AN INTRODUCTION TO THE SCOTT COMPLEXITY OF COUNTABLE STRUCTURES AND A SURVEY OF RECENT RESULTS

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Abstract. Every countable structure has a sentence of the infinitary logic $\mathcal{L}_{\omega_1\omega}$ which characterizes that structure up to isomorphism among countable structures. Such a sentence is called a Scott sentence, and can be thought of as a description of the structure. The least complexity of a Scott sentence for a structure can be thought of as a measurement of the complexity of describing the structure. We begin with an introduction to the area, with short and simple proofs where possible, followed by a survey of recent advances.

§1. Introduction. A central feature of elementary first-order logic is the compactness theorem, which has as a consequence the Ryll–Nardzewski theorem that any countably-categorical structure is relatively simple: for each n, there are only finitely many automorphism orbits of n-tuples. So most structures are not countably categorical, which means that no matter which elementary first-order sentences we write down, we cannot completely characterize the structure up to isomorphism.

If we strengthen our language, Scott [62] showed that we can characterize every countable structure up to isomorphism. The required strengthening of the language allows countably infinite conjunctions and disjunctions, so that for example we can express that an abelian group is torsion by saying that for each element x, either $x = 0$, or $x + x = 0$, or $x + x + x = 0$, and so on. More formally, we use the logic $\mathcal{L}_{\omega_1\omega}$ which is defined as follows:

DEFINITION 1.1. Let τ be a vocabulary. The formulas of the logic $\mathcal{L}_{\omega_1\omega}(\tau)$ are built inductively as follows:

- terms and atomic formulas are defined as in elementary first-order logic;
- if φ is a formula, so is $\neg\varphi$;
- if $\bar{x}$ is a finite tuple of variables and X is a countable set of formulas such that for each $\varphi \in X$, the free variables of φ are contained within $\bar{x}$, then $\bigwedge_{\varphi \in X} \varphi$ and $\bigvee_{\varphi \in X} \varphi$ are formulas; and
- if φ is a formula, then so are $\forall x \varphi$ and $\exists x \varphi$.

$\mathcal{L}_{\omega_1\omega}$ does not satisfy compactness, but it does satisfy downward (though not upward) Löwenheim–Skolem for countable fragments. The book by Marker [45] is an excellent reference on infinitary model theory.

Now that we have strengthened our language, it is possible to show that each structure is characterized up to isomorphism among countable structures by a sentence of this logic [62]. We call this sentence a *Scott sentence*.

DEFINITION 1.2. Let $\mathcal{A}$ be a countable structure. A *Scott sentence* for $\mathcal{A}$ is a sentence φ of $\mathcal{L}_{\omega_1\omega}$ such that, up to isomorphism, $\mathcal{A}$ is the only countably model of φ.

The standard proof that every countable structure has a Scott sentence passes through the back-and-forth relations, showing that they must stabilize at some countable ordinal, and then extracting from this a Scott sentence. This *Scott analysis* of a structure has played an important role in the study of Vaught's conjecture, e.g., in Morley's theorem [55] that the number of non-isomorphic countable models of a theory is either at most $\aleph_1$ or is exactly $2^{\aleph_0}$. There are excellent references for these results on Vaught's conjecture, e.g., in Marker's book [45], and so we will only touch on them briefly.

Fixing a countable structure $\mathcal{A}$, there is some countable ordinal at which the back-and-forth relations stabilize. This introduces a way of assigning an ordinal to $\mathcal{A}$, called its *Scott rank*. There are several slightly different variations of what it means to stabilize, leading to several different definitions of Scott rank. Equivalently the Scott rank can be thought of as measuring both the complexity of the automorphism orbits of the structure and the complexity of describing the structure.

Another way of measuring the complexity of $\mathcal{A}$ is the least complexity of a Scott sentence for $\mathcal{A}$. To measure the complexity of $\mathcal{L}_{\omega_1\omega}$ formulas, we use the natural hierarchy of Σ_α and Π_α sentences in normal form.

- An $\mathcal{L}_{\omega_1\omega}$ formula is both Σ_0 and Π_0 if it is quantifier free and does not contain any infinite disjunctions or conjunctions.
- An $\mathcal{L}_{\omega_1\omega}$ formula is Σ_α if it is a countable disjunction of formulas of the form $\exists x \phi$ where each ϕ is Π_β for some $\beta < \alpha$.
- An $\mathcal{L}_{\omega_1\omega}$ formula is Π_α if it is a countable conjunction of formulas of the form $\forall x \phi$ where each ϕ is Σ_β for some $\beta < \alpha$.

This approach led Montalbán [52] to define a new notion of Scott rank as the least ordinal α for which $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence. He proved that this is robust in the sense that there are a number of different equivalent characterizations, including the complexity of defining automorphism orbits, and the complexity of computing isomorphisms between different copies of $\mathcal{A}$.

An even finer measurement, the *Scott complexity* of $\mathcal{A}$, is the least complexity of a Scott sentence for $\mathcal{A}$. Alvir et al. [1] argued by means of the Wadge degrees that the only (reasonable) possible complexities are Σ_α, Π_α, and $d-\Sigma_\alpha$.

This article will begin with a more detailed look at the above background material. Following this, we will touch on more specialized areas. We begin with effective considerations and structures of high Scott rank. Here we give new elementary proofs, avoiding Barwise compactness and Σ_1^1 bounding,

which we think are simpler than many of those in the literature. Next, we look at results on the Scott complexity of particular structures of interest in mathematics, e.g., free groups. Finally, we discuss theories, including a brief summary of connections with Vaught's conjecture.

We also note that there is work applying the Scott analysis to separable metric spaces, which we will not cover in this survey. See [15, 16, 19, 47, 67]; note that [19] has an erratum [20].

§2. Scott sentences and Scott rank. Our first step is to see why every structure has a Scott sentence. The standard way to see this is to analyse the back-and-forth relations. There are a number of different ways to define the back-and-forth relations, but from a computability-theoretic perspective the most natural is the following due to its connections with Σ_α and Π_α sentences.

DEFINITION 2.1. The *standard asymmetric back-and-forth relations* $\leq_\alpha$ on $\mathcal{A}$, for $\alpha < \omega_1$, are defined by:

1. $\bar{a} \leq_0 \bar{b}$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas from among the first $|\bar{a}|$-many formulas.
2. For $\alpha > 0$, $\bar{a} \leq_\alpha \bar{b}$ if for each $\beta < \alpha$ and $\bar{d}$ there is $\bar{c}$ such that $\bar{b}\bar{d} \leq_\beta \bar{a}\bar{c}$.

We define $\bar{a} \equiv_\alpha \bar{b}$ if $\bar{a} \leq_\alpha \bar{b}$ and $\bar{b} \leq_\alpha \bar{a}$. We also define $\bar{a} \equiv_\infty \bar{b}$ if $\bar{a} \equiv_\alpha \bar{b}$ for all $\alpha < \omega_1$.

It is not hard to prove by induction that for $\alpha \geq 1$, $\bar{a} \leq_\alpha \bar{b}$ if and only if every Σ_α sentence true of $\bar{b}$ is true of $\bar{a}$. Similarly, $\bar{a} \equiv_\infty \bar{b}$ if and only if $\bar{a}$ and $\bar{b}$ satisfy the same $\mathcal{L}_{\omega_1\omega}$ formulas.

For each pair of tuples $\bar{a}$ and $\bar{b}$ such that $\bar{a} \not\equiv_\infty \bar{b}$, choose a formula $\varphi_{\bar{a};\bar{b}}(\bar{x})$ such that $\mathcal{A} \models \varphi_{\bar{a};\bar{b}}(\bar{a})$ but $\mathcal{A} \models \neg\varphi_{\bar{a};\bar{b}}(\bar{b})$. Then define $\psi_{\bar{a}}(\bar{x}) := \bigwedge_{\bar{b} \not\equiv_\infty \bar{a}} \varphi_{\bar{a};\bar{b}}(\bar{x})$. So we have that $\mathcal{A} \models \psi_{\bar{a}}(\bar{b})$ if and only if $\bar{a} \equiv_\infty \bar{b}$. We may also add as a conjunct to $\psi_{\bar{a}}(\bar{x})$ all of the atomic formulas true of $\bar{a}$.

Now define:

$$\rho_\varnothing := \forall x \bigvee_{b \in \mathcal{A}} \psi_b(x)$$

$$\rho_{\bar{a}} := \forall \bar{u} \left(\psi_{\bar{a}}(\bar{u}) \longrightarrow \forall x \bigvee_{b \in \mathcal{A}} \psi_{\bar{a}b}(\bar{u}, x) \right)$$

$$\sigma_\varnothing := \bigwedge_{b \in \mathcal{A}} \exists x \psi_b(x)$$

$$\sigma_{\bar{a}} := \forall \bar{u} \left(\psi_{\bar{a}}(\bar{u}) \longrightarrow \bigwedge_{b \in \mathcal{A}} \exists x \psi_{\bar{a}b}(\bar{u}, x) \right),$$

and

$$\theta := \bigwedge_{\bar{a} \in \mathcal{A}} \rho_{\bar{a}} \wedge \sigma_{\bar{a}}.$$

It is easy to argue by a standard back-and-forth argument that θ is a Scott sentence for $\mathcal{A}$. We call θ the *canonical Scott sentence* for $\mathcal{A}$.

THEOREM 2.2 (Scott [62]). *Every countable structure has a Scott sentence.*

PROOF. It is clear that $\mathcal{A} \models \theta$. Suppose that $\mathcal{B}$ is countable and $\mathcal{B} \models \theta$. Suppose that we have a finite partial map $\mathcal{A} \to \mathcal{B}$ mapping $\bar{a}$ to $\bar{b}$ with $\psi_{\bar{a}}(\bar{b})$. Then for such $\bar{a}$ and $\bar{b}$:

- $\bar{a}$ and $\bar{b}$ have the same atomic type;
- given $a' \in \mathcal{A}$, since $\mathcal{B} \models \sigma_{\bar{a}} \wedge \psi_{\bar{a}}(\bar{b})$, there is $b' \in \mathcal{B}$ with $\mathcal{B} \models \psi_{\bar{a}a'}(\bar{b}b')$; and
- given $b' \in \mathcal{B}$, since $\mathcal{B} \models \rho_{\bar{a}} \wedge \psi_{\bar{a}}(\bar{b})$, there is $a' \in \mathcal{A}$ with $\mathcal{B} \models \psi_{\bar{a}a'}(\bar{b}b')$;

So by a back-and-forth argument we can extend our finite partial isomorphism to an isomorphism $\mathcal{A} \to \mathcal{B}$. Thus θ is a Scott sentence for $\mathcal{A}$. ⊣

As a corollary, by adding constants to the language, we get:

COROLLARY 2.3. *Let $\mathcal{A}$ be a countable structure and $\bar{a}, \bar{b} \in \mathcal{A}$. Then there is an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$ if and only if $\bar{a} \equiv_\infty \bar{b}$.*

There are only countably many tuples $\bar{a}$ in $\mathcal{A}$, so there is a countable ordinal α such, that for $\bar{a}, \bar{b} \in \mathcal{A}$, if $\bar{a} \not\equiv_\infty \bar{b}$ then $\bar{a} \not\leq_\alpha \bar{b}$. We may choose the formulas $\varphi_{\bar{a},\bar{b}}$ to be Π_α, so that $\psi_{\bar{a}}$ are also Π_α. Then the formula θ is $\Pi_{\alpha+2}$. So there is a natural notion of rank that we can assign to $\mathcal{A}$ based on the least such ordinal α. Ash and Knight [4] prefer the following ranks:

DEFINITION 2.4. For $\bar{a} \in \mathcal{A}$, define $r(\bar{a})$ as the least ordinal α such that for all $\bar{b}$, if $\bar{a} \leq_\alpha \bar{b}$ then $\bar{a} \equiv_\infty \bar{b}$. Then define:

- $r(\mathcal{A}) = \sup\{r(\bar{a}) : \bar{a} \in \mathcal{A}\}$ and
- $R(\mathcal{A}) = \sup\{r(\bar{a}) + 1 : \bar{a} \in \mathcal{A}\}$.

If λ is a limit ordinal, it is possible for two structures $\mathcal{A}$ and $\mathcal{B}$ to have $r(\mathcal{A}) = r(\mathcal{B}) = \lambda$, but $R(\mathcal{A}) = \lambda$ and $R(\mathcal{B}) = \lambda + 1$.

Scott's original proof of Theorem 2.2 used symmetric back-and-forth relations $\sim_\alpha$ instead of the asymmetric ones defined above. In the definition below we extend only by single elements c and d rather than by tuples to show another possible variation.

DEFINITION 2.5. The *standard symmetric back-and-forth relations* $\sim_\alpha$ on $\mathcal{A}$, for $\alpha < \omega_1$, are defined by:

1. $\bar{a} \sim_0 \bar{b}$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas.
2. For $\alpha > 0$, $\bar{a} \sim_\alpha \bar{b}$ if for each $\beta < \alpha$ and d there is c such that $\bar{a}c \sim_\beta \bar{b}d$, and for all c there is d such that $\bar{a}c \sim_\beta \bar{b}d$.

Again, one can argue that there is a countable ordinal α such that if $\bar{a} \sim_\alpha \bar{b}$, then $\bar{a} \sim_\beta \bar{b}$ for all β. We can define a rank based on these back-and-forth relations.

DEFINITION 2.6. For $\bar{a} \in \mathcal{A}$, define $\mathrm{sr}(\bar{a})$ as the least ordinal α such that for all $\bar{b}$, if $\bar{a} \sim_\alpha \bar{b}$ then $\bar{a} \equiv_\infty \bar{b}$. Then define:

- $\mathrm{sr}(\mathcal{A}) = \sup\{\mathrm{sr}(\bar{a}) : \bar{a} \in \mathcal{A}\}$ and
- $\mathrm{SR}(\mathcal{A}) = \sup\{\mathrm{sr}(\bar{a}) + 1 : \bar{a} \in \mathcal{A}\}$.

This is what Marker [45] uses as the definition of Scott rank. It is from this connection with Scott's original proof of Theorem 2.2 that the name *Scott rank* is given to these notions of rank, though it is unclear which rank should hold the title. In general, we prefer the ranks R and SR over the ranks r and sr, as they contain more information. These ranks R and SR agree at multiples of ω^2.

There are two other possible variations on the back-and-forth relations: defining the symmetric relations $\sim_\alpha$ to extend by tuples (which is a common definition) and defining the asymmetric relations $\leq_\alpha$ to extend by single elements (which is rarely seen). Each of these definitions also gives rise to a notion of rank.

Another common notion of rank, used, e.g., by Sacks [61] and in Marker's book on model theory [44], is as follows.

DEFINITION 2.7. Let $\mathcal{A}$ be a countable structure. We define fragments $\mathcal{L}_\alpha^{\mathcal{A}}$ of $\mathcal{L}_{\omega_1\omega}$ and $\mathcal{L}_\alpha^{\mathcal{A}}$-theories $T_\alpha^{\mathcal{A}}$ as follows:

1. $\mathcal{L}_0^{\mathcal{A}}$ is finitary first-order logic;
2. $T_\alpha^{\mathcal{A}}$ is the complete theory of $\mathcal{A}$ in $\mathcal{L}_\alpha^{\mathcal{A}}$;
3. $\mathcal{L}_\lambda^{\mathcal{A}}$ is the union of $\mathcal{L}_\alpha^{\mathcal{A}}$ for $\alpha < \lambda$ a limit ordinal; and
4. $\mathcal{L}_{\alpha+1}^{\mathcal{A}}$ is the least fragment of $\mathcal{L}_{\omega_1\omega}$ containing $\mathcal{L}_\alpha^{\mathcal{A}}$ and also containing, for each non-principal n-type p of $T_\alpha^{\mathcal{A}}$, the conjunction $\bigwedge p$ of all the formulas in p.

Essentially, $T_{\alpha+1}^{\mathcal{A}}$ gives a name to all of the non-principal types of $T_\alpha^{\mathcal{A}}$, though this may then introduce new non-principal types. However, one can show that this process stabilizes and for some α, all of the types of $T_\alpha^{\mathcal{A}}$ realized in $\mathcal{A}$ are principal. One can then define the Scott rank of $\mathcal{A}$ to be the least α such that $\mathcal{A}$ is the atomic model of $T_\alpha^{\mathcal{A}}$. This rank agrees with the previous ranks at multiples of ω^2. It can be useful when one wants to apply ideas from finitary model theory.

Gao [23] also introduces a notion of Scott rank based on back-and-forth games, which is similar in spirit to the ranks defined using back-and-forth relations.

A final notion of Scott rank was introduced by Montalbán [52] based on the following list of equivalent properties:

THEOREM 2.8 (Montalbán [52]). *Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. The following are equivalent*:

1. *$\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.*
2. *Every automorphism orbit in $\mathcal{A}$ is Σ_α-definable without parameters.*
3. *$\mathcal{A}$ is uniformly (boldface) $\mathbf{\Delta}_\alpha^0$-categorical.*

Montalbán proposed the following definition of Scott rank:

DEFINITION 2.9. The *Scott rank* of a countable structure $\mathcal{A}$ is the least ordinal α such that $\mathcal{A}$ has a $\Pi_{\alpha+1}$ Scott sentence.

Montalbán argued that this is the correct notion of Scott rank because it is robust in the sense that it has many equivalent characterizations via Theorem 2.8. It is this definition that we use for most of the rest of this article, except when otherwise noted.

There is a clear connection between this notion of Scott rank proposed by Montalbán and the rank R of Definition 2.4 defined using the asymmetric back-and-forth relations $\leq_\alpha$. Given $\bar{a} \in \mathcal{A}$, the formula $\psi_{\bar{a}}$ defines the automorphism orbit of $\bar{a}$; this is because if $\bar{a} \equiv_\infty \bar{b}$, then the structures $(\mathcal{A}, \bar{a})$ and $(\mathcal{A}, \bar{b})$, with $\bar{a}$ and $\bar{b}$ named as constants, are isomorphic. With $\alpha = r(\bar{a})$, the automorphism orbit of $\bar{a}$ is defined by a Π_α formula. Thus the Scott rank as defined by Montalbán and the rank R differ by at most one.

§3. Scott complexity. There is some arbitrariness in choosing to define Scott rank using Π formulas rather than Σ formulas. Indeed, if we defined the Scott rank of a structure to be the least ordinal α such that the structure has a $\Sigma_{\alpha+2}$ Scott sentence, then we would have the desirable property that naming constants does not change Scott rank: $\mathcal{A}$ has the same Scott rank as $(\mathcal{A}, \bar{c})$ for any tuple $\bar{c} \in \mathcal{A}$. This follows from the following fact:

THEOREM 3.1 (Montalbán). *Let $\mathcal{A}$ be a countable structure. Then $\mathcal{A}$ has a $\Sigma_{\alpha+1}$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence.*

The proof of this fact is less obvious than it seems; indeed it was stated as an obvious fact without proof in [52] but a proof was first provided (for Σ_3 sentences) in [53]. A full proof appears in [1].

Moreover, defining Scott rank in this way is still robust; combining Theorems 2.8 and 3.1, we get:

THEOREM 3.2. *Let $\mathcal{A}$ be a countable structure, and α a countable ordinal. The following are equivalent*:

1. *$\mathcal{A}$ has a $\Sigma_{\alpha+2}$ Scott sentence.*
2. *There are parameters $\bar{c}$ such that every automorphism orbit in $\mathcal{A}$ is Σ_α-definable over $\bar{c}$.*
3. *$\mathcal{A}$ is (boldface) Δ^0_α-categorical.*

In [54], Montalbán refers to the Scott rank of Definition 2.9 as *parameterless Scott rank* and the Scott rank defined using Σ formulas as *parametrized Scott rank*, and uses the latter as the definition.

Rather than choosing between these two possibilities, one can give up on assigning a single ordinal rank, and rather say that the *Scott complexity* (or *Scott sentence complexity*) of a structure is the least complexity of a Scott sentence for that structure, e.g., if a structure has a Π_3 Scott sentence but no

Σ_3 Scott sentence, then its Scott complexity is Π_3. (This is not yet a formal definition, as we have not specified the possible complexities of sentences.)

Now these complexities are not totally ordered; Π_α and Σ_α are incomparable. So a natural question is: Can a structure have both a $\Sigma_{\alpha+1}$ Scott sentence and a $\Pi_{\alpha+1}$ Scott sentence, but no Σ_α or Π_α Scott sentence? The answer turns out to be yes—for example, the group $\mathbb{Z}$ has a Σ_3 and a Π_3 Scott sentence, but no Σ_2 or Π_2 Scott sentence—but the following theorem of A. Miller says that the structure must then have a Scott sentence which is a conjunction of a Σ_α sentence and a Π_α sentence. We call such a sentence a $d-\Sigma_\alpha$ sentence, where d stands for *difference*, because it is the difference of two Σ_α sentences.

THEOREM 3.3 (Miller [49]). *Let $\mathcal{A}$ be a countable structure. Then if $\mathcal{A}$ has both a Σ_α and a Π_α Scott sentence, it has a $d-\Sigma_\beta$ Scott sentence for some $\beta < \alpha$.*

So for example the group $\mathbb{Z}$ has a $d-\Sigma_2$ sentence but no Σ_2 or Π_2 sentence. Thus when we say that the Scott complexity of a structure is the least complexity of a Scott sentence for that structure, we have to include not just Σ_α and Π_α as complexities, but $d-\Sigma_2$. But among these complexities, there is always a single least complexity of a Scott sentence.
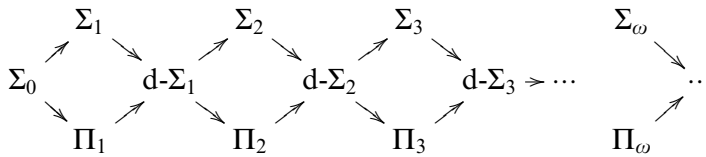
Are there further possible least complexities of Scott sentences? For all of the other complexity classes of sentences we can think of, any structure with a Scott sentence of that complexity also has a simpler Scott sentence. Let us give two examples. First, if a structure has a Scott sentence which is a negation of a $d-\Sigma_\alpha$ sentence, that is, of the form $\varphi \vee \psi$ where φ is Σ_α and ψ is Π_α, then one of either φ or ψ is a Scott sentence on its own, and so the structure has either a Σ_α or Π_α Scott sentence. Second, there are complexities $n-\Sigma_\alpha$ corresponding to the difference hierarchy. Suppose that a structure has a Scott sentence which is $3-\Sigma_\alpha$, that is, of the form $(\varphi \wedge \neg\psi) \vee \theta$ where these sentences are all Σ_α. Then either the $d-\Sigma_\alpha$ sentence $\varphi \wedge \neg\psi$ is a Scott sentence, or the Σ_α sentence ψ is a Scott sentence. A similar argument works for the rest of the difference hierarchy. Empirically through arguments such as these, and by checking examples, it appears that Σ_α, Π_α, and $d-\Sigma_\alpha$ are the only possible Scott complexities. Thus we define:

DEFINITION 3.4. The *Scott sentence complexity* of a structure $\mathcal{A}$ is the least complexity, from among Σ_α, Π_α, and $d-\Sigma_\alpha$, of a Scott sentence for $\mathcal{A}$.

It is desirable to have some kind of more rigorous argument that these should be the possible complexities, and that it is not just that we cannot think of any other possibilities. For this, we use a perspective that began in [49] of looking at the Borel complexity of the set of copies of a structure.

Given a language $\mathcal{L}$, there is a Polish space $\mathrm{Mod}(\mathcal{L})$ of all $\mathcal{L}$-structures with domain ω. Given a structure $\mathcal{A} \in \mathrm{Mod}(\mathcal{L})$, let $\mathrm{Iso}(\mathcal{A})$ be the set of all isomorphic copies of $\mathcal{A}$ in $\mathrm{Mod}(\mathcal{L})$. There is a correspondence between Scott sentences of $\mathcal{A}$ and the complexity of $\mathrm{Iso}(\mathcal{A})$ via the Lopez-Escobar theorem, and its strengthenings by Vaught and Miller.

THEOREM 3.5 (Lopez-Escobar [40], Vaught [65], and Miller [50]). *Let $\mathbb{K}$ be a subclass of $\mathrm{Mod}(\mathcal{L})$ which is closed under isomorphism. Then $\mathbb{K}$ is $\mathbf{\Sigma}_\alpha^0$ if and only if $\mathbb{K}$ is axiomatized by an infinitary Σ_α sentence. The same is true for $\mathbf{\Pi}_\alpha^0$ and Π_α sentences, and for the difference hierarchy.*

Since each structure has a Scott sentence, $\mathrm{Iso}(\mathcal{A})$ is always Borel.

Intuitively, the Lopez-Escobar result should extend to any natural and robust notion of complexity for sentences. So in place of looking at the least complexity of a Scott sentence for $\mathcal{A}$, we look at the complexity of $\mathrm{Iso}(\mathcal{A})$. This gives us access to additional structure via Wadge reducibility.

DEFINITION 3.6 (Wadge). Let A and B be subsets of $\mathrm{Mod}(\mathcal{L})$. We say that A is *Wadge reducible* to B, and write $A \leq_W B$, if there is a continuous function f with $A = f^{-1}[B]$, i.e.,

$$x \in A \iff f(x) \in B.$$

The equivalence classes under this pre-order are called the Wadge degrees; we write $[A]_W$ for the Wadge degree of A. The Wadge hierarchy is the set of Wadge degrees under continuous reductions.

With enough determinacy, the Wadge hierarchy is very well-behaved; it is well-founded and almost totally ordered (in the sense that any anti-chain has size at most two).

THEOREM 3.7 (Martin and Monk, AD). *The Wadge order is well-founded.*

THEOREM 3.8 (Wadge's Lemma, AD, [66]). *Given $A, B \subseteq \omega^\omega$, either $A \leq_W B$ or $B \leq_W \omega^\omega - A$.*

Since determinacy for Borel sets is provable in ZFC, this theorem holds in ZFC for such sets. Since $\mathrm{Iso}(\mathcal{A})$ is Borel, we will not need to use any determinacy assumptions.

In general, for each of the pointclasses Γ from among $\mathbf{\Sigma}_\alpha^0, \mathbf{\Pi}_\alpha^0, \mathbf{\Delta}_\alpha^0, \mathrm{d-}\mathbf{\Sigma}_\alpha^0$, and other pointclasses arising from the Borel or difference hierarchies, if A is Wadge-reducible to a set in Γ, then A itself is in Γ; and moreover, there is a Γ-complete set. We denote by Γ the Wadge degree of a Γ-complete set. So, for example, $\mathbf{\Sigma}_1^0$ is the Wadge degree of open, but not clopen, sets.

Now we define:

DEFINITION 3.9. The *Scott complexity* of a structure $\mathcal{A}$ is the Wadge degree of $\mathrm{Iso}(\mathcal{A})$.

Of course we want to show that this is the same as Definition 3.4. (We have made a slight differentiation between the two by calling one Scott sentence complexity and the other Scott complexity; they are technically different types of objects, the former being a complexity class of sentences and the

latter being a Wadge degree, though by the Lopez-Escobar theorem the two are in correspondence.) The possible Scott complexities are exactly what we expect given the discussion above: $\mathbf{\Pi}^0_\alpha$, $\mathbf{\Sigma}^0_\alpha$, and $\mathrm{d}-\mathbf{\Sigma}^0_\alpha$. There are certain values of α for which some of these are not possible, e.g., Σ_2 is not the Scott complexity of any structure. A complete list is as follows:

THEOREM 3.10 (Miller [49] and Alvir et al. [1]). *The possible Scott complexities of countable structures $\mathcal{A}$ are*:

1. $\mathbf{\Pi}^0_\alpha$ *for $\alpha \geq 1$,*
2. $\mathbf{\Sigma}^0_\alpha$ *for $\alpha \geq 3$ a successor ordinal, and*
3. $\mathrm{d}-\mathbf{\Sigma}^0_\alpha$ *for $\alpha \geq 1$ a successor ordinal.*

There is a countable structure with each of these Wadge degrees.

Miller constructed a number of examples of structures with various Scott complexities, and also ruled out some possibilities, such as $\mathbf{\Sigma}^0_2$ (for relational languages). He left open the case of constructing a structure of Scott complexity $\mathbf{\Sigma}^0_{\alpha+1}$ for α a limit ordinal. Alvir et al. found such examples, and showed that these are the only possibilities.

As a consequence of this theorem, *Scott sentence complexity* as in Definition 3.4 and *Scott complexity* as in Definition 3.9 are really the same thing, under the natural correspondence between Wadge degrees of $\mathrm{Iso}(\mathcal{A})$ and Scott sentences for $\mathcal{A}$, that is, identifying Σ_α and $\mathbf{\Sigma}^0_\alpha$, etc.

§4. Structures of high Scott rank. We now turn to an effective analysis of Scott complexity. We say that an ordinal α is X-computable if there is an X-computable linear order isomorphic to α. As there are only countably many X-computable ordinals, there must be a least non-X-computable ordinal, which we call ω_1^X (or ω_1^{CK} when X is computable). Since any initial segment of an X-computable well-founded linear order has a X-computable presentation, the X-computable ordinals are closed downwards; the X-computable ordinals are exactly the ordinals below ω_1^X.

We will give simpler and more elementary proofs than are usual in the literature, avoiding the use of Σ^1_1 bounding, ordinal notations, or the Barwise compactness theorem. (See, e.g., [4] for proofs of some of these results using the Barwise compactness theorem.) Instead, our main tool will be the fact that if an X-computable tree is well-founded, then its tree rank is an X-computable ordinal. Recall that the tree rank is defined as follows:

- $rk(x) = 0$ if x is a leaf.
- $rk(x)$ is otherwise the least ordinal (or possibly ∞) greater than the ranks of the children of x.

The rank of a tree is the rank of its root node.

THEOREM 4.1. *Let T be an X-computable well-founded tree. Then the tree rank of T is an X-computable ordinal.*

PROOF SKETCH. Given a tree $T \subseteq \omega^{<\omega}$ we will define computably in T a linear order, called the Kleene–Brouwer order. If T is well-founded then the

Kleene–Brouwer order will be as well; and in this case, the order type of the Kleene–Brouwer order will be an ordinal greater than the tree rank of T. The Kleene–Brouwer order is defined on the nodes of T by $s \leq_{KB} t$ if and only if

- $t \preceq s$ (s extends t) or
- there is n such that $s(n) < t(n)$ and $t \upharpoonright n = s \upharpoonright n$ (s is to the left of t).

If an X-computable tree T is well-founded with tree rank α, the Kleene–Brouwer order of T can be computed from T and hence is an X-computable presentation of an ordinal $\beta \geq \alpha$. Thus α itself is X-computable. We leave the details to the reader; they are not difficult. ⊣

4.1. Bounds on Scott complexity. There are countably many computable structures, and hence by a simple counting argument there must be some ordinal $\alpha < \omega_1$ which is an upper bound on the Scott rank of a computable structure. Our goal in this section is to compute this bound, as well as to prove some other results along the way.

For this section, it is helpful to extend the back-and-forth relations to tuples from different structures.

DEFINITION 4.2. Let $\mathcal{A}$ and $\mathcal{B}$ be structures. The *standard asymmetric back-and-forth relations* $\leq_\alpha$, for $\alpha < \omega_1$, are defined for $\bar{a} \in \mathcal{A}$ and $\bar{b} \in \mathcal{B}$ of the same length by:

1. $(\mathcal{A}, \bar{a}) \leq_0 (\mathcal{B}, \bar{b})$ if $\bar{a}$ and $\bar{b}$ satisfy the same quantifier-free formulas from among the first $|\bar{a}|$-many formulas.
2. For $\alpha > 0$, $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ if for each $\beta < \alpha$ and $\bar{d} \in \mathcal{B}$ there is $\bar{c} \in \mathcal{A}$ such that $(\mathcal{B}, \bar{b}\bar{d}) \leq_\beta (\mathcal{A}, \bar{a}\bar{c})$.

We define $\equiv_\infty$ etc. as before.

To begin, suppose that we have a computable structure $\mathcal{A}$.

THEOREM 4.3 (Nadel [56]). *Let $\mathcal{A}$ and $\mathcal{B}$ be X-computable structures. If $\mathcal{A} \equiv_\alpha \mathcal{B}$ for every $\alpha < \omega_1^X$, then $\mathcal{A} \cong \mathcal{B}$.*

PROOF. A finite partial isomorphism maps an element $\bar{a}$ of $\mathcal{A}$ to a tuple $\bar{b}$ of $\mathcal{B}$ such that $(\mathcal{A}, \bar{a}) \equiv_0 (\mathcal{B}, \bar{b})$: $\bar{a}$ and $\bar{b}$ satisfy the same atomic formulas from among the first $|\bar{a}|$-many. We can form a tree of finite partial isomorphisms; the root node is the empty partial isomorphism, and the children of a node are the finite partial isomorphisms extending that node.

Let T be the subtree of the tree of finite partial isomorphisms such that at the nth level, all of the finite partial isomorphism contain the first n elements of $\mathcal{A}$ in their domain, and the first n elements of $\mathcal{B}$ in their range. So T is X-computable. A path through T corresponds to a (total) isomorphism $\mathcal{A} \to \mathcal{B}$.

CLAIM 1. Consider a node σ of T corresponding to a finite partial isomorphism $\bar{a} \mapsto \bar{b}$. If $(\mathcal{A}, \bar{a}) \equiv_{\alpha \cdot 4} (\mathcal{B}, \bar{b})$, then the tree rank of σ is at least α. ⊣

PROOF. We argue by induction on α. We must show that for each $\beta < \alpha$, there is a child of σ of tree rank at least β. Suppose without loss of generality that σ is at the nth level of the tree.

First, let d be the $(n+1)$st element of $\mathcal{B}$. (If d is already contained in $\bar{b}$, then do nothing here; we assume that d is not in $\bar{b}$.) Since $(\mathcal{A}, \bar{a}) \equiv_{\alpha \cdot 4} (\mathcal{B}, \bar{b})$, there is $c \in \mathcal{A}$ such that $(\mathcal{A}, \bar{a}c) \equiv_{\beta \cdot 4 + 2} (\mathcal{B}, \bar{b}d)$. Now let $c' \in \mathcal{A}$ be the $n+1$st element of $\mathcal{A}$. (Again, if c' is already contained in $\bar{a}c$, then do nothing here; we assume that c' is not in $\bar{a}c$.) Let $d' \in \mathcal{B}$ be such that $(\mathcal{A}, \bar{a}cc') \equiv_{\beta \cdot 4} (\mathcal{B}, \bar{b}dd')$. Then $\bar{a}cc' \mapsto \bar{b}dd'$ is a finite partial isomorphism which is a child of σ, and by the induction hypothesis, it has tree rank at least β. ⊣

Now $(\mathcal{A}, \varnothing) \equiv_{\omega_1^X} (\mathcal{B}, \varnothing)$ and so the tree rank of T is at least ω_1^X. Since T is X-computable, T must have a path, which is an isomorphism $\mathcal{A} \to \mathcal{B}$. ⊣

We can easily apply this to tuples in a single structure.

COROLLARY 4.4 (Nadel [56]). *Let $\mathcal{A}$ be an X-computable structures and $\bar{a}, \bar{b} \in \mathcal{A}$. If $\bar{a} \equiv_\alpha \bar{b}$ for every $\alpha < \omega_1^X$, then there is an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$.*

We also get an upper bound on the Scott complexity of a computable structure.

COROLLARY 4.5 (Nadel [56]). *An X-computable structure has a $\Pi_{\omega_1^{CK}+2}$ Scott sentence, and hence has Scott rank at most $\omega_1^{CK} + 1$.*

This is true for all of the different definitions of Scott rank.

PROOF. By the previous corollary, if $\bar{a} \leq_{\omega_1^X} \bar{b}$, then $\bar{a} \equiv_\infty \bar{b}$. Then as in the discussion preceding Definition 2.4, we can choose the formulas $\varphi_{\bar{a}, \bar{b}}$ in the definition of the canonical Scott sentence to be $\Pi_{\omega_1^X}$, so that the resulting Scott sentence is $\Pi_{\omega_1^X + 2}$. ⊣

Next we will consider the computability of the Scott sentences of a computable structure. We give a slightly informal definition of the computable formulas of $\mathcal{L}_{\omega_1 \omega}$. For the fully formal definition, see [4].

DEFINITION 4.6.

- A computable Σ_0 or Π_0 formula is just a finitary quantifier-free formula.
- A computable Σ_α formula is a disjunction of formulas $\exists \bar{x} \psi(\bar{x})$ where the formulas $\psi(\bar{x})$ are computable Π_β for some $\beta < \alpha$, and there is a c.e. set of codes for the formulas over which the disjunction is being taken.
- A computable Π_α formula is a conjunction of formulas $\exists \bar{x} \psi(\bar{x})$ where the formulas $\psi(\bar{x})$ are computable Σ_β for some $\beta < \alpha$, and the there is a c.e. set of codes for the formulas over which the conjunction is being taken.

Let $\mathcal{A}$ be an X-computable structure. We will define, recursively for $\alpha < \omega_1^X$ and tuples $\bar{a} \in \mathcal{A}$, formulas

- $\varphi_{\bar{a},\alpha}$ such that $\mathcal{B} \models \varphi_{\bar{a},\alpha}(\bar{b})$ if and only if $(\mathcal{A}, \bar{a}) \leq_\alpha (\mathcal{B}, \bar{b})$ and
- $\psi_{\bar{a},\alpha}$ such that $\mathcal{B} \models \psi_{\bar{a},\alpha}(\bar{b})$ if and only if $(\mathcal{B}, \bar{b}) \leq_\alpha (\mathcal{A}, \bar{a})$.

We put $\varphi_{\bar{a},0}(\bar{x}) = \psi_{\bar{a},0}(\bar{x})$ to be the conjunction of the first $|\bar{x}|$-many atomic and negated atomic formulas about $\bar{x}$. For $\alpha > 0$, we put

$$\varphi_{\bar{a},\alpha}(\bar{x}) = \bigwedge_{\beta < \alpha} \forall \bar{y} \bigvee_{\bar{c} \in \mathcal{A}} \psi_{\bar{a}\bar{c},\beta}(\bar{x}, \bar{y}),$$

and

$$\psi_{\bar{a},\alpha}(\bar{x}) = \bigwedge_{\beta < \alpha} \bigwedge_{\bar{c} \in \mathcal{A}} \exists \bar{y} \; \varphi_{\bar{a}\bar{c},\beta}(\bar{x}, \bar{y}).$$

Both $\varphi_{\bar{a},\alpha}$ and $\psi_{\bar{a},\alpha}$ are computable $\Pi_{\alpha \cdot 2}$ formulas.

Suppose that α is an X-computable ordinal such that for all $\bar{a}, \bar{b} \in \mathcal{A}$, if $\bar{a} \leq_\alpha \bar{b}$ then $\bar{a} \equiv_\infty \bar{b}$. Then $\mathcal{A} \models \psi_{\bar{a},\alpha}(\bar{b})$ if and only if $\bar{a} \equiv_\infty \bar{b}$. In the definition of the canonical Scott sentence, we may use $\psi_{\bar{a},\alpha}$ as the formula $\psi_{\bar{a}}$. We get that the resulting Scott sentence is a computable $\Pi_{\alpha \cdot 2 + 2}$ sentence. (Note that this argument only works if the ordinal α is X-computable.) Thus we get:

THEOREM 4.7 (Nadel [56]). *Let $\mathcal{A}$ be an X-computable structure. Then $\mathcal{A}$ has Scott rank $< \omega_1^X$ if and only if $\mathcal{A}$ has an X-computable Scott sentence.*

An interesting line of investigation is to study the difference between the least-complexity Scott sentence and the least-complexity computable Scott sentence for a structure. Not much is known, though for example Ho constructed a subgroup of $\mathbb{Q}$ with a $\mathrm{d}-\Sigma_2$ Scott sentence but no computable $\mathrm{d}-\Sigma_2$ Scott sentence [33]. (The structure has computable Σ_3 and Π_3 structures, so that Theorem 3.3 is not true for computable Scott sentences.)

Finally, we will prove effective versions of Theorem 4.3 and Corollary 4.4.

THEOREM 4.8 (Nadel [56]). *Let $\mathcal{A}$ and $\mathcal{B}$ be X-computable structures. If $\mathcal{A}$ and $\mathcal{B}$ satisfy the same X-computable infinitary formulas, then they are isomorphic.*

PROOF. For each $\alpha < \omega_1^X$, we have that $\mathcal{B} \models \varphi_{\varnothing,\alpha}$ where this is the X-computable formula defined above. This implies that $\mathcal{A} \leq_\alpha \mathcal{B}$ for each α. Thus $\mathcal{A}$ and $\mathcal{B}$ are isomorphic by Theorem 4.3.   ⊣

COROLLARY 4.9 (Nadel [56]). *Let $\mathcal{A}$ be an X-computable structure. If $\bar{a}$ and $\bar{b} \in \mathcal{A}$ satisfy the same X-computable formulas, then there is an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$.*

4.2. Construction of structures of high Scott complexity. Corollary 4.5 is the best possible bound on the Scott complexity of an X-computable structure. Harrison [26] constructed a computable linear order of order type $\omega_1^{CK} \cdot (1 + \mathbb{Q})$, which has Scott complexity $\Pi_{\omega_1^{CK} + 2}$. It is a *pseudo-well-ordering* in the sense that it has no hyperarithmetic descending sequence. More generally, we can construct the Harrison linear order $\omega_1^X \cdot (1 + \mathbb{Q})$

relative to any set X, which has Scott complexity $\Pi_{\omega_1^X+2}$. Often, the Barwise compactness theorem is used to construct the Harrison linear order, but we will give a more natural construction found in [34] and which is explained in Lemma III.2.1 of [60]. Our account comes from [14]. We will give the general overview of the argument giving most but not all of the details.

The relation "Y is not hyperarithmetic in X" is $\Sigma_1^1(X)$ and so there is an X-computable tree T whose paths are pairs $\langle Y, f \rangle$ where f witnesses that Y is not hyperarithmetic in X. T is X-computable uniformly in X and has no X-hyperarithmetic path. Take the Kleene-Brouwer order of T. We get an X-computable linear order L with no X-hyperarithmetic descending sequence, as given a descending sequence in L, with one more jump we can compute a path through T. Then we argue that L has order type $\omega_1^X \cdot (1 + \mathbb{Q}) + \alpha$ for some $\alpha < \omega_1^X$. This takes several steps:

- Let γ be the order type of the well-founded initial segment of L. Write $L = L_1 + L_2$ with L_1 the well-founded initial segment of L. If γ were a computable ordinal, then we could hyperarithmetically split L as $L_1 + L_2$, and L_2 has no least element, giving a hyperarithmetic descending sequence in L. Also, we cannot have $\gamma > \omega_1^X$ as then we could fix some element u such that $\{v \in L : v < u\}$ is a computable linear ordering of order type ω_1^X. Thus $\gamma = \omega_1^X$.
- Similarly, given any $u \in L$, either $\{v \in L : v \geq u\}$ is well-founded or the well-founded initial segment of $\{v \in L : v \geq u\}$ has order type ω_1^X. Thus $L \cong \omega_1^X \cdot (1 + L^*) + \alpha$ for some linear order L^* and ordinal α.
- α must be computable, as if u is the initial element of α, $\{v \in L : v \geq u\}$ is a computable linear order.
- We claim that L^* is dense and without endpoints. If L^* had a left endpoint, then $\omega_1^X + \omega_1^X$ would be an initial segment of L, which we have already argued is not the case. If it had a right endpoint, then $\omega_1^X + \alpha$ would be a final segment of L, which cannot happen because $\omega_1^X + \alpha$ is not X-computable. And if L^* was not dense then we could find elements $a, b \in L$ such that $[a, b) \cong \omega_1^X$, and this cannot happen as ω_1^X is not X-computable.

Now if we fix the first element a of α, and restrict L to the elements below a, we obtain an X-computable linear order of order type $\omega_1^X \cdot (1 + \mathbb{Q})$ as desired. (Up until this point, the construction has been uniform in X, and indeed we can use a trick to make this step uniform as well: we instead replace L by $L \cdot \omega$ which has order type $\omega_1^X \cdot (1 + \mathbb{Q})$.)

Now we have to argue that the Scott complexity of $\omega_1^X \cdot (1 + \mathbb{Q})$ is $\Pi_{\omega_1^X+2}$. We must show that it has no $\Sigma_{\omega_1^X+2}$ Scott sentence. Using Theorem 3.2, it suffices to show that there is no tuple $\bar{c}$ such that the automorphism orbits are definable by Σ_α formulas, $\alpha < \omega_1^X$. (Note that if an orbit is definable by a $\Sigma_{\omega_1^X}$ formula, then it is definable by a Σ_α formula for some $\alpha < \omega_1^X$.) The parameters $\bar{c}$ do not help us define the automorphism orbits because, given $\bar{c} = (c_1, \ldots, c_n)$ with $c_1 < c_2 < \cdots < c_n$, between two of the parameters is a linear order of order type $\omega_1^X \cdot (1 + \mathbb{Q}) + \beta$ for some $\beta < \omega_1^X$.

So let us ignore the parameters. (See, for example, Section 15.3.3 of [4] for similar arguments.) Let u be an element of the non-well-founded part of the Harrison linear order, and suppose to the contrary that $\varphi(\bar{x})$ is a Σ_α formula defining the automorphism type of u, with $\alpha < \omega_1^X$. Then (using the formula $\varphi_{u,\alpha}$ from the previous section) there is an X-computable $\Pi_{\alpha \cdot 2}$ formula defining the automorphism orbit of u. So we may assume, after increasing α, that φ is X-computable. So the automorphism orbit $\{v \in L : L \models \varphi(v)\}$ of u is X-hyperarithmetic, and has no least element. Then we can find a hyperarithmetic descending sequence in L.

Thus we obtain:

THEOREM 4.10. *Given $X \in 2^\omega$, there is an X-computable linear order of Scott rank $\omega_1^X + 1$ and Scott complexity $\Pi_{\omega_1^X + 2}$. Moreover, there is a uniform operator Φ such that $\Phi(X)$ has order type $\omega_1^X \cdot (1 + \mathbb{Q})$, so that $\omega_1^X = \omega_1^Y$ if and only if $\Phi^X \cong \Phi^Y$.*

In general, we say that a structure $\mathcal{A}$ has *high Scott rank/high Scott complexity* if it satisfies one of the following equivalent conditions:

- it does not have an X-computable Scott sentence;
- its Scott rank is not X-computable, i.e., at least $\omega_1^{\mathcal{A}}$; and
- it has Scott complexity at least $\Pi_{\omega_1^{\mathcal{A}}}$.

All of the different possible definitions of Scott rank agree on whether a structure has high Scott rank, and if it does have high Scott rank, they agree on whether the Scott rank is ω_1^X or $\omega_1^X + 1$.

In terms of Scott complexity, there are five possible Scott complexities for a structure of high Scott complexity:
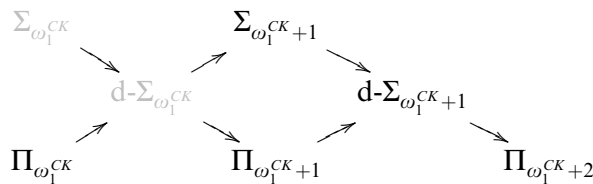
Structures of Scott complexity $\Pi_{\omega_1^{CK}}$ and $\Pi_{\omega_1^{CK}} + 1$ have Scott rank ω_1^{CK}, and structures of the other three Scott complexities have Scott rank $\omega_1^{CK} + 1$.

While a computable structure with Scott rank $\omega_1^{CK} + 1$ was known in the 1960s, it was not until the 2000s that a computable structure of Scott rank ω_1^{CK} was constructed. [41] gave a complicated construction of a Δ_2^0 structure with Scott rank ω_1^{CK}, and later Knight and Millar gave a simpler construction while also obtaining a computable structure [37]. The structures they build have Scott complexity $\Pi_{\omega_1^{CK}}$.

THEOREM 4.11 (Knight and Millar [37]). *There is a computable structure of Scott rank ω_1^{CK} and Scott complexity $\Pi_{\omega_1^{CK}}$. Moreover, given X, there is an X-computable structure of Scott rank ω_1^X and Scott complexity $\Pi_{\omega_1^X}$.*

PROOF SKETCH. The construction of Knight and Millar uses the Barwise compactness theorem to build a computable thin homogeneous tree, where

these terms are defined below. We will give a proof along the same lines, building a computable thin homogeneous tree, but we will build this tree in a more constructive way and avoid using the Barwise compactness theorem. We will give a computable construction but the construction relativizes to any X.

DEFINITION 4.12. A tree T is *thin* if there is a computable ordinal bound on the ordinal tree ranks at each level of the tree. ⊣

Note that there can be nodes on the tree of rank ∞; the bound is just on the nodes with ordinal rank.

DEFINITION 4.13. A tree T is *homogeneous* if:

- Whenever x has a successor of rank α, it has infinitely many successors of rank α.
- If some element at level n has a successor of rank α, every element at level n with rank $> \alpha$ has a successor of rank α.

Such a tree is called homogeneous because any two nodes at the same level of the tree and with the same rank are automorphic, and indeed the isomorphism type of the tree is determined by which tree ranks appear at each level. One can show that a computable thin homogeneous tree has a $\Pi_{\omega_1^{CK}}$ Scott sentence; one writes down the conjunctions of the sentences, one for each level, which say which tree ranks show up at that level. Each of these sentences is $\Pi_{\omega_1^{CK}}$ because there is a bound on the tree ranks at that level. Though the tree ranks at each level are bounded, if they are unbounded below ω_1^{CK} overall, the tree has Scott complexity $\Pi_{\omega_1^{CK}}$. See [37] for details.

So we want to construct a thin homogeneous tree with unbounded ranks. To construct such a tree, first fix a computable presentation $\mathcal{H}$ of the Harrison linear order. By the trick of replacing each element of $\mathcal{H}$ by a copy of ω, we may assume that we can compute successors, predecessors, and limits in $\mathcal{H}$. We construct for each $n = 1, 2, \ldots$ a set $S_n \subseteq \mathcal{H}$ such that:

1. $S_1 \subseteq S_2 \subseteq S_3 \subseteq \cdots$,
2. $\bigcup_i S_i = \mathcal{H}$,
3. S_1 is cofinal in $\mathcal{H}$,
4. each S_n is bounded above inside the well-founded part of $\mathcal{H}$,
5. for each $a \in S_n$ which is a limit element of $\mathcal{H}$, $\{b \in S_{n+1} : b < a\}$ is unbounded below a,
6. for each $a \in S_n$ which is a successor element of $\mathcal{H}$, the predecessor of a is in S_{n+1}.

Construct S_1 as follows. Computably list out the elements of $\mathcal{H}$, e.g., as $h_1, h_2, h_3, \ldots$ Consider the elements of $\mathcal{H}$ one-by-one in order, and put each element which is larger than all of the previous elements into S_1. For example, if the first nine elements are ordered as

$$h_9 < h_2 < h_5 < h_1 < h_6 < h_7 < h_3 < h_4 < h_8,$$

then we put h_1 in S_1, omit h_2 because it is smaller than h_2, put h_3 and h_4 in S_1, then omit h_5, h_6, and h_7 because they are lower than h_4, put h_8 in S_1, and

omit h_9. So we get $S_1 = \{h_1, h_3, h_4, h_8, \ldots\}$. Note that the order type of S_1 is ω. Only finitely many of the elements of S_1 are in the well-founded part of $\mathcal{H}$; this is for (4).

Given $S_1, \ldots, S_n$, we construct S_{n+1} as follows. Each element of S_n will be an element of S_{n+1}, satisfying (1). For each element y of $S_n - S_{n-1}$ which is a limit element of $\mathcal{H}$, we can find $x \in S_n$ such that there are no element z of S_{n+1} with $x < z < y$. Then, in the open interval $\{z \in \mathcal{H} : x < z < y\}$ we can do the same construction as we did for S_1, listing out the elements in this interval and putting each element which is greater than the previous elements into S_{n+1}. Thus we satisfy (5). For each successor element of S_n, we put its predecessor into S_{n+1}, satisfying (6). To satisfy (2), we also have to add to S_{n+1} the first element of $\mathcal{H}$ which does not yet appear in S_n. (4) is easy to verify, and (3) is just about S_1.

Let T a subtree of the tree of all descending sequences in $\mathcal{H}$, with the nodes at level n being from S_n; i.e., the path $\bar{h} = \langle h_1, \ldots, h_n \rangle$ has $h_1 \in S_1, h_2 \in S_2, \ldots, h_n \in S_n$. Given a node corresponding to the sequence $\bar{h} = \langle h_1, \ldots, h_n \rangle$, it is easy to argue using (5) and (6) that the tree rank of $\bar{h}$ is the order type of $\{x \in \mathcal{H} : x < h_n\}$ if h_n is in the well-founded part of $\mathcal{H}$, and ∞ otherwise. By choice of the S_n—namely (4)—T is a thin tree. To make it homogeneous, we just need to duplicate each node infinitely many times. By (3), the root node—which is the empty sequence—has tree rank ∞, but by (2), there is no bound on the tree ranks at all levels. $\dashv$

The construction of the Harrison linear order was uniform in the sense that given X, one could effectively construct the Harrison linear order relative to X. Moreover, if $\omega_1^X = \omega_1^Y$, then the Harrison linear order relative to X is the same as the Harrison linear order relative to Y. One can view this as saying that the construction produces a natural/canonical structure, *the* Harrison linear order. On the other hand, the construction just given of an X-computable structure of Scott rank ω_1^X is not uniform in this way. The structure one gets depends on the sets $S_1, S_2, \ldots$, and these in turn depend on the particular presentation of the Harrison linear order $\mathcal{H}$.

Chan [14] asked whether there is some other uniform construction of a computable structure of Scott rank ω_1^{CK}. Becker [6] and Chan et al. (unpublished) independently showed that there is no uniform construction.

THEOREM 4.14 (Becker [6] and Chan et al.,). *There is no Borel operator Φ such that*:

- *for all $X, Y \in 2^\omega$, $\Phi(X) \cong \Phi(Y) \Longleftrightarrow \omega_1^X = \omega_1^Y$.*
- *for all $X \in 2^\omega$, $\Phi(X)$ is a computable structure of Scott complexity strictly less than $\Pi_{\omega_1^X + 2}$.*

PROOF. The proof we give is due to an anonymous referee. Suppose to the contrary that there was such a Φ. Since Φ is Borel, for sufficiently large β, the inverse image of a Σ_β^0 set under Φ is Σ_β^0. Choose X such that $\omega_1^X > \beta$. Let $\mathcal{A} = \Phi(X)$. If $\mathrm{Iso}(\mathcal{A})$ is $\Sigma_{\omega_1^X + 2}^0$, then so is $\Phi^{-1}[\mathrm{Iso}(\mathcal{A})]$. But by Theorem 2.3 of [42], the set $\Phi^{-1}[\mathrm{Iso}(\mathcal{A})] = \{Y : \omega_1^Y = \omega_1^X\}$ is not $\Sigma_{\omega_1^X + 2}^0$. $\dashv$

Becker [7] later proved a number of strengthenings of this result.

Theorem 4.8 says that two computable structures that satisfy the same computable infinitary formulas are isomorphic. Given a computable structure $\mathcal{A}$, the computable infinitary theory of $\mathcal{A}$ is the collection of all the computable $\mathcal{L}_{\omega_1\omega}$ sentences true of $\mathcal{A}$. The structure of Scott rank ω_1^{CK} constructed by Knight and Millar (Theorem 4.11) is the only model, computable or not, of its computable infinitary theory; we say that its computable infinitary theory is countably categorical. On the other hand the Harrison linear order, which has Scott rank $\omega_1^{CK} + 1$, is not the only model of its computable infinitary theory. Millar and Sacks [48] asked whether there is a computable structure of Scott rank ω_1^{CK} such that the computable infinitary theory is not countably categorical. This is equivalent to finding a structure of Scott complexity $\Pi_{\omega_1^{CK}+1}$. (In one direction, if the computable infinitary theory of a computable structure is countably categorical, then the structure has a $\Pi_{\omega_1^{CK}}$ Scott sentence, namely the conjunction of all the computable sentences true of it. The converse is slightly more difficult; see [1].)

Millar and Sacks [48] produced such a structure $\mathcal{A}$ which is not hyperarithmetic, but which does have $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. Freer [22] proved the analogous result higher up: for any admissible α, there is a structure $\mathcal{A}$ with $\omega_1^{\mathcal{A}} = \alpha$ and $\mathcal{A}$ has Scott complexity $\Pi_{\omega_1^{\mathcal{A}}+1}$. Finally, Harrison-Trainor et al. [31] constructed a computable example.

THEOREM 4.15 (Harrison-Trainor et al. [31]). *There is a computable structure with Scott complexity $\Pi_{\omega_1^{CK}+1}$.*

There are two remaining complexities, $\Sigma_{\omega_1^{CK}+1}$ and $d-\Sigma_{\omega_1^{CK}+1}$. These are exactly the structures with the property that they have Scott rank $\omega_1^{CK} + 1$, but after naming a constant, they have Scott rank ω_1^{CK}. This follows from Theorem 3.2 and the following corresponding result for $d-\Sigma_\alpha$ sentences:

THEOREM 4.16 (Alvir et al. [1]). *Let $\mathcal{A}$ be a countable structure. Then $\mathcal{A}$ has a $d-\Sigma_\alpha$ Scott sentence if and only if for some $\bar{c} \in \mathcal{A}$, $(\mathcal{A}, \bar{c})$ has a Π_α Scott sentence and the automorphism orbit of $\bar{c}$ in $\mathcal{A}$ is Σ_α-definable.*

Turetsky asked whether such structures exist. Alvir et al. [1] showed that they do. Thus there are computable structures of all five possible high Scott complexities.

THEOREM 4.17 (Alvir et al., [1]). *There are computable structures of all possible high Scott complexities: $\Pi_{\omega_1^{CK}}$, $\Pi_{\omega_1^{CK}+1}$, $\Sigma_{\omega_1^{CK}+1}$, $d-\Sigma_{\omega_1^{CK}+1}$, and $\Pi_{\omega_1^{CK}+2}$.*

4.3. Computably approximable structures. Theorem 4.8 says that two computable structures that satisfy the same computable infinitary formulas are isomorphic. Thus the computable infinitary theory of a computable structure characterizes the structure among computable structures. (If a computable structure does not have a countably categorical computably infinitary theory, for example the Harrison linear order or the structure of

Theorem 4.15, then the other models of its computable infinitary theory are necessarily non-computable.) If we restrict our attention to computable structures only, is there a single sentence characterizing every computable structure? We call such a sentence a *pseudo-Scott sentence*.

DEFINITION 4.18. Let $\mathcal{A}$ be a computable structure. A *pseudo-Scott sentence* for $\mathcal{A}$ is a sentence whose computable models are just the computable copies of $\mathcal{A}$.

Calvert and Knight [11] asked whether every computable structure has a computable infinitary pseudo-Scott sentence. A counterexample would have to be of high Scott rank, as structures of computable Scott rank have computable Scott sentences, and any Scott sentence is a pseudo-Scott sentence.

QUESTION 4.19 (Calvert and Knight [11]). Is there a computable structure of noncomputable Scott rank with a computable infinitary pseudo-Scott sentence?

This question is still open. A structure without a computable Scott sentence is said to be weakly computably approximable.

DEFINITION 4.20. A computable structure $\mathcal{A}$ of non-computable Scott rank is *weakly computably approximable* if every computable infinitary sentence φ true in $\mathcal{A}$ is also true in some computable $\mathcal{B} \not\cong \mathcal{A}$.

The Harrison linear order has the property that any computable sentence true of it is also true of some other structure which is not just computable, but which has computable Scott rank. Moreover, the Harrison order has the following stronger property:

DEFINITION 4.21. A computable structure $\mathcal{A}$ of high Scott rank is *strongly computably approximable* if for all Σ_1^1 sets S, there is a uniformly computable sequence $(\mathcal{C}_n)$ such that if $n \in S$, then $\mathcal{C}_n \cong \mathcal{A}$, and if $n \notin S$, then $\mathcal{C}_n$ has computable Scott rank.

One can see that the Harrison order is strongly computably approximable by a simple modification of the construction given earlier.

Calvert et al. also showed that there is also such a structure of Scott rank ω_1^{CK}. This can also be seen using our construction for Theorem 4.11 combined with the fact that the Harrison order is strongly computably approximable.

THEOREM 4.22 (Calvert et al. [12]). *There is a computable tree T of Scott rank ω_1^{CK} that is strongly computably approximable.*

This theorem is very useful for other constructions; for example it was used in the proofs of Theorems 4.15 and 4.17.

Not every computable structure of high Scott rank is strongly computably approximable; indeed, there is a single computable sentence all of whose models have high Scott rank.

THEOREM 4.23 (Harrison-Trainor [27]). *For $\alpha = \omega_1^{CK}$ or $\alpha = \omega_1^{CK} + 1$: There is a computable model $\mathcal{A}$ of Scott rank α and a Π_2^c sentence ψ such that $\mathcal{A} \models \psi$, and whenever $\mathcal{B}$ is any structure and $\mathcal{B} \models \psi$, $\mathcal{B}$ has Scott rank α.*

Another application of the existence of strongly computably approximable structures is a computation of the index set complexity of being of high Scott rank.

THEOREM 4.24 (Calvert et al. [9]). *The index set*

$$\{i : \mathcal{A}_i \text{ has high Scott rank}\}$$

of computable structures with high Scott rank is Π_1^1 m-complete.

They also computed the complexity of the index sets of computable structures of Scott rank ω_1^{CK} and $\omega_1^{CK} + 1$; these are complete for differences of Π_1^1 sets and differences of Σ_1^1 sets respectively.

4.4. Computable categoricity and high Scott complexity. A computable structure $\mathcal{A}$ is said to be *computably categorical* if whenever $\mathcal{B} \cong \mathcal{A}$ is a computable copy of $\mathcal{A}$, there is a computable isomorphism from $\mathcal{A}$ to $\mathcal{B}$. For a long time, an important problem was to characterize the structures which are computably categorical. It turned out that the more well-behaved notion was that of *relative computable categoricity*: a computable structure $\mathcal{A}$ is relatively computably categorical if for every copy $\mathcal{B}$ of $\mathcal{A}$, not necessarily computable, there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ computable from $\mathcal{B}$. Relative computable categoricity is characterised by the following theorem:

THEOREM 4.25 (Ash et al. [3] and Chisholm [17]). *A computable structure $\mathcal{A}$ is relatively computably categorical if and only if it has a computable Σ_1 Scott family: there is a tuple $\bar{c}$ of constants and a c.e. collection Φ of computable Σ_1 formulas such that*

1. *each tuple $\bar{a}$ in $\mathcal{A}$ satisfies $\varphi(\bar{a}, \bar{c})$ for some formula $\varphi \in \Phi$ and*
2. *whenever two tuples $\bar{a}$ and $\bar{b}$ in $\mathcal{A}$ satisfy the same formula $\varphi(\cdot, \bar{c})$ from Φ, there is an automorphism of $\mathcal{A}$ taking $\bar{a}$ to $\bar{b}$.*

It is not hard to see that if $\mathcal{A}$ has a computable Σ_1 Scott family, then it is relatively computably categorical; the other direction is a forcing argument. So in some sense if a structure is relatively computably categorical, then there is a good structural reason for this.

As for (plain) computable categoricity, in certain classes there are good structural ways to determine whether a structure is computably categorical—e.g., a linear order is computably categorical if and only if it has finitely many adjacencies [25]—but there is no good characterization in general. Goncharov [24] showed that there are structures that are computably categorical but not relatively computably categorical. Later, Downey et al. [21] showed that the set of (indices for) computably categorical structures is Π_1^1 m-complete; this can be interpreted as meaning that there is no good characterization of computable categoricity that is simpler than the naive definition. As part of this proof, they constructed for each computable

ordinal $\alpha < \omega_1^{CK}$ a computably categorical structure $\mathcal{A}$ which is not *relatively Δ_α^0-categorical*, i.e., it is not the case that for every $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ which is Δ_α^0 in $\mathcal{B}$.

This left a gap: Is every computably categorical structure Δ_α^0-categorical for some computable ordinal $\alpha < \omega_1^{CK}$? Or even, is every computably categorical structure Δ_1^1-categorical? (A structure is relatively Δ_1^1-categorical if for every $\mathcal{B} \cong \mathcal{A}$ there is an isomorphism from $\mathcal{A}$ to $\mathcal{B}$ which is Δ_1^1 in $\mathcal{B}$.) If $\mathcal{A}$ is a computable structure of computable Scott rank $\alpha < \omega_1^{CK}$, then it is relatively $\Delta_{\alpha \cdot 2}^0$-categorical; this follows from a similar argument as Theorem 4.8. In the other direction, one can argue that a computable structure which is relatively Δ_1^1-categorical must have computable Scott rank. (See Proposition 1 of [64].) So a computable structure is relatively Δ_1^1-categorical if and only if it has computable Scott rank. Turetsky [64] showed:

THEOREM 4.26 (Turetsky [64]). *There is a computably categorical structure which has high Scott complexity and thus is not relatively Δ_1^1-categorical.*

Turetsky was also able to adapt this construction to build a structure of computable dimension 2:

THEOREM 4.27 (Turetsky [64]). *There is a structure of computable dimension 2 with two computable copies which are not hyperarithmetically isomorphic. This structure has high Scott complexity.*

This is of particular interest because up until Turetsky's construction the only existing constructions [18, 24, 32] of a structure of computable dimension 2, using a method called the special component technique, build two computable copies which are not computably isomorphic, but which are Δ_3^0 isomorphic.

§5. Scott complexity for certain structures. Scott complexity gives a good measuring stick for studying particular structures that we are interested in. Scott complexity measures the difficulty of characterizing a structure up to isomorphism, as well as (using Theorem 2.8) the difficulty of understanding the automorphisms of the structure, and computing isomorphisms between different copies of the structure. By finding the Scott complexity of a particular structure, we come to understand the simplest way of characterizing it. Moreover, the problem of computing the Scott complexity of a structure gives us a way of testing our understanding of that structure; indeed, one longstanding open questions is:

QUESTION 5.1. What is the Scott complexity of the pure transcendental field $\mathbb{Q}(x_1, x_2, ...)$?

This highlights the fact that we do not have a good understanding of *pure transcendence bases*, i.e., transcendence bases which also generate the field extension. In contrast, Nielson transformations give us an excellent understanding of bases for free groups, allowing us to compute their Scott complexity; see Theorem 5.7.

To compute the Scott complexity of a structure, one usually writes down a Scott sentence of a particular complexity, and then one proves a lower bound showing that this complexity is optimal. So, for example, one might write down a Σ_4 Scott sentence for a structure $\mathcal{A}$, and then give a Wadge reduction f from a Σ_4^0-complete set S to $\mathrm{Iso}(\mathcal{A})$, i.e., a continuous operator f such that if $x \in S$ then $f(x) \cong \mathcal{A}$, and if $x \notin S$ then $f(x) \ncong \mathcal{A}$.

Much of the literature takes the approach of using index sets rather than Wadge reductions.

DEFINITION 5.2. Fix a listing $\mathcal{M}_i$ of the partial computable structures in the language of $\mathcal{A}$. Then

$$\mathcal{I}_{\mathcal{A}} = \{i \in \omega : \mathcal{M}_i \cong \mathcal{A}\}$$

is the index set of $\mathcal{A}$.

If $\mathcal{A}$ has a Scott sentence which is a computable Σ_α sentence then $\mathcal{I}_{\mathcal{A}}$ will be of complexity Σ_α^0 as a subset of ω. The same is true for Π_α and Π_α^0, and $\mathrm{d}{-}\Sigma_\alpha$ and $\mathrm{d}{-}\Sigma_\alpha^0$ as well. So if we can prove, for example, that $\mathrm{Iso}(\mathcal{A})$ is Σ_α^0 m-complete, then $\mathcal{A}$ cannot have a computable Π_α Scott sentence.

This strategy does not always work. Knight and McCoy [36] showed that there is a structure (a particular subgroup of $\mathbb{Q}$) for which the index set is m-complete $\mathrm{d}{-}\Sigma_\alpha^0$, but there is no computable $\mathrm{d}{-}\Sigma_\alpha$ Scott sentence. The structure does however have an X-computable $\mathrm{d}{-}\Sigma_\alpha$ Scott sentence, for a particular low c.e. set X used to build the subgroup. So at the computable level, the index set complexity and the Scott complexity do not always match up. To fix this, we must relativize.

DEFINITION 5.3. Fix a listing $\mathcal{M}_i^X$ of the partial X-computable structures in the language of $\mathcal{A}$. Then

$$\mathcal{I}_{\mathcal{A}}^X = \{i \in \omega : \mathcal{M}_i^X \cong \mathcal{A}\}$$

is the index set of $\mathcal{A}$ relative to X.

So for example to show that $\mathcal{A}$ does not have any Scott sentence which is Π_α or simpler, we must show that there is a set X such that for every $Y \geq_T X$, the index set $\mathcal{I}_{\mathcal{A}}^Y$ relative to Y is Σ_α^Y m-complete. This essentially works out to the same things as giving a Wadge reduction.

We now turn to some examples.

THEOREM 5.4 (Calvert et al. [10] and Calvert [8]). *Let $\mathcal{A}$ be a $\mathbb{Q}$-vector space. Then*:

1. *If $\dim(\mathcal{A}) = 1$, then $\mathcal{A}$ has Scott complexity Π_2.*
2. *If $\infty > \dim(\mathcal{A}) > 1$, then $\mathcal{A}$ has Scott complexity $\mathrm{d}{-}\Sigma_2$.*
3. *If $\dim(\mathcal{A}) = \infty$, then $\mathcal{A}$ has Scott complexity Π_3.*

THEOREM 5.5 (Calvert [8]). *Let $\mathcal{A}$ be an algebraically closed field. Then*:

1. *If $\mathcal{A}$ has finite transcendence degree, then $\mathcal{A}$ has Scott complexity $\mathrm{d}{-}\Sigma_2$.*
2. *If $\mathcal{A}$ has infinite transcendence degree, then $\mathcal{A}$ has Scott complexity Π_3.*

Both of these classes are examples of strongly minimal theories and hence admit a pregeometry of algebraic closure. Csima et al. (unpublished) have shown that the hardness portion of the results above are consequences of the existence of this pregeometry. Other classes of structures also admit a pregeometry with the right properties; another example is the class of real closed ordered fields. Calvert et al. [10] first computed the complexity of Archimedean real closed ordered fields.

THEOREM 5.6 (Calvert et al. [10]). *Let $\mathcal{A}$ be an Archimedean real closed ordered field.*

1. *If the transcendence degree is* 0, *then $\mathcal{A}$ has Scott complexity Π_2.*
2. *If the transcendence degree of $\mathcal{A}$ is finite but not* 0, *then $\mathcal{A}$ has Scott complexity* $\mathrm{d}-\Sigma_2$.
3. *If the transcendence degree of $\mathcal{A}$ is infinite, then $\mathcal{A}$ has Scott complexity Π_3.*

They also prove a number of results on reduced abelian *p*-groups.

Groups have been a very fruitful area for proving interesting results about Scott complexity. This began with free groups, where Carson et al. [13] and McCoy and Wallbaum [46] used Nielson transformations to compute the Scott complexities of free groups on various generators.

THEOREM 5.7 [13, 46]. 1. *For $n \geq 1$, the free group F_n on n generators has Scott complexity* $\mathrm{d}-\Sigma_2$.
2. *The free group F_∞ on infinitely many generators has Scott complexity Π_4.*

Knight and Saraph [38] continued the study of the Scott complexity of groups by looking at torsion-free abelian groups of finite rank and finitely generated groups. For groups of finite rank, Knight and Saraph show that there is always a Σ_3 Scott sentence.

THEOREM 5.8 (Knight and Saraph [38]). *Let G be a torsion-free abelian group of finite rank. Then G has a Σ_3 Scott sentence.*

Moreover, they show that there are computable torsion-free abelian groups of rank one which have Scott complexity Σ_3.

THEOREM 5.9 (Knight and Saraph [38]). *If G is a subgroup of $\mathbb{Q}$ in which* 1 *is not infinitely divisible by any prime, but in which* 1 *is finitely divisible by infinitely many primes, then G has Scott complexity Σ_3.*

One can also analyse the computability of these Scott sentences; see [38] as well as [33].

Knight and Saraph also show that every finitely generated group has a Σ_3 Scott sentence, and indeed the same proof works for structures of any language.

THEOREM 5.10 (Knight and Saraph [38]). *Every finitely generated structure has a Σ_3 Scott sentence.*

By Theorem 5.7, every finitely generated free group has a $d-\Sigma_2$ Scott sentence. Moreover, the same is true of any finitely generated abelian group and of the dihedral group.

THEOREM 5.11 (Knight and Saraph [38]). *Every finitely generated abelian group has a* $d-\Sigma_2$ *Scott sentence.*

THEOREM 5.12 (Knight and Saraph [38]). *The infinite dihedral group has Scott complexity* $d-\Sigma_2$.

Next, Ho [33] showed that polycyclic groups (which include nilpotent groups and abelian groups) and certain solvable groups also all have computable $d-\Sigma_2^0$ Scott sentence.

THEOREM 5.13 (Ho [33]). *Every finitely generated polycyclic group has a* $d-\Sigma_2$ *Scott sentence.*

At the time, every finitely generated group that had been checked had a $d-\Sigma_2$ Scott sentence, and so Knight and Saraph asked whether every finitely generated group has a $d-\Sigma_2$ Scott sentence. Harrison-Trainor and Ho [29] answered this question by showing that there is a finitely generated group with no $d-\Sigma_2$ Scott sentence and hence with Scott complexity Σ_3.

THEOREM 5.14 (Harrison-Trainor and Ho [29]). *There is a finitely-generated computable group with Scott complexity* Σ_3.

Thus the upper bound given by Theorem 5.10 is best possible.

The proof of this last theorem relied on a characterizations using a number of different equivalent conditions on when a finitely generated structure has a $d-\Sigma_2$ Scott sentence.

THEOREM 5.15 (Miller [49], Harrison-Trainor and Ho [29], and Alvir et al. [2]). *Let* $\mathcal{A}$ *be a finitely generated structure. The following are equivalent*:

1. $\mathcal{A}$ *has a* $d-\Sigma_2$ *Scott sentence*;
2. $\mathcal{A}$ *has a* Π_3 *Scott sentence*;
3. $\mathcal{A}$ *is the only model of its* Σ_2 *theory*;
4. *some generating tuple of* $\mathcal{A}$ *is defined by a* Π_1 *formula*;
5. *every generating tuple of* $\mathcal{A}$ *is defined by a* Π_1 *formula*; *and*
6. $\mathcal{A}$ *does not contain a copy of itself as a proper* Σ_1-*elementary substructure.*

(2) is due to Miller [49], (3) is due to Harrison-Trainor and Ho [30], (4) and (5) are due to Alvir et al. [2], and (6) is due to Harrison-Trainor and Ho [29]. It is condition (6) which seems to be the most useful for constructing examples, as it has a more semantic flavour. Alvir et al. also have a nice characterization of when a structure has a *computable* $d-\Sigma_2$ Scott sentence.

THEOREM 5.16 (Alvir et al. [2]). *Let* $\mathcal{A}$ *be a finitely generated structure. The following are equivalent*:

1. $\mathcal{A}$ *has a computable* $d-\Sigma_2$ *Scott sentence*;
2. *some generating tuple of* $\mathcal{A}$ *is defined by a computable* Π_1 *formula*;
3. *every generating tuple of* $\mathcal{A}$ *is defined by a computable* Π_1 *formula*.

There are indeed computable structures with a $d-\Sigma_2$ Scott sentence, but no computable $d-\Sigma_2$ Scott sentence; for example Theorem 2.8 of [28] gives an example of a computable finitely generated principal ideal domain which, as a module over itself, has a $d-\Sigma_2$ Scott sentence but no computable $d-\Sigma_2$ Scott sentence.

These general tools give us a way to study Scott complexity in other classes of finitely generated structures. The known result are shown in the table below. Those results which have not yet been cited are due to Harrison-Trainor and Ho [29] and Harrison-Trainor [28].

Some structure has no $d-\Sigma_2$ Scott sentence	Every structure has a $d-\Sigma_2$ Scott sentence
Rings	Commutative rings
Modules	Modules over Noetherian rings
	Fields
	Vector spaces
Groups	Abelian groups
	Free groups
	Torsion-free hyperbolic groups

One important class of structures for which we do not know the answer is finitely presented groups.

QUESTION 5.17 (Question 1.7 of [29] and Question 1 of [2]). Does every finitely presented group have a $d-\Sigma_2$ Scott sentence?

The best progress towards answering this question is in [28]. There, Harrison-Trainor proves the following purely group-theoretic characterization of the finitely presented groups with a $d-\Sigma_2$ Scott sentence:

THEOREM 5.18 (Harrison-Trainor [28]). *Let G be a finitely presented group. Then the following are equivalent*:

1. *G does not have a $d-\Sigma_2$ Scott sentence and*
2. *G contains a proper subgroup $H \cong G$ with the property that for every finite set of non-identity elements $a_1, \ldots, a_n \in G$, there is a normal subgroup $H' \subseteq G$ such that $G = H' \rtimes H$ and $a_1, \ldots, a_k \notin H'$.*

One can immediately see from this that any finitely presented Hopfian group has a $d-\Sigma_2$ Scott sentence. (A Hopfian group is one which is not isomorphic to any of its proper quotients.) This includes cases such as finitely generated abelian groups and free groups for which we already knew this, as well as a few new examples such as finitely presented residually finite groups and torsion-free hyperbolic groups. We can also show that the Baumslag-Solitar groups $B(1, n)$, which were originally constructed specifically as examples of non-Hopfian finitely generated groups, have $d-\Sigma_2$ Scott sentences.

Paolini [58] gave a sufficient condition, a weakening of Hopfianity, for an arbitrary finitely presented structure to have a $d-\Sigma_2$ Scott sentence. We say that $\mathcal{A}$ is quasi-Hopfian if there is a finite generating set $\bar{a}$ of $\mathcal{A}$ such that whenever $f : \mathcal{A} \to \mathcal{A}$ is a surjective homomorphism of $\mathcal{A}$ which is injective on $\bar{a}$, f is injective.

THEOREM 5.19 (Paolini [58]). *Let $\mathcal{A}$ be a finitely presented structure. If $\mathcal{A}$ is quasi-Hopfian, then $\mathcal{A}$ has a $\mathrm{d}-\Sigma_2$ Scott sentence.*

Paolini also gives a condition on the automorphism group of a quasi-Hopfian computable structure which is sufficient to have a computable $\mathrm{d}-\Sigma_2$ Scott sentence. Paolini applies this to show that the free projective plane of rank 4, which is quasi-Hopfian but not Hopfian, has a computable $\mathrm{d}-\Sigma_2$ Scott sentence. He also considers Coxeter groups of finite rank; these are Hopfian, and hence have $\mathrm{d}-\Sigma_2$ Scott sentences, but it takes more work to show that these Scott sentences are computable.

The complexity of saying that a structure is finitely generated is Σ_3. When a finitely generated structure has Scott complexity Σ_3, is this just because it is Σ_3 to say that it is finitely generated? Or is the complexity of picking out the structure from among other finitely generated structures actually Σ_3? We can ask the analogous question whenever the complexity of a structure in a class $\mathcal{K}$ is the same or less than the complexity of $\mathcal{K}$. For such cases, we would like to separate the complexity of describing the class of structure from the complexity of describing the structure inside of the class.

DEFINITION 5.20. Let $\mathcal{A}$ be a structure in a class $\mathcal{K}$. We say that φ is a *quasi Scott sentence for $\mathcal{A}$ within $\mathcal{K}$* if, for countable $\mathcal{B} \in \mathcal{K}$,

$$\mathcal{B} \models \varphi \iff \mathcal{B} \cong \mathcal{A}.$$

The *Scott complexity of $\mathcal{A}$ within $\mathcal{K}$* is $\Gamma \in \{\Sigma_\alpha, \Pi_\alpha, \mathrm{d}-\Sigma_\alpha\}$ if Γ is the least complexity of a quasi Scott sentence for $\mathcal{A}$ within $\mathcal{K}$.

To prove an upper bound, we just need to exhibit a quasi Scott sentence. We again prove lower bounds using a reduction. For example, to show that $\mathcal{A}$ has Scott complexity at least Σ_α within $\mathcal{K}$, we give a Wadge reduction f from a Σ_α^0-complete set S to $\mathrm{Iso}(\mathcal{A})$ with the additional property that $f(x) \in \mathcal{K}$ for all x.

For finitely generated structures, we have the following interesting result:

THEOREM 5.21 (Harrison-Trainor and Ho [30]). *Every finitely generated structure has a Π_3 quasi-Scott sentence within the class of finitely generated structures.*

Recall that this means that for each finitely generated structure $\mathcal{A}$, there is a Π_3 sentence φ such that $\mathcal{A}$ is the only finitely generated model of φ. In [30], Harrison-Trainor and Ho prove that there are finitely generated groups with no $\mathrm{d}-\Sigma_2$ quasi Scott sentence within the class of finitely generated structures. Thus the analogue for quasi Scott sentences of Theorem 3.3 is not true: if a structure has a Σ_3 and a Π_3 quasi Scott sentence within the class of finitely generated structures, it is not necessarily true that it has a $\mathrm{d}-\Sigma_2$ quasi Scott sentence. There are two questions that are still open about quasi Scott sentences:

QUESTION 5.22 (Question 5.7 of [29]). Give a characterization of the finitely generated structures which have a $\mathrm{d}-\Sigma_2$ quasi Scott sentence within the class of finitely generated structures.

QUESTION 5.23 (See Section 5.3 of [30]). Is there a finitely generated structure with a d$-\Sigma_2$ quasi Scott sentence within the class of finitely generated structures, but no d$-\Sigma_2$ Scott sentence?

Recall from Theorem 5.7 that the free group F_∞ on infinitely many generators has Scott complexity Π_4. McCoy and Walbaum [46] show that the class of free groups is axiomatizable by a Π_4 sentence. Within this class, F_∞ is simpler to describe. (F_1 and F_2 also become simpler to describe within this class.)

THEOREM 5.24 [13, 46]. *Let $\mathcal{FG}$ be the class of free groups.*

1. *The free group $F_1 \cong \mathbb{Z}$ on one generator has Scott complexity Π_1 within $\mathcal{FG}$.*
2. *The free group F_2 on two generators has Scott complexity Π_2 within $\mathcal{FG}$.*
3. *For $n > 2$, the free group F_n on n generators has Scott complexity d$-\Sigma_2^0$ within $\mathcal{FG}$.*
4. *The free group F_∞ on infinitely many generators has Scott complexity Π_3 within $\mathcal{FG}$.*

Quasi Scott sentences are also useful for talking about very simple structures, such as the one-dimensional $\mathbb{Q}$-vector space; then Theorem 5.4 says that the Scott complexity of $\mathcal{V}$ is Π_2. However even just saying that $\mathcal{V}$ is a vector space is already Π_2; perhaps all the complexity is in saying that $\mathcal{V}$ is a vector space, and it is simpler than that to say that a vector space is one-dimensional? However Calvert et al. [10] show that the one-dimensional $\mathbb{Q}$-vector space has Scott complexity Π_2 within the class of $\mathbb{Q}$-vector spaces.

Quasi Scott sentences within the class of finitely generated structures are similar to *quasi-finite axiomatizability*. A structure is quasi-finitely axiomatizable if there is a sentence φ of finitary first-order logic such that the structure is the only finitely generated structure satisfying φ, that is, if it has a finitary quasi Scott sentence within the class of finitely generated structures. See the survey [57].

§6. Theories. Given an $\mathcal{L}_{\omega_1\omega}$ sentence φ we can think of φ as a theory defining the class $\{\mathcal{A} : \mathcal{A} \models \varphi\}$. This generalizes the notion of an elementary first-order theory, as given such a theory T, the conjunction $\bigwedge T$ of all the sentences in T is an $\mathcal{L}_{\omega_1\omega}$ sentence.

6.1. Vaught's conjecture. As previously mentioned, we give only the briefest overview of Vaught's conjecture and its connections with Scott sentences.

A central question in model theory is given a theory T, how many countable models can T have? The continuum hypothesis would imply that if T has infinitely many countable models, then it can only have countably many or continuum many countable models. In descriptive set theory, one often finds (e.g., for analytic sets) that even without the continuum hypothesis, one can prove that naturally occurring sets must have cardinality

countable or continuum. Vaught's conjecture [65] asks whether this is true for the number of countable models of a theory.

CONJECTURE 6.1 (Vaught). *The number of countable models of an elementary first-order theory is either countable or continuum.*

It is natural to extend this conjecture to theories given by sentences of $\mathcal{L}_{\omega_1\omega}$.
Morley was able to eliminate almost all of the other possibilities.

THEOREM 6.2 (Morley [55]). *The number of countable models of an elementary first-order theory is either $\aleph_0$, $\aleph_1$, or continuum.*

The standard proof of this theorem rests on what is called the Scott analysis together with Silver's theorem. The Scott analysis looks at the back-and-forth relations on all of the models of the theory, or, as in [44], the analysis as in Definition 2.7. This gives a level-by-level splitting of the models of the theory, e.g., at level α one splits the models into $\equiv_\alpha$-classes. Silver's theorem says that co-analytic equivalence relations have either countably many or continuum many equivalence classes [63]. So at each level α there are either countably many or continuum many equivalence classes. If the theory has fewer than continuum-many models, at each level α there must only be countably many equivalence classes of models. Then there are ω_1 levels, hence at most $\aleph_1$ countable models.

Of course Vaught's conjecture is automatically true if the continuum hypothesis is. Thus Vaught's conjecture is often stated in a way that is independent of the continuum hypothesis. We say a sentence φ of $\mathcal{L}_{\omega_1\omega}$ is *scattered* if it does not have a perfect set of countable models (in the space $\mathrm{Mod}(\mathcal{L})$). Then one can state Vaught's conjecture as saying that any scattered theory has countably many countable models. Using the Scott analysis, one can show:

THEOREM 6.3. *The following are equivalent*:

1. *φ is scattered,*
2. *for each countable α, there are only countably many $\equiv_\alpha$-equivalence classes of models of φ, and*
3. *for each countable α, there are only countably many models of φ of Scott rank less than α.*

The Scott analysis, though not necessarily Scott rank or complexity, plays a central role in a large body of results about what a counterexample to Vaught's conjecture would have to look like. Often, these results are about the uncountable models of the theory. One example mentioning Scott rank explicitly is:

THEOREM 6.4 (Harrington). *A counterexample to Vaught's Conjecture has models of arbitrarily large Scott rank below ω_2.*

In this theorem, we need the Scott rank for uncountable structures, e.g., by extending the Scott ranks based on back-and-forth relations. Harrington's

proof was never published, but recently new proofs have independently been given by Baldwin et al. [5], Knight et al. [35], and Larson [39].

Montalbán showed that there are computability-theoretic ways of stating Vaught's conjecture; the proof also uses an analysis of the back-and-forth relations.

THEOREM 6.5. (ZFC+PD; Montalbán [51]). *Let φ be an $\mathcal{L}_{\omega_1\omega}$ sentence with uncountably many countable models. The following are equivalent*:

$(V1)$ *φ is a counterexample to Vaught's conjecture.*

$(V2)$ *φ satisfies hyperarithmetic-is-recursive on a cone, i.e., for all sufficiently powerful oracles X, any hyperarithmetic-in-X model of φ has an X-computable copy.*

$(V3)$ *There is an oracle relative to which,*

$$\{DgSp(\mathcal{A}) \ : \ \mathcal{A} \models \varphi\} = \{\{X \in 2^\omega \ : \ \omega_1^X \geq \alpha\} : \alpha \in \omega_1\},$$

where $DgSp(\mathcal{A})$ is the degree spectrum of $\mathcal{A}$, that is, the collection of sets which can compute a copy of $\mathcal{A}$.

As an example of an open question relating Scott rank and Vaught's conjecture, we have:

QUESTION 6.6 (Montalbán, Berkeley Vaught's Conjecture Workshop). A *minimal theory T* has the property that for any $\mathcal{L}_{\omega_1\omega}$-sentence φ, exactly one of $T \wedge \varphi$ or $T \wedge \neg\varphi$ has models of arbitrarily high Scott rank. Does the existence of a minimal theory imply the existence of a counterexample to Vaught's conjecture?

6.2. Scott spectra. The results of this section were proved using the notion of Scott rank defined using symmetric back-and-forth relations (as in Definition 2.5) except with arbitrary tuples, replacing (2) of that definition with:

(2) For $\alpha > 0$, $\bar{a} \sim_\alpha \bar{b}$ if for each $\beta < \alpha$ and $\bar{d}$ there is $\bar{c}$ such that $\bar{a}\bar{c} \sim_\beta \bar{b}\bar{d}$, and for all $\bar{c}$ there is $\bar{d}$ such that $\bar{a}\bar{c} \sim_\beta \bar{b}\bar{d}$.

It is likely that the same or similar results can be proved for any of the definitions of Scott rank.

Let φ be a Π_2 sentence which we think of as similar to a theory in the sense that it defines a class of models. Many natural theories are of this form, e.g., most algebraic classes of interest such as algebraically closed fields or torsion groups. Montalbán asked whether there must be a model of φ of somewhat low Scott complexity, say Scott rank 2 or less. One might expect that it would be possible to construct a model of any Π_2 theory without introducing any complicated types, resulting in a structure of low Scott rank. This turns out not to be the case; surprisingly, there are Π_2 sentences all of whose models are very complicated.

THEOREM 6.7 (Harrison-Trainor [27]). *Fix $\alpha < \omega_1$. There is a Π_2 sentence T whose models all have Scott rank α.*

More generally, we can consider the collection of Scott ranks of all of the models of φ.

DEFINITION 6.8. Let φ be an $\mathcal{L}_{\omega_1\omega}$ sentence. The *Scott spectrum* of φ is the collection

$$\{\mathrm{SR}(\mathcal{A}) : \mathcal{A} \text{ is countable and } \mathcal{A} \models \varphi\}$$

of Scott ranks of countable models of φ.

There are many natural questions about Scott spectra. For example, what kind of gaps can one have—for which $\alpha < \beta$ can a theory have models of Scott rank α and β, but no models of Scott rank in between? Is there any relation between the complexity of the theory and the possible Scott spectrum? The following result of Sacks says that the Scott spectrum of a class is either bounded, or the class contains models of high Scott rank.

THEOREM 6.9 (Sacks [59]). *Let φ be an $\mathcal{L}_{\omega_1\omega}$ sentence. Suppose that the Scott rank of $\mathcal{M}$ is $\leq \omega_1^{\mathcal{M}\oplus\varphi}$ for all countable models $\mathcal{M}$ of φ. Then there is $\alpha < \omega_1^{\varphi}$ such that the Scott rank of every model of φ is at most α.*

Harrison-Trainor [27] gave a complete descriptive set-theoretic classification of the possible Scott spectra. To give this characterization, we need a few definitions.

DEFINITION 6.10. Let $(L, \leq)$ be a linear order. The *well-founded part* $\mathrm{wf}(L)$ of L is the largest initial segment of L which is well-founded. The *well-founded collapse* of L, $\mathrm{wfc}(L)$, is the order type of L after we collapse the non-well-founded part $L \setminus \mathrm{wf}(L)$ to a single element.

We can identify $\alpha \in \mathrm{wf}(L)$ with the ordinal which is the order type of $\{\beta \in L : \beta < \alpha\}$. We can also identify $\mathrm{wf}(L)$ with its order type. If L is well-founded, with order type α, then $\mathrm{wfc}(L) = \mathrm{wf}(L) = \alpha$. If L is not well-founded, $\mathrm{wfc}(L) = \mathrm{wf}(L) + 1$. We extend these operations to collections of linear orders by applying the operation to each order in the collection.

Then we get the following characterization of the Scott spectra:

THEOREM 6.11 (Harrison-Trainor [27]; ZFC + PD). *The Scott spectra of $\mathcal{L}_{\omega_1\omega}$-sentences are exactly the sets of the form:*

1. $\mathrm{wf}(\mathcal{C})$,
2. $\mathrm{wfc}(\mathcal{C})$, *or*
3. $\mathrm{wf}(\mathcal{C}) \cup \mathrm{wfc}(\mathcal{C})$,

where $\mathcal{C}$ is a $\mathbf{\Sigma}_1^1$ class of linear orders of ω.

The more difficult and most useful part of this characterization is to show that each such collection of ordinals is the Scott spectrum of a theory. This direction is provable within ZFC and does not use the set-theoretic assumption of projective determinacy. One can use this direction to construct interesting examples of Scott spectra, such as the collection of successor ordinals, or the collection of admissible ordinals.

Harrison-Trainor also proved that Π_2 theories are enough to get all of the Scott spectra.

THEOREM 6.12 (Harrison-Trainor [27]). *Each Scott spectrum is the Scott spectrum of a Π_2^{in} sentence.*

For example, for every pair of ordinals $\alpha < \beta$, $\{\alpha, \beta\}$ is a Scott spectrum.

6.3. Scott height. Of the countably many computable $\mathcal{L}_{\omega_1\omega}$ sentences, some of them have a Scott spectrum which is unbounded below ω_1, and others are bounded below ω_1. Since there are only countably many computable sentences, of all those that are bounded below ω_1, there must be some countable ordinal which is the least common upper bound for all of them. Using $\mathcal{L}_{\omega_1\omega}^c$ for the computable $\mathcal{L}_{\omega_1\omega}$ sentences, we define:

DEFINITION 6.13. The Scott height $sh(\mathcal{L}_{\omega_1\omega}^c)$ of $\mathcal{L}_{\omega_1\omega}^c$ is the least ordinal $\alpha < \omega_1$ such that if φ is a computable $\mathcal{L}_{\omega_1\omega}$ sentence whose Scott spectrum is bounded below ω_1, then the Scott spectrum of φ is bounded below α.

The Scott height is a way of measuring how far the computable sentences can "reach" up the class of countable ordinals.

Sacks and Marker made significant progress towards computing the Scott height of $\mathcal{L}_{\omega_1\omega}^c$. Sacks [59] showed that $sh(\mathcal{L}_{\omega_1,\omega}^c) \leq \delta_2^1$, where δ_2^1 is the least ordinal which has no Δ_2^1 presentation. Marker [43] was able to resolve this question for pseudo-elementary classes.

DEFINITION 6.14. A class $\mathcal{C}$ of structures in a language $\mathcal{L}$ is an $\mathcal{L}_{\omega_1\omega}$-pseudo-elementary class ($PC_{\mathcal{L}_{\omega_1\omega}}$-class) if there is an $\mathcal{L}_{\omega_1\omega}$-sentence φ in an expanded language $\mathcal{L}' \supseteq \mathcal{L}$ such that the structures in $\mathcal{C}$ are the reducts to $\mathcal{L}$ of the models of φ. $\mathcal{C}$ is a computable $PC_{\mathcal{L}_{\omega_1\omega}}$-class if φ is a computable sentence.

We can define the Scott height of $PC_{\mathcal{L}_{\omega_1\omega}}$ in a similar way to the Scott height of $\mathcal{L}_{\omega_1\omega}^c$, except that now we consider all $\mathcal{L}_{\omega_1\omega}$-pseudo-elementary classes which are the reducts of the models of a computable sentence. Marker [43] showed that $sh(PC_{\mathcal{L}_{\omega_1\omega}^c}) = \delta_2^1$. Finally, Harrison-Trainor [27] showed that every Scott spectrum of a pseudo-elementary $\mathcal{L}_{\omega_1\omega}$-class is the Scott spectrum of an $\mathcal{L}_{\omega_1\omega}$-sentence.

THEOREM 6.15 (Harrison-Trainor [27]). *Every Scott spectrum of a pseudo-elementary $\mathcal{L}_{\omega_1\omega}$-class is the Scott spectrum of an $\mathcal{L}_{\omega_1\omega}$-sentence. Moreover, every Scott spectrum of a computable pseudo-elementary $\mathcal{L}_{\omega_1\omega}$-class is the Scott spectrum of a computable $\mathcal{L}_{\omega_1\omega}$-sentence.*

Combined, all of these results allow us to compute the Scott height.

COROLLARY 6.16 (Sacks [59], Marker [43], and Harrison-Trainor [27]). $sh(\mathcal{L}_{\omega_1,\omega}^c) = \delta_2^1$.

Marker also studies Scott heights for other types of classes. We say that a class is co-pseudo-elementary $\mathcal{L}_{\omega_1\omega}$, or $c\mathcal{PC}_{\mathcal{L}_{\omega_1\omega}}$, if it is the complement of a pseudo-elementary class. Then Marker [43] shows that:

THEOREM 6.17 (Marker [43]). $sh(c\mathcal{PC}_{\mathcal{L}_{\omega_1\omega}}) = \gamma_2^1$.

Here, γ_2^1 is the least ordinal such that any non-empty Π_2^1 class of ordinals contains an ordinal below γ_2^1. (By a Π_2^1 class of ordinals, we mean a Π_2^1 class of presentations of linear orders on domain ω.) We have $\gamma_2^1 > \delta_2^1$.

REFERENCES

[1] R. ALVIR, N. GREENBERG, M. HARRISON-TRAINOR, and D. TURETSKY, *Scott complexity of countable structures*. **The Journal of Symbolic Logic**, 2021, DOI: 10.1017/jsl.2021.4 .

[2] R. ALVIR, J. F. KNIGHT, and C. McCOY, *Complexity of Scott sentences*. **Fundamenta Mathematicae**, vol. 251(2020), no. 2, pp. 109–129.

[3] C. ASH, J. KNIGHT, M. MANASSE, and T. SLAMAN, *Generic copies of countable structures*. **Annals of Pure and Applied Logic**, vol. 42(1989), no. 3, pp. 195–205.

[4] C. J. ASH and J. KNIGHT, **Computable Structures and the Hyperarithmetical Hierarchy , vol.** 144, Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 2000.

[5] J. T. BALDWIN, S. D. FRIEDMAN, M. KOERWIEN, and M. C. LASKOWSKI, *Three red herrings around Vaught's conjecture*. **Transactions of the American Mathematical Society**, vol. 368(2016), no. 5, pp. 3673–3694.

[6] H. BECKER, *Strange structures from computable model theory*. **Notre Dame Journal of Formal Logic**, vol. 58(2017), no. 1, pp. 97–105.

[7] ———, *Assigning an isomorphism type to a hyperdegree*. **The Journal of Symbolic Logic**, vol. 85(2020), no. 1, pp. 325–337.

[8] W. CALVERT, *The isomorphism problem for classes of computable fields*. **Archive for Mathematical Logic**, vol. 43(2004), no. 3, pp. 327–336.

[9] W. CALVERT, E. FOKINA, S. S. GONCHAROV, J. F. KNIGHT, O. KUDINOV, A. S. MOROZOV, and V. PUZARENKO, *Index sets for classes of high rank structures*. **The Journal of Symbolic Logic**, vol. 72(2007), no. 4, pp. 1418–1432.

[10] W. CALVERT, V. S. HARIZANOV, J. F. KNIGHT, and S. MILLER, *Index sets of computable models*. **Algebra i Logika**, vol. 45(2006), no. 5, pp. 538–574, 631–632.

[11] W. CALVERT and J. F. KNIGHT, *Classification from a computable viewpoint*. this JOURNAL, vol. 12(2006), no. (2), pp. 191–218.

[12] W. CALVERT, J. F. KNIGHT, and J. MILLAR, *Computable trees of Scott rank ω_1^{CK}, and computable approximation*. **The Journal of Symbolic Logic**, vol. 71(2006), no. 1, pp. 283–298.

[13] J. CARSON, V. HARIZANOV, J. KNIGHT, K. LANGE, C. McCOY, A. MOROZOV, S. QUINN, C. SAFRANSKI, and J. WALLBAUM, *Describing free groups*. **Transactions of the American Mathematical Society**, vol. 364(2012), no. 11, pp. 5715–5728.

[14] W. CHAN, *The countable admissible ordinal equivalence relation*. **Annals of Pure and Applied Logic**, vol. 168(2017), no. 6, pp. 1224–1246.

[15] ———, *Bounds on Scott ranks of some Polish metric spaces*. **Journal of Mathematical Logic**, vol. 21 (2021), no. 1, Paper No. 2150001, 23 pp.

[16] W. CHAN and R. CHEN, *Bounds on continuous Scott rank*. **Proceedings of the American Mathematical Society**, vol. 148(2020), no. 8, pp. 3591–3605.

[17] J. CHISHOLM, *Effective model theory vs. recursive model theory*. **The Journal of Symbolic Logic**, vol. 55(1990), no. 3, pp. 1168–1191.

[18] P. CHOLAK, S. GONCHAROV, B. KHOUSSAINOV, and R. A. SHORE, *Computably categorical structures and expansions by constants*. **The Journal of Symbolic Logic**, vol. 64(1999), no. 1, pp. 13–37.

[19] M. DOUCHA, *Scott rank of Polish metric spaces*. **Annals of Pure and Applied Logic**, vol. 165(2014), no. 12, pp. 1919–1929.

[20] ———. *Erratum to: "Scott rank of Polish metric spaces"* [*Ann. Pure Appl. Logic* 165 (12) (2014) 1919–1929]. **Annals of Pure and Applied Logic**, vol. 168(2017), no. 7, p. 1490.

[21] R. G. DOWNEY, A. M. KACH, S. LEMPP, A. E. M. LEWIS-PYE, A. MONTALBÁN, and D. D. TURETSKY, *The complexity of computable categoricity*. **Advances in Mathematics**, vol. 268(2015), pp. 423–466.

[22] C. E. FREER, *Models with high Scott rank*, Ph.D. thesis, Harvard University, ProQuest LLC, Ann Arbor, MI, 2008.

[23] S. GAO, *Complexity ranks of countable models*. **Notre Dame Journal of Formal Logic**, vol. 48(2007), no. 1, pp. 33–48.

[24] S. S. GONČAROV, *The number of nonautoequivalent constructivizations*. **Algebra i Logika**, vol. 16(1977), no. 3, pp. 257–282, 377.

[25] S. S. GONČAROV and V. D. DZGOEV, *Autostability of models*. **Algebra i Logika**, vol. 19(1980), no. 1, pp. 45–58, 132.

[26] J. HARRISON, *Recursive pseudo-well-orderings*. **Transactions of the American Mathematical Society**, vol. 131(1968), pp. 526–543.

[27] M. HARRISON-TRAINOR, *Scott ranks of models of a theory*. **Advances in Mathematics**, vol. 330(2018), pp. 109–147.

[28] ———, *Describing finitely presented algebraic structures*, preprint, 2020, http://www-personal.umich.edu/~matthhar/papers/finitely-presented-structures.pdf.

[29] M. HARRISON-TRAINOR and M.-C. HO, *On optimal Scott sentences of finitely generated algebraic structures*. **Proceedings of the American Mathematical Society**, vol. 146(2018), no. 10, pp. 4473–4485.

[30] ———, *Finitely generated groups are universal among finitely generated structures*. **Annals of Pure and Applied Logic**, vol. 172(2021), no. 1, p. 102855.

[31] M. HARRISON-TRAINOR, G. IGUSA, and J. F. KNIGHT, *Some new computable structures of high rank*. **Proceedings of the American Mathematical Society**, vol. 146(2018), no. 7, pp. 3097–3109.

[32] D. R. HIRSCHFELDT, B. KHOUSSAINOV, and R. A. SHORE, *A computably categorical structure whose expansion by a constant has infinite computable dimension*. **The Journal of Symbolic Logic**, vol. 68(2003), no. 4, pp. 1199–1241.

[33] M.-C. HO, *Describing groups*. **Proceedings of the American Mathematical Society**, vol. 145(2017), no. 5, pp. 2223–2239.

[34] S. C. KLEENE. *On the forms of the predicates in the theory of constructive ordinals. II*. **American Journal of Mathematics**, vol. 77(1955), pp. 405–428.

[35] J. KNIGHT, A. MONTALBÁN, and N. SCHWEBER, *Computable structures in generic extensions*. **The Journal of Symbolic Logic**, vol. 81(2016), no. 3, pp. 814–832.

[36] J. F. KNIGHT and C. MCCOY, *Index sets and Scott sentences*. **Archive for Mathematical Logic**, vol. 53(2014), nos. 5–6, pp. 519–524.

[37] J. F. KNIGHT and J. MILLAR, *Computable structures of rank ω_1^{CK}*. **Journal of Mathematical Logic**, vol. 10(2010), nos. 1–2, pp. 31–43.

[38] J. F. KNIGHT and V. SARAPH, *Scott sentences for certain groups*. **Archive for Mathematical Logic**, vol. 57(2018), nos. 3–4, pp. 453–472.

[39] P. B. LARSON, *Scott processes*, **Beyond First Order Model Theory** (José Iovino, editors), CRC Press, Boca Raton, FL, 2017, pp. 23–76.

[40] E. G. K. LOPEZ-ESCOBAR, *An interpolation theorem for denumerably long formulas*. **Fundamenta Mathematicae**, vol. 57(1965), pp. 253–272.

[41] M. MAKKAI, *An example concerning Scott heights*. **The Journal of Symbolic Logic**, vol. 46(1981), no. 2, pp. 301–318.

[42] D. MARKER, *An analytic equivalence relation not arising from a Polish group action*. **Fundamenta Mathematicae**, vol. 130(1988), no. 3, pp. 225–228.

[43] ———, *Bounds on Scott rank for various nonelementary classes*. **Archive for Mathematical Logic**, vol. 30(1990), no. 2, pp. 73–82.

[44] ———. **Model Theory: An Introduction, vol. 217**, Graduate Texts in Mathematics, Springer-Verlag, New York, 2002.

[45] ———. **Lectures on Infinitary Model Theory, volume 46**, Lecture Notes in Logic, Association for Symbolic Logic, Chicago, IL; Cambridge University Press, Cambridge, 2016.

[46] C. MCCOY and J. WALLBAUM, *Describing free groups, Part II*: Π_4^0 *hardness and no Σ_2^0 basis*. **Transactions of the American Mathematical Society**, vol. 364(2012), no. 11, pp. 5729–5734.

[47] A. G. MELNIKOV and A. NIES, *The classification problem for compact computable metric spaces*, **The Nature of Computation, vol.** 7921 (Paola Bonizzoni, Vasco Brattka and Benedikt Löwe, editors), Lecture Notes in Computer Science, Springer, Heidelberg, 2013, pp. 320–328.

[48] J. MILLAR and G. E. SACKS, *Atomic models higher up*. **Annals of Pure and Applied Logic**, 155(2008), no. 3, pp. 225–241.

[49] A. W. MILLER, *On the Borel classification of the isomorphism class of a countable model*. **Notre Dame Journal of Formal Logic**, vol. 24(1983), no. 1, pp. 22–34.

[50] D. E. MILLER, *The invariant Π_α^0 separation principle*. **Transactions of the American Mathematical Society**, vol. 242(1978), pp. 185–204.

[51] A. MONTALBÁN, *A computability theoretic equivalent to Vaught's conjecture*. **Advances in Mathematics**, vol. 235(2013), pp. 56–73.

[52] ———, *A robuster Scott rank*. **Proceedings of the American Mathematical Society**, vol. 143(2015), no. 12, pp. 5427–5436.

[53] ———, *Effectively existentially-atomic structures*, **Computability and Complexity** (A. Day, M. Fellows, N. Greenberg, B. Khoussainov, A. Melnikov, and F. Rosamond, editor), Springer, Cham, 2017, pp. 221–237.

[54] ———, **Computable Structure Theory: Beyond the Arithmetic**, Draft.

[55] M. MORLEY, *The number of countable models*. **The Journal of Symbolic Logic**, vol. 35(1970), pp. 14–18.

[56] M. NADEL, *Scott sentences and admissible sets*. **Annals of Mathematics Logic**, vol. 7(1974), pp. 267–294.

[57] A. NIES, *Describing groups*. this JOURNAL, vol. 13(2007), no. 3, 305–339.

[58] G. PAOLINI, *Computable Scott sentences for quasi-Hopfian finitely presented structures*, preprint, 2020, arXiv:2010.13167.

[59] G. E. SACKS. *On the number of countable models*, **Southeast Asian Conference on Logic (Singapore, 1981)**, **vol. 111** (Chi Tat Chong and Malcolm John Wicks, editors), Studies in Logic and the Foundations of Mathematics. North-Holland, Amsterdam, 1983, pp. 185–195.

[60] ———, **Higher Recursion Theory**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin, 1990.

[61] ———, *Bounds on weak scattering*. **Notre Dame Journal of Formal Logic**, vol. 48(2007), no. 1, pp. 5–31.

[62] D. SCOTT, *Logic with denumerably long formulas and finite strings of quantifiers*, **Theory of Models (Proceedings of the 1963 International Symposium at Berkeley)** (J. W. Addison, Leon Henkin, Alfred Tarski, editors), North-Holland, Amsterdam, 1965, pp. 329–341.

[63] J. H. SILVER, *Counting the number of equivalence classes of Borel and Coanalytic equivalence relations*. **Annals of Mathematical Logic**, vol. 18(1980), no. 1, pp. 1–28.

[64] D. TURETSKY, *Coding in the automorphism group of a computably categorical structure*. **Journal of Mathematical Logic**, vol. 20(2020), p. 2050016.

[65] R. L. VAUGHT. *Denumerable models of complete theories*, **Infinitistic Methods (Proceedings of the Symposium on Foundations of Mathematics. Warsaw, 2–9 September 1959)**, Pergamon, Oxford; Państwowe Wydawnictwo Naukowe, Warsaw, 1961, pp. 303–321.

[66] W. W. WADGE, **Reducibility and determinateness on the Baire space**, Ph.D. thesis, University of California, Berkeley, ProQuest LLC, Ann Arbor, MI, 1983.

[67] I. B. YAACOV, M. DOUCHA, A. NIES, and T. TSANKOV, *Metric Scott analysis*. **Advances in Mathematics**, vol. 318(2017), pp. 46–87.

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