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ON LARGE DEVIATION RATES FOR SUMS ASSOCIATED WITH GALTON–WATSON PROCESSES

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Abstract

Given a supercritical Galton–Watson process $\{Z_n\}$ and a positive sequence $\{\varepsilon_n\}$, we study the limiting behaviors of $\mathbb{P}(S_{Z_n}/Z_n \geq \varepsilon_n)$ with sums S_n of independent and identically distributed random variables X_i and $m = \mathbb{E}[Z_1]$. We assume that we are in the Schröder case with $\mathbb{E}Z_1 \log Z_1 < \infty$ and X_1 is in the domain of attraction of an α-stable law with $0 < \alpha < 2$. As a by-product, when Z_1 is subexponentially distributed, we further obtain the convergence rate of Z_{n+1}/Z_n to m as $n \to \infty$.

Keywords: Galton–Watson process; domain of attraction; stable distribution; slowly varying function; large deviation; Lotka–Nagaev estimator; Schröder constant

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1. Introduction and main results

1.1. Motivation

Let $Z = (Z_n)_{n\geq 1}$ be a supercritical Galton–Watson process with $Z_0 = 1$ and offspring distribution $\{p_k : k \geq 0\}$. Define $m = \sum_{k\geq 1} kp_k > 1$. We assume in this paper that $p_0 = 0$ and $0 < p_1 < 1$.

It is known that $Z_{n+1}/Z_n \to m$ almost surely (a.s.) and Z_{n+1}/Z_n is the so-called Lotka–Nagaev estimator of m; see [13]. This estimator has been used in studying the amplification rate and the initial number of molecules for an amplification process in a quantitative polymerase chain reaction experiment; see [11], [12], and [17]. Concerning the Bahadur efficiency of the estimator leads to the investigation of the large deviation behaviors of Z_{n+1}/Z_n. In fact, it was proved in [13] that if $\sigma^2 = \mathrm{var}(Z_1) \in (0, \infty)$ then

$$\lim_{n\to\infty} \mathbb{P}\bigg(m^{n/2}\bigg(\frac{Z_{n+1}}{Z_n} - m\bigg) < x\bigg) = \int_0^\infty \Phi\bigg(\frac{x\sqrt{u}}{\sigma}\bigg)\omega(u)\,\mathrm{d}u, \tag{1}$$

where Φ is the standard normal distribution function and ω denotes the continuous density function of $W := \lim_{n\to\infty} Z_n/m^n$ a.s. In [1], Athreya showed that if $p_1 m^r > 1$ and $\mathbb{E}[Z_1^{2r+\delta}] < \infty$ for some $r \geq 1$ and $\delta > 0$, then

$$\lim_{n\to\infty} \frac{1}{p_1^n}\mathbb{P}\bigg(\bigg|\frac{Z_{n+1}}{Z_n} - m\bigg| \geq \varepsilon\bigg) \quad \text{exists finitely;}$$

see also [2]. Later, Ney and Vidyashankar [15] weakened the assumption and were able to obtain the rate of convergence of a Lotka–Nagaev estimator by studying the asymptotic properties of

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the harmonic moments of Z_n, where it was assumed that $\mathbb{P}(Z_1 \geq x) \sim ax^{1-\eta}$ for some $\eta > 2$ and $a > 0$. See [16] for some further results.

Recently, Fleischmann and Wachtel [10] considered a generalization of the above problem by studying sums indexed by Z; see also [16]. More precisely, let $X = (X_n)_{n\geq 1}$ denote a family of independent and identically distributed (i.i.d.) real-valued random variables. They investigated the large deviation probabilities for S_{Z_n}/Z_n: the convergence rate of

$$\mathbb{P}\biggl(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\biggr) \quad \text{as } n \to \infty,$$

where $\varepsilon_n \to 0$ is a positive sequence and $S_n := X_1 + X_2 + \cdots + X_n$. In fact, if $X_1 \overset{\mathrm{D}}{=} Z_1 - m$, then $S_{Z_n}/Z_n \overset{\mathrm{D}}{=} Z_{n+1}/Z_n - m$. The assumption in [10] is that $\mathbb{E}[Z_1 \log Z_1] < \infty$, $\mathbb{E}[X_1^2] < \infty$ and $\mathbb{P}(X_1 \geq x) \sim ax^{-\eta}$ for some $\eta > 2$, which implies that X_1 is in the domain of attraction of normal distributions.

Motivated by the above mentioned works, the main purpose of this paper is to study the convergence rates of Z_{n+1}/Z_n under weaker conditions. We shall use the framework of [10] but we assume that $\mathbb{E}[Z_1 \log Z_1] < \infty$ and X_1 is in the domain of attraction of a stable law; see Assumptions 1 and 2 below. Then we answer the question raised in [10, Remark 11(a)]. In particular, we further obtain the convergence rate of Z_{n+1}/Z_n under the assumption $\mathbb{P}(Z_1 > x) \sim L(x)x^{-\beta}$ for some $1 < \beta < 2$ and some slowly varying function L, which partially improves upon [15, Theorem 3].

For proofs, we shall use the strategy of [10]. However, our arguments are deeply involved because of the lack of high moments and the perturbations of slowly varying functions. We overcome those difficulties by using Fuk–Nagaev inequalities, estimation of growth of random walks, large deviation probabilities for sums under subexponentiality, and establishing the asymptotic properties of

$$\mathbb{E}[Z_n^{-t} L(\varepsilon_n Z_n)], \qquad t > 0, \text{ as } n \to \infty. \tag{2}$$

In the next section, Section 1.2, we will give our basic assumptions on Z and X. Our main results will be presented in Section 1.3. We prove Fuk–Nagaev inequalities and establish the asymptotic properties of (2) in Section 2. The proofs of the main results will be given in Section 3. With C, c, for example, we denote positive constants which might change from line to line.

1.2. Basic assumptions

Define $F(x) = \mathbb{P}(X_1 \leq x)$.

Assumption 1. *We make the following assumptions:*

- $\mathbb{P}(X_1 \geq x) \sim x^{-\beta} L(x)$, *where* $\beta > 0$ *and* L *is a slowly varying function;*
- *if* $\varepsilon_n \to 0$ *then* L *is bounded away from* 0 *and* ∞ *on every compact subset of* $[0, \infty)$*;*
- X_1 *is in the domain of attraction of an* α*-stable law with* $0 < \alpha < 2$*;*
- $\mathbb{E}[X_1] = 0$ *if* $1 < \alpha < 2$*;*
- $\mathbb{E}[Z_1 \log Z_1] < \infty$, $p_0 = 0$, $p_1 > 0$.

From the assumption, it is easy to see that $\alpha \leq \beta$. The last point in the assumption means that we are in the Schröder case. In fact, we only need to assume that $0 < p_0 + p_1 < 1$.

Remark 1. The second point in the assumption is technical. In fact, by [3, Theorem 1.5.6] for any $\eta > 0$ and $a > 0$, there exist two positive constants C_η such that, for any $y > a, z > a$,

$$\frac{L(z)}{L(y)} \le C_\eta \max\biggl(\Bigl(\frac{z}{y}\Bigr)^{\eta}, \Bigl(\frac{z}{y}\Bigr)^{-\eta}\biggr). \tag{3}$$

If L is bounded away from 0 and ∞ on every compact subset of $[0, \infty)$, then (3) holds for any $y > 0, z > 0$.

Remark 2. Under Assumption 1 it follows that there exists a function $b(k)$ of regular variation of index $1/\alpha$ such that

$$b(k)^{-1} S_k \xrightarrow{\text{D}} U_s, \tag{4}$$

where U_s is an α-stable random variable; see [8] and [20]. Without loss of generality, we may and will assume that function b is continuous and monotonically increasing from $\mathbb{R}^+$ onto $\mathbb{R}^+$ and $b(0) = 0$; see [8]. We also have

$$b(x) = x^{1/\alpha} s(x), \qquad x > 0,$$

where $s\colon (0, \infty) \to (0, \infty)$ is a slowly varying function. Then (3) also holds for s with $y \ge 1, z \ge 1$.

Define

$$\mu(1; x) = \int_{-x}^{x} y F(\mathrm{d}y), \qquad \mu(2; x) = \int_{-x}^{x} y^2 F(\mathrm{d}y). \tag{5}$$

Under Assumption 1, by the arguments in [8], we have, as $x \to +\infty$,

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to p_+, \qquad \frac{F(-x)}{1 - F(x) + F(-x)} \to p_-, \qquad p_+ + p_- = 1,$$

and

$$\frac{x^2[1 - F(x) + F(-x)]}{\mu(2; x)} \to \frac{2 - \alpha}{\alpha},$$

$$\mu(2; x) \sim \begin{cases} \dfrac{\alpha}{2 - \alpha} x^{2-\alpha} R(x) & \text{if } p_+ = 0, \\ \dfrac{\beta p_+}{2 - \beta} x^{2-\beta} L(x) & \text{if } 0 < p_+ < 1, \\ \dfrac{\beta}{2 - \beta} x^{2-\beta} L(x) & \text{if } p_+ = 1, \end{cases} \tag{6}$$

where R is a slowly varying function. Furthermore, the function b in (4) must satisfy, as $x \to +\infty$,

$$x[1 - F(b(x))] \to C p_+ \frac{2 - \alpha}{\alpha}, \qquad x F(-b(x)) \to C p_- \frac{2 - \alpha}{\alpha}; \tag{7}$$

see [8, Equation (5.25)]. In particular, it is implied in the above that if $p_+ = 0$ then $F(-x) \sim x^{-\alpha} R(x)$ as $x \to +\infty$.

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Assumption 2. *For technical reasons, we also need to make the following assumptions:*

- U_s *is strictly stable;*
- *if* $1 < \alpha < 2$, *we assume that* $\liminf_{x\to+\infty} s(x) \in (0, +\infty]$*;*
- *if* $0 < p_+ < 1$ *and* $\alpha = 1$, *we assume that* $\mu(1; x) = 0$ *for all* $x > 0$*;*
- *if* $p_+ = 0$, *we assume that* $\alpha < \beta$*;*
- *if* $1 < \alpha < 2$ *and* $p_+ > 0$, *we assume that*

$$\limsup_{n\to+\infty} \frac{F(-b(n)/[\log n]^{1/\alpha})}{(\log n)F(-b(n))} \le 1.$$

Remark 3. The assumption that U_s is strictly stable implies that, when $\alpha = 1$, we must have $\alpha = \beta$ and *the skewness parameter* of U_s is 0. The second point in Assumption 2 will be used to deduce (32) which is required in Lemma 5. The third point is used in Step 2 in Lemma 6 to find a good upper bound for $\mathbb{P}(x)$, which appears in [14, Theorem 1.2]. The last two points are required in [6, Theorems 9.2 and 9.3], which are needed in our proofs.

From now on, Assumptions 1 and 2 are in force.

1.3. Main results

Before presenting the main results, we first introduce some notation. Recall $b(x)$ from (4). Define $J(x) = xb(x)^{-1}$ and $l(x) = \inf\{y \in [0, \infty) \colon J(y) > x\}$. According to [3, Theorem 1.5.12], $l(x)$ is an asymptotic inverse of J, i.e.

$$l(J(x)) \sim J(l(x)) \sim x \quad \text{as } x \to +\infty.$$

Define $l(\varepsilon_n^{-1}) = l_n$. Note that l is also regular varying function with index $(\alpha - 1)/\alpha$. Denote by $f(s)$ the generating function of our offspring law. Define γ (Schröder constant) by

$$f'(0) = m^{-\gamma} = p_1.$$

For $1 < \alpha < 2$ and $\alpha < \beta$, let

$$\chi_n := \frac{l_n^{\gamma-\beta} m^{(\beta-1-\gamma)n} b(l_n)^\beta}{L(l_n^{-1} b(l_n) m^n)} = \frac{b(l_n)^\gamma}{(\varepsilon_n m^n)^{\gamma-\beta} L(\varepsilon_n m^n) m^n}.$$

For $0 \le t < \gamma + 1$, define

$$I_t = \int_0^\infty u^{1-t} \omega(u)\,\mathrm{d}u.$$

Remark 4. As $u \to 0+$, there exist constants $0 < C_1 < C_2 < \infty$ such that

$$C_1 < \frac{\omega(u)}{u^{\gamma-1}} < C_2. \tag{8}$$

See [4] and [7] and the references therein for related results. The assumption $\mathbb{E}[Z_1 \log Z_1] < \infty$, together with (8), implies that I_t is finite; see [3, Theorem 8.12.7].

We are ready to present our main results. As illustrated in [15], there is a 'phase transition' in rates depending on γ. Thus, we will have three different cases in regard to γ and β. We first consider the case of $\gamma > \beta - 1$.

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Theorem 1. *Let* $0 < \alpha < 1$. *Assume that* $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$ *and* $\varepsilon_n \to +\infty$ *as* $n \to \infty$. *If* $\gamma > \beta - 1$ *then*

$$\lim_{n\to\infty} m^{(\beta-1)n} \varepsilon_n^{\beta} L(\varepsilon_n m^n)^{-1} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) = I_\beta. \tag{9}$$

Theorem 2. *Let* $1 \leq \alpha < 2$. *Assume that* $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$ *as* $n \to \infty$ *and* $\gamma > \beta - 1$.

(i) *Assume that* $1 < \alpha < 2$, $p_+ = 0$, *and* $\varepsilon_n \to 0$. *If* $\lim_{n\to\infty} \chi_n = 0$ *then (9) holds.*

(ii) *Assume that* $1 < \alpha < 2$, $p_+ = 0$, *and* $\varepsilon_n \to 0$. *If* $\lim_{n\to\infty} \chi_n = \infty$ *then*

$$V_I \leq \varliminf_{n\to\infty} l_n^{-\gamma} m^{\gamma n} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) \leq \varlimsup_{n\to\infty} l_n^{-\gamma} m^{\gamma n} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) \leq V_S, \tag{10}$$

where

$$V_I = \varliminf_{u\downarrow 0} u^{1-\gamma} \omega(u) \int_0^\infty u^{\gamma-1} \mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha})\, \mathrm{d}u,$$
$$V_S = \varlimsup_{u\downarrow 0} u^{1-\gamma} \omega(u) \int_0^\infty u^{\gamma-1} \mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha})\, \mathrm{d}u.$$

(iii) *Assume that* $1 < \alpha < 2$, $p_+ = 0$, *and* $\varepsilon_n \to 0$. *If* $\lim_{n\to\infty} \chi_n = y \in (0, \infty)$ *then*

$$V_I + yI_\beta \leq \varliminf_{n\to\infty} l_n^{-\gamma} m^{\gamma n} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) \leq \varlimsup_{n\to\infty} l_n^{-\gamma} m^{\gamma n} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) \leq V_S + yI_\beta.$$

(iv) *Assume that* $p_+ > 0$ *and* $\varepsilon_n \to \varepsilon \in (0, \infty)$. *Then (9) holds.*

Remark 5. The assumption $p_+ = 0$ implies that U_s is a spectrally negative α-stable random variable with mean 0 and skewness parameter -1. By [20, Equation (1.2.11)], we have

$$\int_0^\infty u^{\gamma-1} \mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha})\, \mathrm{d}u < \infty.$$

As an application of Theorem 2(iv) by taking $\varepsilon_n = \varepsilon$, we immediately obtain the following result, which improves the corresponding result in [15, Theorem 3], where it is assumed that L is a constant function.

Corollary 1. *If* $\mathbb{P}(Z_1 > x) \sim x^{-\beta} L(x)$ *for* $1 < \beta < 2$ *and* $\gamma > \beta - 1$, *then*

$$\lim_{n\to\infty} m^{(\beta-1)n} L(m^n)^{-1} \mathbb{P}\bigg(\frac{Z_{n+1}}{Z_n} - m \geq \varepsilon\bigg) = I_\beta \varepsilon^{-\beta}. \tag{11}$$

Remark 6. In fact, by (37) below, one may prove that

$$\lim_{n\to\infty} m^{(\beta-1)n} L(m^n)^{-1} \mathbb{P}\bigg(m - \frac{Z_{n+1}}{Z_n} \geq \varepsilon\bigg) = 0.$$

Proof. Theorem 2(iv) implies (11). □

Next, we consider the case of $\gamma = \beta - 1$. Let d be the greatest common divisor of the set $\{j - i : i \neq j, p_j p_i > 0\}$.

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Theorem 3. *Suppose that* $0 < \alpha < 1$ *and* $\beta > 1$. *Assume that* $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$ *and* $\varepsilon_n \to +\infty$ *as* $n \to \infty$. *If* $\gamma = \beta - 1$ *then*

$$d \liminf_{u\downarrow 0} u^{1-\gamma}\omega(u) \le \liminf_{n\to\infty} \frac{\varepsilon_n^{\beta}\mathbb{P}(S_{Z_n}/Z_n \ge \varepsilon_n)}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/km^{\gamma n}} \le \limsup_{n\to\infty} \frac{\varepsilon_n^{\beta}\mathbb{P}(S_{Z_n}/Z_n \ge \varepsilon_n)}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/km^{\gamma n}} \le d \limsup_{u\downarrow 0} u^{1-\gamma}\omega(u). \tag{12}$$

Define

$$\pi_n = \frac{l_n^{\gamma}\varepsilon_n^{\beta}}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/k}.$$

Theorem 4. *Let* $1 < \alpha < 2$. *Assume that* $\varepsilon_n \to 0$, $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$.

(i) *Assume that* $p_+ = 0$ *and* $\gamma = \beta - 1$. *If* $\pi_n \to 0$ *then (12) holds.*

(ii) *Assume that* $p_+ = 0$ *and* $\gamma = \beta - 1$. *If* $\pi_n \to +\infty$ *then (10) holds.*

(iii) *Assume that* $p_+ = 0$ *and* $\gamma = \beta - 1$. *If* $\pi_n \to y \in (0, \infty)$ *then*

$$V_I + yd \liminf_{u\downarrow 0} u^{1-\gamma}\omega(u) \le \liminf_{n\to\infty} l_n^{-\gamma} m^{\gamma n}\mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \ge \varepsilon_n\bigg) \le \limsup_{n\to\infty} l_n^{-\gamma} m^{\gamma n}\mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \ge \varepsilon_n\bigg) \le V_S + yd \limsup_{u\downarrow 0} u^{1-\gamma}\omega(u).$$

(iv) *Assume that* $p_+ > 0$ *and* $\gamma = \beta - 1$. *Then (12) holds with* ε_n *replaced by any* $\varepsilon > 0$.

Remark 7. If L is a constant function then (12) can be replaced by

$$\lim_{n\to\infty} n^{-1}\varepsilon_n^{\beta} m^{\gamma n}\mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \ge \varepsilon_n\bigg) = \frac{1}{\Gamma(\beta-1)}\int_1^m Q(\mathbb{E}[\mathrm{e}^{-vW}])v^{\beta-2}\,\mathrm{d}v,$$

where

$$Q(s) = \sum_{k=1} q_k s^k = \lim_{n\to\infty} \frac{f_n(s)}{m^{-\gamma n}}, \qquad 0 \le s < 1, \qquad q_k = \lim_{n\to\infty} \mathbb{P}(Z_n = k)m^{\gamma n}$$

and f_n denotes the iterates of f. See [1, Proposition 2] for $Q(s)$ and $(q_k)_{k\ge 1}$. The key is the limiting behavior of $\mathbb{E}[Z_n^{-\gamma} L(\varepsilon_n Z_n)]$ as $n \to \infty$; see [15, Theorem 1] and Remark 10 below in this paper.

Finally, we consider the case of $\gamma < \beta - 1$.

Theorem 5. *If* $1 < \alpha < 2$ *and* $\gamma < \beta - 1$ *or* $\mathbb{E}[X_1^{1+\gamma}\mathbf{1}_{\{X_1>0\}}] < \infty$, *where* $\mathbf{1}$ *is the indicator function, then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} m^{\gamma n}\mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \ge \varepsilon\bigg) = \sum_{k\ge 1} q_k \mathbb{P}(S_k \ge \varepsilon k).$$

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Remark 8. It is difficult for us to find a simpler form of the right-hand side of this equation. In fact, $\mathbb{P}(S_k \geq \varepsilon k)$ could be represented via a convolution formula and q_k is determined as $Q(s)$ is the unique solution of

$$Q(f(s)) = p_1 Q(s), \qquad 0 \leq s < 1.$$

See [1, Proposition 2].

Corollary 2. *If* $\mathbb{P}(Z_1 > x) \sim x^{-\beta} L(x)$ *for* $1 < \beta < 2$ *and* $\gamma < \beta - 1$ *or* $\mathbb{E}[Z_1^{1+\gamma}] < \infty$*, then*

$$\lim_{n\to\infty} m^{\gamma n} \mathbb{P}\bigg(\bigg|\frac{Z_{n+1}}{Z_n} - m\bigg| \geq \varepsilon\bigg) = \sum_{k\geq 1} q_k \phi(k, \varepsilon),$$

where $\phi(k, \varepsilon) = \mathbb{P}(|(1/k)\sum_{i=1}^{k} \xi_i - m| > \varepsilon)$ *and* $(\xi_i)_{i\geq 1}$ *are i.i.d. random variables with the same distribution as* Z_1.

Remark 9. When L is a constant function and $\mathbb{P}(Z_1 > x) \sim x^{-\beta} L$, the above result has been proved in [15]. Athreya [1, Theorem 1 and Corollary 1] also proved the same result under the assumption $\mathbb{E}[Z_1^{2a+\delta}] < \infty$ and $p_1 m^a > 1$ for some $a \geq 1$ and $\delta > 0$.

We also generalize (1) to the stable setting.

Theorem 6. *Assume that* $0 < \alpha < 2$. *If* $\varepsilon_n m^n b(m^n)^{-1} \to x \in (-\infty, +\infty)$ *then*

$$\lim_{n\to\infty} \mathbb{P}\bigg(\frac{S_{Z_n}}{Z_n} \geq \varepsilon_n\bigg) = \int_0^\infty \mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha} x)\omega(u)\, du.$$

As an application of the above theorem, the following result generalizes (1); see [13, Theorem 3].

Corollary 3. *Assume that* $1 < \beta < 2$ *and* $\mathbb{P}(Z_1 > x) \sim x^{-\beta} L(x)$ *as* $x \to +\infty$. *Then for every* $x \in (-\infty, +\infty)$,

$$\lim_{n\to\infty} \mathbb{P}\bigg(\frac{m^n}{b(m^n)}\bigg(\frac{Z_{n+1}}{Z_n} - m\bigg) \leq x\bigg) = \int_0^\infty \mathbb{P}(U_s \leq u^{(\beta-1)/\beta} x)\omega(u)\, du. \tag{13}$$

Proof. Obviously, $Z_1 - m$ is in the domain of attraction of a β-stable law. Using Theorem 6 with $\varepsilon_n = xb(m^n)m^{-n}$ gives (13). □

2. Preliminaries

2.1. Fuk–Nagaev inequalities

The following result is parallel to [10, Lemma 14] where X_1 has finite variance.

Lemma 1. *For any* $0 < \alpha < 1$, $r > 0$, *and* $k \geq 1$,

$$\mathbb{P}(S_k \geq \varepsilon_n k) \leq \begin{cases} k\mathbb{P}(X_1 \geq r^{-1}\varepsilon_n k) + c_r \varepsilon_n^{-\beta r} k^{(1-\beta)r}, & \beta < 1, \\ k\mathbb{P}(X_1 \geq r^{-1}\varepsilon_n k) + c_r \varepsilon_n^{-tr} k^{(1-t)r}, & \beta \geq 1, \end{cases} \tag{14}$$

hold for $t \in (\alpha, 1] \cap (\alpha, \beta)$.

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Proof. By [14, Theorem 1.1], we have, for any $0 < t \le 1$,

$$\mathbb{P}(S_k \ge \varepsilon_n k) \le k\mathbb{P}(X_1 \ge r^{-1}\varepsilon_n k) + \bigg(\frac{\mathrm{e}\mathbb{E}[X_1^t; \mathbf{1}_{\{0\le X_1 \le r^{-1}\varepsilon_n k\}}]}{r^{1-t}\varepsilon_n^t k^{t-1}}\bigg)^r. \tag{15}$$

Noting that as $x \to +\infty$, $\mathbb{P}(X_1 \ge x) \sim x^{-\beta}L(x)$, we have, for $x > 1$,

$$\mathbb{E}[X_1^t; \mathbf{1}_{\{0\le X_1\le x\}}] \le \begin{cases} Cx^{t-\beta}, & \beta < t, \\ C_t, & t < \beta. \end{cases} \tag{16}$$

And if $x \le 1$, obviously we have

$$\mathbb{E}[X_1^t; \mathbf{1}_{\{0\le X_1\le x\}}] \le C(1 \vee x^{t-\beta}). \tag{17}$$

Then if $\beta < 1$, applying (15) with $\beta < t$, together with (16) and (17), yields (14). If $\beta \ge 1$, with the help of (16) and (17), taking any $\alpha < t \le 1$ also implies (14). □

2.2. Harmonic moments

It is well known that $W_n := m^{-n}Z_n \to W$ a.s. Furthermore, we have the global limit theorem

$$\lim_{n\to\infty} \mathbb{P}(Z_n \ge xm^n) = \int_x^\infty \omega(t)\,\mathrm{d}t, \qquad x > 0. \tag{18}$$

In particular, one can deduce that, for $0 < \delta < 1 < A < \infty$,

$$\mathbb{E}[(W_n)^t \mathbf{1}_{\{W_n<\delta\}}] \to \int_0^\delta u^t\omega(u)\,\mathrm{d}u, \qquad t > -\gamma. \tag{19}$$

We also recall here a result from [10, Lemma 13]. There exists a constant $C > 0$ such that

$$\mathbb{P}(Z_n = k) \le C\bigg(\frac{1}{k} \wedge \frac{k^{\gamma-1}}{m^{\gamma n}}\bigg), \qquad k, n \ge 1. \tag{20}$$

Lemma 2. *Assume that $\varepsilon_n m^n \to \infty$. Then as $n \to \infty$,*

$$\mathbb{E}[Z_n^t L(\varepsilon_n Z_n)] \sim m^{nt}L(\varepsilon_n m^n)\int_0^\infty u^t\omega(u)\,\mathrm{d}u, \qquad -\gamma < t < 1, \tag{21}$$

and

$$\begin{aligned} d\,\underline{\lim}_{u\downarrow 0}\, u^{1-\gamma}\omega(u) &\le \underline{\lim}_{n\to\infty} \frac{\mathbb{E}[Z_n^{-\gamma}L(\varepsilon_n Z_n)]}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/km^{\gamma n}} \\ &\le \overline{\lim}_{n\to\infty} \frac{\mathbb{E}[Z_n^{-\gamma}L(\varepsilon_n Z_n)]}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/km^{\gamma n}} \\ &\le d\,\overline{\lim}_{u\downarrow 0}\, u^{1-\gamma}\omega(u). \end{aligned} \tag{22}$$

Proof. We first prove (21). Recall that $W_n = Z_n/m^n$. Note that

$$\mathbb{E}[Z_n^t L(\varepsilon_n Z_n)] = m^{nt}L(\varepsilon_n m^n)\mathbb{E}\bigg[(W_n)^t\frac{L(\varepsilon_n m^n W_n)}{L(\varepsilon_n m^n)}\bigg]. \tag{23}$$

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Then for $0<\delta<1<A$, by (3) and (19), we have for some $0<\eta<\gamma$ small enough,

$$\mathbb{E}\bigg[(W_n)^t\frac{L(\varepsilon_n m^n W_n)}{L(\varepsilon_n m^n)}\mathbf{1}_{\{W_n<\delta\}}\bigg]\le C\mathbb{E}[(W_n)^{t-\eta}\mathbf{1}_{\{W_n<\delta\}}]=(1+o(1))C\int_0^\delta u^{t-\eta}\omega(u)\,\mathrm{d}u.$$

Meanwhile by the dominated convergence theorem, we have

$$\mathbb{E}\bigg[(W_n)^t\frac{L(\varepsilon_n m^n W_n)}{L(\varepsilon_n m^n)}\mathbf{1}_{\{\delta\le W_n\le A\}}\bigg]\to\int_\delta^A u^t\omega(u)\,\mathrm{d}u.$$

Finally, using (3) with $\eta=1-t$, we have

$$\mathbb{E}\bigg[(W_n)^t\frac{L(\varepsilon_n m^n W_n)}{L(\varepsilon_n m^n)}\mathbf{1}_{\{W_n>A\}}\bigg]\le C\mathbb{E}[W_n\mathbf{1}_{\{W_n>A\}}]=(1+o(1))C\int_A^\infty u\omega(u)\,\mathrm{d}u.$$

Letting $\delta\to 0$ and $A\to\infty$, together with (23), we obtain (21).

The remainder of this proof is devoted to (22). Let $\{k_n\}$ be a sequence such that $k_n\to\infty$ and $k_n=o(m^n)$. Then, for any $0<\delta\le 1$,

$$\mathbb{E}[Z_n^{-\gamma}L(\varepsilon_n Z_n)]=\bigg(\sum_{k<k_n}+\sum_{k_n\le k\le\delta m^n}+\sum_{k>\delta m^n}\bigg)\frac{L(\varepsilon_n k)}{k^\gamma}\mathbb{P}(Z_n=k)=:I_0+I_1+I_2.$$

By [9, Corollary 5], we have

$$I_1=(1+o(1))d\sum_{k_n\le k\le\delta m^n}\frac{L(\varepsilon_n k)}{k^\gamma}m^{-n}\omega\bigg(\frac{k}{m^n}\bigg),$$

which is larger than

$$(1+o(1))d\inf_{u\le\delta}u^{1-\gamma}\omega(u)\sum_{k_n\le k\le\delta m^n}\frac{L(\varepsilon_n k)}{km^{\gamma n}}$$

and less than

$$(1+o(1))d\sup_{u\le\delta}u^{1-\gamma}\omega(u)\sum_{k_n\le k\le\delta m^n}\frac{L(\varepsilon_n k)}{km^{\gamma n}}.$$

On the other hand, by the dominated convergence theorem, together with (3), we have

$$I_2\sim m^{-\gamma n}L(\varepsilon_n m^n)\int_\delta^\infty u^{-\gamma}\omega(u)\,\mathrm{d}u.$$

We also have

$$\frac{Z_n^{-\gamma}L(\varepsilon_n Z_n)}{m^{-\gamma n}L(\varepsilon_n m^n)}\mathbf{1}_{\{Z_n\le\delta m^n\}}\to W^{-\gamma}\mathbf{1}_{\{W\le\delta\}}\quad\text{a.s.}$$

whose expectation is infinite by (8). Then Fatou's lemma yields

$$\frac{\limsup_{n\to\infty}I_2}{I_0+I_1}=0. \tag{24}$$

By (20), we also have

$$I_0\le\sum_{k<k_n}\frac{L(\varepsilon_n k)}{km^{\gamma n}}. \tag{25}$$

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Then one may choose k_n such that

$$\sum_{k<k_n} \frac{L(\varepsilon_n k)}{k} \bigg[\sum_{k<m^n} \frac{L(\varepsilon_n k)}{k} \bigg]^{-1} \to 0. \tag{26}$$

Meanwhile, one can also deduce that

$$(1+o(1))d \inf_{u\le\delta} u^{1-\gamma}\omega(u) \sum_{\delta m^n \le k \le m^n} \frac{L(\varepsilon_n k)}{k m^{\gamma n}} \le \mathbb{E}[Z_n^{-\gamma} L(\varepsilon_n Z_n) \mathbf{1}_{\{\delta m^n \le Z_n \le m^n\}}]$$
$$\sim m^{-\gamma n} L(\varepsilon_n m^n) \int_\delta^1 u^{-\gamma}\omega(u)\,\mathrm{d}u,$$

which, together with (24)–(26), gives $\limsup_{n\to\infty} I_0/I_1 = \limsup_{n\to\infty} I_2/I_1 = 0$. Thus,

$$\begin{aligned} d \inf_{u<\delta} u^{1-\gamma}\omega(u) &\le \varliminf_{n\to\infty} \frac{\mathbb{E}[Z_n^{-\gamma} L(\varepsilon_n Z_n)]}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/k m^{\gamma n}} \\ &\le \varlimsup_{n\to\infty} \frac{\mathbb{E}[Z_n^{-\gamma} L(\varepsilon_n Z_n)]}{\sum_{1\le k\le m^n} L(\varepsilon_n k)/k m^{\gamma n}} \\ &\le d \sup_{u<\delta} u^{1-\gamma}\omega(u) \end{aligned}$$

holds for any $\delta > 0$. Letting $\delta \to 0$ implies (22). We have completed the proof. □

Remark 10. Lemma 2 could be compared with [15, Theorem 1] where $L = 1$. Under the assumption $\mathbb{E}[Z_1 \ln Z_1] < \infty$, when $-\gamma < t < 0$, our result completes the result in [15]. However, when $t = -\gamma$, a precise limit is obtained in [15].

3. Proofs

We only prove Theorems 1, 2, 5, and 6. The ideas to prove Theorems 3 and 4 are similar to Theorems 1 and 2, respectively. We omit the details here.

3.1. Proof of Theorem 1

Lemma 3. *Suppose that all the assumptions in Theorem 1 hold. Then there exists $\eta > 0$ small enough such that, for any $0 < \delta < 1 < A$,*

$$\limsup_{n\to\infty} \frac{\varepsilon_n^\beta (m^n)^{(\beta-1)}}{L(\varepsilon_n m^n)} \sum_{k\le\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C\delta^{\gamma-\beta+1-\eta}; \tag{27}$$

$$\limsup_{n\to\infty} \frac{\varepsilon_n^\beta (m^n)^{(\beta-1)}}{L(\varepsilon_n m^n)} \sum_{k\ge A m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C \int_A^\infty u\omega(u)\,\mathrm{d}u. \tag{28}$$

Proof. We first prove (27). Consider the case of $\beta < 1$. Applying (3) with $0 < \eta < \gamma-\beta+1$, together with (14) and (20), gives

$$\begin{aligned} &\sum_{k\le\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &\quad \le C \sum_{k\le\delta m^n} \mathbb{P}(Z_n = k)(k\mathbb{P}(X_1 \ge r^{-1}\varepsilon_n k) + k^{(1-\beta)r}\varepsilon_n^{-\beta r}) \\ &\quad \le C(L(\varepsilon_n m^n)\varepsilon_n^{-\beta}(m^n)^{1-\beta}\delta^{\gamma-\beta+1-\eta} + \delta^{(1-\beta)r+\gamma}(m^n)^{(1-\beta)r}\varepsilon_n^{-\beta r}). \end{aligned}$$

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Choosing $r > 1$ and noting $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$, one can check that

$$(m^n)^{(1-\beta)r}\varepsilon_n^{-\beta r}\frac{L(\varepsilon_n m^n)}{\varepsilon_n^{\beta}(m^n)^{(\beta-1)}} = o(1). \tag{29}$$

Then (27) follows readily if $\beta < 1$. The $\beta \geq 1$ case can be proved similarly by applying (14) again with $r = \alpha\beta/(1-\alpha) + \beta + 1$ and $(1-t)r = 1$.

Following a similar reasoning also yields (28) by applying (3) with $\eta = \beta$. In fact, if $\beta < 1$, (14) and (20) imply that

$$\sum_{k\geq Am^n}\mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq k\varepsilon_n) \leq C(1+o(1))\frac{L(\varepsilon_n m^n)}{\varepsilon_n^{\beta} m^{n(\beta-1)}}\int_A^{\infty} u\omega(u)\,\mathrm{d}u + CA^{(1-\beta)r}(m^n)^{(1-\beta)r}\varepsilon_n^{-\beta r},$$

which, together with (29), proves (28) in the $\beta < 1$ case. Applying (3), (14), and (20) suitably also proves the $\beta \geq 1$ case. We omit the details here. □

Lemma 4. *Suppose that all the assumptions in Theorem 1 hold. Then there exists* $\eta > 0$ *small enough such that, for any* $0 < \delta < 1 < A$,

$$\limsup_{n\to\infty}\bigg|m^{(\beta-1)n}\varepsilon_n^{\beta}L(\varepsilon_n m^n)^{-1}\sum_{k=\delta m^n}^{Am^n}\mathbb{P}(S_k \geq \varepsilon_n k)\mathbb{P}(Z_n = k) - I_\beta\bigg| \leq C\bigg(\int_A^{\infty} u\omega(u)\,\mathrm{d}u + \delta^{\gamma-\beta+1-\eta}\bigg).$$

Proof. Using [6, Theorem 9.3] for $\alpha < \beta$ and [5, Theorem 3.3] for $\alpha = \beta$, it follows that

$$\lim_{n\to\infty}\sup_{x\geq x_n}\bigg|\frac{\mathbb{P}(S_n \geq x)}{n\mathbb{P}(X_1 \geq x)} - 1\bigg| = 0$$

holds for any x_n satisfying

$$nF(-x_n) = o(1) \quad \text{if } \alpha < \beta \quad \text{or} \quad n(1-F(x_n)) = o(1) \quad \text{if } \alpha = \beta.$$

Since $\varepsilon_n m^n b(m^n)^{-1} \to \infty$, we have

$$m^n F(-\varepsilon_n m^n) = o(1) \quad \text{if } \alpha < \beta \quad \text{and} \quad m^n(1-F(\varepsilon_n m^n)) = o(1) \quad \text{if } \alpha = \beta.$$

In fact, if $\alpha < \beta$, we could denote by b^{-1} the inverse of b. Then $\varepsilon_n m^n b(m^n)^{-1} \to \infty$ implies $m^n/b^{-1}(\varepsilon_n m^n) \to 0$ and, hence, by (7), we have

$$m^n F(-\varepsilon_n m^n) = \frac{m^n}{b^{-1}(\varepsilon_n m^n)}b^{-1}(\varepsilon_n m^n)F(-\varepsilon_n m^n) \to 0.$$

If $\alpha = \beta$, the argument is similar. Define

$$\eta_n := \sup_{\delta m^n < k < Am^n}\sup_{x\geq \varepsilon_n k}\bigg|\frac{\mathbb{P}(S_k \geq x)}{k\mathbb{P}(X_1 \geq x)} - 1\bigg|.$$

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Then one can check that $\eta_n = o(1)$ as $n \to \infty$. Thus, as $n \to \infty$,

$$\sum_{k=\delta m^n}^{Am^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) = (1 + o(1)) \sum_{k=\delta m^n}^{Am^n} k\mathbb{P}(Z_n = k)\mathbb{P}(X_1 \geq \varepsilon_n k)$$

$$= (1 + o(1))\varepsilon_n^{-\beta} \sum_{k=\delta m^n}^{Am^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k)$$

$$= (1 + o(1))\varepsilon_n^{-\beta} \sum_{k=\delta m^n}^{Am^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k). \tag{30}$$

Meanwhile, applying (3) with some $0 < \eta < \gamma - \beta + 1$ and (20) yields

$$L(\varepsilon_n m^n)^{-1} m^{(\beta-1)n} \sum_{k<\delta m^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k) \leq C\delta^{\gamma-\beta+1-\eta}$$

and applying (3) with $\eta = \beta$ and (20) gives

$$L(\varepsilon_n m^n)^{-1} m^{(\beta-1)n} \sum_{k>Am^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k) \leq (1 + o(1))C \int_A^{\infty} u\omega(u)\,\mathrm{d}u.$$

Thus, by Lemma 2, we have

$$\left| m^{(\beta-1)n} L(\varepsilon_n m^n)^{-1} \sum_{k=\delta m^n}^{Am^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k) - I_\beta \right|$$
$$\leq (1 + o(1))C\left(\int_A^{\infty} u\omega(u)\,\mathrm{d}u + \delta^{\gamma-\beta+1-\eta}\right).$$

Then by (30), as $n \to \infty$,

$$\left| m^{(\beta-1)n} \varepsilon_n^{\beta} L(\varepsilon_n m^n)^{-1} \sum_{k=\delta m^n}^{Am^n} \mathbb{P}(S_k \geq \varepsilon_n k)\mathbb{P}(Z_n = k) - I_\beta \right|$$
$$= \left| (1 + o(1)) m^{(\beta-1)n} \sum_{k=\delta m^n}^{Am^n} L(\varepsilon_n k)k^{1-\beta}\mathbb{P}(Z_n = k) - I_\beta \right|$$
$$\leq (1 + o(1))C\left(\int_A^{\infty} u\omega(u)\,\mathrm{d}u + \delta^{\gamma-\beta+1-\eta}\right).$$

The desired result follows readily. □

Proof of Theorem 1. Letting $\delta \to 0$ and $A \to \infty$ in Lemma 3 and Lemma 4 gives the theorem. □

3.2. Proof of Theorem 2

Recall that $l(x)$ is an asymptotic inverse of $x \mapsto J(x) = xb(x)^{-1}$ and $l(\varepsilon_n^{-1}) = l_n$. If $\alpha < \beta$, we may write

$$l(x) = x^{\alpha/(\alpha-1)} s'(x) \tag{31}$$

for some slowly varying function s'. Note that Assumption 2 implies that

$$\liminf_{x\to+\infty} s'(x) > 0. \tag{32}$$

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Lemma 5. *Assume that* $1 < \alpha < 2$, $p_+ = 0$, $\gamma > \beta - 1$, $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$, *and* $\varepsilon_n \to 0$. *Then, for any* $0 < \delta < 1 < A$,

$$\sum_{1 \le k \le \delta l_n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C\delta^\gamma l_n^\gamma m^{-\gamma n}, \tag{33}$$

$$\sum_{k \le \delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C\delta^{\gamma+1-\beta-\eta} \frac{L(\varepsilon_n m^n)}{\varepsilon_n^\beta m^{(\beta-1)n}} + Cl_n^\gamma m^{-\gamma n}, \tag{34}$$

$$\sum_{k \ge Am^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C\varepsilon_n^{-\beta} m^{(1-\beta)n} L(\varepsilon_n m^n) + CA^{-2\gamma} l_n^\gamma m^{-\gamma n}, \tag{35}$$

and for any large enough A,

$$\begin{aligned}\sum_{Al_n < k \le Am^n} & \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &\le C(1 + A^{\gamma+1-\beta+\eta})\varepsilon_n^{-\beta} m^{(1-\beta)n} L(\varepsilon_n m^n) + CA^{-2\gamma} l_n^\gamma m^{-\gamma n}.\end{aligned} \tag{36}$$

Proof. The proof will be divided into three parts. (i) We shall prove (33), by noting (20), whose left-hand side is less than

$$\sum_{1 \le k \le \delta l_n} \mathbb{P}(Z_n = k) \le \frac{C}{m^{\gamma n}} \sum_{1 \le k \le \delta l_n} k^{\gamma-1} \le C\delta^\gamma l_n^\gamma m^{-\gamma n}.$$

(ii) We shall first prove (34) and (36). By [14, Corollary 1.6], if $s > 1$, $1 \le t < \beta$, and $k > (4\mathbb{E}[X_1^t \mathbf{1}_{\{X_1 \ge 0\}}]s^t)^{1/(t-1)} \varepsilon_n^{t/(1-t)}$, then

$$\mathbb{P}(S_k \ge k\varepsilon_n) \le k\mathbb{P}(X_1 \ge s^{-1}k\varepsilon_n) + C(\varepsilon_n)^{-ts/2} k^{(1-t)s/2}. \tag{37}$$

Furthermore, (32) implies that there exists $A_l > 0$ such that (37) holds for $t = \alpha$ and all $k > A_l l_n$. Thus,

$$\begin{aligned}\sum_{k \le \delta m^n} & \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &\le \sum_{1 \le k \le A_l l_n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) + \sum_{A_l l_n < k \le \delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &=: I_1 + I_2.\end{aligned} \tag{38}$$

Applying (20) again gives

$$I_1 \le \sum_{1 \le k \le A_l l_n} \mathbb{P}(Z_n = k) \le \frac{c}{m^{\gamma n}} \sum_{1 \le k \le A_l l_n} k^{\gamma-1} \le CA_l l_n^\gamma m^{-\gamma n}. \tag{39}$$

Note that $l_n \varepsilon_n \to \infty$. Applying (37) with $t = \alpha$, (3) with $\eta < \gamma - \beta + 1$, and (20), we have

$$\begin{aligned} I_2 &\le \sum_{A_l l_n < k \le \delta m^n} \mathbb{P}(Z_n = k)(k\mathbb{P}(X_1 \ge s^{-1}k\varepsilon_n) + C(\varepsilon_n)^{-\alpha s/2} k^{(1-\alpha)s/2}) \\ &\le \frac{C}{m^{\gamma n}} \bigg(\sum_{A_l l_n < k \le \delta m^n} k^\gamma \mathbb{P}(X_1 \ge s^{-1}k\varepsilon_n) + \sum_{k > A_l l_n} (\varepsilon_n)^{-\alpha s/2} k^{(1-\alpha)s/2+\gamma-1} \bigg) \\ &\le \frac{C}{m^{\gamma n}} \bigg(L(\varepsilon_n m^n) \sum_{k \le \delta m^n} \varepsilon_n^{-\beta} k^{\gamma-\beta} \Big(\frac{k}{m^n}\Big)^{-\eta} + \sum_{k > A_l l_n} (\varepsilon_n)^{-\alpha s/2} k^{(1-\alpha)s/2+\gamma-1} \bigg) \\ &\le C\delta^{\gamma+1-\beta-\eta} \varepsilon_n^{-\beta} m^{(1-\beta)n} L(\varepsilon_n m^n) + CA_l^{-2\gamma} l_n^\gamma m^{-\gamma n} s'(\varepsilon_n^{-1})^{-2\gamma}, \end{aligned} \tag{40}$$

where in the last inequality, we use (31), (32), and choose $s = 4\gamma/(\alpha - 1)$. Substituting (39) and (40) into (38), together with (32), gives (34). Replacing A_l and δ by A and modifying the last two steps in (40) accordingly, we immediately obtain (36).

(iii) We shall prove (35). Note that $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$ implies that $l_n \le m^n$. Using (37) with $s = 4\gamma/(\alpha - 1)$ and (3) with $\eta = \beta$, we have

$$\begin{aligned}
&\sum_{k\ge Am^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\
&\quad \le \sum_{k\ge Am^n} \mathbb{P}(Z_n = k)(k\mathbb{P}(X_1 \ge s^{-1}k\varepsilon_n) + C(\varepsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2}) \\
&\quad \le C\bigg(\sum_{k\ge Am^n} \mathbb{P}(Z_n = k)\varepsilon_n^{-\beta}k^{1-\beta}L(s^{-1}k\varepsilon_n) + \sum_{k>Al_n} (\varepsilon_n)^{-\alpha s/2}k^{(1-\alpha)s/2+\gamma-1}m^{-\gamma n}\bigg) \\
&\quad \le C\varepsilon_n^{-\beta}m^{(1-\beta)n}L(\varepsilon_n m^n) + CA^{-2\gamma}l_n^{\gamma}m^{-\gamma n}s'(\varepsilon_n^{-1})^{-2\gamma},
\end{aligned}$$

where the second term in the last inequality is deduced according to similar reasonings for (40). Then (35) follows readily. □

Lemma 6. *Assume that* $1 \le \alpha < 2$, $p_+ > 0$, $\gamma > \beta - 1$, $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$, *and* $\varepsilon_n \to \varepsilon \in [0, \infty)$. *Then there exists* $\eta > 0$ *small enough such that, for any* $0 < \delta < 1$,

$$\sum_{k\le\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C\delta^{\gamma-\beta+1-\eta}L(\varepsilon_n m^n)\varepsilon_n^{-\beta}m^{n(1-\beta)}. \tag{41}$$

Proof. The proof will be divided into three steps. (i) Note that $p_+ > 0$ implies $\alpha = \beta$. We first prove that

$$\mathbb{P}(S_k \ge \varepsilon_n k) \le C(k\mathbb{P}(X_1 \ge r^{-1}\varepsilon_n k) + \varepsilon_n^{-\beta}k^{(1-\beta)}L(\varepsilon_n k)), \qquad k \ge 1. \tag{42}$$

Recall (5). By [18, Lemma], we have for $k \ge 1$ and $x > 0$,

$$\mathbb{P}(S_k \ge x) \le Ck\bigg(\mathbb{P}(|X_1| \ge x) + \frac{\mu(2;x)}{x^2} + \frac{|\mu(1;x)|}{x}\bigg). \tag{43}$$

Then (6) implies, for $1 < \beta < 2$,

$$\mu(2;x) = \mathbb{E}[|X_1|^2\mathbf{1}_{\{|X_1|\le x\}}] \le cx^{2-\beta}L(x), \qquad x > 0. \tag{44}$$

According to [8, Chapter XVII, Equations (5.17), (5.21), and (5.22)], as $x \to \infty$,

$$\frac{x}{\mu(2;x)}\mathbb{E}[|X_1|\mathbf{1}_{\{|X_1|>x\}}] \to c \ne 0,$$

which, together with $\mathbb{E}[X_1] = 0$, yields, for $1 < \beta < 2$,

$$|\mu(1;x)| = |\mathbb{E}[X_1\mathbf{1}_{\{|X_1|\le x\}}]| \le \mathbb{E}[|X_1|\mathbf{1}_{\{|X_1|>x\}}] \sim cx^{1-\beta}L(x).$$

Thus, for $1 < \beta < 2$, we have $|\mu(1;x)| \le cx^{1-\beta}L(x)$ for all $x > 0$. Then according to (43), it follows that (42) holds for $1 < \beta < 2$.

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(ii) We shall prove (42) for $\alpha = \beta = 1$. By [14, Theorem 1.2] and Assumption 2, we have

$$\mathbb{P}(S_k \geq x) \leq k\mathbb{P}(X_1 > x) + \frac{\mathrm{e}k\mu(2; x)}{x^2}. \tag{45}$$

Then (45) and (44) yield (42).

(iii) We shall prove (41). By using (3), (20), and (42) accordingly,

$$\begin{aligned}\sum_{k\leq\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) &\leq C\varepsilon_n^{-\beta} \sum_{k\leq\delta m^n} \mathbb{P}(Z_n = k)k^{1-\beta}L(\varepsilon_n k)\\ &\leq CL(\varepsilon_n m^n)\varepsilon_n^{-\beta} m^{(-\gamma+\eta)n} \sum_{k\leq\delta m^n} k^{\gamma-\beta-\eta}\\ &\leq C\delta^{\gamma-\beta+1-\eta}L(\varepsilon_n m^n)\varepsilon_n^{-\beta} m^{n(1-\beta)}.\end{aligned}$$

We have completed the proof. □

Lemma 7. *Suppose that* $1 < \alpha < 2$, $\gamma > \beta - 1$, $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$, *and* $\varepsilon_n \to \varepsilon \in [0, \infty)$. *If* $\beta > \alpha$, *we further assume that*

$$\lim_{n\to\infty} \chi_n = y \in [0, \infty). \tag{46}$$

Then there exists $\eta > 0$ *small enough such that, for any* $0 < \delta < 1$,

$$\limsup_{n\to\infty} \left| \frac{m^{(\beta-1)n}\varepsilon_n^{\beta}}{L(\varepsilon_n m^n)} \sum_{k>\delta m^n} \mathbb{P}(S_k \geq \varepsilon_n k)\mathbb{P}(Z_n = k) - I_\beta \right| \leq C\delta^{\gamma-\beta+1-\eta}. \tag{47}$$

Proof. First, if $1 < \alpha < 2$ and $p_+ = 0$, then by [6, Theorem 9.2], it holds that

$$\lim_{k\to\infty} \sup_{x\geq x_k} \left| \frac{\mathbb{P}(S_k \geq x)}{k\mathbb{P}(X_1 \geq x)} - 1 \right| = 0 \quad \text{for any } x_k = t\left(\frac{\beta-\alpha}{\alpha-1}\log k\right)^{(\alpha-1)/\alpha} b(k),\ t > 0. \tag{48}$$

Define

$$\eta_n := \sup_{k>\delta m^n} \sup_{x\geq\varepsilon_n k} \left| \frac{\mathbb{P}(S_k \geq x)}{k\mathbb{P}(X_1 \geq x)} - 1 \right|.$$

Then one can apply (48) with $x_k = k\varepsilon_n$ to ensure that $\eta_n = o(1)$. To apply (48) it suffices to show that

$$\liminf_{n\to\infty} \frac{m^n\varepsilon_n}{b(\delta m^n)(\ln(m^n))^{(\alpha-1)/\alpha}} \to +\infty. \tag{49}$$

In fact, since L and s are slowly varying functions, then for any $\eta, \eta' > 0$, there exists $C_\eta, C_{\eta'}$ such that $L(l_n^{-1}b(l_n)m^n) \leq C_\eta l_n^{-\eta} b(l_n)^\eta m^{\eta n}$ and

$$\frac{l_n^{\gamma-\beta} m^{(\beta-1-\gamma)n} b(l_n)^\beta}{L(l_n^{-1}b(l_n)m^n)} \geq C_\eta \frac{l_n^{\gamma-\beta+\beta/\alpha+\eta-\eta/\alpha}}{m^{(\gamma-\beta+1+\eta)n}} \frac{s(l_n)^\beta}{s(l_n)^\eta} \geq C_\eta C_{\eta'} \frac{l_n^{\gamma-\beta+\beta/\alpha+\eta-\eta/\alpha-\beta\eta'-\eta\eta'}}{m^{(\gamma-\beta+1+\eta)n}}. \tag{50}$$

Since $\alpha < \beta$, then one could choose η, η' small enough such that

$$0 < \lambda := \frac{\gamma-\beta+1+\eta}{\gamma-\beta+\beta/\alpha+\eta-\eta/\alpha-\beta\eta'-\eta\eta'} < 1. \tag{51}$$

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Thus, (46) and (50) imply that

$$\limsup_{n\to\infty} \frac{m^{\lambda n}}{l_n} \in (0, +\infty]. \tag{52}$$

We also note that, for any $\eta'' > 0$,

$$\frac{\delta m^n \varepsilon_n}{b(\delta m^n)} = \left(\frac{\delta m^n}{l^n}\right)^{(\alpha-1)/\alpha} \frac{s(l_n)}{s(\delta m^n)} \geq C_{\eta''}\left(\frac{\delta m^{\lambda n}}{l_n}\right)^{(\alpha-1)/\alpha} (\delta m^{(1-\lambda)n})^{(\alpha-1)/\alpha}\left(\frac{l_n}{\delta m^n}\right)^{\eta''}.$$

Choosing η'' small enough in the above, together with (52) and (51), yields that (49) holds. We obtain $\eta_n = o(1)$. The remainder of the proof for the case of $1 < \alpha < 2$ and $\beta > \alpha$ is similar to Lemma 4. We omit it here.

When $1 < \alpha = \beta < 2$ and $p_+ = 1$, (48) holds for x_k satisfying $x_k/b(k) \to \infty$; see [19] and the references therein. Obviously, in this case $\eta_n = o(1)$.

When $1 \leq \alpha = \beta < 2$ and $0 < p_+ < 1$, (48) holds for x_k satisfying $k\mathbb{P}(X_1 > x_k) \to 0$ and $(k/x_k)\int_{-x_k}^{x_k} x\,\mathrm{d}F(x) \to 0$; see [5, Theorem 3.3]. By using (3), (7), and the fact $\varepsilon_n m^n b(m^n)^{-1} \to \infty$, one can check that $\eta_n = o(1)$. Then the desired result can be proved similarly. □

Lemma 8. *Assume that* $1 < \alpha < 2$, $p_+ = 0$, $\gamma > \beta - 1$, $\varepsilon_n m^n b(m^n)^{-1} \to +\infty$, *and* $\varepsilon_n \to 0$. *Then*

$$\begin{aligned} V_I(\delta, A) &\leq \varliminf_{n\to\infty} m^{\gamma n} l_n^{-\gamma} \sum_{\delta l_n < k < A l_n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq k\varepsilon_n) \\ &\leq \varlimsup_{n\to\infty} m^{\gamma n} l_n^{-\gamma} \sum_{\delta l_n < k < A l_n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq k\varepsilon_n) \\ &\leq V_S(\delta, A), \end{aligned} \tag{53}$$

where

$$V_I(\delta, A) = \varliminf_{u\to 0} u^{1-\gamma}\omega(u) \int_\delta^A u^{\gamma-1}\mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha})\,\mathrm{d}u,$$

$$V_S(\delta, A) = \varlimsup_{u\to 0} u^{1-\gamma}\omega(u) \int_\delta^A u^{\gamma-1}\mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha})\,\mathrm{d}u.$$

Proof. Define

$$H_2 = \{\delta l_n < k < A l_n : k = (\mathrm{mod})d\}. \tag{54}$$

By [9, Corollary 5] and (20), we have

$$\begin{aligned} &(1 + o(1))d \inf_{u \leq A l_n m^{-n}} u^{1-\gamma}\omega(u) \sum_{k\in H_2} \frac{k^{\gamma-1}}{m^{\gamma n}}\mathbb{P}(S_k \geq \varepsilon_n k) \\ &\leq \sum_{k\in H_2} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) \\ &= (1 + o(1))d \sum_{k\in H_2} m^{-n}\omega\left(\frac{k}{m^n}\right)\mathbb{P}(S_k \geq \varepsilon_n k) \\ &\leq (1 + o(1))d \sup_{u \leq A l_n m^{-n}} u^{1-\gamma}\omega(u) \sum_{k\in H_2} \frac{k^{\gamma-1}}{m^{\gamma n}}\mathbb{P}(S_k \geq \varepsilon_n k). \end{aligned} \tag{55}$$

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Recall (4). Then, for any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{k\in H_2} \left| \mathbb{P}(S_k \ge k\varepsilon_n) - \mathbb{P}\left(U_s \ge \frac{k\varepsilon_n}{b(k)}\right)\right| = 0.$$

Recall that $J(x) = xb(x)^{-1}$ and l is the asymptotic inverse function of J. Then, as $n \to \infty$,

$$\begin{aligned}\sum_{k\in H_2} k^{\gamma-1}\mathbb{P}(S_k \ge k\varepsilon_n) &= (1+o(1)) \sum_{k\in H_2} k^{\gamma-1}\mathbb{P}\left(U_s \ge \frac{k\varepsilon_n}{b(k)}\right) \\ &= (1+o(1))l_n^{\gamma} \sum_{k\in H_2} (kl_n^{-1})^{\gamma-1}\mathbb{P}\left(U_\alpha \ge \frac{k\varepsilon_n}{b(k)}\right)l_n^{-1} \\ &= (1+o(1))d^{-1}l_n^{\gamma} \int_\delta^A u^{\gamma-1}\mathbb{P}(U_s \ge u^{1-1/\alpha})\,\mathrm{d}u, \end{aligned} \tag{56}$$

where the last equality follows from the facts that

$$\frac{k\varepsilon_n}{b(k)} = \frac{k^{(\alpha-1)/\alpha}}{\varepsilon_n^{-1}s(k)} \sim \frac{k^{(\alpha-1)/\alpha}}{J(l_n)s(k)} = \frac{s(l_n)}{s(k)}\left(\frac{k}{l_n}\right)^{1-1/\alpha}, \qquad \lim_{n\to\infty} \sup_{k\in H_2} \frac{s(l_n)}{s(k)} = 1.$$

Then by letting $n \to \infty$ in (55) and (56) implies the desired result by noting the fact that $l_n m^{-n} \to 0$. □

Proof of Theorem 2. (i) If $\chi_n \to 0$ then we have

$$l_n^{\gamma} m^{-\gamma n} m^{(\beta-1)n} \varepsilon_n^{\beta} L(\varepsilon_n m^n)^{-1} = o(1).$$

Thus, letting $\delta \to 0$ in (34) and Lemma 7 yields the desired result.

(ii) Recall H_2 from (54). By taking A large enough in (35) and (36), we have

$$\begin{aligned}&\sum_{k\notin H_2} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &\quad = \left(\sum_{1\le k\le \delta l_n} + \sum_{Al_n<k<Am^n} + \sum_{k\ge Am^n}\right)\mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \\ &\quad \le C(2 + A^{\gamma+1-\beta+\eta})\varepsilon_n^{-\beta} m^{(1-\beta)n} L(\varepsilon_n m^n) + C(A^{-2\gamma} + \delta^{\gamma})l_n^{\gamma} m^{-\gamma n}.\end{aligned}$$

Since $\chi_n \to \infty$, we have $\varepsilon_n^{-\beta} m^{(1-\beta)n} L(\varepsilon_n m^n) = o(l_n^{\gamma} m^{-\gamma n})$. Thus,

$$\overline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} \sum_{k\notin H_2} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \ge k\varepsilon_n) \le C(A^{-2\gamma} + \delta^{\gamma}).$$

Furthermore, by (53), we have

$$\begin{aligned}V_I(\delta, A) &\le \underline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} \mathbb{P}(S_{Z_n} \ge Z_n\varepsilon_n) \\ &\le \overline{\lim_{n\to\infty}}\, l_n^{-\gamma} m^{\gamma n} \mathbb{P}(S_{Z_n} \ge Z_n\varepsilon_n) \le C(A^{-2\gamma} + \delta^{\gamma}) + V_S(\delta, A).\end{aligned}$$

Letting $\delta \to 0$ and $A \to \infty$ yields (10).

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(iii) Note that $\chi_n \to y \in (0, \infty)$ implies that

$$l_n^\gamma m^{-\gamma n} \sim y\varepsilon_n^\beta m^{(\beta-1)n} L(\varepsilon_n m^n)^{-1}.$$

Then the desired result follows from (36), (41), (47), and (53).

(iv) Combining Lemmas 6 and 7 and letting $\delta \to 0$ yields the desired result. We have completed the proof of Theorem 2. □

Proof of Theorem 6. First, note that $\int_0^\infty \mathbb{P}(U_s \geq u^{(\alpha-1)/\alpha}x)\omega(u)\,\mathrm{d}u < \infty$. Then by (4), for any $\delta > 0$,

$$\lim_{n\to\infty} \sup_{k\geq \delta m^n} \bigg| \mathbb{P}(S_k \geq \varepsilon_n k) - \mathbb{P}\bigg(U_s \geq \frac{\varepsilon_n k}{b(k)}\bigg)\bigg| = 0.$$

Thus,

$$\sum_{k\geq\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) = (1 + o(1)) \sum_{k\geq\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}\bigg(U_s \geq \frac{\varepsilon_n k}{b(k)}\bigg).$$

Denote $\bar{F}_s(x) = \mathbb{P}(U_s \geq x)$. Then, we have

$$\begin{aligned}
&\sum_{k\geq\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) \\
&\quad = (1 + o(1)) \sum_{k\geq\delta m^n} \mathbb{P}(Z_n = k)\bar{F}_s\bigg(\varepsilon_n m^n b(m^n)^{-1}\bigg(\frac{k}{m^n}\bigg)^{(\alpha-1)/\alpha} \frac{s(m^n)}{s(k)}\bigg) \\
&\quad = (1 + o(1))\mathbb{E}\bigg[\bar{F}_s\bigg(\varepsilon_n m^n b(m^n)^{-1}(W_n)^{(\alpha-1)/\alpha} \frac{s(m^n)}{s(W_n m^n)}\bigg)\mathbf{1}_{\{W_n\geq\delta\}}\bigg] \\
&\quad \to \int_\delta^\infty \bar{F}_s(u^{(\alpha-1)/\alpha}x)\omega(u)\,\mathrm{d}u.
\end{aligned}$$

On the other hand, by (18), as $n \to \infty$,

$$\sum_{k\leq\delta m^n} \mathbb{P}(Z_n = k)\mathbb{P}(S_k \geq \varepsilon_n k) \leq \sum_{k\leq\delta m^n} \mathbb{P}(Z_n = k) = (1 + o(1)) \int_0^\delta \omega(u)\,\mathrm{d}u.$$

Letting δ go to 0 yields the desired result. □

Proofs of Theorem 5 and Corollary 2. Applying (37) with $\varepsilon_n = \varepsilon$, $k > C_s\varepsilon^{t/(1-t)} =: A(s, t, \varepsilon)$ and $s = (2\gamma + 2)/(t - 1) > 1$ implies that

$$\begin{aligned}
&m^{\gamma n} \sum_{k\geq1} \mathbb{P}(Z_n = k)\mathbb{P}(S_n \geq \varepsilon k) \\
&\quad \leq C \sum_{k\geq1} k^{\gamma-1}\mathbb{P}(S_n \geq \varepsilon k) \\
&\quad \leq C \sum_{k\leq A(s,t,\varepsilon)} k^{\gamma-1} + C \sum_{k>A(s,t,\varepsilon)} k^{\gamma-1}\mathbb{P}(S_n \geq \varepsilon k) \\
&\quad \leq CA(s, t, \varepsilon)^\gamma + C \sum_{k\geq1} k^\gamma \mathbb{P}(X_1 \geq s^{-1}\varepsilon k) + C \sum_{k>A(s,t,\varepsilon)} \varepsilon^{-ts/2}k^{-2},
\end{aligned}$$

which is finite under the assumption of the theorem. Then the dominated convergence theorem yields Theorem 5. Corollary 2 follows from the same argument as above. □

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