

On a property of the nodal set of least energy sign-changing solutions for quasilinear elliptic equations

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In this note, we prove the Payne-type conjecture about the behaviour of the nodal set of least energy sign-changing solutions for the equation $-\Delta_p u = f(u)$ in bounded Steiner symmetric domains $\Omega \subset \mathbb{R}^N$ under the zero Dirichlet boundary conditions. The nonlinearity f is assumed to be either superlinear or resonant. In the latter case, least energy sign-changing solutions are second eigenfunctions of the zero Dirichlet p -Laplacian in Ω . We show that the nodal set of any least energy sign-changing solution intersects the boundary of Ω . The proof is based on a moving polarization argument.

Keywords: p -Laplacian; superlinear; second eigenvalue; least energy nodal solution; nodal set; Payne conjecture; polarization

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1. Introduction

Consider the boundary value problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathcal{D})$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p -Laplacian, $p > 1$. We impose the following assumptions on Ω :

(O₁) $\Omega \subset \mathbb{R}^N$ is a bounded connected open set, $N \geq 2$.

(O₂) Ω is Steiner symmetric with respect to the hyperplane $H_0 := \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1 = 0\}$, see [21, p. 14]. Equivalently, Ω is convex with respect to the axis e_1 (i.e., any line segment parallel to e_1 with endpoints in Ω is contained in Ω) and symmetric with respect to H_0 .

(O₃) If Ω satisfies the interior ball condition at $x \in \partial\Omega$, then $\partial\Omega$ is of class $C^{1,\varsigma}$ in a neighbourhood of x for some $\varsigma \in (0, 1)$.

(O₄) Ω satisfies the exterior ball condition at any $x \in \partial\Omega$.

For instance, an open ball centred at the origin or a convex polytope which is symmetric with respect to H_0 satisfies (O₁) – (O₄), but a spherical shell does not satisfy (O₂). While (O₁) and (O₂) are principal for our arguments, the assumptions (O₃) and (O₄) can potentially be relaxed.

We will consider two main types of the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$:

I. The first type contains f with *superlinear* and subcritical behaviour and can be described by the following four assumptions:

(A₁) $f \in C^1(\mathbb{R} \setminus \{0\}) \cap C_{loc}^{0,\gamma}(\mathbb{R})$ for some $\gamma \in (0, 1)$.

(A₂) There exist $q \in (p, p^*)$ and $C > 0$ such that $|sf'(s)|, |f(s)| \leq C(|s|^{q-1} + 1)$ for all $s \in \mathbb{R} \setminus \{0\}$. Here $p^* = Np/(N - p)$ if $p < N$ and $p^* = +\infty$ if $p \geq N$.

(A₃) $f'(s) > (p - 1)((f(s))/(s)) > 0$ for all $s \in \mathbb{R} \setminus \{0\}$, and $\limsup_{s \rightarrow 0} f(s)/(|s|^{p-2}s) < \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first eigenvalue of the zero Dirichlet p -Laplacian in Ω , that is,

$$\lambda_1(\Omega) = \min_{u \in \mathcal{S}} \int_{\Omega} |\nabla u|^p \, dx,$$

where $\mathcal{S} := \{u \in \mathring{W}_p^1(\Omega) : \|u\|_{L^p(\Omega)} = 1\}$.

(A₄) There exist $s_0 > 0$ and $\theta > p$ such that $0 < \theta F(s) \leq sf(s)$ for all $|s| > s_0$, where

$$F(s) := \int_0^s f(t) \, dt.$$

The model case of the nonlinearity which satisfies (A₁) – (A₄) is $f(u) = C|u|^{q-2}u$ for any $q \in (p, p^*)$ and $C > 0$.

II. The second type of the nonlinearity f is the *resonant* case $f(u) = \lambda_2(\Omega)|u|^{p-2}u$, where $\lambda_2(\Omega)$ is the second eigenvalue of the zero Dirichlet p -Laplacian in Ω which can be characterized as (see [13])

$$\lambda_2(\Omega) = \inf_{\mathcal{A} \subset \mathcal{F}_2} \sup_{u \in \mathcal{A}} \int_{\Omega} |\nabla u|^p \, dx,$$

where $\mathcal{F}_2 := \{\mathcal{A} \subset \mathcal{S} : \text{there exists a continuous odd surjection } h : S^1 \rightarrow \mathcal{A}\}$ and S^1 stands for a circle in \mathbb{R}^2 .

Weak solutions of (D) are critical points of the energy functional $E : \mathring{W}_p^1(\Omega) \rightarrow \mathbb{R}$ defined as

$$E[u] = \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx - \int_{\Omega} F(u) \, dx.$$

The functional E is weakly lower semicontinuous and belongs to $C^1(\mathring{W}_p^1(\Omega))$.

REMARK 1.1. Let $u \in \mathring{W}_p^1(\Omega)$ be a weak solution of (D). Then $u \in L^\infty(\Omega)$ (it can be shown using a bootstrap argument, see, for example, [14, lemma 3.14]), and hence $u \in C_{\text{loc}}^{1,\delta}(\Omega)$, see [27]. Moreover, if $x \in \partial\Omega$ has a neighbourhood $B(x, \varepsilon)$ such that $\partial\Omega \cap B(x, \varepsilon)$ is of class $C^{1,\varsigma}$ (e.g., under the assumption (O_3)), then $u \in C_{\text{loc}}^{1,\kappa}(\Omega \cup (\partial\Omega \cap B(x, \varepsilon)))$, as it follows from [22, § 3].

In this note, we will study weak nodal (or, equivalently, sign-changing) solutions of (D), that is, solutions $u \in \mathring{W}_p^1(\Omega)$ such that $u = u^+ + u^-$, where $u^+ := \max\{u, 0\}$, $u^- := \min\{u, 0\}$, and $u^\pm \not\equiv 0$ in Ω . In the resonant case $f(u) = \lambda_2(\Omega)|u|^{p-2}u$, any nontrivial solution (which is necessarily nodal) can be naturally referred as *second eigenfunction*. Noting that $u^\pm \in \mathring{W}_p^1(\Omega)$, it is not hard to see that any nodal solution of (D) belongs to the *nodal Nehari set*

$$\mathcal{M}(\Omega) := \left\{ u \in \mathring{W}_p^1(\Omega) : u^\pm \not\equiv 0, E'[u]u^\pm \equiv \int_{\Omega} |\nabla u^\pm|^p \, dx - \int_{\Omega} u^\pm f(u^\pm) \, dx = 0 \right\}.$$

In the superlinear case (i.e., f satisfies $(A_1) - (A_4)$), among the set of all nodal solutions of (D), we will be interested in the *least energy* nodal solutions, which can be obtained as follows. Any minimizer of the problem

$$E[u] \rightarrow \min; \quad u \in \mathcal{M}(\Omega) \tag{1.1}$$

is a least energy nodal solution of (D), see [4, 6, 9].

An analogous property also holds in the resonant case $f(u) = \lambda_2(\Omega)|u|^{p-2}u$. Namely, for any function $u \in \mathcal{M}(\Omega)$ there exist nonzero $\alpha, \beta \in \mathbb{R}$ such that $\alpha u^+ + \beta u^-$ is a second eigenfunction of (D). Indeed, since in the resonant case the problem (D) is homogeneous, we can assume that $\|u^\pm\|_{L^p(\Omega)} = 1$. Therefore, $u \in \mathcal{M}(\Omega)$ reads as $\lambda_2(\Omega) = \int_{\Omega} |\nabla u^\pm|^p \, dx$. Consider the set

$$\mathcal{A} := \{v \in \mathring{W}_p^1(\Omega) : v = \alpha u^+ + \beta u^-, |\alpha|^p + |\beta|^p = 1\}.$$

It is not hard to see that $\mathcal{A} \subset \mathcal{F}_2$ and $\lambda_2(\Omega) = \int_{\Omega} |\nabla v|^p \, dx$ for any $v \in \mathcal{A}$. If we suppose now that there is no second eigenfunction which belongs to \mathcal{A} , then we apply the deformation lemma of Ghoussoub [16, lemma 3.7] (see also the particular statement for our case in [3, proposition 2.2]) to generate $\tilde{\mathcal{A}} \subset \mathcal{F}_2$ such that

$$\lambda_2(\Omega) \leq \sup_{v \in \tilde{\mathcal{A}}} \int_{\Omega} |\nabla v|^p \, dx < \sup_{v \in \mathcal{A}} \int_{\Omega} |\nabla v|^p \, dx = \lambda_2(\Omega),$$

a contradiction. Since any second eigenfunction changes its sign in Ω , we conclude that $\alpha, \beta \neq 0$.

The above-mentioned facts indicate that least energy nodal solutions of (\mathcal{D}) under the assumptions $(A_1) - (A_4)$ and second eigenfunctions of the zero Dirichlet p -Laplacian are conceptually the same objects. Analogous observation links ground states of (\mathcal{D}) (i.e., least energy solutions) with first eigenfunctions. See [18] for rigorous results in this direction.

Let us define the *nodal set* of $u \in C(\Omega)$ as

$$\mathcal{Z}(u) = \overline{\{x \in \Omega : u(x) = 0\}}.$$

If u is a nodal solution of (\mathcal{D}) , then $\mathcal{Z}(u)$ is nonempty. Moreover, the strong maximum principle (see, for instance, [28]) implies that $\mathcal{Z}(u)$ does not have isolated points. However, we are not aware of the unique continuation property for nodal solutions of (\mathcal{D}) , that is, the fact that $\text{Int}(\mathcal{Z}(u)) = \emptyset$.

Connected components of $\Omega \setminus \mathcal{Z}(u)$ are called nodal domains of u . Note that each least energy nodal solution of (\mathcal{D}) has exactly two nodal domains. Indeed, the arguments of [9, p. 1051] can be easily adopted for the superlinear case, and the resonant case was treated in [11].

In this note, we intend to prove that the nodal set $\mathcal{Z}(u)$ intersects the boundary of Ω . Namely, we prove the following result.

THEOREM 1.2. *Assume that Ω satisfies $(O_1) - (O_4)$, and either $f(u) = \lambda_2(\Omega)|u|^{p-2}u$ or $(A_1) - (A_4)$ holds. Let u be a least energy nodal solution of (\mathcal{D}) . Then*

$$\text{dist}(\mathcal{Z}(u), \partial\Omega) = 0. \tag{1.2}$$

In the *resonant* case with the classical Laplace operator, such type of results is connected with the *Payne conjecture* [24, p. 467] which stated that the nodal set of second eigenfunctions of the zero Dirichlet Laplacian in any domain cannot be closed. It is known that the conjecture is not generally true, see counterexamples constructed in [20] and [15] for special domains. However, determination of classes of domains for which the Payne conjecture is valid remains an attractive problem. The interested reader will easily find various results in this direction in the literature. Let us specially emphasize the work [25] where Payne proved his conjecture for planar sets which satisfy (O_1) and (O_2) . Under the same assumptions, the general higher-dimensional case was treated in [12]. Being proved by entirely different arguments, our theorem 1.2 represents the direct generalization of these results for the nonlinear settings under the additional assumptions (O_3) and (O_4) . In the resonant case with the p -Laplacian, the validity of the Payne conjecture was proposed as an open problem even in the case of a ball, see [3, remark 4.2]. Easily, a ball satisfies $(O_1) - (O_4)$, and hence theorem 1.2 applies.

In the *superlinear* case with the Laplace operator, the Payne conjecture was proved in [1] for a ball in \mathbb{R}^N with $N \geq 2$ and for a two-dimensional annulus. Considering the homogeneous superlinear case $f(u) = |u|^{q-2}u$ and smooth $\Omega \subset \mathbb{R}^2$, the authors of [19] were able to prove (1.2) by appealing to [2] when q is sufficiently close to 2 and Ω is convex; the case of large q was considered in [17]. See also [7] for related questions. However, we were not aware of the results on the Payne conjecture for the superlinear case with the general p -Laplace operator.

This note is organized as follows. In §2, we give several necessary facts about the polarization of functions and sets. In §3, we prove theorem 1.2. The proof of Theorem 1.2 consists of two steps. In the first step, we show that there *always* exists a least energy nodal solution of (D) which satisfies (1.2). Such a weak form of the Payne conjecture is proved using (O₁) and (O₂) only. The arguments are based on the analysis of the behaviour of the polarization of least energy nodal solutions with respect to moving hyperplanes. In the second step, we prove that if (1.2) does not hold for some least energy nodal solution, then the least energy nodal solution obtained in the previous step has a contradictory behaviour on ∂Ω. Here the arguments are based on the Hopf maximum principle which requires us to use (O₃) and (O₄). This finally implies that *any* least energy nodal solution of (D) satisfies (1.2). In §4, we discuss possible relaxations of the assumptions (A₁) – (A₄).

2. Auxiliary facts

First, we recall the notion of *polarization* (or, equivalently, *two-point rearrangement*) of sets and functions, see, for example, [4, 8]. Consider the hyperplane $H_a := \{x \in \mathbb{R}^N : x_1 = a\}$ where $x := (x_1, x_2, \dots, x_N)$, and let $\sigma_a(x) := (2a - x_1, x_2, \dots, x_N)$ be the reflection of x with respect to H_a . Denote the half-spaces separated by H_a as

$$\Sigma_a^+ := \{x \in \mathbb{R}^N : x_1 > a\} \quad \text{and} \quad \Sigma_a^- := \{x \in \mathbb{R}^N : x_1 < a\}.$$

We define two polarizations of a measurable set Ω with respect to H_a as follows:

$$P_a\Omega = \begin{cases} \Omega \cap \sigma_a(\Omega) & \text{in } \Sigma_a^+, \\ \Omega \cup \sigma_a(\Omega) & \text{in } \Sigma_a^-, \\ \Omega & \text{on } H_a, \end{cases} \quad \tilde{P}_a\Omega = \begin{cases} \Omega \cup \sigma_a(\Omega) & \text{in } \Sigma_a^+, \\ \Omega \cap \sigma_a(\Omega) & \text{in } \Sigma_a^-, \\ \Omega & \text{on } H_a. \end{cases}$$

Here \tilde{P}_a is introduced only for simplicity of further arguments since $\tilde{P}_a\Omega = \mathbb{R}^N \setminus (P_a(\mathbb{R}^N \setminus \Omega))$.

It is not hard to see that P_a and \tilde{P}_a satisfy the following domain monotonicity property.

LEMMA 2.1. *If $\Omega_1 \subset \Omega_2$, then $P_a\Omega_1 \subset P_a\Omega_2$ and $\tilde{P}_a\Omega_1 \subset \tilde{P}_a\Omega_2$.*

The following result links the Steiner symmetry of Ω with polarizations of Ω , see [8, lemma 6.3]. We give its proof for the sake of completeness.

LEMMA 2.2. *Ω satisfies (O₂) if and only if $P_a\Omega = \Omega$ for all $a \geq 0$ and $\tilde{P}_a\Omega = \Omega$ for all $a \leq 0$.*

Proof. Recall that Ω is Steiner symmetric with respect to the hyperplane H_0 if and only if for any point $(b_1, b_2, \dots, b_N) \in \Omega$ the whole line segment $\{(s, b_2, \dots, b_N) \in \mathbb{R}^N : |s| \leq |b_1|\}$ is a subset of Ω . Since every segment of this kind is stable under P_a for $a \geq 0$ and \tilde{P}_a for $a \leq 0$, we get the necessary part of the lemma.

To prove the sufficient part, we suppose, by contradiction, that there exist $b \in \Omega$ and $c \in \{(s, b_2, \dots, b_N) \in \mathbb{R}^N : |s| \leq |b_1|\}$ such that $c \notin \Omega$. Assume, without loss of

generality, that $b_1 > 0$, and take $a = (b_1 + c_1)/2 \geq 0$. Then we see that P_a will exchange the points c and b , that is, $P_a\Omega \neq \Omega$. It is a contradiction. \square

We will use P_a to polarize functions. Let v be a measurable function defined on the whole \mathbb{R}^N . Define the polarization of v with respect to H_a as

$$(P_a v)(x) = \begin{cases} \min\{v(x), v(\sigma_a(x))\}, & x \in \Sigma_a^+, \\ \max\{v(x), v(\sigma_a(x))\}, & x \in \Sigma_a^-, \\ v(x), & x \in H_a. \end{cases}$$

It is known that $(P_a v)^\pm = P_a(v^\pm)$, see [4, lemma 2.1]. Therefore,

$$P_a(v^+ + v^-) = P_a v = (P_a v)^+ + (P_a v)^- = P_a(v^+) + P_a(v^-), \tag{2.1}$$

and we may write $P_a v^\pm$ for short. It is not hard to see that $\text{supp } P_a v^+ = P_a(\text{supp } v^+)$ and $\text{supp } P_a v^- = \tilde{P}_a(\text{supp } v^-)$. Therefore, we deduce from (2.1) that

$$\text{supp } P_a v = P_a(\text{supp } v^+) \cup \tilde{P}_a(\text{supp } v^-). \tag{2.2}$$

The following result easily follows from lemmas 2.1 and 2.2.

COROLLARY 2.3. *Let Ω satisfy (O_2) and v be such that $\text{supp } v \subset \bar{\Omega}$. Then $\text{supp } P_a v^+ \subset \bar{\Omega}$ for all $a \geq 0$ and $\text{supp } P_a v^- \subset \bar{\Omega}$ for all $a \leq 0$.*

It is known that if $v \in W_p^1(\mathbb{R}^N)$, then $P_a v \in W_p^1(\mathbb{R}^N)$, see [8, lemma 5.3] applied to v^\pm . Moreover,

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla P_a v^\pm|^p dx &= \int_{\mathbb{R}^N} |\nabla v^\pm|^p dx, \\ \int_{\mathbb{R}^N} P_a v^\pm f(P_a v^\pm) dx &= \int_{\mathbb{R}^N} v^\pm f(v^\pm) dx, \quad \int_{\mathbb{R}^N} F(P_a v^\pm) dx = \int_{\mathbb{R}^N} F(v^\pm) dx, \end{aligned}$$

see [4, lemmas 2.2 and 2.3]. Furthermore, it is possible to polarize functions from $\mathring{W}_p^1(\Omega)$, where Ω is an open set, by considering their trivial extension to \mathbb{R}^N . Resulting functions belong to $\mathring{W}_p^1(P_a\Omega \cup \tilde{P}_a\Omega)$, as it follows from [8, corollary 5.1]. This result can be clarified in the following way. Let $v \in \mathring{W}_p^1(\Omega) \cap C(\Omega)$ and let $\mathcal{O}^+ := \{x \in \Omega : v(x) > 0\}$ and $\mathcal{O}^- := \{x \in \Omega : v(x) < 0\}$. Then [10, lemma 5.6] implies that $v^\pm \in \mathring{W}_p^1(\mathcal{O}^\pm)$, and therefore $P_a v \in \mathring{W}_p^1(P_a\mathcal{O}^+ \cup \tilde{P}_a\mathcal{O}^-)$. Recalling now (2.2), we arrive at the following fact.

LEMMA 2.4. *Let Ω satisfy (O_1) . Let $u \in \mathcal{M}(\Omega) \cap C(\Omega)$ and $\text{supp } P_a u \subset \bar{\Omega}$ for some $a \in \mathbb{R}$. Then $P_a u \in \mathcal{M}(\Omega)$ and $E[P_a u] = E[u]$. In particular, if u is a minimizer of (1.1) such that $\text{supp } P_a u \subset \bar{\Omega}$, then $P_a u$ is also a minimizer of (1.1), that is, u and $P_a u$ are both least energy nodal solutions of (\mathcal{D}) .*

3. Proof of the main result

Throughout this section, u denotes a least energy nodal solution of (\mathcal{D}) and we suppose, by contradiction, that the nodal set of u does not intersect the boundary of Ω , that is,

$$\text{dist}(\mathcal{Z}(u), \partial\Omega) =: d > 0. \tag{3.1}$$

We will assume, in particular, that $u > 0$ in $\{x \in \Omega : \text{dist}(x, \partial\Omega) < d\}$, and derive a contradiction.

Notice that if u is negative near $\partial\Omega$, then $v := -u$ is a solution of

$$\begin{cases} -\Delta_p v = -f(-v) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Since $g(s) := -f(-s)$ satisfies $(A_1) - (A_4)$, we can apply the arguments from below to v . Alternatively, we can apply the following arguments to u using \tilde{P}_a instead of P_a .

The idea of the proof of Theorem 1.2 is based on the following simple observation. Considering the polarization $P_a u$, we continuously increase $a \geq 0$ until a moment $d_1/2$ (see below) when $\mathcal{Z}(P_{d_1/2} u)$ touches $\partial\Omega$ for the first time. From lemma 2.4 we see that $P_{d_1/2} u$ is a least energy nodal solution of (\mathcal{D}) . However, we show that $P_{d_1/2} u$ cannot be a solution since it contradicts the Hopf maximum principle at a special point of the boundary. The details are as follows.

As the first step of the proof of Theorem 1.2, we show that apart from u , (\mathcal{D}) possesses a least energy nodal solution v such that $\text{dist}(\mathcal{Z}(v), \partial\Omega) = 0$.

Consider the e_1 -distance between $\mathcal{Z}(u)$ and the left part of $\partial\Omega$ (see Fig. 1):

$$d_1 := \min \{ \alpha > 0 : (x_1, x_2, \dots, x_N) \in \mathcal{Z}(u) \quad \text{and} \quad (x_1 - \alpha, x_2, \dots, x_N) \in \partial\Omega \}.$$

We have $d_1 \geq d > 0$. Denote the subset of $\mathcal{Z}(u)$ which delivers the minimum to d_1 as $\mathcal{Y}(u)$, that is,

$$\mathcal{Y}(u) := \{ (x_1, x_2, \dots, x_N) \in \mathcal{Z}(u) : (x_1 - d_1, x_2, \dots, x_N) \in \partial\Omega \}. \tag{3.2}$$

Finally, we define the relative boundary of $\mathcal{Y}(u)$ as

$$\partial\mathcal{Y}(u) = \{ x \in \mathcal{Y}(u) : \text{any neighbourhood of } x \text{ contains } y \in \mathcal{Z}(u) \setminus \mathcal{Y}(u) \}.$$

It is not hard to see that $\mathcal{Y}(u)$ and $\partial\mathcal{Y}(u)$ are not empty.

LEMMA 3.1. *Let (O_1) and (O_2) be satisfied. Then $\text{supp } P_a u \subset \bar{\Omega}$ for all $a \in [0, d_1/2]$ and $\text{dist}(\mathcal{Z}(P_{d_1/2} u), \partial\Omega) = 0$. In particular, $P_a u$ is a least energy nodal solution of (\mathcal{D}) for all $a \in [0, d_1/2]$.*

Proof. Recalling (2.2), we will study the behaviour of $\text{supp } P_a u^+$ and $\text{supp } P_a u^-$ with respect to a . First, corollary 2.3 implies that $\text{supp } P_a u^+ \subset \bar{\Omega}$ for all $a \geq 0$. Consider now $\text{supp } P_a u^-$. Take any $x \in \text{supp } u^-$. Then, in view of (3.1), the assumption that $u > 0$ in $\{x \in \Omega : \text{dist}(x, \partial\Omega) < d\}$, and the definition of d_1 , we see

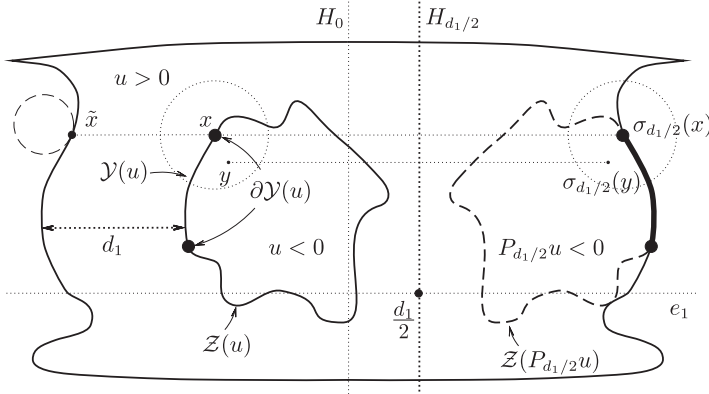


Figure 1. Nodal sets of u and $P_{d_1/2}u$.

that $(x_1 - d_1, x_2, \dots, x_N) \in \bar{\Omega}$. Moreover, $(x_1 - d_1, x_2, \dots, x_N) \in \partial\Omega$ if and only if $x \in \mathcal{Y}(u)$. Hence, from (O_2) , we get

$$\begin{aligned} \sigma_a(x) &= (2a - x_1, x_2, \dots, x_N) \in \bar{\Omega}, \quad \forall a \in [0, d_1/2], \\ \sigma_{d_1/2}(x) &= (d_1 - x_1, x_2, \dots, x_N) \in \partial\Omega \iff x \in \mathcal{Y}(u). \end{aligned} \tag{3.3}$$

Consequently, $\tilde{P}_a x \in \bar{\Omega}$ for all $a \in [0, d_1/2]$. Since $x \in \text{supp } u^-$ is arbitrary, we recall that $\text{supp } P_a u^- = \tilde{P}_a(\text{supp } u^-)$ and hence conclude that $\text{supp } P_a u^- \subset \bar{\Omega}$ for all $a \in [0, d_1/2]$.

Let us take some $x \in \partial\mathcal{Y}(u)$. By definition, any small neighbourhood of x contains $y \in \mathcal{Z}(u) \setminus \mathcal{Y}(u)$. Therefore, $\sigma_{d_1/2}(y) \in \Omega$. On the one hand, since $u > 0$ near $\partial\Omega$, we see that $u(\sigma_{d_1/2}(y)) > 0$. On the other hand, $u(y) = 0$. Thus, $P_{d_1/2}$ will exchange y and $\sigma_{d_1/2}(y)$, and hence $P_{d_1/2}u(\sigma_{d_1/2}(y)) = 0$. Since a neighbourhood of x is arbitrary, we conclude that $\text{dist}(\mathcal{Z}(P_{d_1/2}u), \partial\Omega) = 0$.

Finally, applying lemma 2.4, we see that $P_a u$ is a least energy nodal solution of (D) for all $a \in [0, d_1/2]$. \square

REMARK 3.2. Lemma 3.1 is reminiscent of the moving plane method of Serrin [26], with the difference that, instead of the reflection, the polarization of functions is used.

As the second step of the proof of Theorem 1.2, we show that under the assumption (3.1) the least energy nodal solution $P_{d_1/2}u$ has a contradictory behaviour on $\partial\Omega$. We start with the following auxiliary result.

LEMMA 3.3. Let (O_1) and (O_4) be satisfied, and let $x \in \mathcal{Y}(u)$. Then $|\nabla u(x)| \neq 0$. Moreover, $\mathcal{Z}(u)$ is of class C^2 in a neighbourhood of x , and hence $\text{supp } u^\pm$ satisfies the interior ball condition at x .

Proof. Fix any $x \in \mathcal{Y}(u)$. Then $\tilde{x} := (x_1 - d_1, x_2, \dots, x_N) \in \partial\Omega$, see Fig. 1. Note that in view of (O_4) , we can find a ball $B(\tilde{y}, \varepsilon)$ which touches \tilde{x} from the outside of Ω . Since we assume that (3.1) is satisfied, the translated ball $B(\tilde{y} + d_1 e_1, \varepsilon)$ is a

subset of Ω for $\varepsilon > 0$ small enough. Moreover, from the definition of d_1 it is not hard to deduce that the e_1 -distance between any $z \in B(\tilde{y}, \varepsilon)$ and $\mathcal{Z}(u)$ is greater than or equal to d_1 , and it is equal to d_1 if and only if $z \in \partial\Omega$ and $z + d_1e_1 \in \mathcal{Y}(u)$. Since we assume that $u > 0$ in the neighbourhood of $\partial\Omega$, we conclude that $z + d_1e_1 \in \text{supp } u^+$ and hence $B(\tilde{y} + d_1e_1, \varepsilon) \subset \text{supp } u^+$.

Recalling now that $u \in C_{\text{loc}}^{1,\delta}(\Omega)$ (see remark 1.1), we use the Hopf maximum principle [28, theorem 5] to get $|\nabla u(x)| \neq 0$. This implies that the p -Laplacian is strictly elliptic in a neighbourhood of x , which yields that u is C^2 -smooth in this neighbourhood. Applying the implicit function theorem, we conclude that $\mathcal{Z}(u)$ is a graph of a C^2 -function in a neighbourhood of x , and the interior ball condition on $\text{supp } u^\pm$ at x follows immediately. \square

LEMMA 3.4. *Let $(O_1) - (O_4)$ be satisfied. Then $P_{d_1/2}u$ is not a solution of (\mathcal{D}) .*

Proof. Let us denote, for simplicity, $v := P_{d_1/2}u$. Take any $x \in \partial\mathcal{Y}(u)$. By the definition of $\partial\mathcal{Y}(u)$, we have $\sigma_{d_1/2}(x) \in \partial\Omega$. Fix a small $\varepsilon \in (0, d)$ and consider the neighbourhoods $B(x, \varepsilon)$ and $B(\sigma_{d_1/2}(x), \varepsilon)$, see Fig. 1. By lemma 3.3, the first neighbourhood has nonempty intersection with $\text{supp } u^-$, while the second does not (since we assume that $u > 0$ in the neighbourhood of $\partial\Omega$). Hence, considering any $y \in B(x, \varepsilon) \cap \text{supp } u^-$, we obtain that $v(y) = u(\sigma_{d_1/2}(y)) > 0$ and $v(\sigma_{d_1/2}(y)) = u(y) < 0$. Thus, small neighbourhoods of $\sigma_{d_1/2}(x)$ intersect with $\text{supp } v^-$. Moreover, since $x \in \partial\mathcal{Y}(u)$, we also see that small neighbourhoods of $\sigma_{d_1/2}(x)$ intersect with $\text{supp } v^+$ and

$$\partial\Omega \cap \text{supp } v^+ \cap B(\sigma_{d_1/2}(x), \varepsilon) \neq \emptyset. \tag{3.4}$$

By lemma 3.3, $\text{supp } u^-$ satisfies the interior ball condition at x , which implies that $\text{supp } v^-$ and, consequently, Ω satisfy the interior ball condition at $\sigma_{d_1/2}(x)$. Therefore, due to (O_3) , we get $v \in C_{\text{loc}}^{1,\kappa}(\Omega \cup (\partial\Omega \cap B(\sigma_{d_1/2}(x), \varepsilon)))$ (see remark 1.1), and consequently, the Hopf maximum principle implies

$$\frac{\partial v}{\partial n}(\sigma_{d_1/2}(x)) > 0, \tag{3.5}$$

where n is the outward unit normal to $\partial\Omega$. On the other hand, since (3.4) holds, we get for any sufficiently small $\varepsilon > 0$ that

$$\frac{\partial v}{\partial n}(\hat{x}) \leq 0, \quad \forall \hat{x} \in \partial\Omega \cap \text{supp } v^+ \cap B(\sigma_{d_1/2}(x), \varepsilon). \tag{3.6}$$

Finally, taking a sequence $\{\hat{x}_k\}_{k \in \mathbb{N}} \subset \partial\Omega \cap \text{supp } v^+ \cap B(\sigma_{d_1/2}(x), \varepsilon)$ such that $\hat{x}_k \rightarrow \sigma_{d_1/2}(x)$ as $k \rightarrow +\infty$, we obtain a contradiction between (3.5) and (3.6), since v is regular up to $\partial\Omega \cap B(\sigma_{d_1/2}(x), \varepsilon)$. \square

REMARK 3.5. The idea of a contradiction was inspired by the proof of [3, lemma 3.2], where the authors showed the nonradiality of second eigenfunctions of the zero Dirichlet p -Laplacian in a ball.

Finally, we see that lemmas 3.1 and 3.4 contradict each other, which implies that (3.1) does not hold, and hence theorem 1.2 is valid.

4. Discussion

The assumptions $(A_1) - (A_4)$ can be relaxed. Indeed, all we need for the proof of Theorem 1.2 is that: 1) a minimizer of the variational problem (1.1) exists; 2) any minimizer of (1.1) is a nodal solution of (\mathcal{D}) ; 3) such solutions are sufficiently regular and satisfy the Hopf maximum principle on smooth parts of $\partial\Omega$; 4) polarizations P_a and \tilde{P}_a preserve all functionals present in E and $\mathcal{M}(\Omega)$. For example, the assumption (A_2) is used to guarantee the existence part; that is, since (A_2) holds, the embedding $\mathring{W}_p^1(\Omega) \hookrightarrow L^q(\Omega)$ is compact. If we consider the embedding of $\mathring{W}_p^1(\Omega)$ into suitable Orlicz spaces, we can impose an extension of (A_2) which allows an exponential growth of f for $p = N$ and arbitrary growth for $p > N$, see, for example, [23, (F4) and lemma 5.6]. The assumption (A_3) can be relaxed to allow, for instance, convex-concave nonlinearities. In this case, the variational problem (1.1) has to be restricted to suitable subsets of $\mathcal{M}(\Omega)$ (see, e.g., [5]) and then the arguments from the present note can be applied.

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