless $m=\frac{1}{2}(p-1)(n-2s)-1$ for some s such that $2s\le n$, in which case that coefficient is equal to $(-1)^{s+1}\beta P^s(x)$ where β is the Bockstein operation. A careful comparison of St_{in} with the sum of Steenrod operations as defined in [SE62] shows the constructions are indeed identical except that N. Steenrod deliberately inserted exactly these invertible 'correction factors'. For him, as far as we can tell, the purpose was to ensure that

(i) $P^0=\mathrm{id}$, and
(ii) the total Steenrod, $\sum_{s=0}^\infty P^s$, is a ring endomorphism of $H^*(X)$.

On the one hand, from the perspective of the results of this paper, these two constraints are pure obfuscation; the latter denies the reality that the natural source of the algebra homomorphism is the Tate construction $H^*(X)^{(1)}$ rather than $H^*(X)$, while the former masks the resemblance (not a coincidence!) to the Artin–Schreier morphism. On the other hand, from the perspective of the author, it is rather fortunate as the core innovation of this work would surely have been discovered sooner otherwise.

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2.8 Artin–Schreier

We indicate how the Artin–Schreier map comes naturally out of the above considerations. First note that if n is even then the number

$$(-1)^{qn(n-1)/2}(q!)^{-n}$$

boils down to $(-1)^{n/2}$. It is a standard fact that on a degree 2 class x we have $P^0(x) = x, P^1(x) = x^p$, and higher powers vanish. Therefore

$$St_{\text{in}}(x) = x^p - \hbar^{p-1}x + \hbar^{p-2}\beta(x).$$

Let $X = BT$ for some compact torus T. Since its cohomology is supported in even degrees, the Bockstein operator acts as zero and St_{in}, on the level of k-cohomology, is exactly the $\hbar$-Artin–Schreier map

$$\mathcal{O}(\mathfrak{t}_k)^{(1)} \xrightarrow{AS_\hbar} \mathcal{O}(\mathfrak{t}_k)[\hbar] \subset \mathcal{O}(\mathfrak{t}_k)[a, \hbar]$$

as defined in Fact 2.1.

Recall that if G is a compact Lie group with maximal torus T, and p is large enough with respect to the Weyl group of G, then the projection $BT \to BG$ induces an inclusion

$$H^*(BG, k) \to H^*(BT, k)$$

which is identified with

$$\mathcal{O}(\mathfrak{t}_k//W) \to \mathcal{O}(\mathfrak{t}_k).$$

The $\hbar$-Artin–Schreier map induces a map on subspaces

$$\mathcal{O}(\mathfrak{t}_k//W)^{(1)} \xrightarrow{AS_\hbar} \mathcal{O}(\mathfrak{t}_k//W)[\hbar]$$

which is also important in the theory of Frobenius-constant quantization. The point is that this map is also induced by St_{in}, since it is compatible with pullbacks.

It is entertaining to show more directly how $AS_\hbar$ arises, without relying on any outside facts about Steenrod operations. We can reduce to the rank 1 case $T = S^1$. Let b denote the degree 2 generator (first Chern class of tautological line bundle) of BS^1; we need to show that $St_{\text{in}}(b) = b^p - \hbar^{p-1}b$. Let $C_p \subset S^1$ denote the cyclic group of order p, considered as distinct from μ_p. Consider the projection

$$BC_p \to BS^1.$$

It induces an injective map

$$k[b] \to k[s, b]$$

in cohomology, where s is a degree 1 generator. By functoriality it is enough to prove the equality when b is regarded as a cohomology class of BC_p. Note that amongst degree $2p$ elements of $k[b, \hbar]$, the desired element $b^p - \hbar^{p-1}b$ is the unique one which gives 0 when we set b to any multiple in $\mathbb{F}_{\mathrm{p}}$ of $\hbar$, and gives b^p when we set $\hbar = 0$. The latter statement is automatic, so we have to check the former. So fix some $t \in \mathbb{F}_{\mathrm{p}}$. Having chosen an isomorphism $\mu_p \cong C_p$, t determines a group homomorphism $\mu_p \to C_p$.

The constant sheaf $\Sigma^n k$ of BC_p is contained in the full subcategory

$$D^b(k[C_p]\text{-mod}) = D_{C_p}(*) \subset D(BC_p).$$

Our coefficients are k, which we drop from the notation. It is easier for our purpose to work in $D^b(k[C_p]\text{-mod})$. Compatible with the functor St_D out of $D(BC_p)$ we have the functor

$$St_D : D^b(k[C_p]\text{-mod}) \to D^b(k[\mu_p \ltimes (C_p)^{\times p}]\text{-mod}).$$

This is then composed with the diagonal restriction

$$D^b(k[\mu_p \ltimes (C_p)^{\times p}]\text{-mod}) \xrightarrow{\Delta^*} D^b(k[\mu_p \times C^p]\text{-mod}).$$

By definition $St_{\text{in}}(b)$ is given by applying that composition to the morphism $k \xrightarrow{b} k[2]$, where k is the trivial C_p-module. We want further to set $b = t\hbar$; this corresponds to restricting along the map

$$\mu_p \xrightarrow{\text{id}\times t} \mu_p \times C_p.$$

Write

$$(\text{id} \times t)^* : D^b(k[\mu_p \times C_p]\text{-mod}) \to D^b(k[\mu_p]\text{-mod})$$

for the corresponding restriction map. We need to show that $(\text{id} \times t)^* \circ \Delta^* \circ St(b) = 0$. But actually there is an isomorphism of functors

$$(\text{id} \times t)^* \circ \Delta^* \circ St_D \cong St_D \circ i^*$$

where i^* is the forgetful functor $D^b(k[C_p]\text{-mod}) = D_{C_p}(*) \to D(*)$. Indeed, for an object $A^\bullet$ of $D^b(k[C_p]\text{-mod})$, the underlying complex of both functors is $(A^\bullet)^{\otimes \mu_p}$, and the automorphism which sends each summand

$$A^{i_1} \otimes \cdots \otimes A^{i_p}$$

to itself by $1 \otimes \sigma^t \otimes \sigma^{2t} \otimes \cdots \otimes \sigma^{(p-1)t}$ intertwines the two actions of μ_p. Here σ is some generator of μ_p. But the functor $St \circ i^*$ kills b, since i^* does.

3. Coulomb branch

3.1 Prelude: Frobenius-constant quantizations

Let k be a field of characteristic p and let $\mathcal{C}$ be a symmetric monoidal category over k. The reader may assume that $\mathcal{C}$ is the category of comodules of some commutative Hopf algebra in SVect_k. Let A be a commutative (and associative) algebra in $\mathcal{C}$. Let

$$F : A^{(1)} \to A$$

be the Frobenius map. Let Q be an augmented commutative algebra in $\mathcal{C}$ with augmentation $\epsilon : Q \to k$. Following [BK08], we make the following definition.

DEFINITION 3.1.

(i) A *Q-quantization* of A is a flat associative Q-algebra A_Q in $\mathcal{C}$ such that $A_Q \otimes_Q k = A$.

(ii) A *Frobenius-constant Q-quantization* of A is a Q-quantization A_Q of A together with a map

$$F_Q : A^{(1)} \to Z(A_Q)$$

of algebras which lifts the Frobenius map, that is, such that $\epsilon \circ F_Q \equiv F$. Here $Z(A_Q)$ denotes the center of A_Q.

The main example for us is as follow. We take K to be some $\mathbb{G}_m$-equivariant algebraic group in Vect_k, and view $\mathcal{O} := \mathcal{O}(K \rtimes \mathbb{G}_m)$ as a Hopf algebra in SVect_k concentrated in degree 0. We take $\mathcal{C} = \mathcal{O}\text{-comod}$. Let $\hbar$ be a basis vector of the one-dimensional representation of $K \rtimes \mathbb{G}_m$ in which K acts trivially and $\mathbb{G}_m$ acts with weight 2. Let $Q = k[\hbar]$. In this case, we will call a Q-quantization simply an *$\hbar$-quantization*, or just a *quantization* if the meaning is clear.

Fact 3.2. (i) Let X be a smooth affine algebraic variety over k. Then the ring of asymptotic crystalline differential operators, $\mathcal{D}_\hbar(X)$, is a canonical $\hbar$-quantization of $\mathcal{O}(T^*X)$. Here $\mathbb{G}_m$ acts

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trivially on $\mathcal{O}(X)$ and on vector field with weight 2. Let ∂ be a vector field on X. Then ∂^p acts as a derivation on $\mathcal{O}(X)$, so that $\partial^p - \partial^{[p]}$ annihilates $\mathcal{O}(X)$ for a unique vector field $\partial^{[p]}$. Then $\mathcal{D}_\hbar(X)$ has a canonical Frobenius-constant structure determined by

$$F_\hbar : \begin{array}{ccll} x & \mapsto & x^p & x \in \mathcal{O}(X) \\ \partial & \mapsto & \partial^p - \hbar^{p-1}\partial^{[p]} & \partial \in Vect(X). \end{array}$$

(ii) Let J be a smooth algebraic group over k. Then $F_\hbar$ as above is $K = J \times J$-equivariant (induced by left and right regular actions). In particular, if we take invariants for the left factor, we obtain a Frobenius-constant structure for the quantization $\mathcal{U}_\hbar(J)$ of $\mathcal{O}(\mathrm{Lie}(J)^*)$.

(iii) Let T be a complex torus and let $T^\vee$ be the Langlands dual split torus over k, that is,

$$T^\vee = \mathrm{Spec}(k[\mathbb{X}_\bullet(T)])$$

where $\mathbb{X}_\bullet(T)$ is the cocharacter lattice of T and $k[\mathbb{X}_\bullet(T)]$ is its group algebra. We have canonical identifications

$$\mathcal{O}((\mathfrak{t}^\vee)^*) = \mathcal{O}(\mathfrak{t}_k),$$
$$\mathcal{U}_\hbar(\mathfrak{t}^\vee) = \mathcal{O}(\mathfrak{t}_k \times \mathbb{G}_a).$$

If we take Spec of the Frobenius-constant structure we recover the $\hbar$-Artin–Schreier map

$$F_\hbar = AS_\hbar : \mathfrak{t}_k \times \mathbb{G}_a \to \mathfrak{t}_k^{(1)}$$

of Fact 2.1.

Remark 3.3. If a commutative algebra and its quantization contain in a natural way $H^*_G(*, k)$ for some complex reductive group G with maximal torus T, then when searching for a Frobenius-constant structure it is natural to look for one which is compatible with the $\hbar$-Artin–Schreier map.

3.2 Interlude

The next eight subsections are intended to be a self-contained description of an extension of the theory of Beilinson–Drinfeld Grassmannians, an amazing invention first described in the canonical [BD91]. Our extension required a little more geometric machinery than was present in [BD91], but we were fortunately able to find what was required in Raskin's excellently concise paper [Ras15] on placid ind-schemes. This was actually originally written with $\mathcal{D}$-modules in mind, but the underlying geometry is the same. These eight subsections are also intended to be a self-contained account of the necessary details of [Ras15], and we have attempted to present things in a slightly more geometric and less categorical way, but highly recommend reading the paper. We have tried to make it clear in the text what is new and what is essentially to be found in [BD91] and [Ras15]. What follows is an overview.

We give the following very brief and incomplete overview of the purpose of Beilinson–Drinfeld Grassmannians.

(i) If G is a complex algebraic group, its affine Grassmannian Gr_1 is the algebro-geometric incarnation of the E_2-group $\Omega^1 K$ of based loops in a compact form K of G.
(ii) Gr_1 comes naturally equipped with a 'convolution diagram' which is the algebro-geometric incarnation of the pointwise multiplication structure on $\Omega^1 K$.
(iii) The Beilinson–Drinfeld Grassmannian Gr_2 of G is both a deformation of Gr_1 and the algebro-geometric incarnation of the loop-concatenation structure on $\Omega^1 K$.

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(iv) The Beilinson–Drinfeld Grassmannian fits inside a deformation of the aforementioned 'convolution diagram', called the 'global convolution diagram' (see [BD91, MV07]), which is the algebro-geometric incarnation of the Eckmann–Hilton argument.

Now we review in similar terms what we do that is *new*. (i) We construct a 'global convolution diagram' in the context of the 'variety of triples' (which is the geometric basis of the quantum Coulomb branch; see [BFN16]). This construction is also summarized in the appendix to [BFN17]. The Beilinson–Drinfeld Grassmannian was already constructed in this wider context in [BFN16], but the global convolution diagram (and hence the Eckmann–Hilton argument) had proved elusive.

(ii) We construct a geometric analogue of the Kudo–Araki–Dyer–Lashof operations, in the same way that the Beilinson–Drinfeld Grassmannian itself is a geometric analogue of the loop-concatenation structure of loop groups. We do this for the variety of triples,but it is new even for the ordinary Grassmannian.

(iii) We produce a gadget that is to our geometric analogue of the Kudo–Araki–Dyer–Lashof operations as the global convolution diagram is to the Beilinson–Drinfeld Grassmannian (for the variety of triples as well as the basic Grassmannian case). This provides the geometric incarnation of a principle which is similar in spirit to the Eckmann–Hilton argument, but hitherto unknown, namely, that any S^1-framed E_3-algebra over $\mathbb{F}_\mathrm{p}$ defines a Frobenius-constant quantization (to be elucidated in a forthcoming work; the topological insight is that one can move a ball past – or rather, through – a donut in an S^1-equivariant way).

3.3 Formal neighborhoods

Let X be a smooth complex curve and let S be a finite set. Given a commutative ring R and an R-point x of X^S, we denote the coordinates of x by x_s $(s \in S)$, write $\Gamma(x_s)$ for the graph of x_s in X_R and write $I(x_s)$ for its ideal. We write $\Delta_S(x)$ for the formal neighborhood of the union of the graphs of x_s $(s \in S)$. That is, $\Delta_S(x)$ is the direct limit in affine schemes over X:

$$\Delta_S(x) = \varinjlim_i \Delta_{S,i}(x)$$

where

$$\Delta_{S,i}(x) = \operatorname{Spec}\left(\mathcal{O}_{X_R}\Big/ \prod_{s\in S} I(x_s)^i\right).$$

Given a subset $S' \subset S$ and an R-point x of X^S, we will write

$$\Delta_S^{S'}(x)$$

for the S'-punctured formal neighborhood, that is, the complement of the union of the graphs of x_s $(s \in S')$ in $\Delta_S(x)$. As a sheaf of algebras on $\Delta_S(x)$, $\mathcal{O}(\Delta_S^{S'}(x))$ has an exhaustive increasing filtration:

$$F^j\mathcal{O}(\Delta_S^{S'}(x)) = \mathcal{O}(\Delta_S(x)).\prod_{s\in S'} I(x_s)^{-j}.$$

Suppose we have $S'' \subset S' \subset S$ and $x \in X^S(R)$. The inclusion $S' \subset S$ defines a projection $f : X^S \to X^{S'}$, and we will occasionally write

$$\Delta_{S'}^{S''}(x)$$

for $\Delta_{S'}^{S''}(f(x))$. We have a closed embedding $\Delta_{S'}^{S''}(x) \to \Delta_S^{S''}(x)$. Note, however, that this is in a sense non-uniform in x: for instance, if for every $s \in S$ there exists an $s' \in S'$ such that $x_s = x_{s'}$, then the embedding is an isomorphism; and conversely. This is essentially the fact underlying Beilinson and Drinfeld's 'fusion' Grassmannian [BD91]. We will make more of this when we discuss co-placid morphisms; see Example 3.15.

For notational simplicity, we frequently remove commas and braces from S, S', and also drop the part (x), when it is clear which point we refer to. So, for example, the expression

$$\Delta^{\{1\}}_{\{1,2\}}(x)$$

becomes

$$\Delta^{1}_{12}.$$

Remark 3.4. Exponential objects such as $X^{\Delta^{S'}_S}$ will form the building blocks of our geometric objects. This is not really a new idea. For example, $G^{\Delta^{\varnothing}_S}$ is the 'global group' of [BD91]; moreover, it acts on $G^{\Delta^{S}_S}$ and the quotient is the Beilinson–Drinfeld Grassmannian of degree $|S|$. However, the idea to let S' lie somewhere between $\varnothing$ and S does seem to be new, and is the essential insight in constructing the global convolution diagram for the variety of triples.

3.4 Global groups; pro-smoothness

Now fix an affine algebraic group G over $\mathbb{C}$. Consider the following functor from commutative rings to groups over X^S:

$$G_S(R) := \{(x, f) | x \in X^S(R), f : \Delta_S(x) \to G\}.$$

Then G_S is represented by the limit of an inverse system of smooth affine group schemes over X^S,

$$G_S = \varprojlim_i G_{S,i},$$

such that each transition morphism is a smooth homomorphism. Here $G_{S,i}$ may be taken to represent the functor

$$G_{S,i}(R) = \{(x, f) | x \in X^S(R), f : \Delta_{S,i}(x) \to G\}.$$

Later, the notation $G_{S,i}$ may represent a piece of some other cofiltered system presenting G_S; we will refer to the specific group above as $\mathrm{Map}(\Delta_{S,i}, G)$. The fact that each transition morphism is smooth is directly verified using the valuative criterion. Indeed, let $\mathrm{Spec}(\tilde{R})$ be a square-zero thickening of $\mathrm{Spec}(R)$. A commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(R) & \to & G_{S,i+1} \\ \downarrow & & \downarrow \\ \mathrm{Spec}(\tilde{R}) & \to & G_{S,i} \end{array}$$

is the same thing as a point $\tilde{x} \in X^S(\tilde{R})$, with residue $x \in X^S(R)$, and a commutative diagram

$$\begin{array}{ccc} \Delta_{S,i}(x) & \to & \Delta_{S,i+1}(x) \\ \downarrow & & \downarrow \\ \Delta_{S,i}(\tilde{x}) & \to & G. \end{array}$$

This determines a morphism $P \to G$ where P is the appropriate pushout in affine schemes. Since $\Delta_{S,i}(x)$ is equal to the intersection of $\Delta_{S,i+1}(x)$ with $\Delta_{S,i}(\tilde{x})$ inside $\Delta_{S,i+1}(\tilde{x})$, and $\Delta_{S,i+1}(\tilde{x})$ is a square-zero thickening of $\Delta_{S,i+1}(x)$, it follows that $\Delta_{S,i+1}(\tilde{x})$ is a square-zero thickening of P. Therefore since G is smooth we can extend $P \to G$ to $\Delta_{S,i+1}(\tilde{x}) \to G$, as required. Note that $G_{S,0} = X^S$ so, in particular, each $G_{S,i}$ is smooth over X^S.

Now fix $x \in X^S(\mathbb{C})$. It partitions S into subsets $S_1, \ldots, S_n$ according to coincidence amongst the coordinates. Write y_m for the coordinate x_s for any $s \in S_m$, and z_m for the $\mathbb{C}$-point of X^{S_m} with coordinates y_m. We have

$$\begin{aligned} \Delta_{S,i}(x) &= \operatorname{Spec}\left(\mathcal{O}_X / \textstyle\prod_{m=1}^n I(y_m)^{i|S_m|}\right) \\ &= \textstyle\coprod_{m=1}^n \operatorname{Spec}\left(\mathcal{O}_X / I(y_m)^{i|S_m|}\right). \end{aligned}$$

Therefore we have

$$G_{S,i} \times_{X^S} \{x\} = \prod_{m=1}^n G_{S_m,i} \times_{X^{S_m}} \{z_m\} = \prod_{m=1}^n G_{\{m\},i|S_m|} \times_{X^{\{m\}}} \{y_m\}.$$

The smooth transition map $G_{\{m\},(i+1)|S_m|} \times_{X^{\{m\}}} \{y_m\} \to G_{\{m\},i|S_m|} \times_{X^{\{m\}}} \{y_m\}$ is surjective for all $i \geq 0$ and has a unipotent kernel for all $i \geq 1$. It follows that $G_{S,i+1} \to G_{S,i}$ has the same property. Thus G_S is a *prosaic* affine group scheme over X^S in the following sense.

Definition 3.5.

(i) A scheme T over B is said to be *pro-smooth* over B if it can be written as the limit of an inverse system of schemes T_i smooth over B and with smooth transition morphisms. If T is pro-smooth then it is formally smooth (in particular, flat) over T.

(ii) An affine groupoid scheme $\mathcal{G}$ over B is *pro-smooth* over B if it can be written as the limit of an inverse system of affine groupoid schemes $\mathcal{G}_i$ over B whose structure maps to B are both smooth, and which has smooth transition homomorphisms.

(iii) The scheme T (respectively, affine groupoid scheme $\mathcal{G}$) over B is said to be a *pro-smooth cover* if it is pro-smooth and the transition (respectively, transition and structure) morphisms of (i) (respectively, (ii)) are coverings (in addition to being smooth).

(iv) Let the affine groupoid scheme $\mathcal{G} = \lim_{i \in \mathbb{Z}_{\geq 0}^{op}} \mathcal{G}_i$ over B be a pro-smooth cover. Then each $\mathcal{G}_i$ is the fpqc quotient over B of $\mathcal{G}$ by some pro-smooth affine subgroup K_i. We say that $\mathcal{G}$ is *prosaic* if the K_i can be chosen to be also pro-unipotent.

(v) Let $\mathcal{G}$ be an affine group scheme over the same base B. Then $\mathcal{G}$ is said to be *pro-smooth*, *a pro-smooth cover*, *prosaic* over B if it is so when regarded as a groupoid.

From now on, 'groupoid' will mean 'affine pro-smooth covering groupoid', unless it is clear from the context that this is not the case. All examples of groupoids will actually be prosaic.

Remark 3.6. Recall the construction of G_S. If the affine algebraic group G is replaced by an arbitrary smooth affine variety T over $\mathbb{C}$, we get a pro-smooth affine variety T_S over X^S in exactly the same way.

3.5 Beilinson–Drinfeld Grassmannians; reasonableness

We also consider the functor

$$G_S^{S'}(R) := \{(x, f) | x \in X^S(R), f : \Delta_S^{S'} \to G\}.$$

Then $G_S^{S'}$ is represented by an ind-affine ind-scheme, formally smooth over X^S. It is a group in ind-schemes over X^S, but not an inductive limit of group schemes. It is a *reasonable* ind-scheme in the following sense (taken from [Dri06]).

Definition 3.7.

(i) An ind-scheme T is *reasonable* if it admits a *reasonable presentation*, that is, an expression

$$T = \varinjlim_{j \in \mathcal{J}} T^j$$

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where $\mathcal{J}$ is some (countable) filtered indexing category, and the transition morphisms in the filtered system of schemes $(T^j)_{j\in\mathcal{J}}$ are all finitely presented (f.p.) closed embeddings (that is, they have finitely generated ideal sheaves). Note that any two reasonable presentations admit a common refinement, so that the category of reasonable presentations of T is filtered.

(ii) A closed subscheme of a reasonable ind-scheme T is *reasonable* if it is a term in some reasonable presentation of T.

(iii) A morphism $U \to T$ of reasonable ind-schemes is *co-reasonable* if for some (equivalently, any) reasonable presentation $T = \varinjlim_{j\in\mathcal{J}} T^j$ of T, the presentation $U = \varinjlim_{j\in\mathcal{J}} U \times_T T^j$ of U as an ind-scheme is reasonable. Warning: this is not a relative version of reasonableness for ind-schemes.

Example 3.8. (i) Let T be a reasonable ind-scheme and let $U \to T$ be either ind-f.p. or an ind-flat cover. Then $U \to T$ is co-reasonable.

(ii) In the case of $G_S^{S'}$, one reasonable presentation is given as follows. Fix a finite set $\{a_1, \dots, a_n\}$ of generators of $\mathcal{O}(G)$. Then set $\mathcal{J} = \mathbb{Z}_{\geq 0}$ and set $G_S^{S',j}$ to be the closed subscheme of $G_S^{S'}$ which on the level of R-points is given by

$$G_S^{S',j}(R) = \{(x,f) | x \in X^S(R), f : \Delta_S^{S'} \to G, a_k \circ f \in H^0 F^j \mathcal{O}(\Delta_S^{S'})\}.$$

Here we have taken $G_S^{S',0} = G_S$. The left- and right-regular actions of the subgroup G_S preserve the inductive structure, meaning that each $G_S^{S',j}$ has a free action on both sides by G_S over X^S, even though it is not itself a group. Moreover, the fpqc quotient $G_S^{S',j}/G_S$ is of finite type over X^S, and flat, although generally quite singular. The result is that the fpqc quotient

$$G_S^{S'}/G_S$$

has the structure of ind-finite-type ind-flat ind-scheme over X^S. In particular, it is reasonable, and $G_S^{S'} \to G_S$ is an ind-flat cover and thus co-reasonable.

On R-points, we may identify

$$G_S^{S'}/G_S(R) = \left\{ (x, \mathcal{E}, f) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \end{array} \right\} /\!\sim.$$

Here the symbol '$/\sim$' means 'taken up to isomorphism', that is, we identify two R-points

$$(x, \mathcal{E}, f) \sim (x', \mathcal{E}', f')$$

if $x = x'$ and there exists an isomorphism of $\mathcal{E}$ with $\mathcal{E}'$ which intertwines f, f'. Such an isomorphism is unique if it exists. The following fact is due to [BD91].

LEMMA 3.9.

(i) *$G_S^{S'}/G_S$ is ind-projective over X^S if and only if G is reductive.*

(ii) *$G_S^{S'}/G_S$ is ind-reduced if and only if G has no non-trivial characters.*

Remark 3.10. Ultimately we are concerned only with the analytifications of these ind-schemes, so point (ii) appears merely for interest's sake. But point (i) is crucial for the definition of convolution in Borel–Moore homology.

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We may reidentify the R-points of $G_S^{S'}$ in a way more compatible with the above identification of $G_S^{S'}/G_S(R)$:

$$G_S^{S'}(R) = \left\{ (x, \mathcal{E}, f, g) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \\ g \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S(x) \end{array} \right\} /\sim.$$

Notice that the inclusion $S' \subset S$ induces a closed embedding $\Delta_{S'}^{S''} \to \Delta_S^{S''}$ for any $S'' \subset S'$. This in turn induces restriction homomorphisms

$$G_S^{S''} \to G_{S'}^{S''}.$$

These maps are co-reasonable. One readily checks by looking at points that the induced maps

$$G_S^{S''}/G_S \to X^S \times_{S'} G_{S'}^{S''}/G_{S'}$$

are isomorphisms. In particular, we have $G_S^{S''}/G_S \xrightarrow{\sim} X^S \times_{S''} G_{S''}^{S''}/G_{S''}$.

Remark 3.11. G_S^S/G_S is known as the Beilinson–Drinfeld Grassmannian Gr_S (on $|S|$ points). In particular, the fibers of $G_S^{S'}/G_S$ over X^S are products of copies of the ordinary affine Grassmannian Gr_G of G.

3.6 Jet bundles; placidity

We will use the following notion, due to Raskin [Ras15].

Definition 3.12.

(i) A scheme T is called *placid* if it admits a *placid presentation*, that is, an expression

$$T = \varprojlim_{i \in \mathcal{I}^{op}} (T_i)$$

for some filtered (countable) indexing category $\mathcal{I}$, such that each T_i is of finite type over $\mathbb{C}$ and each transition morphism $T_i \to T_{i'}$ is a smooth affine covering. We will denote this placid presentation by $T_{\mathcal{I}}$.

(ii) An ind-scheme T is called *placid* if it admits a *placid presentation*, that is, an expression

$$T = \varinjlim_{j \in \mathcal{J}} \varprojlim_{i \in (\mathcal{I}^j)^{op}} (T_i^j).$$

Here $\mathcal{J}, \mathcal{I}^j$ are filtered (countable) indexing categories, $T_{\mathcal{I}^j}^j$ is a placid presentation of its limit scheme $T^j := \varprojlim_{i \in (\mathcal{I}^j)^{op}} (T_i^j)$, and the transition morphisms $T^j \to T^{j'}$ are ind-f.p. closed embeddings.

Remark 3.13. (i) If the placid ind-scheme T maps to some base B of finite type over $\mathbb{C}$, then the placid presentation may be taken over B.

(ii) Let T be a placid ind-scheme and let $T = \varinjlim_{j \in \mathcal{J}} (T_j)$ be a reasonable presentation of T. Then each T_j is a placid scheme, so that this reasonable presentation can be extended to a placid presentation.

(iii) Any two placid presentations of the placid ind-scheme T admit a common refinement (which is again a placid presentation). Thus the collection of placid presentations of T forms a filtered category $\mathcal{P}(T)$.

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(iv) Suppose that T is a placid ind-scheme and U is an ind-scheme with an ind-f.p. map $f : U \to T$. Then U is automatically placid. The short explanation is 'by Noetherian approximation'. We spell it out: given any reasonable presentation

$$T = \underset{j\in\mathcal{J}}{\underrightarrow{\mathrm{colim}}}(T^j)$$

of T, we set $U_j := U \times_T T_j$ and obtain the reasonable presentation

$$U = \underset{j\in\mathcal{J}}{\underrightarrow{\mathrm{colim}}}(U^j)$$

of U. Then, given any placid presentation

$$T^j = \underset{i\in(\mathcal{I}^j)^{op}}{\underleftarrow{\lim}} T_i^j$$

of T^j, there exists some index a of $\mathcal{I}^j$ such that there is a T_a^j-scheme U_a^j fitting into a Cartesian diagram

$$\begin{array}{ccc} U^j & \to & T^j \\ \downarrow & & \downarrow \\ U_a^j & \to & T_a^j. \end{array}$$

Moreover, since $T^j \to T_a^j$ is a covering, the choice of T_a^j-scheme U_a^j is unique. Thus if we replace $\mathcal{I}^j$ by its final subcategory based at a, we can present $U^j \to T^j$ as the limit of a cofiltered system of f.p. morphisms

$$(U_i^j \to T_i^j)_{i\in(\mathcal{I}^j)^{op}}$$

such that for each $i \to i'$ in $\mathcal{I}$, the square

$$\begin{array}{ccc} U_{i'}^j & \to & T_{i'}^j \\ \downarrow & & \downarrow \\ U_i^j & \to & T_i^j \end{array}$$

is Cartesian. Placid presentations of this form will be called *Cartesian*.

(v)The product (over B) of placid ind-schemes is placid.

(vi) Consider placid presentations

$$T = \underset{j\in\mathbb{Z}_{\geq 0}}{\underrightarrow{\mathrm{colim}}} \underset{i\in(\mathbb{Z}_{\geq j})^{op}}{\underleftarrow{\lim}} (T_i^j)$$

of T with the property that T_i^j is formed out of $T_i^{j'}$ as in point (iv) whenever $i \geq j' \geq j$. We call such placid presentations *neat*. This is possibly a technically useless notion. But every placid presentation admits a refinement which is neat up to replacing the indexing categories $\mathcal{J}, \mathcal{I}^j$ by final subcategories. Thus many constructions on placid ind-schemes can be phrased in terms of neat presentations.

Note that for any morphism $f : U \to T$ of placid schemes and any placid presentations $U = \underleftarrow{\lim}_{i\mathcal{I}_U^{op}} U_i$, $T = \underleftarrow{\lim}_{i\mathcal{I}_T^{op}} T_i$, for any $i \in \mathcal{I}_T$ there exist $i' \in \mathcal{I}_U$ and a unique map $U_{i'} \to T_i$ making the square

$$\begin{array}{ccc} U & \to & T \\ \downarrow & & \downarrow \\ U_{i'} & \to & T_i \end{array}$$

commutative. Thus, by changing the indexing sets appropriately we can choose placid presentations of U, T with a common indexing set $\mathcal{I}$ and write $f = \underleftarrow{\lim}_{i\in\mathcal{I}}(U_i \to T_i)$. Such a presentation

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will be called *compatible*. This notion extends immediately to morphisms of placid ind-schemes. A Cartesian presentation is a compatible presentation in which all appropriate squares are Cartesian. If $\mathcal{G}$ is an affine groupoid scheme over some base B of finite type over $\mathbb{C}$ and f is $\mathcal{G}$-equivariant (over B), then we can find a $\mathcal{G}$-equivariant compatible presentation. If in addition f is f.p. so that it admits a Cartesian presentation, then this can also be chosen to be $\mathcal{G}$-equivariant.

Definition 3.14.

(i) A morphism $f : U \to T$ between placid schemes is called *co-placid* if for some (equivalently, every) pair of placid presentations $U_{\mathcal{I}_U}$, $T_{\mathcal{I}_T}$ of U, T and for every index $i \in \mathcal{I}_T$, then for some (equivalently, every) index $i' \in \mathcal{I}_U$ such that we have a commutative square

$$\begin{array}{ccc} U & \xrightarrow{f} & T \\ \downarrow & & \downarrow \\ U_{i'} & \xrightarrow{f'} & T_i, \end{array}$$

the morphism f' is a smooth covering.

(ii) A morphism $f : U \to T$ between placid ind-schemes is called *co-placid* if it is co-reasonable and for some (equivalently, every) reasonable presentation $T = \varinjlim_{j\in\mathcal{J}} T^j$ of T, the map of placid schemes $U \times_T T^j \to T^j$ is co-placid.

The notion of a co-placid morphism has been defined previously in the literature: see [Ras15], where they are called simply 'placid'. As noted in [Ras15], it is *not* a relative version of placidity for ind-schemes. This is the reason for the present renaming.

Example 3.15. (i) Let T be a placid ind-scheme. Let $U \to T$ be either ind-smooth or an ind-pro-smooth cover. Then it is co-placid.

(ii) $G_S^{S'}$ is a placid ind-scheme, and the fpqc quotient map

$$G_S^{S'} \to G_S^{S'}/G_S$$

is an ind-pro-smooth ind-affine affine cover of a placid (indeed, ind-finite type) ind-scheme, so is co-placid.

(iii) Given $S'' \subset S' \subset S$, the morphism

$$f : G_S^{S''} \to X^S \times_{X^{S'}} G_{S'}^{S''}$$

is co-placid. Indeed, consider $G_{S,i} := \mathrm{Map}(\Delta_{S,i}, G)$. We have $G_S = \varprojlim_{i\in\mathbb{Z}_{\geq 0}} G_{S,i}$ and $X^S \times_{X^{S'}} G_{S'} = \varprojlim_{i\in\mathbb{Z}_{\geq 0}} X^S \times_{X^{S'}} G_{S,i}$ and the morphism

$$G_{S,i} \to X^S \times_{X^{S'}} G_{S',i}$$

induced by the closed embeddings $\Delta_{S',i}(x) \subset \Delta_{S,i}(x)$ for any $x \in X^S(R)$. This is a smooth covering, by the same argument of § 3.4 for the prosaicness of G_S. This shows that

$$g : G_S \to X^S \times_{X^{S'}} G_{S'}$$

is co-placid. To conclude, note that the morphism f is an ind-locally trivial g-bundle over the ind-finite type ind-scheme $G_S^{S''}/G_S = X^S \times_{X^{S'}} G_{S'}^{S''}/G_{S'}$.

Warning 3.16. Morphism g is *not* a pro-smooth cover, and f is *not* an ind-pro-smooth cover. It is tempting to imagine that it is the quotient map by some group scheme $\ker(G_S \to X^S \times_{X^{S'}}$

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$G_{S'}$), but there is no such group scheme. To see this, fix some section $S \to S'$ and consider the corresponding 'multi-diagonal' embedding $X^{S'} \to X^S$. Then g is an isomorphism over $X^{S'}$. However, over a generic point of X^S, g is a non-trivial projection from $G(\mathcal{O})^S \to G(\mathcal{O})^{S'}$; see § 3.9 for the notation.

(iv) Fix a representation N of G of dimension d. Let $\mathbf{N} := \mathrm{Spec}(\mathrm{Sym}(N^*))$ be the corresponding G-module, that is, vector space in the category of schemes over $\mathbb{C}$ with G-action. Then $\mathbf{N}_S$ is a G_S-module; indeed, each $\mathrm{Map}(\Delta_{S,i}, \mathbf{N})$ is a $\mathrm{Map}(\Delta_{S,i}, G)$-module and the transition maps are G_S-equivariant. We have the placid ind-scheme

$$\widetilde{\mathcal{T}}_S^{S'} := G_S^{S'} \times_{X^S} \mathbf{N}_S.$$

This is an ind-pro-smooth covering of its fpqc quotient (relative to X^S) ind-scheme

$$\mathcal{T}_S^{S'} := G_S^{S'} \frac{\times_{X^S}}{G_S} \mathbf{N}_S.$$

This is an infinite-dimensional vector bundle over $G_S^{S'}/G_S$. On the level of R-points we identify

$$\mathcal{T}_S^{S'}(R) = \left\{ (x, \mathcal{E}, f, \tilde{v}) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \\ \tilde{v} \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \end{array} \right\} /\sim.$$

Here by 'an $\mathbf{N}$-section of $\mathcal{E}$' we mean a section of the associated $\mathbf{N}$-bundle. Of course $\mathcal{T}_S^{S'}$ is the inverse limit of vector bundles over $G_S^{S'}/G_S$,

$$\mathcal{T}_S^{S'} = \varprojlim_i \mathcal{T}_{S,i}^{S'},$$

where $\mathcal{T}_{S,i}^{S'}$ is the associated bundle of $\mathbf{N}_{S,i}$, a vector bundle of rank $di|S|$. In particular, $\mathcal{T}_S^{S'}$ is a placid ind-scheme, with $\mathcal{T}_S^{S',j}$ being the infinite-dimensional vector bundle $\mathcal{T}_S^{S'}|_{G_S^{S',j}/G_S}$ over $G_S^{S',j}/G_S$. We identify the R-points of $\widetilde{\mathcal{T}}_S^{S'}$ compatibly as follows:

$$\widetilde{\mathcal{T}}_S^{S'}(R) = \left\{ (x, \mathcal{E}, f, g, \tilde{v}) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \\ g \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S(x) \\ \tilde{v} \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \end{array} \right\} /\sim.$$

(v) $\mathbf{N}_S^{S'}$ is a $G_S^{S'}$-module (in ind-schemes). Therefore multiplication gives a map between the placid ind-schemes

$$\mathcal{T}_S^{S'} \to \mathbf{N}_S^{S'}.$$

We define $\mathcal{R}_S^{S'}$ to be the fiber product

$$\mathcal{R}_S^{S'} := \mathcal{T}_S^{S'} \times_{\mathbf{N}_S^{S'}} \mathbf{N}_S.$$

This is an ind-scheme over $G_S^{S'}/G_S$, with $\mathcal{R}_S^{S',j} := \mathcal{T}_S^{S',j} \times_{\mathbf{N}_S^{S'}} \mathbf{N}_S = \mathcal{R}_S^{S'}|_{G_S^{S',j}/G_S}$. Moreover, $\mathcal{R}_S^{S'}$ is a *vector space* over $G_S^{S'}/G_S$, but unlike $\mathcal{T}_S^{S'}$ it is not a vector bundle because the fibers jump. Furthermore, $\mathcal{R}_S^{S',j}$ contains $\ker(\mathcal{T}_S^{S',j} \to \mathcal{T}_{S,i}^{S',j})$ for i large enough (depending on j), that is, we have a diagram

$$\ker(\mathcal{T}_S^{S',j} \to \mathcal{T}_{S,i}^{S',j}) \subset \mathcal{R}_S^{S',j} \subset \mathcal{T}_S^{S',j}$$

of vector spaces over $G_S^{S',j}/G_S$. Therefore $\mathcal{R}_S^{S'}$ is placid: we may take $\mathcal{R}_{S,i}^{S',j}$ to be the image in $\mathcal{T}_{S,i}^{S',j}$ of $\mathcal{R}_S^{S',j}$, for i large enough. Also, $\mathcal{R}_S^{S'}$ is of ind-finite codimension in $\mathcal{T}_S^{S'}$, that is, it is an ind-f.p. closed sub-ind-scheme. On the level of R-points, we have

$$\mathcal{R}_S^{S'}(R) = \left\{ (x, \mathcal{E}, f, v) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \\ v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends to } \Delta_S(x) \end{array} \right\} /\sim.$$

Here $f(v)$ is a section of the trivial $\mathbf{N}$-bundle on $\Delta_S^{S'}(x)$, and we require that it extends to a section of the trivial $\mathbf{N}$-bundle over $\Delta_S(x)$. Such an extension is unique if it exists. We denote the preimage of $\mathcal{R}_S^{S'}$ in $\widetilde{\mathcal{T}}_S^{S'}$ as $\widetilde{\mathcal{R}}_S^{S'}$. It is an ind-f.p. closed sub-ind-scheme of $\widetilde{\mathcal{T}}_S^{S'}$, an ind-pro-smooth cover of $\mathcal{R}_S^{S'}$, its fpqc quotient (locally on X^S) by G_S, and on R-points we identify

$$\widetilde{\mathcal{R}}_S^{S'}(R) = \left\{ (x, \mathcal{E}, f, g, v) \;\middle|\; \begin{array}{l} x \in X^S(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \Delta_S(x) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S^{S'}(x) \\ g \text{ a trivialization of } \mathcal{E} \text{ over } \Delta_S(x) \\ v \text{ a section of the trivial } \mathbf{N}\text{-bundle on } \Delta_S(x) \\ \quad \text{such that } fg^{-1}(v) \text{ extends to } \Delta_S(x) \end{array} \right\} /\sim.$$

(vi) The product (over B) of co-placid maps is co-placid.

Remark 3.17. (i) Suppose $X = \mathbb{G}_a$ with parameter t. Then $I(x_s)$ is trivialized by $t - t_{x_s}$. There is an isomorphism from $\mathcal{T}_S^{S'}$ to the kernel of the covering $\mathcal{T}_S^{S'} \to \mathcal{T}_{S,i}^{S'}$, given on R-points by

$$(x, f, \mathcal{E}, \tilde{v}) \mapsto \left(x, f, \mathcal{E}, \prod_{s \in S} (t - t_{x_s}) \tilde{v} \right).$$

We call this isomorphism the *fiberwise shift map* of $\mathcal{T}$.

(ii) The placid ind-scheme $T = G_S^{S'}/G_S, \mathcal{T}_S^{S'}, \mathcal{R}_S^{S'}$ is *special* in that one may take the smooth affine covering maps $T_i^j \to T_{i'}^j$ to be vector bundles. But placidity seems to be the more flexible definition; for instance, I do not know if point (iii) of Remark 3.13 holds if we replace 'placid' by 'special'.

3.7 Equivariance

(i) Note that $G_S^{S'}/G_S, \mathcal{T}_S^{S'}, \mathcal{R}_S^{S'}$ are all acted on by G_S, and the various maps between them are G_S-equivariant. In fact, each 'approximation' $G_S^{S',j}/G_S$, $\mathcal{T}_i^j$, $\mathcal{R}_i^j$ is acted on by some quotient $G_{S,i'}$ of G_S, and the transition morphisms are all G_S-equivariant.

(ii) Suppose that $X = \mathbb{G}_a$. Then we have the action of $\mathbb{G}_m$ on X by multiplication. It also acts diagonally on X^S. Therefore we may consider $\mathbb{C}^* \times X^S$ as a smooth groupoid over X^S. The group $G_S^{S'}$ over X^S is $\mathbb{C}^* \times X^S$-equivariant, so we may form the semidirect product

$$G_S^{S'} \rtimes \mathbb{C}^*,$$

a placid affine groupoid ind-scheme over X^S. The special case, $G_S \rtimes \mathbb{C}^*$, is a prosaic affine groupoid scheme over X^S. Then the G_S-equivariant structures of $G_S^{S'}/G_S, \mathcal{T}_S^{S'}, \mathcal{R}_S^{S'}$ and their above approximations upgrade to $G_S \rtimes \mathbb{C}^*$-equivariant structures (over X^S). Again all morphisms, transition or otherwise, of the previous subsection are $G_S \rtimes \mathbb{C}^*$-equivariant.

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Remark 3.18. (i) Let T be a placid ind-scheme over some base B and $\mathcal{G}$ be a groupoid scheme over B which acts on T. Then we can always choose a $\mathcal{G}$-equivariant placid presentation of T, simply by 'smoothing out' any placid presentation by the action of $\mathcal{G}$. The category $\mathcal{P}_{\mathcal{G}}(T)$ of $\mathcal{G}$-equivariant placid presentations of T is filtered.

(ii) Now suppose that T is special. I do not know whether we can choose the special presentation of T to be $\mathcal{G}$-equivariant. However, if T is special by virtue of being a placid vector space over some intermediate $\mathcal{G}$-equivariant ind-scheme U of ind-finite type, $T \to U \to B$, and $\mathcal{G}$ acts linearly, then we can do it. This is what happens for $G_S^{S'}, \mathcal{T}_S^{S'}, \mathcal{R}_S^{S'}$. In the latter two cases, we can take $U = G_S^{S'}/G_S$. In the former case, we can take U to be the quotient of $G_S^{S'}$ by the kernel of the surjection $G_S \to \mathrm{Map}(\Delta_{S,1}, G)$.

3.8 Dimension theories

The following definitions can be found in [Ras15], and essentially in the earlier work [Dri06], and probably in many other texts.

Definition 3.19. (i) Let T be a placid scheme, with a placid presentation $T = \varprojlim_{i\in(\mathcal{I})^{op}}(T_i)$. A *dimension theory* on $T_{\mathcal{I}}$ is a function $d : \mathcal{I} \to \mathbb{Z}$, whose value on $i \in \mathcal{I}$ will be written $d(T_i)$, satisfying the condition

$$d(T_{i'}) - d(T_i) = \dim(T_{i'}) - \dim(T_i)$$

whenever $i \to i'$ in $\mathcal{I}$. The set of dimension theories on $T_{\mathcal{I}}$ is denoted

$$\mathrm{dimth}(T_{\mathcal{I}}).$$

(ii) Take a placid presentation $T = \varprojlim_{i\in(\mathcal{I})^{op}}(T_i)$ of the scheme T and a finer placid presentation $T = \varprojlim_{i_1\in(\mathcal{I}_1)^{op}}(T_{i_1})$, $\mathcal{I} \subset \mathcal{I}_1$. We may extend a dimension theory d on $T_{\mathcal{I}}$ to a unique dimension theory, denoted d, on $T_{\mathcal{I}_1}$ by setting

$$d(T_{i_1}) := d(T_{i'}) - \dim(T_{i'}) + \dim(T_i)$$

for any $i' \in \mathcal{I}$ such that $i_1 \to i'$ in $\mathcal{I}_1$. We have thus constructed a filtered system

$$(\{\text{dimension theories on } T_{\mathcal{I}}\})_{T_{\mathcal{I}}\in\mathcal{P}}$$

indexed by the filtered category $\mathcal{P}$ of placid presentations of T.

(iii) A *dimension theory* on T is an element of the colimit of the above filtered system. The set of dimension theories on $T_{\mathcal{I}}$ is denoted

$$\mathrm{dimth}(T).$$

Now let $f : U \to T$ be an f.p. map of placid schemes and choose Cartesian placid presentations indexed by $\mathcal{I}$ as in Remark 3.13. In this presentation, a dimension theory d on $T_{\mathcal{I}}$ defines one on $U_{\mathcal{I}}$, denoted f^*d and given by the formula

$$f^*d(U_i) := d(T_i).$$

That is, we have a map $\mathrm{dimth}(T_{\mathcal{I}}) \to \mathrm{dimth}(U_{\mathcal{I}})$. The composition of this with the map $\mathrm{dimth}(U_{\mathcal{I}}) \to \mathrm{dimth}(U)$ factors through the map $\mathrm{dimth}(T_{\mathcal{I}}) \to \mathrm{dimth}(T)$, yielding a map

$$f^* : \mathrm{dimth}(T) \to \mathrm{dimth}(U)$$

independent of any choices of presentation. If context prevents confusion, we may denote f^*d by d.

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Similarly, suppose $f : U \to T$ is a co-placid map of placid schemes and fix a compatible presentation $f = \varprojlim_{i\in\mathcal{I}^{op}}(U_i \to T_i)$. The dimension theory d on $T_{\mathcal{I}}$ defines one on $U_{\mathcal{I}}$, denoted $f^!d$ and given by the formula

$$f^!(d)(U_i) := d(T_i) + \dim(U_i) - \dim(T_i).$$

This procedure again determines a map

$$f^! : \mathrm{dimth}(T) \to \mathrm{dimth}(U)$$

independent of any choices of presentation.

Definition 3.20. (i) Let T be a placid ind-scheme. Fix a reasonable presentation

$$T = \varinjlim_{j\in\mathcal{J}} T^j.$$

We write this as $T^{\mathcal{J}}$. A *dimension theory* on T is an element of the limit of the cofiltered system

$$(\mathrm{dimth}(T^j))_{j\in\mathcal{J}^{op}}$$

with transition morphisms given by the $*$-pullback. For different reasonable presentations these limits are canonically isomorphic.

(ii) Given an ind-f.p. map $f : U \to T$ of placid ind-schemes, we get a map

$$f^* : \mathrm{dimth}(T) \to \mathrm{dimth}(U)$$

determined by the condition that the square

$$\begin{array}{ccc} \mathrm{dimth}(T) & \xrightarrow{f^*} & \mathrm{dimth}(U) \\ \downarrow & & \downarrow \\ \mathrm{dimth}(T_j) & \xrightarrow{(f^j)^*} & \mathrm{dimth}(U \times_T T_j) \end{array}$$

commutes for every reasonable closed subscheme T_j of T.

(iii) Given a co-placid map $f : U \to T$ of placid ind-schemes, we get a map

$$f^! : \mathrm{dimth}(T) \to \mathrm{dimth}(U)$$

determined by the condition that the square

$$\begin{array}{ccc} \mathrm{dimth}(T) & \xrightarrow{f^!} & \mathrm{dimth}(U) \\ \downarrow & & \downarrow \\ \mathrm{dimth}(T_j) & \xrightarrow{(f^j)^!} & \mathrm{dimth}(U \times_T T_j) \end{array}$$

commutes for every reasonable closed subscheme T_j of T.

(iv) Given placid ind-schemes U, T over B, we get a map

$$(-) +_B (-) : \mathrm{dimth}(U) \times \mathrm{dimth}(T) \to \mathrm{dimth}(U \times_B T).$$

Indeed, if $U = \varinjlim_{j\in\mathcal{J}_U} \varprojlim_{i\in(\mathcal{I}_U^j)^{op}} U_i^j$, $T = \varinjlim_{j\in\mathcal{J}_T} \varprojlim_{i\in(\mathcal{I}_T^j)^{op}} T_i^j$ are placid presentations over B and d_U, d_T are dimension theories on U, T then

$$U \times_B T = \varinjlim_{(j_U,j_T)\in\mathcal{J}_U\times\mathcal{J}_T} \varprojlim_{(i_U,i_T)\in(\mathcal{I}_U^{j_U})^{op}\times(\mathcal{I}_T^{j_T})^{op}} U_{i_U}^{j_U} \times_B T_{i_T}^{j_T}$$

is a placid presentation and we define

$$(d_U + d_T)(U_{i_U}^{j_U} \times_B T_{i_T}^{j_T}) := d_U(U_{i_U}^{j_U}) + d_T(T_{i_T}^{j_T}).$$

If $f : U \times_B T \to U \times B$ is the ind-f.p. closed embedding, then we have $(-) +_B (-) = f^*((-) +_{\operatorname{Spec} C} (-))$. We will usually write $(-) +_B (-)$ simply as $(-) + (-)$.

Remark 3.21. (i) The set of dimension theories on a connected placid ind-scheme T is a $\mathbb{Z}$-torsor. Thus the set of dimension theories on a general placid ind-scheme T is a $\mathbb{Z}^{\pi_0(T)}$-torsor. For an ind-f.p. (respectively, co-placid) map $f : U \to T$ of placid ind-schemes, f^* (respectively, $f^!$) is a map of $\mathbb{Z}^{\pi_0(T)}$-sets.

(ii) In fact, there is a sheaf (in an appropriate sense) of $\mathbb{Z}$-torsors on any placid ind-scheme T whose set of global sections equals the set of dimension theories on T. We do not need to consider it since in every example of this paper, this sheaf is trivial.

Example 3.22. (i) Recall that $\mathcal{T}_S^{S'}$ is an infinite-dimensional vector bundle over $G_S^{S'}/G_S$, with 'approximations' $\mathcal{T}_{S,i}^{S',j}$ for $j \in \mathbb{Z}_{\geq 0}$ and $i \in \mathbb{Z}_{\geq ?}$ for some positive integer ? depending on j (see Example 3.15). The approximation $\mathcal{T}_{S,i}^{S',j}$ is a vector bundle over $G_S^{S',j}/G_S$ of rank $di|S|$. Thus $f : \mathcal{T}_S^{S'} \to G_S^{S'}/G_S$ is co-placid, and since $G_S^{S'}/G_S$ is ind-finite type it has a dimension theory d_0 with constant value 0. We will denote the dimension theory $f^! d_0$ on $\mathcal{T}_S^{S'}$ by $\operatorname{rank}(\mathcal{T}_S^{S'})$. We have

$$\operatorname{rank}(\mathcal{T}_S^{S'})(\mathcal{T}_{S,i}^{S',j}) := di|S|.$$

It would perhaps be safer to call this $\operatorname{rank}_{G_S^{S'}/G_S}(\mathcal{T}_S^{S'})$, but the notation becomes too unwieldy. The reader should bear this in mind.

(ii) Let $f : \mathcal{R}_S^{S'} \to \mathcal{T}_S^{S'}$ be the defining ind-f.p. closed embedding. Then we have the dimension theory $f^* \operatorname{rank}(\mathcal{T}_S^{S'})$ on $\mathcal{R}_S^{S'}$. We will call this simply $\operatorname{rank}(\mathcal{T}_S^{S'})$.

(iii) We also have the dimension theory on $\widetilde{\mathcal{T}}_S^{S'}$ obtained as the !-pullback of $\operatorname{rank}(\mathcal{T}_S^{S'})$ (or of d_0 directly). Its $*$-pullback to $\tilde{R}_S^{S'}$ coincides with the !-pullback of the dimension theory $\operatorname{rank}(\mathcal{T}_S^{S'})$ on $\mathcal{R}_S^{S'}$. These dimension theories on $\widetilde{\mathcal{T}}_S^{S'}$, $\widetilde{\mathcal{R}}_S^{S'}$ will both be denoted $\operatorname{rank}(\widetilde{\mathcal{T}}_S^{S'})$.

(iv) We will denote the !-pullback to $G_S^{S'}$ of the constantly 0 dimension theory on $G_S^{S'}/G_S$ by $\operatorname{rank}(G_S^{S'})$. It satisfies

$$\operatorname{rank}(G_S^{S'})(G_{S,i}^{S',j}) := \dim_{X^S}(G_{S,i}) = \dim(G_{S,i}) - |S|.$$

3.9 Notational remark

We will use the same notational simplification for $G_S^{S'}$, $\mathcal{T}_S^{S'}$, $\mathcal{R}_S^{S'}$, $\widetilde{\mathcal{T}}_S^{S'}$, $\widetilde{\mathcal{R}}_S^{S'}$ as for formal neighborhoods: for instance,

$$\mathcal{R}_{\{1,2\}}^{\{1\}}$$

may be written as

$$\mathcal{R}_{12}^{1}.$$

Fix a $\mathbb{C}$-point $x \in X$ and a local parameter t at x. This determines isomorphisms $\Delta_1(x) = \operatorname{Spec}(\mathcal{O})$, $\Delta_1^1(x) = \operatorname{Spec}(\mathcal{K})$ where $\mathcal{O} = \mathbb{C}[[t]]$, $\mathcal{K} = \mathbb{C}((t))$. The groups of $\mathbb{C}$-points of $(G_1)_x$, $(G_1^1)_x$ are put in isomorphism with $G(\mathcal{O})$, $G(\mathcal{K})$. Note that since G_1 is pro-smooth over X, we have

$$(G_1^1/G_1)_x = (G_1^1)_x/(G_1)_x.$$

We will write informally

$$(G_1)_x = G(\mathcal{O}),$$

$$(G_1^1)_x = G(\mathcal{K}),$$

$$(G_1^1/G_1)_x = Gr_G.$$

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We will write $(\mathcal{T}_1^1)_x$, $(\mathcal{R}_1^1)_x$ as $\mathcal{T},\mathcal{R}$. Since these are ind-f.p. sub-ind-schemes of $\mathcal{T}_1^1$, they have dimension theories given by pulling back $\operatorname{rank}(\mathcal{T}_1^1)$. We will write the resulting dimension theories both as $\operatorname{rank}(\mathcal{T})$. These dimension theories correspond to the $\dim \mathbf{N}(\mathcal{O})$ of [BFN16] (although dimension theories are not explicitly used in [BFN16]).

3.10 Borel–Moore homology

Rather than recall the general formalism of equivariant constructible derived categories on placid ind-schemes (see [Ras15]), we content ourselves with the following definition.

Definition 3.23. (i) Let R be a commutative ring. Let T be a scheme of finite type over the base B of finite type over $\mathbb{C}$ with an action of the affine algebraic groupoid $\mathcal{G}$ over B. Then T^{an} has a $\mathcal{G}^{an}$-equivariant constant sheaf R and dualizing complex ω with coefficients in R, and we will write

$$H^n_{\mathcal{G}}(T,R) := H^n_{\mathcal{G}^{an}}(T^{an},R) = \operatorname{Hom}_{D^b_{\mathcal{G}^{an}}(T^{an})}(R,\Sigma^n R),$$
$$H^{\mathrm{BM},\mathcal{G}}_n(T,R) := H^n_{\mathcal{G}^{an}}(T^{an},\omega) = \operatorname{Hom}_{D^b_{\mathcal{G}^{an}}(T^{an})}(R,\Sigma^n \omega).$$

(ii) Now suppose that $\mathcal{G}$ is an affine groupoid scheme over B. We may write $\mathcal{G}$ as the limit of its fpqc quotient affine algebraic groupoids $(\mathcal{G}_i)_{i\in\mathcal{I}^{op}}$. We may assume that action of $\mathcal{G}$ on T factors through each $\mathcal{G}_i$. We set

$$H^*_{\mathcal{G}}(T,R) := \varinjlim_{i\in\mathcal{I}} H^*_{\mathcal{G}_i}(T,R)$$

and

$$H^{\mathrm{BM},\mathcal{G}}_*(T,R) := \varinjlim_{i\in\mathcal{I}} H^{\mathrm{BM},\mathcal{G}_i}_*(T,R).$$

Here we have used the fact that for any $i\to i'$ in $\mathcal{I}$, the $\mathcal{G}_{i'}$-equivariant complexes obtained from the $\mathcal{G}_i$-equivariant constant sheaf (respectively, dualizing complex) are canonically isomorphic to their $\mathcal{G}_{i'}$-equivariant counterparts. Thus these restriction functors determine the maps of equivariant cohomology (respectively, Borel–Moore homology) which we take colimits over.

(iii) Let T be a placid scheme over some base B of finite type over $\mathbb{C}$ and let $\mathcal{G}$ be an affine algebraic groupoid over B which acts on T. Let d be a dimension theory on T. Then the *$2d$-shifted $\mathcal{G}$-equivariant Borel–Moore homology of* T, $H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R)$, is defined as follows. Let $T = \varprojlim_{i\in(\mathcal{I})^{op}}(T_i)$ be a $\mathcal{G}$-equivariant placid presentation of T. Observe that pullback defines a graded map of R-modules

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d(T_i)}(T_i,R)\to H^{\mathrm{BM},\mathcal{G}}_{*-2d(T_{i'})}(T_{i'},R)$$

whenever $i\to i'$ in $\mathcal{I}$, since $T_{i'}\to T_i$ is a $(d(T_{i'})-d(T_i))$-dimensional smooth covering. We then set

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R) := \varinjlim_{i\in\mathcal{I}} H^{\mathrm{BM},\mathcal{G}}_{*-2d(T_i)}(T_i,R).$$

For different choices of placid presentation, we get canonically isomorphic answers, which justifies the definition.

(iv) Let $f: T^0\to T^1$ be a $\mathcal{G}$-equivariant ind-f.p. embedding of placid ind-schemes over B. Let d be a dimension theory on T^1. Then there is a pushforward map

$$f_*: H^{\mathrm{BM},\mathcal{G}}_{*-2f^*d}(T^0,R)\to H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T^1,R)$$

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defined by choosing Cartesian placid presentations of T^0, T^1 indexed by $\mathcal{I}$ and observing that for each $i \in \mathcal{I}$ the diagram

$$\begin{array}{ccc} H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^0_{i'})}(T^0_{i'}, R) & \xrightarrow{f_*} & H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^1_{i'})}(T^1_{i'}, R) \\ \uparrow & & \uparrow \\ H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^0_{i})}(T^0_{i}, R) & \xrightarrow{f_*} & H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^0_{i})}(T^1_{i}, R) \end{array}$$

commutes. Here the horizontal maps are pushforwards maps along the closed embeddings $T^0_i \to T^1_i$, while the vertical maps are the pullback maps of point (1). The resulting map is independent of any choices we have made.

(v) Now let T be a placid ind-scheme over some base B of finite type over $\mathbb{C}$ and let $\mathcal{G}$ be an affine algebraic groupoid over B which acts on T. Let d be a dimension theory on T. Then we set

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) := \varinjlim_{j \in \mathcal{J}} H^{\mathrm{BM},\mathcal{G}}_{*-2f^*d}(T^j, R)$$

using the pushforward maps of point (ii) of this definition, for any choice $T = \varinjlim_{j\in\mathcal{J}}(T^j)$ of $\mathcal{G}$-equivariant reasonable presentation of T. For different presentations we get canonically isomorphic colimits. Here we have written d for the unique dimension theory on T^j compatible with the dimension d on T.

Remark 3.24. (i) Since $\mathcal{G}$ is a pro-smooth covering groupoid, its action on any finite-type approximation T^j_i to the placid ind-scheme factors through the quotient $\mathcal{H}$ by some (pro-smooth covering) subgroup. We then have

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^j_i)}(T^j_i, R) = H^*_{\mathcal{G}}(B, R) \otimes_{H^*_{\mathcal{H}}(B,R)} H^{\mathrm{BM},\mathcal{H}}_{*-2d(T^j_i)}(T^j_i, R).$$

(ii) If, moreover, $\mathcal{G}$ is prosaic, then we can choose the subgroup in question to be also pro-unipotent, in which case we have $H^*_{\mathcal{G}}(B, R) \cong H^*_{\mathcal{H}}(B, R)$ so that

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d(T^j_i)}(T^j_i, R) = H^{\mathrm{BM},\mathcal{H}}_{*-2d(T^j_i)}(T^j_i, R).$$

(iii) It may even happen that we can choose a section $\mathcal{H} \to \mathcal{G}$. Take, for example, $X = \mathbb{G}_a$, $\mathcal{G} = G_1 \rtimes \mathbb{C}^*$, $\mathcal{H} = G \times \mathbb{C}^* \times X$ where G embeds in G_1 as the subgroup of constant functions. In this case, $\mathcal{H}$ acts on all of T, and we have

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) = H^{\mathrm{BM},\mathcal{H}}_{*-2d}(T, R).$$

Though it may give a psychological advantage since $\mathcal{H}$ is an actual smooth algebraic groupoid, this reduction is usually technically unhelpful.

The following procedures in ordinary equivariant Borel–Moore homology are also defined in the world of placid ind-schemes. Fix a $\mathcal{G}$-equivariant placid ind-scheme T over B (of finite type over $\mathbb{C}$) and a dimension theory d on T.

(i) *Change of groupoid base.* Suppose that we have some finite-type $\mathcal{G}$-space A over B. Then there exists a semi-direct groupoid scheme $\mathcal{G} \ltimes_B A$, affine pro-smooth over A. Suppose that $T \to B$ factors through A. Then T is also $\mathcal{G} \ltimes_B A$-equivariant (over A) and we have a canonical isomorphism

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) \xrightarrow{\sim} H^{\mathrm{BM},\mathcal{G}\ltimes_B A}_{*-2d}(T, R).$$

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(ii) *Open restriction.* Let $A \subset B$ be a $\mathcal{G}$-equivariant open subscheme and let $j : T|_A \to T$ be the $\mathcal{G}$-equivariant ind-f.p. ind-open embedding of placid ind-schemes over A. The restriction to A of a $\mathcal{G}$-equivariant placid presentation of T is a $\mathcal{G}|_A$-equivariant placid presentation of $T|_A$, and, applying the ordinary open restriction in Borel–Moore homology, one obtains a map

$$j^! : H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2j^*d}(T|_A, R).$$

One usually goes on to compose this map with the isomorphism

$$H^{\mathrm{BM},\mathcal{G}}_{*-2j^*d}(T|_A, R) \cong H^{\mathrm{BM},\mathcal{G} \ltimes_B A}_{*-2j^*d}(T|_A, R).$$

(iii) *Proper pushforward.* Let $f : U \to T$ be a $\mathcal{G}$-equivariant ind-proper (in particular, ind-f.p.) map of placid ind-schemes over B. By choosing a $\mathcal{G}$-equivariant Cartesian placid presentation and applying the ordinary proper pushforward in Borel–Moore homology, one obtains a map

$$f_* : H^{\mathrm{BM},\mathcal{G}}_{*-2f^*d}(U, R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R).$$

(iv) *Restriction of equivariance.* Let $\mathcal{H} \to \mathcal{G}$ be a morphism of groupoid schemes over B. Then we have 'restriction of equivariance' maps

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) \to H^{\mathrm{BM},\mathcal{H}}_{*-2d}(T, R).$$

(v) *Co-placid restriction.* Let $f : U \to T$ be a $\mathcal{G}$-equivariant co-placid map of placid ind-schemes over B. By choosing a $\mathcal{G}$-equivariant compatible presentation of f and applying the ordinary smooth pullback in Borel–Moore homology, one obtains a map

$$f^! : H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2f^!d}(U, R).$$

(vi) *Restriction with supports.* Let $p : T' \to U'$ be a $\mathcal{G}$-equivariant co-placid map of placid ind-schemes over B, and let $f' : U' \to T'$ be a $\mathcal{G}$-equivariant section of p. Let $g : T \to T'$ be ind-f.p. and let $U = T \times_{T'} U'$, so that we have a Cartesian square

$$\begin{array}{ccc} U & \xrightarrow{f} & T \\ \downarrow g' & & \downarrow g \\ U' & \xrightarrow{f'} & T' \end{array}$$

Suppose we have a dimension theory d' on U' such that $g^* p^! d' = d$, and set $d^g := (g')^* d'$. We can choose a $\mathcal{G}$-equivariant compatible presentation of this Cartesian square, that is, write it as

$$\varinjlim_{j \in \mathcal{J}} \varprojlim_{i \in \mathcal{I}^j} \begin{pmatrix} U_i^j & \xrightarrow{f_i^j} & T_i^j \\ \downarrow & & \downarrow \\ (U')_i^j & \xrightarrow{(f')_i^j} & (T')_i^j \end{pmatrix}$$

such that the induced presentations of the vertical morphisms $U \to U'$, $T \to T'$ are Cartesian, and such that there exists a $\mathcal{G}$-equivariant presentation $\varinjlim_{j \in \mathcal{J}} \varprojlim_{i \in \mathcal{I}^j} ((T')_i^j \xrightarrow{p_i^j} (U')_i^j)$ of p. Then p_i^j is a smooth covering and $(f')_i^j$ is its section, and we have a 'restriction with supports' morphism $(f_i^j)^! : H^{\mathrm{BM},\mathcal{G}}_*(T_i^j, R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2\dim(p_i^j)}(U_i^j, R)$, which assembles to give a morphism

$$f^! : H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T, R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2d^g}(U, R).$$

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(vii) *Averaging.* Suppose that $\mathcal{H} \subset \mathcal{G}$ is a normal subgroup over B of finite index. Then we have the averaging maps

$$H^{\mathrm{BM},\mathcal{H}}_{*-2d}(T,R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R).$$

(viii) *Specialization.* Suppose now that $B = \mathbb{G}_a$ and $\mathcal{G}$ is a group scheme over $B = \mathbb{G}_a$. We have the closed subscheme $i : \{0\} \to \mathbb{G}_a$, and the complementary open subscheme $j : \mathbb{G}_m \to \mathbb{G}_a$. Let us write $T^*, \mathcal{G}^*$ for the restrictions to $\mathbb{G}_m$ and $T|_0, \mathcal{G}|_0$ for the restrictions to $\{0\}$. Choose a $\mathcal{G}$-equivariant placid presentation

$$T = \varinjlim_{j\in\mathcal{J}} \varprojlim_{i\in(\mathcal{I}^j)^{op}} T^j_i$$

of T over $\mathbb{G}_a$. Then by restriction we obtain a $\mathcal{G}^*$-equivariant placid presentation

$$T^* = \varinjlim_{j\in\mathcal{J}} \varprojlim_{i\in(\mathcal{I}^j)^{op}} (T^j_i)^*$$

of T^* over $\mathbb{G}_a$ and a $\mathcal{G}|_0$-equivariant placid presentation

$$T|_0 = \varinjlim_{j\in\mathcal{J}} \varprojlim_{i\in(\mathcal{I}^j)^{op}} (T^j_i)|_0$$

of $T|_0$ over $\{0\}$, both of which are Cartesian with the original placid presentation. Since $\mathcal{G}$ is pro-smooth covering, we have specialization maps

$$s^j_i : H^{\mathrm{BM},\mathcal{G}^*}_*((T^j_i)^*, R) \to H^{\mathrm{BM},\mathcal{G}|_0}_{*+2}((T^j_i)|_0, R)$$

which are compatible, so yield

$$s : H^{\mathrm{BM},\mathcal{G}^*}_{*-2j^*d}(T^*, R) \to H^{\mathrm{BM},\mathcal{G}|_0}_{*+2-2i^*d}(T|_0, R).$$

We will write $d^* := j^*d$, $d|_0 := i^*d$.

(ix) *Steenrod's construction.* Let $\mathcal{G}$ be an affine algebraic group and $B = *$ (we will not need the relative situation). Write pd for the dimension theory $\underbrace{d + \cdots + d}_{p\text{ times}}$ on the $\mathcal{G}^{\mu_p} \rtimes \mu_p$-equivariant placid ind-scheme T^{μ_p}. We have nonlinear maps

$$St^{\mathrm{BM}} : H^{\mathrm{BM},\mathcal{G}}_{n-2d}(T,R) \to H^{\mathrm{BM},\mathcal{G}^{\mu_p}\rtimes\mu_p}_{pn-2pd}(T^{\mu_p}, R)$$

which are monoidal with respect to the map $St^H : H^n_{\mathcal{G}}(*,R) \to H^{pn}_{\mathcal{G}^{\mu_p}\rtimes\mu_p}(*,R)$ and whose discrepancy from additivity is averaged from $H^{\mathrm{BM},\mathcal{G}^{\mu_p}}_{pn-2pd}(T^{\mu_p}, R)$. If R is a perfect field k of characteristic p, we have the nonlinear graded St_{ex}-monoidal maps

$$St^{\mathrm{BM}}_{\mathrm{ex}} : H^{\mathrm{BM},\mathcal{G}}_*(T,k)^{(1)} \to H^{\mathrm{BM},\mathcal{G}^{\mu_p}\rtimes\mu_p}_*(T^{\mu_p}, k).$$

The following facts carry over from the ordinary case.

(i) *Descent.*

(a) Suppose that $\mathcal{H}$ is a normal (pro-smooth covering) subgroup of $\mathcal{G}$ over B, and that the fpqc quotient group $\mathcal{G}/\mathcal{H}$ exists as a pro-smooth covering group over B. The main example here is $\mathcal{G} = \mathcal{H} \times_B \mathcal{L}$ for pro-smooth covering groups $\mathcal{H}$, $\mathcal{L}$ over B. Suppose that $\mathcal{H}$ acts freely on T, so that, in particular, the fpqc quotient (relative to B) $T/\mathcal{H}$ exists as a $\mathcal{G}/\mathcal{H}$-equivariant placid ind-scheme over B. So the quotient map $f : T \to T/\mathcal{H}$ is $\mathcal{G}$-equivariant and co-placid. Suppose that there exists a dimension theory d' on $T/\mathcal{H}$ satisfying $f^!d' = d$.

Then the composition

$$H^{\mathrm{BM},\mathcal{G}/\mathcal{H}}_{*-2d'}(T/\mathcal{H},R) \to H^{\mathrm{BM},\mathcal{G}}_{*-2d'}(T/\mathcal{H},R) \xrightarrow{f^!} H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R)$$

is an isomorphism.

(b) Suppose that Q is an algebraic group acting regularly on B in the sense that the stabilizer groupoid $Q_B := B \times_B (Q \ltimes B)$ is smooth over B, and $\mathcal{L}$ is a Q-equivariant pro-smooth covering group over B. Set $\mathcal{G} = \mathcal{L} \rtimes Q$. Thus the maximal subgroup $\mathcal{H} = \mathcal{L} \rtimes_B Q_B$ is a pro-smooth covering group over B, and the quotient groupoid $P := \mathcal{G}/\mathcal{H} = (Q \ltimes B)/Q_B$ is smooth over B. Let $A = B/P$, $\pi : B \to A$. Suppose that we are given a slice $i : A \subset B$. Consider the placid $\mathcal{G} \times_B \pi^*(\mathcal{H}|_A)$-equivariant ind-scheme

$$\mathcal{G}|_A \times_A T|_A.$$

On the one hand, the normal subgroup $\pi^*(\mathcal{H}|_A)$ acts freely and the quotient is T, giving

$$H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R) \xrightarrow{\sim} H^{\mathrm{BM},\mathcal{G}\times_A \mathcal{H}|_A}_{*-2d'}(\mathcal{G}|_A \times_A T|_A, R)$$

where d' is the !-pullback of d. On the other hand, the normal subgroup $\mathcal{L}$ acts freely, and the quotient is $Q \times T|_A$ with its residual action of $(Q \ltimes B) \times_B \pi^*(\mathcal{H}|_A) = (Q \times \mathcal{H}|_A) \ltimes_A B$, where $\mathcal{H}|_A$ acts trivially on B over A. Thus we have the isomorphisms

$$\begin{array}{ccc} H^{\mathrm{BM},\mathcal{H}|_A}_{*-2d''}(T|_A,R) & \xrightarrow{\sim} & H^{Q\times\mathcal{H}|_A}_{*-2d''-2\dim Q}(Q\times T|_A,R) \\ & & \downarrow \wr \\ H^{\mathrm{BM},\mathcal{G}\times_A\mathcal{H}|_A}_{*-2d'}(\mathcal{G}|_A \times_A T|_A,R) & \xleftarrow{\sim} & H^{(Q\times\mathcal{H}|_A)\ltimes_A B}_{*-2d''-2\dim Q}(Q\times T|_A,R) \end{array}$$

if d'' is a dimension theory whose !-pullback along $\mathcal{G}|_A \times_A T|_A \to Q \times T|_A \to T|_A$ equals d'. Thus we obtain an isomorphism

$$H^{\mathrm{BM},\mathcal{H}|_A}_{*-2d''}(T|_A,R) \xrightarrow{\sim} H^{\mathrm{BM},\mathcal{G}}_{*-2d}(T,R).$$

(ii) *Compatibilities.* These various maps between Borel–Moore homology groups all commute with each other, whenever this makes sense. Consider the following examples.

(a) If $f : U \to T$ (resp. $p : V \to T$) is an ind-proper (resp. co-placid) $\mathcal{G}$-equivariant map, so that we have a $\mathcal{G}$-equivariant Cartesian square

$$\begin{array}{ccc} W & \xrightarrow{f'} & V \\ \downarrow p' & & \downarrow p \\ U & \xrightarrow{f} & T \end{array}$$

then we have $p^! f_* = f'_*(p')^!$.

(b) Specialization commutes with proper pushforward, restriction of equivariance, co-placid restriction and averaging. To spell this out in the most complicated case, take a 'restriction with supports framework' as in (vi) above with $B = \mathbb{G}_a$ and $\mathcal{G}$ a group. We may restrict all the data over $\{0\}$ or over $\mathbb{G}_a - \{0\}$, and obtain again 'restriction with supports frameworks'. We therefore obtain a square

$$\begin{array}{ccc} H^{\mathrm{BM},\mathcal{G}^*}_{*-2d^*}(T^*,R) & \xrightarrow{s_T} & H^{\mathrm{BM},\mathcal{G}|_0}_{*-2d|_0}(T|_0,R) \\ \downarrow (f|_{\mathbb{G}_a-\{0\}})^! & & \downarrow (f|_{\{0\}})^! \\ H^{\mathrm{BM},\mathcal{G}^*}_{*-2(d^g)^*}(U^*,R) & \xrightarrow{s_U} & H^{\mathrm{BM},\mathcal{G}|_0}_{*-2(d^g)|_0}(U|_0,R) \end{array}$$

which is commutative.

(c) Averaging commutes with change of groupoid base, open restriction, proper pushforward, co-placid restriction. It also commutes with restriction of equivariance in the following sense. If $\mathcal{H} \subset \mathcal{G}$ is of finite index over B and $\mathcal{G}' \to \mathcal{G}$ is a map, the fiber product $\mathcal{H} \times_{\mathcal{G}} \mathcal{G}'$ is of finite index in $\mathcal{G}'$, and the two possible maps from the $\mathcal{H}$-equivariant Borel–Moore homology to the $\mathcal{G}'$-equivariant homology coincide. For some reason, we have steadfastly avoided using quotient stacks. If we had used them, this would be an example of proper base change.

3.11 The branch

We assume from now on that $X = \mathbb{G}_a$ with global parameter t. Thus the 0-fibers $G(\mathcal{O})$ of G_1, $\mathcal{R}$ of $\mathcal{R}_1^1$, etc., have actions of $\mathbb{C}^*$. We assume also that G is reductive, so that $G_S^{S'}/G_S$ is ind-projective over X^S. We recall the definitions of [BFN16] with respect to our notation.

Definition 3.25.

(i) The *Coulomb branch* (over R) is the graded $H^*_{G(\mathcal{O})}(*, R)$-module

$$A^* := H^{\mathrm{BM}, G(\mathcal{O})}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, R).$$

(ii) The *quantum Coulomb branch* (over R) is the graded $H^*_{G(\mathcal{O}) \rtimes \mathbb{C}^*}(*, R) = H^*_{G(\mathcal{O})}(*, R)[\hbar]$-module

$$A^*_\hbar := H^{\mathrm{BM}, G(\mathcal{O}) \rtimes \mathbb{C}^*}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, R).$$

Here $\hbar$ has degree 2.

(iii) We will often write $A^*, A^*_\hbar$ as simply $A, A_\hbar$.

Lemma 3.26. *A^*, $A^*_\hbar$ are evenly graded and free over $H^*_{G(\mathcal{O})}(*, R)$, $H^*_{G(\mathcal{O})}(*, R)[\hbar]$. We have a canonical isomorphism $A^*_\hbar/\hbar \cong A_\hbar$.*

Proof. The proof in [BFN16] in the case $R = \mathbb{C}$ works for any R. The essential point is that the equivariant parameters are in even degrees, and an equivariant placid presentation may be chosen such that each 'approximation' has a complex cell decomposition. □

We will also consider $\mathcal{A}^* := H^{\mathrm{BM}, G(\mathcal{O}) \rtimes \mu_p}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, R)$. The same proof shows that the natural map

$$H^*_{\mu_p}(*, R) \otimes_{R[\hbar]} A^*_\hbar \to \mathcal{A}^*$$

is an isomorphism. In particular, in the case $R = \mathbb{F}_\mathrm{p}$ we have $\mathcal{A}^* = A^*_\hbar[a]$. We have an averaging map $A^* \to \mathcal{A}^*$, which after identifying $A^* = R \otimes_{R[\hbar]} A^*_\hbar$, $\mathcal{A}^* = H^*_{\mu_p}(*, R) \otimes_{R[\hbar]} A^*_\hbar$ is induced by the averaging map of $R[\hbar]$-modules $R \to H^*_\mu(*, R)$. This is the map which multiplies by p in degree 0. Therefore in the case $R = \mathbb{F}_\mathrm{p}$, the averaging map equals 0.

Remark 3.27. (i) In [BFN16] $A^*, A^*_\hbar$ are given ring structures by a form of convolution in Borel–Moore homology; $\mathcal{A}^*$ is also a ring in the same way. They show that A^* is commutative and $A^*_\hbar$ is an $\hbar$-quantization of A^*. We will recall the construction in the course of the proof of our main theorem. The idea (due originally to Beilinson and Drinfeld [BD91]) is to express the multiplication in A^* by a manifestly commutative specialization map.

(ii) Recall that $\mathcal{T}$ is the fpqc quotient $G(\mathcal{K}) \frac{\times}{G(\mathcal{O})} \mathbf{N}(\mathcal{O})$ of the placid ind-scheme $\tilde{T} := G(\mathcal{K}) \times \mathbf{N}(\mathcal{O})$. Both are ind-pro-smooth covers of (in fact, ind-pro-smooth ind-fiber bundles over) Gr_G, and have respective dimension theories $\mathrm{rank}(\mathcal{T})$, $\mathrm{rank}(\widetilde{\mathcal{T}})$ given by their ranks over Gr_G. Let us denote by $\widetilde{\mathcal{R}}$ the corresponding $G(\mathcal{O})$-bundle over $\mathcal{R}$; it has a compatible dimension theory rank($\widetilde{\mathcal{T}}$). By descent, we have the isomorphism

$$A^*_\hbar = H^{\mathrm{BM},G(\mathcal{O})\rtimes\mathbb{C}^*}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R},R) \xrightarrow{\sim} H^{\mathrm{BM},(G(\mathcal{O})\times G(\mathcal{O}))\rtimes\mathbb{C}^*}_{*-2\,\mathrm{rank}(\widetilde{\mathcal{T}})}(\widetilde{\mathcal{R}},R),$$

and similarly for A^*. This shows that $A^*, A^*_\hbar$ have two module structures over $H^*_{G(\mathcal{O})}(*,R)$. In fact, these module structures coincide with the left- and right-multiplication by a subalgebra $H^*_{G(\mathcal{O})}(*,R) \subset A^*, A^*_\hbar$. Since A^* is commutative, these two module structures coincide everywhere. However, $H^*_{G(\mathcal{O})}(*,R)$ is *not* in the center of $A^*_\hbar$, so these two module structures are different there.

3.12 The 'large center' map

We will set $R = \mathbb{F}_\mathrm{p}$ from now on. The rest of this section is devoted to the proof (and explanation) of the following theorem.

Theorem 3.28. *$A_\hbar$ is a Frobenius-constant quantization of A.*

We will construct the requisite map $F_\hbar$ using Steenrod's construction and a specialization map. First we introduce some new notation. We will fix $X = \mathbb{G}_a$, with parameter t. We will use another base curve $Y = \mathbb{G}_a$ with parameter t^p. We have $Y = X//\mu_p$, where $\mu_p \subset \mathbb{C}^*$ acts on X through the character χ given by restricting the weight 1 action of $\mathbb{C}^*$. Let $\pi : X \to Y$ be the quotient map; under the identifications $X = \mathbb{G}_a = Y$, π is the pth-power map, and is $\mathbb{C}^*$-equivariant when $\mathbb{C}^*$ acts on Y with weight p. For $y \in Y^S(R)$, we will use t^p to identify the coordinates y_s with elements of R. Then, for $y \in Y^S(R)$, we have the affine scheme

$$\pi^*\Delta^{S'}_S(y) = \mathrm{Spec}\Bigg(R[t]\Bigg[\Bigg[\prod_{s\in S}(t^p - y_s)\Bigg]\Bigg]\Bigg[\prod_{s\in S'}(t^p - y_{s'})^{-1}\Bigg]\Bigg).$$

Now χ determines a 'twisted-diagonal' embedding $X \subset X^{\mu_p}$ as the χ-eigenline for the cyclic action of μ_p. Let α be one of the symbols G, $\mathbf{N}$, $\mathcal{T}$, $\mathcal{R}$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{R}}$. Then we will set

$$\alpha_{(p)} := \alpha_{\mu_p}//_\chi\mu_p,$$
$$\alpha^{(p)}_{(p)} := \alpha^{\mu_p}_{\mu_p}//_\chi\mu_p.$$

Here the symbol '$//_\chi\mu_p$' means 'restrict along the twisted-diagonal then take categorical quotient by μ_p'. The action of μ_p in question is the one that does not involve loop rotation, that is, that which scales $x \in X(R)$ but does not change any of the data $\mathcal{E}, f, v$ etc. These are all placid ind-schemes over Y, and behave in essentially the same way as their earlier counterparts: $G_{(p)}$ is an affine pro-smooth covering group scheme over Y, $Gr_{(p)} := G^{(p)}_{(p)}/G_{(p)}$ is an ind-projective ind-scheme over Y, $\mathcal{T}^{(p)}_{(p)}$ is an infinite-dimensional vector bundle over $Gr_{(p)}$, $\mathcal{R}^{(p)}_{(p)}$ is its sub-vector space of ind-finite codimension over $Gr_{(p)}$. They are all $G_{(p)} \rtimes \mathbb{C}^*$-equivariant. They also have chosen dimension theories, denoted rank($G^{(p)}_{(p)}$), rank($\mathcal{T}^{(p)}_{(p)}$), rank($\widetilde{\mathcal{T}^{(p)}_{(p)}}$) etc., which are compatible with each other in the same way as for the $\alpha^{S'}_S$, and compatible with the dimension theories on $\alpha^{\mu_p}_{\mu_p}$ in the natural way. That is, the $*$- (or !-)pullback along the μ_p-fppf quotient map of the chosen dimension theory on $\alpha^{(p)}_{(p)}$ coincides with the $*$-pullback along the ind-f.p. closed embedding $X \times_{X^{\mu_p}} \alpha^{\mu_p}_{\mu_p} \to \alpha^{\mu_p}_{\mu_p}$ of the chosen dimension theory on $\alpha^{\mu_p}_{\mu_p}$. Unlike $\alpha^{S'}_S$ which is globally trivial over the coincidence-free open subset of X^S, these spaces $\alpha_{(p)}$, $\alpha^{(p)}_{(p)}$ are only *locally* trivial away from $\{0\}$. On the level of R-points, they admit similar interpretations to $\alpha^{S'}_S$, only involving

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$\pi^*\Delta_1(y)$, $\pi^*\Delta_1^1(y)$. For example, we have

$$\mathcal{R}_{(p)}^{(p)}(R) := \left\{ (y, \mathcal{E}, f, v) \;\middle|\; \begin{array}{l} y \in Y(R) \\ \mathcal{E} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \\ f \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1^1(y) \\ v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends to } \pi^*\Delta_1(y) \end{array} \right\} /\!\sim.$$

The fiber of this space over $\{0\}$ is a copy of $\mathcal{R}$, while the fibers over $Y - \{0\} := Y^*$ are copies of $\mathcal{R}^p$. Another example is

$$G_{(p)}(R) := \left\{ (y, g) \;\middle|\; \begin{array}{l} y \in Y(R) \\ g : \pi^*\Delta_1(y) \to G \end{array} \right\}.$$

The action of $G_{(p)} \rtimes \mathbb{C}^*$ on $\mathcal{R}_{(p)}^{(p)}$ is given as

$$\begin{aligned} (y, g).(y, \mathcal{E}, f, v) &= (y, \mathcal{E}, g \circ f, v), \\ z.(y, \mathcal{E}, f, v) &= (z^p y, z_*\mathcal{E}, z_* f, z_* v) \end{aligned}$$

where for $z \in R^\times$, z_* denotes the pushforward along the multiplication-by-z endomorphism of X_R, $t \mapsto zt$, which transforms $\pi^*\Delta_1(y)$ into $\pi^*\Delta_1(z^p y)$. The key new feature is that $G_{(p)} \rtimes \mu_p$ is a *subgroup* of $G_{(p)} \rtimes \mathbb{C}^*$. In fact, $G_{(p)} \rtimes \mu_p$ is one component of the maximal subgroup of $G_{(p)} \rtimes \mathbb{C}^*$, the other being $\{0\} \times \mathbb{C}^*$. Contrast with $G_S \rtimes \mathbb{C}^*$, whose maximal subgroup is $G_S \cup (\{0\} \times \mathbb{C}^*)$.

Note that the 0-fiber of $G_{(p)} \rtimes \mathbb{C}^* \circlearrowright \mathcal{R}_{(p)}^{(p)}$ is identified with $G(\mathcal{O}) \rtimes \mathbb{C}^* \circlearrowright \mathcal{R}$ ($\mathbb{C}^*$ acting in the usual way, that is, with weight 1 on $t \in \mathcal{O}$). Meanwhile, the 1-fiber of $G_{(p)} \rtimes \mu_p \circlearrowright \mathcal{R}_{(p)}^{(p)}$ is identified with $G(\mathcal{O})^{\mu_p} \rtimes \mu_p \circlearrowright \mathcal{R}^{\mu_p}$, where now μ_p acts in the usual cyclic way of § 2 (without any loop rotation). The latter identification comes from the defining identification of $\pi^*\{1\}$ with μ_p.

Warning 3.29. The notation $\mathcal{R}^{\mu_p}$ means $\mathrm{Map}(\mu_p, \mathcal{R})$. This μ_p superscript is not to be confused with the μ_p superscript in $\mathcal{R}_{\mu_p}^{\mu_p}$, where it indicates a set of allowed poles as in $\mathcal{R}_S^{S'}$.

Consider the following composition $F'_\hbar : A^n \to \mathcal{A}^{pn}$.

$$\begin{array}{ccc} A^n = H^{\mathrm{BM}, G(\mathcal{O})}_{n-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, \mathbb{F}_{\mathrm{p}}) & \xrightarrow{\text{Steenrod}} & H^{\mathrm{BM}, G(\mathcal{O})^{\mu_{\mathrm{p}}} \rtimes \mu_{\mathrm{p}}}_{pn-2p\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}^{\mu_{\mathrm{p}}}, \mathbb{F}_{\mathrm{p}}) \\ & & \downarrow{\scriptstyle \text{Descent isomorphism (b)}} \\ H^{\mathrm{BM}, G^*_{(p)} \rtimes \mu_{\mathrm{p}}}_{pn-2-2p\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\mathrm{p}}) & \xleftarrow[\text{Restrict}]{} & H^{\mathrm{BM}, G^*_{(p)} \rtimes \mathbb{C}^*}_{pn-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\mathrm{p}}) \\ \downarrow{\scriptstyle \text{Specialize}} & & \\ H^{\mathrm{BM}, G(\mathcal{O}) \rtimes \mu_{\mathrm{p}}}_{pn-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, \mathbb{F}_{\mathrm{p}}) = \mathcal{A}^{pn} & & \end{array} \tag{3.1}$$

For degree reasons it factors through the inclusion $A^*_\hbar \subset A^*_\hbar[a] = \mathcal{A}^*$. We may form the graded $H^*_{G(\mathcal{O})}(*, \mathbb{F}_{\mathrm{p}})^{(1)}$-module $(A^*)^{(1)}$, and we have a map of $\mathbb{Z}$-graded multiplicative $\mathbb{F}_{\mathrm{p}}$-sets

$$F_\hbar : (A^*)^{(1)} \to A^*_\hbar.$$

3.13 Linearity

We have the following proposition.

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Proposition 3.30. *$F_\hbar$ is $St_{\rm in}$-linear. That is, it is linear and transports multiplication by $r \in H^{m}_{G(\mathcal{O})}(*, \mathbb{F}_{\rm p})$ to multiplication by $St_{\rm in}(r) \in H^{pm}_{G(\mathcal{O})\rtimes\mathbb{C}^*}(*, \mathbb{F}_{\rm p}) \subset H^{pm}_{G(\mathcal{O})\rtimes\mathbb{C}^*}(*, \mathbb{F}_{\rm p})$.*

Proof. First we show that $F_\hbar$ is $St_{\rm in}$-multiplicative. Recall that restriction of equivariance commutes with specialization. We have a closed embedding from the Y-version of $G_{\{1\}}$ to $G_{(p)}$ determined by the formula

$$(y, g : \Delta_{\{1\}}(y) \to G) \mapsto (y, g \circ \pi : \pi^*\Delta_{\{1\}}(y) \to G).$$

Over 0, this is identified with the embedding $G(\mathcal{O}) \to G(\mathcal{O})$, $t \mapsto t$. Over 1 this is identified with the diagonal embedding $G(\mathcal{O}) \to G(\mathcal{O})^{\mu_p}$. Since the restriction of equivariance along the former embedding,

$$H^{\mathrm{BM}, G(\mathcal{O})\rtimes\mu_p}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, \mathbb{F}_{\rm p}) \to H^{\mathrm{BM}, G(\mathcal{O})\rtimes\mu_p}_{*-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}, \mathbb{F}_{\rm p}),$$

is an isomorphism, it follows that the map $H^{\mathrm{BM}, G^*_{(p)}\rtimes\mu_p}_{*-2-2p\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) \to \mathcal{A}^*$ factors as

$$H^{\mathrm{BM}, G^*_{(p)}\rtimes\mu_p}_{*-2-2p\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) \xrightarrow{\text{Restrict}} H^{\mathrm{BM}, G^*_{\{1\}}\rtimes\mu_p}_{*-2-2p\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) \xrightarrow{\text{Specialize}} \mathcal{A}^*.$$

Since $G_{\{1\}} \rtimes \mu_p$ is a constant group over Y with fibers $G(\mathcal{O}) \rtimes \mu_p$, the latter specialization map is $H^*_{G(\mathcal{O})\rtimes\mu_p}(*, \mathbb{F}_{\rm p})$-linear. Certainly the descent isomorphism (b) and the restriction of equivariance from $\mathbb{C}^*$ to μ_p in the diagram defining $F'_\hbar$ commute with restriction of equivariance from $G_{(p)}$ to $G_{\{1\}}$. It follows that $F_\hbar$ is $St_{\rm in}$-multiplicative, by definition of $St_{\rm in}$.

Now we show linearity. Averaging (over μ_p) commutes with the descent isomorphism (b) and restriction of equivariance in that we have the following commutative diagram.

$$\begin{array}{ccc}
H^{\mathrm{BM}, G(\mathcal{O})^{\mu_{\rm p}}}_{pn-2p\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}^{\mu_{\rm p}}, \mathbb{F}_{\rm p}) & \xrightarrow{\text{Averaging}} & H^{\mathrm{BM}, G(\mathcal{O})^{\mu_{\rm p}}\rtimes\mu_{\rm p}}_{pn-2p\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}^{\mu_{\rm p}}, \mathbb{F}_{\rm p}) \\
\downarrow{\scriptstyle \text{Descent isom.}} & & \downarrow{\scriptstyle \text{Descent isom.}} \\
H^{\mathrm{BM}, \pi^*G^*_{(p)}\rtimes\mathbb{C}^*}_{pn-2-2\pi^*\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}(\pi^*(\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) & \xrightarrow{\text{Averaging}} & H^{\mathrm{BM}, G^*_{(p)}\rtimes\mathbb{C}^*}_{pn-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) \\
\downarrow{\scriptstyle \text{Restriction}} & & \downarrow{\scriptstyle \text{Restriction}} \\
H^{\mathrm{BM}, \pi^*G^*_{(p)}\rtimes\mu_{\rm p}}_{pn-2-2\pi^*\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}(\pi^*(\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p}) & \xrightarrow{\text{Averaging}} & H^{\mathrm{BM}, G^*_{(p)}\rtimes\mu_{\rm p}}_{pn-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p})
\end{array}$$

Here for the second two averaging maps we have identified $H^{\mathrm{BM}, G^*_{(p)}\rtimes ?}_{pn-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p})$ with $H^{\mathrm{BM}, (\pi^*G^*_{(p)})\rtimes(?\times\mu_p)}_{pn-2-2\pi^*\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}(\pi^*(\mathcal{R}^{(p)}_{(p)})^*, \mathbb{F}_{\rm p})$ for $? = \mathbb{C}^*, \mu_p$.

Since averaging commutes also with specialization, it follows that the discrepancy from additivity lies in the image of the averaging map $A^* \to \mathcal{A}^*$. But this map is equal to 0. □

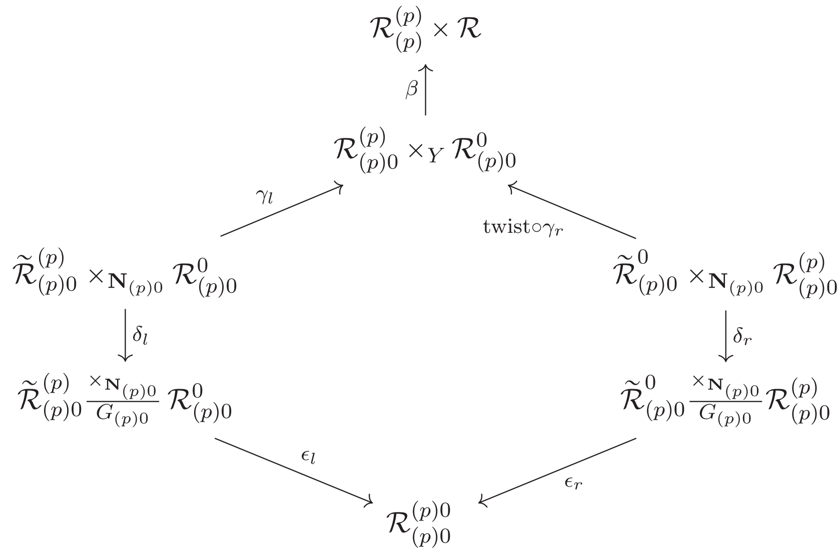
3.14 Centrality

We have the following proposition.

Proposition 3.31. *$F_\hbar$ maps into the center of $A^*_\hbar$.*

Proof. The idea is to adapt the proof of commutativity of A^* given in the Appendix to [BFN17] to the present setup. That proof is itself an adaptation of the construction, using Beilinson–Drinfeld Grassmannians, of the commutativity constraint on the Satake category; see [MV07].

Consider the following diagram.



Let α be as before, that is, any of the symbols G, $\mathbf{N}$, $\mathcal{T}$, $\mathcal{R}$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{R}}$. We set

$$\alpha^{(p)0}_{(p)0} := \left(X \times_{(X^{\mu_p \cup \{0\}})} \alpha^{\mu_p \cup \{0\}}_{\mu_p \cup \{0\}}\right) // \mu_p.$$

Here the fiber product is taken using the map $X \to X^{\mu_p \cup \{0\}}$ given by the product of the 'twisted-diagonal' embedding $X \to X^{\mu_p}$ determined χ and the inclusion of $\{0\}$ in $\mathbb{G}_a$ under the identification $X^{\mu_p \cup \{0\}} = X^{\mu_p} \times X^{\{0\}} = X^{\mu_p} \times \mathbb{G}_a$. The action of μ_p is again the one which does not involve any loop rotation. Removing the superscript (p) (respectively, 0, $(p)0$) from $\alpha^{(p)0}_{(p)0}$ corresponds to removing the superscript $\mu_p \cup$ (respectively, $\cup\{0\}$, $\mu_p \cup \{0\}$) from $\alpha^{\mu_p \cup \{0\}}_{\mu_p \cup \{0\}}$ in its defining equation. We may write the spaces of the 'left path' as follows:

$$\mathcal{R}^{(p)}_{(p)} \times \mathcal{R}(R) = \left\{ \begin{array}{l|l} (y, \mathcal{E}, f, v, & y \in Y(R) \\ \mathcal{E}_0, f_0, v_0) & \mathcal{E} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \\ & f \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1^1(y) \\ & v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends} \\ & \quad \text{to } \pi^*\Delta_1(y) \\ & \mathcal{E}_0 \text{ a principal } G\text{-bundle over } \Delta_1(\{0\}) \\ & f_0 \text{ a trivialization of } \mathcal{E}_0 \text{ over } \Delta_1^1(\{0\}) \\ & v_0 \text{ an } \mathbf{N}\text{-section of } \mathcal{E}_0 \text{ such that } f_0(v_0) \text{ extends} \\ & \quad \text{to } \Delta_1(\{0\}) \end{array} \right\} /\sim,$$

$$\mathcal{R}^{(p)}_{(p)0} \times_Y \mathcal{R}^{0}_{(p)0}(R) = \left\{ \begin{array}{l|l} (y, \mathcal{E}, f, v, & y \in Y(R) \\ \mathcal{E}_0, f_0, v_0) & \mathcal{E} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & f \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1^1(y) \cup \Delta_1(\{0\}) \\ & v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends} \\ & \quad \text{to } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & \mathcal{E}_0 \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & f_0 \text{ a trivialization of } \mathcal{E}_0 \text{ over } \pi^*\Delta_1(y) \cup \Delta_1^1(\{0\}) \\ & v_0 \text{ an } \mathbf{N}\text{-section of } \mathcal{E}_0 \text{ such that } f_0(v_0) \text{ extends} \\ & \quad \text{to } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \end{array} \right\} /\sim,$$

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$$\widetilde{\mathcal{R}}^{(p)}_{(p)0} \times_{\mathbf{N}_{(p)0}} \mathcal{R}^0_{(p)0}(R) = \left\{ \begin{array}{c|l} & y \in Y(R) \\ & \mathcal{E} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & f \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1^1(y) \cup \Delta_1(\{0\}) \\ (y, \mathcal{E}, f, & g \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ g, v, \mathcal{E}_0, & v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends} \\ f_0, v_0) & \quad \text{to } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & \mathcal{E}_0 \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & f_0 \text{ a trivialization of } \mathcal{E}_0 \text{ over } \pi^*\Delta_1(y) \cup \Delta_1^1(\{0\}) \\ & v_0 \text{ an } \mathbf{N}\text{-section of } \mathcal{E}_0 \text{ such that } f_0(v_0) = g(v) \end{array} \right\} /\sim,$$

$$\widetilde{\mathcal{R}}^{(p)}_{(p)0} \frac{\times_{\mathbf{N}_{(p)0}}}{G_{(p)0}} \mathcal{R}^0_{(p)0}(R) = \left\{ \begin{array}{c|l} & y \in Y(R) \\ & \mathcal{E} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & f \text{ a trivialization of } \mathcal{E} \text{ over } \pi^*\Delta_1^1(y) \cup \Delta_1(\{0\}) \\ & v \text{ an } \mathbf{N}\text{-section of } \mathcal{E} \text{ such that } f(v) \text{ extends} \\ (y, \mathcal{E}, f, v, & \quad \text{to } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ \mathcal{E}_0, h_0, v_0) & \mathcal{E}_0 \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ & h_0 \text{ an isomorphism of } \mathcal{E}_0 \text{ with } \mathcal{E} \\ & \quad \text{over } \pi^*\Delta_1(y) \cup \Delta_1^1(\{0\}) \\ & v_0 \text{ an } \mathbf{N}\text{-section of } \mathcal{E}_0 \text{ such that } h_0(v_0) = v \end{array} \right\} /\sim$$

and

$$\mathcal{R}^{(p)0}_{(p)0}(R) = \left\{ (y, \mathcal{F}, e, w) \;\middle|\; \begin{array}{l} y \in Y(R) \\ \mathcal{F} \text{ a principal } G\text{-bundle over } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \\ e \text{ a trivialization of } \mathcal{F} \text{ over } \pi^*\Delta_1^1(y) \cup \Delta_1^1(\{0\}) \\ w \text{ an } \mathbf{N}\text{-section of } \mathcal{F} \text{ such that } e(w) \text{ extends} \\ \quad \text{to } \pi^*\Delta_1(y) \cup \Delta_1(\{0\}) \end{array} \right\} /\sim.$$

Here the $\{0\}$ of $\Delta_1^?(\{0\})$ denotes the fixed R-point $\{0\}$ of X. By definition, β is the product of the co-placid map $\mathcal{R}^{(p)}_{(p)0} \to \mathcal{R}^{(p)}_{(p)}$ induced by the embedding $\pi^*\Delta_1^1(y) \to \pi^*\Delta_1^1(y) \cup \Delta_1(\{0\})$, and the co-placid map $\mathcal{R}^0_{(p)0} \to Y \times \mathcal{R}$ induced by the embedding $\Delta_1^1(\{0\}) \to \pi^*\Delta_1(y) \cup \Delta_1^1(\{0\})$. The map γ_l factors as

$$\widetilde{\mathcal{R}}^{(p)}_{(p)0} \times_{\mathbf{N}_{(p)0}} \mathcal{R}^0_{(p)0} \xrightarrow{u_l} \widetilde{\mathcal{R}}^{(p)}_{(p)0} \times_Y \mathcal{R}^0_{(p)0} \xrightarrow{v_l} \mathcal{R}^{(p)}_{(p)0} \times_Y \mathcal{R}^0_{(p)0}$$

where v_l is a $G_{(p)0}$-torsor and u_l fits into a 'restriction with supports framework': a Cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{R}}^{(p)}_{(p)0} \times_{\mathbf{N}_{(p)0}} \mathcal{R}^0_{(p)0} & \xrightarrow{u_l} & \widetilde{\mathcal{R}}^{(p)}_{(p)0} \times_Y \mathcal{R}^0_{(p)0} \\ \downarrow & & \downarrow \\ \widetilde{\mathcal{T}}^{(p)}_{(p)0} \times_{\mathbf{N}_{(p)0}} \mathcal{R}^0_{(p)0} & \xrightarrow{u'_l} & \widetilde{\mathcal{T}}^{(p)}_{(p)0} \times_Y \mathcal{R}^0_{(p)0} \end{array}$$

such that u'_l is a section of a vector bundle map

$$\widetilde{\mathcal{T}}^{(p)}_{(p)0} \times_Y \mathcal{R}^0_{(p)0} = G^{(p)}_{(p)0}/G_{(p)0} \times_Y \mathbf{N}_{(p)0} \times_Y \mathcal{R}^0_{(p)0} \to G^{(p)}_{(p)0}/G_{(p)0} \times_Y \mathcal{R}^0_{(p)0}$$

and whose vertical arrows are ind-f.p closed embeddings. The map δ_l is a $G_{(p)0}$-torsor, defined by

$$(y, \mathcal{E}, f, g, v, \mathcal{E}_0, f_0, v_0) \mapsto (y, \mathcal{E}, f, v, \mathcal{E}_0, h_0 = g^{-1} f_0, v_0).$$

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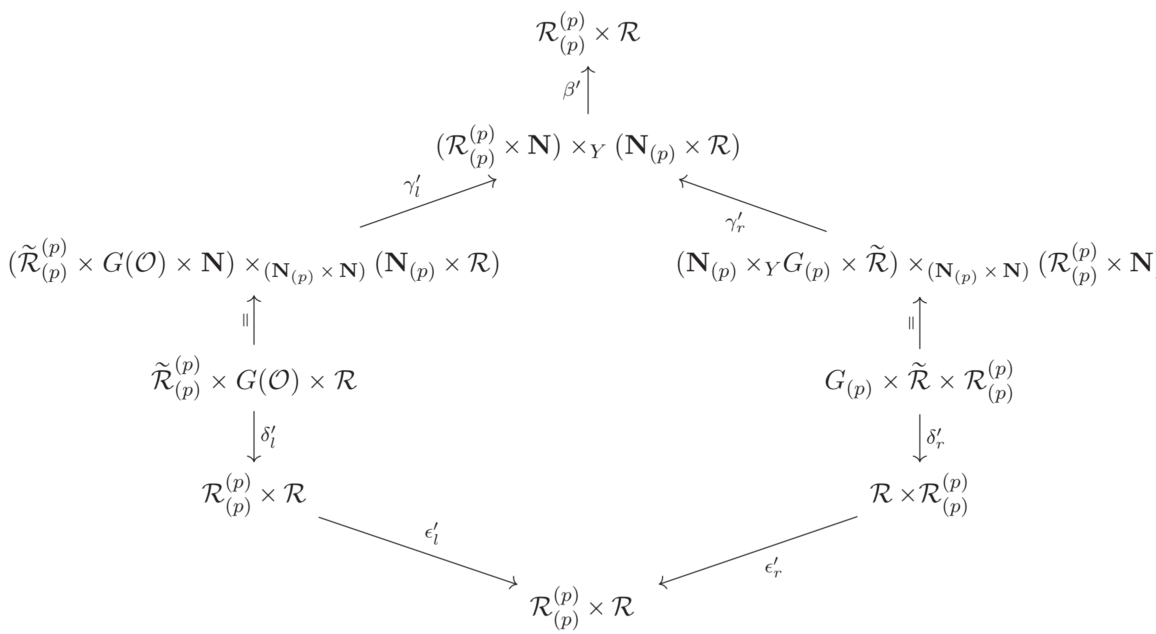
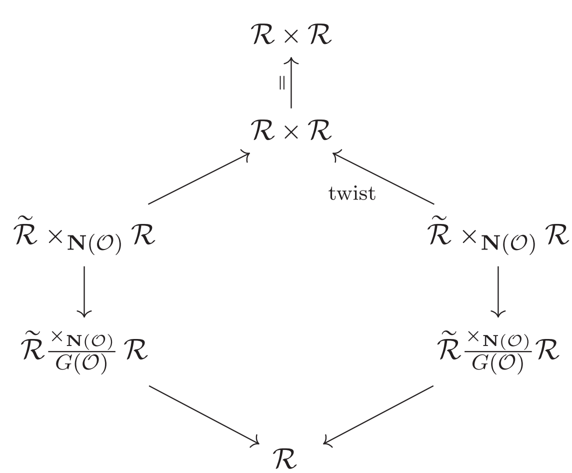
The map ϵ_l is ind-proper, defined by

$$(y, \mathcal{E}, f, v, \mathcal{E}_0, h_0, v_0) \mapsto (y, \mathcal{F} = \mathcal{E}_0, e = hf_0, w = v_0).$$

This describes the 'left path'. The 'right path' exactly mirrors it and has all the same properties: just exchange superscripts μ_p, 0, and on the level of points exchange $\mathcal{E}$ with $\mathcal{E}_0$, f with f_0, v with v_0, g with g_0, h with h_0, etc. We have labeled our data $\mathcal{F}$, e, w in $\mathcal{R}^{(p)0}_{(p)0}$ because in the 'left path' we have $(\mathcal{F}, e, w) = (\mathcal{E}_0, fg^{-1}f_0, v_0)$ while in the 'right path' we have $(\mathcal{F}, e, w) = (\mathcal{E}, f_0 g_0^{-1} f, v)$. If we restrict our diagram to Y^*, then the subscripts (p), 0 'split apart', and the result is rather degenerate. That is, it coincides with the restriction to Y^* of



Here the maps from the fourth row to the top are the obvious projection maps, while the maps from the fourth row to the bottom are the obvious action maps. If instead we restrict our diagram to $\{0\} \subset Y$, the subscripts (p), 0 'fuse' and the result is again degenerate: we get



We leave it to the reader to write out the appropriate equivariant structures implicit in the following chain of maps, and to check that the quoted dimension theories are appropriately

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compatible:

$$\begin{array}{ccc}
H^{\mathrm{BM},G_{(p)}\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})}(\mathcal{R}^{(p)}_{(p)})\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a] & \xrightarrow{\quad = \quad} & H^{\mathrm{BM},G_{(p)}\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})}(\mathcal{R}^{(p)}_{(p)})\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}H^{\mathrm{BM},G(\mathcal{O})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}\,\mathcal{T}}(\mathcal{R}) \\
 & & \Big\downarrow \wr\wr \\
H^{\mathrm{BM},(G_{(p)0}\times_Y G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}^{(p)}_{(p)}\times\mathcal{R}) & \xleftarrow[\text{restrict}]{} & H^{\mathrm{BM},(G_{(p)}\times G(\mathcal{O}))\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})-2\,\mathrm{rank}(\mathcal{T})}(\mathcal{R}^{(p)}_{(p)}\times\mathcal{R}) \\
\Big\downarrow \beta^! & & \\
H^{\mathrm{BM},(G_{(p)0}\times_Y G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)0})-2\,\mathrm{rank}(\mathcal{T}^0_{(p)0})}(\mathcal{R}^{(p)}_{(p)0}\times_Y\mathcal{R}^0_{(p)0}) & \xrightarrow{\text{descent}} & H^{\mathrm{BM},(G_{(p)0}\times_Y G_{(p)0}\times_Y G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\widetilde{\mathcal{T}}^{(p)}_{(p)0})-2\,\mathrm{rank}(\mathcal{T}^0_{(p)0})}(\widetilde{\mathcal{R}}^{(p)}_{(p)0}\times_Y\mathcal{R}^0_{(p)0}) \\
 & & \Big\downarrow \text{restrict on } id\times_Y\text{diag} \\
H^{\mathrm{BM},(G_{(p)0}\times_Y G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\delta^!_l\epsilon^*_l\,\mathrm{rank}(\widetilde{\mathcal{T}}^{(p)0}_{(p)0})}(\widetilde{\mathcal{R}}^{(p)}_{(p)0}\times_{\mathbf{N}_{(p)0}}\mathcal{R}^0_{(p)0}) & \xleftarrow[u^!_l]{} & H^{\mathrm{BM},(G_{(p)0}\times_Y G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\widetilde{\mathcal{T}}^{(p)}_{(p)0})-2\,\mathrm{rank}(\mathcal{T}^0_{(p)0})}(\widetilde{\mathcal{R}}^{(p)}_{(p)0}\times_Y\mathcal{R}^0_{(p)0}) \\
\Big\downarrow \text{descent} & & \\
H^{\mathrm{BM},(G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\epsilon^*_l\,\mathrm{rank}(\widetilde{\mathcal{T}}^{(p)0}_{(p)0})}(\widetilde{\mathcal{R}}^{(p)}_{(p)0}\frac{\times_{\mathbf{N}_{(p)0}}}{G_{(p)0}}\mathcal{R}^0_{(p)0}) & \xrightarrow{\quad(\epsilon_l)_*\quad} & H^{\mathrm{BM},(G_{(p)0})\rtimes\mu_{\mathrm{p}}}_{*-2\,\mathrm{rank}(\widetilde{\mathcal{T}}^{(p)0}_{(p)0})}(\mathcal{R}^{(p)0}_{(p)0})
\end{array}$$

Here we have dropped the homological coefficients '$\mathbb{F}_{\mathrm{p}}$' for brevity. This is called the 'homological left path over Y'. Similarly, there is a 'homological right path over Y'. Moreover, we have versions of both 'homological paths' for the restrictions of our original diagram to Y^*, $\{0\}$, and specialization map of paths

$$\text{'homological left path over } Y^*\text{'} \to \text{'homological left path over } \{0\}\text{'},$$

$$\text{'homological right path over } Y^*\text{'} \to \text{'homological right path over } \{0\}\text{'}$$

since every step of both paths is compatible with specialization. One the one hand, both 'homological paths over Y^*' give as their composition the identity map

$$H^{\mathrm{BM},G_{(p)}\rtimes\mu_p}_{*-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*)\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a]\to H^{\mathrm{BM},G_{(p)}\rtimes\mu_p}_{*-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*)\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a].$$

On the other hand, the 'left homological path over $\{0\}$' gives as its composition the multiplication map (indeed, this is the definition of convolution from [BFN16])

$$A^*_\hbar[a]\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a]\xrightarrow{\text{'convolution'}}A^*_\hbar[a],$$

while the 'right homological path over $\{0\}$' gives as its composition the *twisted* multiplication map

$$A^*_\hbar[a]\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a]\xrightarrow{\text{'convolution'}\circ\text{twist}}A^*_\hbar[a].$$

It follows that in fact the image of the specialization map

$$H^{\mathrm{BM},G_{(p)}\rtimes\mu_p}_{*-2-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}((\mathcal{R}^{(p)}_{(p)})^*)\otimes_{\mathbb{F}_{\mathrm{p}}[a,\hbar]}A^*_\hbar[a]\to A^*_\hbar[a]$$

is in the center of $A^*_\hbar[a]$. By its very definition, $F_\hbar$ factors through this map. □

3.15 Completion of the proof

(i) *$F_\hbar$ is multiplicative.* The proof is essentially the same as the proof of centrality, but instead of keeping one copy of $\mathcal{R}$ fixed and allowing the other to deform to $\mathcal{R}^{\mu_p}$ with its cyclic μ_p-action,

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we allow *both* copies of $\mathcal{R}$ to deform in that way. In fact it is easier because we only need one 'path'. We will content ourselves with drawing the defining diagram; the conscientious reader can plug in the method of specialization:

$$\mathcal{R}^{(p)}_{(p)} \times \mathcal{R}^{(p)}_{(p)} \leftarrow \widetilde{\mathcal{R}}^{(p)}_{(p)} \times_{\mathbf{N}_{(p)}} \mathcal{R}^{(p)}_{(p)} \to \widetilde{\mathcal{R}}^{(p)}_{(p)} \frac{\times_{\mathbf{N}_{(p)}}}{G_{(p)}} \mathcal{R}^{(p)}_{(p)} \to \mathcal{R}^{(p)}_{(p)}.$$

(ii) $F_\hbar$ *sends* 1 *to* 1. Note that $1 \in A^*$ is the fundamental class of the fiber $\mathbf{N}(\mathcal{O})$ of $\mathcal{R}$ over the base point of Gr. Certainly Steenrod's construction sends this to the fundamental class of the fiber $\mathbf{N}(\mathcal{O})^{\mu_p}$ of $\mathcal{R}^{\mu_p}$ over the base point of Gr^{μ_p}, and this is sent by the 'descent isomorphism (b)' construction to the fundamental class of the fiber $\mathbf{N}^*_{(p)}$ of $(\mathcal{R}^{(p)}_{(p)})^*$ over the base section of $Gr^*_{(p)}$. But this section extends to a base section of $Gr_{(p)}$, namely, the trivial G-bundle with the trivial trivialization. The fiber of $\mathcal{R}^{(p)}_{(p)}$ over this section is $\mathbf{N}_{(p)}$. Since this is a vector bundle over Y, specialization sends its fundamental class to the fundamental class of its 0-fiber $\mathbf{N}(\mathcal{O})$, as required.

(iii) $F_\hbar$ mod $\hbar$ *is the Frobenius map.* This is essentially clear from the construction. It amounts to showing that the specialization (over Y) of the class b in

$$H^{\mathrm{BM},G_{(p)}}_{*-2\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}\big((\mathcal{R}^{(p)}_{(p)})^*\big)$$

obtained by applying Steenrod's construction to $a \in (A^*)^{(1)}$, then applying descent isomorphism (b) and then restricting all the way to $G_{(p)}$-equivariance, equals x^p (recall diagram (3.1)). But by a general property of specialization, it is equal to the specialization over X of the class $\pi^*(b)$ in

$$H^{\mathrm{BM},\pi^*G^*_{(p)}}_{*-2\pi^*\,\mathrm{rank}(\mathcal{T}^{(p)}_{(p)})^*}\big(\pi^*(\mathcal{R}^{(p)}_{(p)})^*\big).$$

Under the identifications $\pi^*(\mathcal{R}^{(p)}_{(p)})^* = \mathcal{R}^{\mu_p} \times Y^*$, $\pi^*G^*_{(p)} = G(\mathcal{O})^{\mu_p} \times Y^*$, $\pi^*(b)$ is just the pull-back of $a^{\boxtimes p}$ along the projection away from Y^*. Thus it is enough to prove the more general statement that for $a_1, \dots, a_p \in A^*$, the convolution product $a_1 \dots a_p$ is equal to the specialization in $\pi^*\mathcal{R}^{(p)}_{(p)}$ of $a_1 \boxtimes \cdots \boxtimes a_p$. That is achieved by choosing an enumeration $\mu_p = \{1, \dots, p\}$ and considering the restriction along the 'twisted-diagonal' embedding $X \to X^{\mu_p}$ of the global convolution diagram (see the Appendix to [BFN17]):

$$\begin{array}{ccc}
\mathcal{R}^{\{1\}}_{\{1\}} \times \cdots \times \mathcal{R}^{\{p\}}_{\{p\}} & \longrightarrow & \widetilde{\mathcal{R}}^{\{1\}}_{\mu_p} \times_{\mathbf{N}_{\mu_p}} \cdots \times_{\mathbf{N}_{\mu_p}} \widetilde{\mathcal{R}}^{\{p-1\}}_{\mu_p} \times_{\mathbf{N}_{\mu_p}} \mathcal{R}^{\{p\}}_{\mu_p} \\
 & & \downarrow \\
 & & \widetilde{\mathcal{R}}^{\{1\}}_{\mu_p} \frac{\times_{\mathbf{N}_{\mu_p}}}{G_{\mu_p}} \cdots \frac{\times_{\mathbf{N}_{\mu_p}}}{G_{\mu_p}} \widetilde{\mathcal{R}}^{\{p-1\}}_{\mu_p} \frac{\times_{\mathbf{N}_{\mu_p}}}{G_{\mu_p}} \mathcal{R}^{\{p\}}_{\mu_p} \\
 & & \downarrow \\
 & & \mathcal{R}^{(\mu_p)}_{(\mu_p)}
\end{array}$$

3.16 Closing remarks

(i) There is a closed embedding

$$\begin{array}{ccc}
\mathcal{R}^1_1 & \to & \mathcal{R}^{(p)}_{(p)} \\
(y, \mathcal{E}, f, v) & \mapsto & (y, \pi^*\mathcal{E}, \pi^*f, \pi^*v)
\end{array}$$

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and similarly compatible closed embeddings $\alpha_1^1 \to \alpha_{(p)}^{(p)}$ for any symbol α (see § 3.12). We also have the compatible closed embeddings of groups $G_1 \to G_{(p)}$, $\mathbf{N}_1 \to \mathbf{N}_{(p)}$. In fact we have already used one of these to prove linearity of $F_\hbar$ in § 3.13.

(ii) For large p, the $\mathbf{N} = 0$ version of $A_\hbar$ has a name: it is the *quantum Toda lattice*, denoted $Toda_\hbar$ and given as the two-sided quantum Hamiltonian reduction $N_\psi^\vee \backslash\backslash \mathcal{D}_\hbar(G^\vee)//_\psi$ of the Rees algebra of crystalline differential operators, $\mathcal{D}_\hbar(G^\vee)$, of the Langlands dual group $G^\vee$ over $\mathbb{F}_\mathrm{p}$, with respect to a regular character ψ of a maximal unipotent $N^\vee \subset G^\vee$. As a quantum Hamiltonian reduction of a ring of differential operators, it has a canonical Frobenius-constant structure. It follows from a torus localization argument that this Frobenius-constant structure coincides with the one we have produced in this paper. The ind-f.p. closed embedding $\mathcal{R} \to \mathcal{T}$ induces a pushforward map of $H^*_{G\times\mathbb{C}^*}(*, \mathbb{F}_\mathrm{p})$-algebras

$$A_\hbar^* \to H_{*-2\,\mathrm{rank}\,\mathcal{T}}^{\mathrm{BM}, G\times\mathbb{C}^*}(\mathcal{T}, \mathbb{F}_\mathrm{p}) \cong H_*^{\mathrm{BM}, G\times\mathbb{C}^*}(Gr, \mathbb{F}_\mathrm{p}).$$

For all p, this map is compatible with the Frobenius-constant structure. For large p this map is an embedding. So for large p, Theorem 3.28 can be understood as saying that the subalgebra $A_\hbar^*$ of the quantum Toda lattice contains the image of $A^* \subset Toda := Toda_\hbar/\hbar$ under the canonical Frobenius-constancy map $Toda^{(1)} \to Toda_\hbar$.

(iii) *An example.* Let $G = \mathbb{C}^*$, $\mathbf{N} = \mathbb{C}_{-r}$, $r \geq 0$. Then on $\mathbb{C}$-points we identify:

(a) $Gr = \mathbb{Z}$;
(b) $\mathcal{T} = \mathbb{Z} \times \mathbb{C}_{-r}[[t]]$;
(c) $\mathcal{R} = \mathbb{Z}_{\leq 0} \times \mathbb{C}_{-r}[[t]] \cup \{1\} \times t^r\mathbb{C}_{-r}[[t]] \cup \{2\} \times t^{2r}\mathbb{C}_{-r}[[t]] \cup \dots$.

For $\mathbf{N} = 0$, $A_\hbar$ is the Weyl algebra $\mathbb{F}_\mathrm{p}[\hbar]\langle x^\pm, \partial\rangle/([\partial, x] = \hbar)$. The equivariant Borel–Moore homology of a point $n \in \mathbb{Z}$ is identified with $\mathbb{F}_\mathrm{p}[\hbar, x\partial].x^n$. It is a direct calculation that $F_\hbar$ is the map

$$\begin{aligned} x^{(1)} &\mapsto x^p, \\ y^{(1)} &\mapsto \partial^p, \\ (xy)^{(1)} &\mapsto x^p\partial^p = \prod_{i=0}^{p-1}(x\partial - i\hbar) = (x\partial)^p - \hbar^{p-1}x\partial = AS_\hbar(x\partial). \end{aligned}$$

Here $y = \partial \bmod \hbar$. For $\mathbf{N} = \mathbb{C}_{-r}$ with $r \geq 0$, $A_\hbar$ is the reduction modulo p of the subalgebra of the integral Weyl algebra $\mathbb{Z}[\hbar]\langle x^\pm, \partial\rangle/([\partial, x] = \hbar)$ with $\mathbb{F}_\mathrm{p}[\hbar, x\partial]$-basis

$$\dots x^{-2}, x^{-1}, 1, \left(\prod_{i=1}^{r}(rx\partial - i\hbar)\right)x, \left(\prod_{i=1}^{2r}(rx\partial - i\hbar)\right)x^2, \dots.$$

It satisfies $(\prod_{i=1}^{nr}(rx\partial - i\hbar)x^n)(\prod_{i=1}^{mr}(rx\partial - i\hbar)x^m) = \prod_{i=1}^{(m+n)r}(rx\partial - i\hbar)x^{m+n}$. Note that this is a subalgebra of the mod p Weyl algebra if and only if p does not divide r. It is again a direct computation that $F_\hbar$ is the map

$$\begin{aligned} (x^{-1})^{(1)} &\mapsto x^{-p}, \\ ((rxy)^r x)^{(1)} &\mapsto \prod_{i=1}^{pr}(rx\partial - i\hbar)x^p, \\ (xy)^{(1)} &\mapsto \prod_{i=0}^{p-1}(x\partial - i\hbar). \end{aligned}$$

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It is an interesting exercise to check that these are really central (of course their images in the mod p Weyl algebra are).

(iv) For large p one can prove the centrality and multiplicativity of the map $F_\hbar$, constructed in § 3.12, via torus localization in conjunction with the above example. But Theorem 3.28 is true for all odd primes p. In fact, the same construction works for $p = 2$ but one has to be a little careful to account for the fact that $a^2 = \hbar$ in that case.

(v) Suppose that the action of G on N extends to an action of a normalizing supergroup $\tilde{G}$ of G. Then the same proof shows that the corresponding flavor deformation is also a Frobenius-constant quantization.

(vi) In [BFN17], an algebra ind-object $\Omega_\hbar$ of the Satake category $D^b_{G(\mathcal{O})\rtimes\mathbb{C}^*}(Gr, \mathbb{C})$ is constructed; its cohomology algebra is the quantum Coulomb branch. The commutativity of the Coulomb branch corresponds to commutativity of the image algebra Ω of $\Omega_\hbar$ in $D^b_{G(\mathcal{O})}(Gr, \mathbb{C})$. It is also possible, by essentially the same method given in the Appendix to [BFN17], to tell the same story with $\mathbb{F}_\mathrm{p}$ coefficients.

4. *K*-theoretic version

4.1 *K*-theory and *K*-homology

Let X be a scheme over some base B over $\mathbb{C}$ and let $\mathcal{G}$ be an (affine, pro-smooth) groupoid scheme over B acting on X. We have the $\mathcal{G}$-equivariant K-homology of X,

$$K^{\mathcal{G}}(X) := K_0(D^b_{\mathcal{G}}\,\mathrm{Coh}(X)),$$

which is by definition the Grothendieck group of $D^b_{\mathcal{G}}\,\mathrm{Coh}(X)$, the $\mathcal{G}$-equivariant derived category of complexes of sheaves on X with bounded, coherent cohomology sheaves. We have also the *$\mathcal{G}$-equivariant K-theory* of X,

$$K_{\mathcal{G}}(X) := K_0(\mathrm{Perf}_{\mathcal{G}}(X)),$$

which is by definition the Grothendieck group of the full subcategory $\mathrm{Perf}_{\mathcal{G}}(X)$ of $D^b_{\mathcal{G}}\,\mathrm{Coh}(X)$ consisting of perfect complexes. We recall some basic facts (see [CG97]).

(i) $K_{\mathcal{G}}(X)$ forms a ring, since $\mathrm{Perf}_{\mathcal{G}}(X)$ is monoidal. The unit element is given by the class of the structure sheaf.

(ii) $K^{\mathcal{G}}(X)$ is a module over $K_{\mathcal{G}}(X)$, since $D^b_{\mathcal{G}}\,\mathrm{Coh}(X)$ is a module category over $\mathrm{Perf}_{\mathcal{G}}(X)$. Under suitable conditions, for instance, if X is smooth and $\mathcal{G}$ is a connected linear algebraic group, every equivariant coherent sheaf has a bounded equivariant resolution by vector bundles, so that the defining functor $\mathrm{Perf}_{\mathcal{G}}(X) \to D^b_{\mathcal{G}}(X)$ is an equivalence, and in particular the map from $K_{\mathcal{G}}(X)$ to $K^{\mathcal{G}}(X)$ is an isomorphism. But this will certainly not be the case in most of our examples.

(iii) Let $f : X \to Y$ be a $\mathcal{G}$-equivariant map of schemes over B. We have a monoidal pullback map

$$f^* : \mathrm{Perf}_{\mathcal{G}}(Y) \to \mathrm{Perf}_{\mathcal{G}}(X)$$

hence the ring map $f^* : K_{\mathcal{G}}(Y) \to K_{\mathcal{G}}(X)$. If the derived functor $f^* : D^b_{\mathcal{G}}\,\mathrm{QCoh}(Y) \to D^b_{\mathcal{G}}\,\mathrm{QCoh}(X)$ sends $D^b_{\mathcal{G}}\,\mathrm{Coh}(Y)$ to $D^b_{\mathcal{G}}\,\mathrm{Coh}(X)$ (e.g. if f may be flat or is a regular closed embedding), then we also get a map

$$f^* : K^{\mathcal{G}}(Y) \to K^{\mathcal{G}}(X)$$

of $K_{\mathcal{G}}(Y)$-modules.

(iv) If instead the derived functor $f_* : D^+_{\mathcal{G}} \operatorname{Sh}(X) \to D^+_{\mathcal{G}} \operatorname{Sh}(Y)$ sends $D^b_{\mathcal{G}} \operatorname{Coh}(X)$ to $D^b_{\mathcal{G}} \operatorname{Coh}(Y)$ (e.g., if f is proper and Y is of finite type), then we have a map

$$f_* : K^{\mathcal{G}}(X) \to K^{\mathcal{G}}(Y)$$

of $K_{\mathcal{G}}(Y)$-modules.

(v) There is also a version of specialization in equivariant K-homology, due to [VV03]. Let $f : X \to B \times \mathbb{G}_a$ be a $\mathcal{G}$-equivariant map, where the factor $\mathbb{G}_a$ is a 'multiplicity space' ignored by the action of $\mathcal{G}$. Let $i : X_0 \to X$ denote the inclusion of the fiber of $B \times \{0\}$, and $j : X^o \to X$ denote the inclusion of the complement. Assume that i is a regular embedding. Then the map $i^* i_*$ on K-homology vanishes, and so we get a map

$$K^{\mathcal{G}}(X)/i_* K^{\mathcal{G}}(X_0) \to K^{\mathcal{G}}(X_0).$$

Note that the restriction map $j^* : K^{\mathcal{G}}(X) \to K^{\mathcal{G}}(X^o)$ has kernel $i_* K^{\mathcal{G}}(X_0)$, so we get an injection $K^{\mathcal{G}}(X)/i_* K^{\mathcal{G}}(X_0) \to K^{\mathcal{G}}(X^o)$. Assume that this injection is also a surjection (e.g. if X is quasi-projective). Then we have obtained a map

$$s : K^{\mathcal{G}}(X^o) \to K^{\mathcal{G}}(X_0)$$

which is the promised *specialization* map.

(vi) There is also a version of restriction with supports. Suppose $f : X \to Y$ is a $\mathcal{G}$-equivariant regular closed embedding, and $g : Z \to Y$ is an arbitrary G-equivariant map. Then $f_* \mathcal{O}_X$ is isomorphic to an object of $\operatorname{Perf}_{\mathcal{G}}(Y)$, so that $g^* f_* \mathcal{O}_X$ is isomorphic to an object of $\operatorname{Perf}_{\mathcal{G}}(Z)$. Moreover, this perfect complex is set-theoretically supported on $W := X \times_Y Z$, that is, its restriction to the complement of W in Z is isomorphic to 0. Thus tensoring with $g^* f_* \mathcal{O}_X$ gives an exact functor:

$$(-) \otimes^{\mathbb{L}}_{\mathcal{O}_Z} g^* f_* \mathcal{O}_X : D^b_{\mathcal{G}} \operatorname{Coh}(Z) \to D^b_{\mathcal{G}} \operatorname{Coh}(Z)_W.$$

The right-hand side is the full subcategory of $D^b_{\mathcal{G}} \operatorname{Coh}(Z)$ consisting of complexes set-theoretically supported on W; the pushforward functor $D^b_{\mathcal{G}} \operatorname{Coh}(W) \to D^b_{\mathcal{G}} \operatorname{Coh}(Z)$ factors through this category. Since the embedding $W \to Z$ is f.p., being the base change of the f.p. embedding $X \to Y$, the resulting functor $D^b_{\mathcal{G}} \operatorname{Coh}(W) \to D^b_{\mathcal{G}} \operatorname{Coh}(Z)_W$ induces an isomorphism in K-homology:

$$K^{\mathcal{G}}(W) \xrightarrow{\sim} K_0(D^b_{\mathcal{G}} \operatorname{Coh}(Z)_W).$$

Thus, we have produced a map

$$K^{\mathcal{G}}(Z) \to K^{\mathcal{G}}(W)$$

which is the promised *restriction with supports*.

(vii) Change of groupoid base, restriction of equivariance, and averaging work exactly as for equivariant Borel–Moore homology; see § 3.10.

The same compatibilities which were used in the previous section to extend the analogous procedures in equivariant Borel–Moore homology to the case of ind-schemes hold just as well for equivariant K-homology. Thus we are able to define the K-theoretic versions of the Coulomb branch and the quantum Coulomb branch by using precisely the same underlying geometry: we have rings (under convolution)

$$KA := K^{G(\mathcal{O})}(\mathcal{R})$$

and

$$KA_q := K^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(\mathcal{R})$$

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which receive ring maps from respectively

$$K_{G(\mathcal{O})}(*) = K_G(*) = R(G)$$

and

$$K_{G(\mathcal{O})\rtimes\mathbb{C}^*}(*) = K_{G\times\mathbb{C}^*}(*) = R(G)[q, q^{-1}].$$

Here $R(G)$ is the (integral) representation ring of G. For a maximal torus T of G, we have $R(G) = \mathbb{Z}[\mathbb{X}^\bullet(T)]^W$ where W is the Weyl group. Moreover, the various compatibilities between specialization and the other procedures hold here as in the case of Borel–Moore homology (see § 3.10), so that the ring structure on KA may also be defined using specialization on the appropriate Beilinson–Drinfeld Grassmannian, and is commutative. Also, KA is free over $R(G)$, KA_q is free over $R(G)[q, q^{-1}]$ with respect to both left and right multiplication, q is in the center $Z(KA_q)$ of KA_q, and $KA = KA_1 := KA_q|_{q=1}$. We will show the following theorem.

Theorem 4.1. *Fix a positive integer n and a primitive nth root of unity ζ. Then there is an injective map of algebras*

$$KA \to Z(KA_\zeta).$$

Here $KA_\zeta := KA_q/\Phi_n(q)$ where Φ_n is the nth cyclotomic polynomial. Equivalently, since KA_q is free over $\mathbb{Z}[q, q^{-1}]$, this is the same as the subalgebra $1 \otimes KA_q$ of $\mathbb{C}_\zeta \otimes_{\mathbb{Z}[q,q^{-1}]} KA_q$, where $\mathbb{C}_\zeta$ is the $\mathbb{Z}[q, q^{-1}]$-algebra whose underlying ring is $\mathbb{C}$ and in which q acts as ζ.

The proof is essentially the same as for the quantum Coulomb branch, except for the following observation.

Remark 4.2. We must generalize from $\mathcal{R}^{\mu_p}$, $\mathcal{R}^{(p)}_{(p)}$ etc. to $\mathcal{R}^{\mu_n}$, $\mathcal{R}^{(n)}_{(n)}$, which are defined exactly as before by replacing any instance of p with n. Indeed, we never used that p was prime in any of our previous constructions, nor in any of our proofs *except for questions of linearity*. So, for instance, it is true that we have maps from the mod n rings

$$F_{\hbar;n} : A^* \to Z(A^*_\hbar)$$

which lift the nth power map $A^* \to A^*$; to linearize these maps, we have to kill all non-unit factors of n. If n is not a prime power, this means we have to kill everything, so we do not obtain an interesting linear map. If $n = p^d$ is a prime power, it amounts to killing p, and the resulting map $F_{\hbar;p^d}$ mod p is the composition of $F_{\hbar;p}$ with the $(d-1)$th power of the Frobenius endomorphism of A mod p, so gives nothing new. However, the map of Theorem 4.1 is a linear map between algebras free over $\mathbb{Z}$, and is something genuinely different for all q.

4.2 Adams operations

Let X, B, $\mathcal{G}$ be as in the previous subsection, and n be a positive integer. We have a monoidal (nonlinear) functor

$$St : D^b_{\mathcal{G}}Coh(X) \to D^b_{\mathcal{G}^{\mu_n}\rtimes\mu_n}Coh(X^{\mu_n})$$

whose composition with the functor $D^b_{\mathcal{G}^{\mu_n}\rtimes\mu_n}Coh(X^{\mu_n}) \to D^b_{\mathcal{G}^{\mu_n}}Coh(X^{\mu_n})$ which forgets the μ_n-equivariant structure coincides with the nth external tensor power functor. The construction is exactly the same as Steenrod's construction of § 2.2, except we work with coherent complexes rather than constructible ones (and with n rather than p). Proposition 2.4 also holds in this situation with the sole caveat that by 'is an induced map' we mean 'is a sum of maps induced from various proper subgroups of μ_n' (rather than only from the trivial subgroup). The analogous

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fact holds also for objects, so we get linear maps

$$Ad^n : K^{\mathcal{G}}(X) \to K^{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})/I$$

where I is the subgroup of $K^{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})$ spanned by all classes of $\mathcal{G}^{\mu_n} \rtimes \mu_n$-equivariant complexes induced from $\mathcal{G}^{\mu_n} \rtimes \Gamma$-equivariant complexes, for some proper subgroup Γ of μ_n. Furthermore, the functor St preserves perfectness: we have

$$St : \mathrm{Perf}_{\mathcal{G}}(X) \to \mathrm{Perf}_{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})$$

and thus linear maps

$$Ad_n : K_{\mathcal{G}}(X) \to K_{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})/J$$

where J is the subgroup of $K_{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})$ spanned by all classes of $\mathcal{G}^{\mu_n} \rtimes \mu_n$-equivariant complexes induced from $\mathcal{G}^{\mu_n} \rtimes \Gamma$-equivariant *perfect* complexes, for some proper subgroup Γ of μ_n. In fact, J is an ideal (by the projection formula), I is a J-stable submodule for the same reason, Ad_n is a map of rings, and Ad^n is a map of Ad_n-modules.

Remark 4.3 (True Adams operations). Induction commutes with restriction, so we have a ring map

$$K_{\mathcal{G}}(X) \xrightarrow{Ad_n} K_{\mathcal{G}^{\mu_n} \rtimes \mu_n}(X^{\mu_n})/J \xrightarrow{\Delta^*} K_{\mathcal{G} \times \mu_n}(X)/J' = K_{\mathcal{G}}(X)[q, q^{-1}]/\Phi_n(q).$$

Here J' is the subgroup of $K_{\mathcal{G} \times \mu_n}(X) = K_{\mathcal{G}}(X)[q, q^{-1}]/(q^n - 1)$ spanned by induced classes. This equals the ideal generated by the elements $\sum_{j=1}^{d} q^{nj/d}$ for all $d > 1$ dividing n; and the lowest common factor of these is $\Phi_n(q)$. By the splitting principle, the image of this map is contained in

$$K_{\mathcal{G}}(X) \subset K_{\mathcal{G}}(X)[q, q^{-1}]/\Phi_n(q).$$

The resulting ring endomorphism of $K_{\mathcal{G}}(X)$ is the nth Adams operation. The relevant example for us is with $\mathcal{G} = G(\mathcal{O})$, $X = *$. Fix a maximal torus T of G; then the nth Adams operation is identified with the ring map

$$\mathbb{Z}[\mathbb{X}^\bullet(T)]^W \to \mathbb{Z}[\mathbb{X}^\bullet(T)]^W$$

which sends a W-invariant sum $\sum_i \chi_i$ of characters χ_i to the W-invariant sum $\sum_i \chi_i^n$.

4.3 Proof of Theorem 4.1

The map in question is constructed as in (3.1), as the composition

$$\begin{array}{ccc}
KA = K^{G(\mathcal{O})}(\mathcal{R}) & \xrightarrow{\quad Ad^n \quad} & K^{G(\mathcal{O})^{\mu_n} \rtimes \mu_n}(\mathcal{R}^{\mu_n})/I \\
 & & \Big\downarrow \text{Descent isomorphism (b)} \\
K^{G^*_{(n)} \rtimes \mu_n}((\mathcal{R}^{(n)}_{(n)})^*)/I'' & \xleftarrow[\text{Restrict}]{\qquad} & K^{G^*_{(n)} \rtimes \mathbb{C}^*}((\mathcal{R}^{(n)}_{(n)})^*)/I' \\
\Big\downarrow \text{Specialize} & & \\
K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R})/I''' & &
\end{array} \tag{4.1}$$

where:

(i) I is the ideal of $K^{G(\mathcal{O})^{\mu_n} \rtimes \mu_n}(\mathcal{R}^{\mu_n})$ spanned by all classes induced from $K^{G(\mathcal{O})^{\mu_n} \rtimes \mu_{n/d}}(\mathcal{R}^{\mu_n})$, for some $d > 1$ dividing n.

(ii) I' is the ideal of $K^{G^*_{(n)} \rtimes \mathbb{C}^*}((\mathcal{R}^{(n)}_{(n)})^*)$ corresponding to I under the descent isomorphism. It is equal to the ideal spanned by all classes pushed forward from $K^{\mathbb{C} \times_{\mathbb{C}} G^*_{(n)} \rtimes \mathbb{C}^*}((\mathbb{C} \times_{\mathbb{C}} \mathcal{R}^{(n)}_{(n)})^*)$,

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for some non-trivial $\mathbb{C}^*$-equivariant cover $\mathbb{C} \to \mathbb{C}$. Recall that in this situation, the base copy of $\mathbb{C}$ has the action of $\mathbb{C}^*$ of weight n, so a $\mathbb{C}^*$-equivariant non-trivial cover $\mathbb{C} \to \mathbb{C}$ is the dth-power map for some $d > 1$ dividing n.

(iii) I'' is the image of I' under restriction. It is the ideal of $K^{G^*_{(n)} \rtimes \mu_n}((\mathcal{R}^{(n)}_{(n)})^*)$ spanned by all classes pushed forward from

$$K^{\mathbb{C} \times_{\mathbb{C}} G^*_{(n)} \rtimes \mu_n}((\mathbb{C} \times_{\mathbb{C}} \mathcal{R}^{(n)}_{(n)})^*)$$

for some non-trivial $\mathbb{C}^*$-equivariant cover $\mathbb{C} \to \mathbb{C}$. Note that for the degree d equivariant cover we have $K^{\mathbb{C} \times_{\mathbb{C}} G^*_{(n)} \rtimes \mu_n}((\mathbb{C} \times_{\mathbb{C}} \mathcal{R}^{(n)}_{(n)})^*) = K^{G^*_{(n)} \rtimes \mu_{n/d}}((\mathcal{R}^{(n)}_{(n)})^*)$, and pushing forward along $\mathbb{C} \to \mathbb{C}$ corresponds to inducing from $\mu_{n/d}$-equivariance to μ_n-equivariance.

(iv) I''' is the image of I'' under specialization. Using the second description of I'' given above, we see that this is the ideal of $K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R})$ spanned by all classes induced from $K^{G(\mathcal{O}) \rtimes \mu_{n/d}}(\mathcal{R})$ for some $d > 1$ dividing n.

Now on the one hand, by the projection formula we see that the composition

$$K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R}) \xrightarrow{\text{restriction}} K^{G(\mathcal{O}) \rtimes \mu_{n/d}}(\mathcal{R}) \xrightarrow{\text{induction}} K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R})$$

coincides with multiplication by the class $\sum_{j=1}^{d} q^{nj/d}$. On the other hand, it is a consequence of the 'cellularity' of $\mathcal{R}$ that the restriction of equivariance $K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R}) \to K^{G(\mathcal{O}) \rtimes \mu_{n/d}}(\mathcal{R})$ is surjective; so the ideal I''' coincides with the ideal generated by the sums $\sum_{j=1}^{d} q^{nj/d}$, whose lowest common factor is $\Phi_n(q)$. For the same reason, the restriction

$$K^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(\mathcal{R}) \to K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R})$$

is also surjective, and realizes $K^{G(\mathcal{O}) \rtimes \mu_n}(\mathcal{R}) \cong K^{G(\mathcal{O}) \rtimes \mathbb{C}^*}(\mathcal{R})/(q^n - 1)$. Therefore we have produced the map

$$KA \to KA_q/\Phi_n(q) = KA_\zeta.$$

This map is linear by construction. The proof that it is a map of Ad_n-algebras, and lands in the center, is just as for the Borel–Moore homology.

Example 4.4. We return to the situation of the example in § 3.16, Example (iii). We have $R(G) = \mathbb{Z}[y, y^{-1}]$ and, for $\mathbf{N} = 0$, KA_q is the $\mathbb{Z}[y, y^{-1}, q, q^{-1}]$-algebra

$$\mathcal{O}(T^{\vee\vee}) \#_q \mathcal{O}(T^\vee) := \mathbb{Z}[y, y^{-1}, q, q^{-1}]\langle x, x^{-1} \rangle / (yx = qxy).$$

For $\mathbf{N} = \mathbb{C}_{-r}$, $r \geq 0$, KA_q is given instead by the subalgebra with basis

$$\ldots, x^{-2}, x^{-1}, 1, \left(\prod_{i=0}^{r-1} (1 - y^r/q^i) \right) x, \left(\prod_{i=0}^{2r-1} (1 - y^r/q^i) \right) x^2, \ldots$$

as a left (or right) $\mathbb{Z}[y, y^{-1}, q, q^{-1}]$-module. The map of Theorem 4.1 sends

$$\begin{aligned} x &\mapsto x^n, \\ y &\mapsto y^n, \end{aligned}$$

which are central when $q^n = 1$, in particular when $q = \zeta$ is a primitive nth root of unity. It sends $(1 - y^r)^r x$ to

$$(1 - y^{nr})^r x^n = \left(\prod_{i=0}^{n-1} (1 - y^r/\zeta^i)^r \right) x^n = \left(\prod_{i=0}^{nr-1} (1 - y^r/\zeta^i) \right) x^n,$$

which is indeed an element of the appropriate subalgebra.

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5. The Frobenius twist

In this short section we use the previous considerations to give a geometric construction of the Frobenius twist functor. First recall the notation of § 3.12: we have an ind-projective flat complex ind-scheme

$$Gr_{(p)}$$

over $\mathbb{A}^1$, which is the moduli space of triples $(y, \mathcal{E}, f)$ where $y \in \mathbb{A}^1$, $\mathcal{E}$ is a principal G-bundle on the formal neighborhood of the set $\pi^{-1}\{y\}$ of pth roots of y, and f is a trivialization of $\mathcal{E}$ away from $\pi^{-1}\{y\}$. There is a pro-smooth algebraic group

$$G_{(p)}$$

over $\mathbb{A}^1$, which is the moduli space of pairs (y, g) where $y \in \mathbb{A}^1$ and g is a map from the formal neighborhood on $\pi^{-1}\{y\}$ to G, and $G_{(p)}$ acts on $Gr_{(p)}$ by changing the trivialization f. The group $\mathbb{C}^*$ acts on $Gr_{(p)}$ by

$$z.(y, \mathcal{E}, f) = (z^p y, z_*\mathcal{E}, z_* f)$$

and on $G_{(p)}$ (as a group) by

$$z.(y, g) = (z^p y, z_* g).$$

Thus we have an action of the pro-smooth algebraic groupoid $G_{(p)} \rtimes \mathbb{C}^*$ over $\mathbb{C}^*$ on $Gr_{(p)}$. Now we have an equivalence

$$D^b_{G_{(p)} \rtimes \mathbb{C}^*}(Gr_{(p)}|_{\mathbb{A}^1 - \{0\}}, \mathbb{F}_p) \cong D^b_{G(\mathcal{O})^{\mu_p} \rtimes \mu_p}(Gr^{\mu_p}, \mathbb{F}_p)$$

where by the bounded derived category of an ind-scheme we mean simply the 2-colimit of the bounded derived categories of its closed subschemes. Thus by applying the Steenrod functor St_D we have a functor

$$St'_D : D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to D^b_{G_{(p)} \rtimes \mathbb{C}^*}(Gr_{(p)}|_{\mathbb{A}^1 - \{0\}}, \mathbb{F}_p).$$

Now note that St'_D is exact for the perverse t-structure, up to a homological shift of degree 1. Thus we consider the shifted functor

$$F : Perv_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to Perv_{G_{(p)} \rtimes \mathbb{C}^*}(Gr_{(p)}|_{\mathbb{A}^1 - \{0\}}, \mathbb{F}_p).$$

We will compose this functor with the IC functor, and so get

$$IC \circ F : Perv_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to Perv_{G_{(p)} \rtimes \mathbb{C}^*}(Gr_{(p)}, \mathbb{F}_p).$$

Now let $i : Gr \to Gr_{(p)}$ denote the embedding of the fiber at 0. There is a 'Thom homomorphism'

$$Th : i^*[-1] \to i^![1],$$

so called because in the case where i is the section of a smooth fibration, Th is an isomorphism on the constant sheaf, and the consequence in cohomology is the Thom isomorphism theorem. But $i^*[-1]$, $i^![1]$ send IC sheaves to perverse sheaves, and so we may take the image of Th.

Theorem 5.1. *The functor*

$$im(Th) \circ IC \circ F : Perv_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to Perv_{G(\mathcal{O}) \rtimes \mathbb{C}^*}(Gr, \mathbb{F}_p)$$

corresponds under geometric Satake to the Frobenius twist. In particular, it is linear in spite of the nonlinearity of F.

Proof. Since we are working with perverse sheaves, we are free to work non-equivariantly, and we will do so for the entire proof; in particular, the functor St'_D secretly means the composition

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of usual St'_D with the functor which forgets the equivariance. Let us put ourselves in the general situation of a variety X with a proper map $f : X \to \mathbb{A}^1$. We recall some general facts; see [SGA7, Beĭ87] for detailed discussions.

(i) The Thom homomorphism factors as

$$i^*[-1] \to \psi \to i^![1].$$

Here $\psi = \psi_f$ is the nearby cycles functor, the map $i^*[-1] \to \psi$ is the adjunction map, and the map $\psi \to i^![1]$ comes from the defining map by self-duality of ψ. Recall that ψ is perverse t-exact, and that it is equipped with a monodromy automorphism $T = T_f$. Now let $\mathcal{F}$ be a perverse sheaf on $X|_{\mathbb{A}^1-\{0\}}$. Then $i^*IC(\mathcal{F})[-1]$ and $i^!IC(\mathcal{F})[1]$ are also perverse, and indeed we have

$$i^*IC(\mathcal{F})[-1] \cong \psi(\mathcal{F})^T,$$
$$i^!IC(\mathcal{F})[1] \cong \psi(\mathcal{F})_T,$$

and the Thom homomorphism is simply the natural map.

(ii) Let $\pi : \mathbb{A}^1 \to \mathbb{A}^1$ be the pth power map. Then we consider $\pi^*X := \mathbb{A}^1 \times_{\mathbb{A}^1} X$ as mapping to the copy of $\mathbb{A}^1$ which is the source of π, via the morphism π^*f. Then we have a canonical isomorphism

$$(\psi_{\pi^*f} \circ \pi^*, T_{\pi^*f}) \cong (\psi_f, T_f^p).$$

The same is true with p replaced by an arbitrary integer.

Now let us write S for the geometric Satake equivalence. Then it suffices to identify $\psi \circ St'_D \circ S(V)$ with $S(V^{\otimes p})$, and T with its usual cyclic automorphism of order p. The identification of $\psi \circ St'_D \circ S(V)$ with $S(V^{\otimes p})$ is essentially by definition after applying point (ii) above. Finally, let H denote the global (non-equivariant) cohomology functor $Perv_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to Vect$. By the geometric Satake equivalence, it suffices to show that the action of T on $H(\psi \circ St'_D \circ S(V)) \cong V^{\otimes p}$ is by the usual cyclic automorphism of order p. Since nearby cycles (and associated monodromy) commute with proper pushforward, it suffices to show it for

$$\psi \circ f_* \circ St'_D \circ S(V)$$

where $f : Gr_{(p)}|_{\mathbb{A}^1-\{0\}} \to \mathbb{A}^1 - \{0\}$ is the structure map. But f_* commutes with St'_D, so we need to show it for

$$\psi \circ St'_D \circ f_* \circ S(V) = \psi \circ St'_D \circ V.$$

The result now follows from the definition of St'_D and the (tautological) fact that for a local system $\mathcal{L}$ on $\mathbb{C}^*$ with fiber (say, at 1) V and monodromy T, we have

$$(\psi_{\mathrm{id}}\mathcal{L}, T_{\mathrm{id}}) \cong (V, T).$$

The proof is complete. □

Remark 5.2. In fact, it is possible to show that the composition of St'_D with

$$\psi' : D^b_{G_{(p)} \rtimes \mathbb{C}^*}(Gr_{(p)}|_{\mathbb{A}^1-\{0\}}, \mathbb{F}_p) \to D^b_{G_{(p)} \rtimes \mu_p}(Gr_{(p)}|_{\mathbb{A}^1-\{0\}}, \mathbb{F}_p) \xrightarrow{\psi} D^b_{G(\mathcal{O}) \rtimes \mu_p}(Gr, \mathbb{F}_p)$$

may be equipped with a central structure for the convolution monoidal structures. That is, we can give $\psi' \circ St'_D$ a monoidal structure, and can give an isomorphism

$$(\psi' \circ St'_D) * \mathrm{id} \cong (\mathrm{id} * (\psi' \circ St'_D)) \circ (twist)$$

as functors

$$D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \boxtimes D^b_{G(\mathcal{O}) \rtimes \mu_p}(Gr, \mathbb{F}_p) \to D^b_{G(\mathcal{O}) \rtimes \mu_p}(Gr, \mathbb{F}_p)$$

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which is compatible with this monoidal structure. It will also be compatible with the commutativity constraint for the convolution monoidal structure on $D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p)$ constructed in [MV07] (using similar considerations).

This functor has the advantage of making sense on the entire derived category, not just for perverse sheaves. However, it has the disadvantages of being nonlinear and involving a loss of equivariance from $\mathbb{C}^*$ to μ_p. The former disadvantage can be forcibly removed by inverting $\hbar$; we are left with a central monoidal triangulated functor

$$\psi' \circ St'_D[\hbar^{-1}] : D^b_{G(\mathcal{O})}(Gr, \mathbb{F}_p) \to D^b_{G(\mathcal{O}) \rtimes \mu_p}(Gr, \mathbb{F}_p)[\hbar^{-1}]$$

which is compatible with the Artin–Schreier map on the level of equivariant parameters. It also corresponds to the Frobenius twist functor for perverse sheaves, in the sense that when applied to a perverse sheaf, it yields the image of our functor $im(Th) \circ IC \circ F$ under the operation of demoting equivariance from $\mathbb{C}^*$ to μ_p and inverting $\hbar$. Indeed, in our analysis of $im(Th) \circ IC \circ F$ we saw that it differs from $\psi' \circ St'_D$ only by μ_p-induced objects, which are killed off by inverting $\hbar$.

In Conjecture 1.6, we attempt to describe this functor in terms of representations of the Langlands dual group and the p-center of its universal enveloping algebra. Our conjecture is predicated on a characteristic p version of the derived geometric Satake theorem of [BF08], which at the time of writing is unproven. Therefore we do not prove it here. Additionally, we foresee a complication arising from the loss of equivariance from $\mathbb{C}^*$ to μ_p. We hope very much that it will prove possible to remove this, but do not yet know how to go about it.

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References

Beĭ87 A. A. Beĭlinson, *How to glue perverse sheaves*, in *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, Lecture Notes in Mathematics, vol. 1289 (Springer, Berlin, 1987), 42–51.

BD91 A. Beĭlinson and V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, Preprint (1991).

BL06 J. Bernstein and V. Lunts, *Equivariant sheaves and functors* (Springer, New York, 2006).

BF08 R. Bezrukavnikov and M. Finkelberg, *Equivariant Satake category and Kostant-Whittaker reduction*, Mosc. Math. J. **8** (2008), 39–72.

BK08 R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in positive characteristic*, J. Amer. Math. Soc. **21** (2008), 409–438.

BEF20 A. Braverman, P. Etingof and M. Finkelberg, *Cyclotomic double affine Hecke algebras (with an appendix by Hiraku Nakajima and Daisuke Yamakawa)*, Ann. Sci. Éc. Norm. Supér. (4) **53** (2020), 1249–1312.

BFN16 A. Braverman, M. Finkelberg and H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories, II*, Adv. Theor. Math. Phys. **22** (2016), 1071–1147.

BFN17 A. Braverman, M. Finkelberg and H. Nakajima, *Ring objects in the equivariant derived Satake category arising from Coulomb branches (with an appendix by Gus Lonergan)*, Adv. Theor. Math. Phys. **23** (2017), 253–344.

CE99 H. Cartan and S. Eilenberg, *Homological algebra*, vol. 41 (Princeton University Press, Princeton, NJ, 1999).

CG97 N. Chriss and V. Ginzburg, *Representation theory and complex geometry* (Birkhäuser, Boston, MA, 1997).

DBB83 P. Deligne, A. A. Beĭlinson and J. Bernstein, *Faisceaux pervers*, Astérisque **100** (1983).

Dri06 V. Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry*, in *The unity of mathematics* (Springer, 2006), 263–304.

DL62 E. Dyer and R. K. Lashof, *Homology of iterated loop spaces*, Amer. J. Math. **84** (1962), 35–88.

KA56 T. Kudo and S. Araki, *Topology of H_n-spaces and H-squaring operations*, Mem. Fac. Sci. Kyushu Univ. Ser. A **10** (1956), 85–120.

MV07 I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), 95–143.

Ras15 S. Raskin, D-modules on infinite dimensional varieties, Preprint (2015).

SE62 N. E. Steenrod and D. B. A. Epstein, *Cohomology operations*: *Lectures by N. E. Steenrod. Written and revised by D. B. A. Epstein*, Annals of Mathematics Studies, vol. 50 (Princeton University Press, Princeton, NJ, 1962).

SGA7 A. Grothendieck, M. Raynaud and D. S. Rim, *Séminaire de Géométrie Algébrique du Bois Marie – 1967–69 – Groupes de monodromie en géométrie algébrique (SGA 7), vol. 1*, Lecture Notes in Mathematics, vol. 288 (Springer, 1972).

VV03 G. Vezzosi and A. Vistoli, *Higher algebraic K-theory for actions of diagonalizable groups*, Invent. Math. **153** (2003), 1–44.

Web19 B. Webster, *Representation theory of the cyclotomic Cherednik algebra via the Dunkl-Opdam subalgebra*, New York J. Math. **25** (2019), 1017–1047.

Gus Lonergan guslonergan@gmail.com
121 Glen Ridge Avenue, Glen Ridge, NJ 07028

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