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INVARIANT MEANS AND ACTIONS OF SEMITOPOLOGICAL SEMIGROUPS ON COMPLETELY REGULAR SPACES AND APPLICATIONS

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Abstract

In this paper, we extend the study of fixed point properties of semitopological semigroups of continuous mappings in locally convex spaces to the setting of completely regular topological spaces. As applications, we establish a general fixed point theorem, a convergence theorem and an application to amenable locally compact groups.

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1. Introduction

Given a semitopological semigroup S and any translation-invariant closed subspace (containing the real-valued constant functions) of $C_b(S)$, the Banach algebra of all bounded continuous real-valued functions on S, we establish a characterisation of the existence of a left invariant mean on it in terms of a nonlinear property for semigroups of continuous mappings on a completely regular topological space. As applications, we provide a general fixed point theorem for semigroups of continuous mappings on a nonempty compact convex set in a separated locally convex space generalising many known results (see, for example, [8, 10]). We also establish a convergence theorem ensuring in certain conditions the existence of a common fixed point for a semigroup of continuous mappings on a compact convex subset of a Hausdorff locally convex space. Furthermore, we also show that an application of our results yields a new proof of existence of a left Haar measure for the class of amenable locally compact groups.

2. Preliminaries and notation

A *semigroup* S together with a Hausdorff topology is termed *semitopological* if its operation is separately continuous. We shall denote by $C_b(S)$ the Banach algebra of all bounded continuous real-valued functions on S. Given $s \in S$ and $f \in C_b(S)$, the left (respectively right) *translate* of f is denoted and defined by $\ell_s f(t) = f(st)$ (respectively $r_s f(t) = f(ts)$). Let Φ denote a closed subspace of $C_b(S)$ containing constants. The subspace Φ is said to be *translation invariant* if it is both both left and right translation invariant. In other words, if $\ell_s(\Phi) \subset \Phi$ and $r_s(\Phi) \subset \Phi$ for all $s \in S$. The subspace Φ is called *left amenable* if there exists an element $m \in \Phi^*$ (the continuous dual of Φ) such that:

(1) $m(e) = 1 = \|m\|$, where $e : S \to \mathbb{R}$ stands for the constant 1 function;
(2) $m(\ell_s f) = m(f)$ for all $s \in S$ and $f \in \Phi$.

We shall write that Φ has a LIM (to stand for the subspace Φ has a left invariant mean). An element m satisfying property (1) is called a *mean*; and if in addition it has property (2), then it is said to be a *left invariant mean*. Given $f \in C_b(S)$, we shall assign the map $\theta_f : S \to C_b(S)$ defined by $\theta_f(s) = \ell_s f$. Throughout this paper we shall deal with the following subspaces of $C_b(S)$; see [1] for details.

- WLUC(S) denotes the subspace of $C_b(S)$ of all f such that θ_f is continuous when $C_b(S)$ is given the weak topology. Members of this space are called *weakly left uniformly continuous functions* on S.
- LMC(S) is the collection of all $f \in C_b(S)$ such that θ_f is continuous with respect to the topology induced by βS on $C_b(S)$. Here βS stands for the set of all means m on $C_b(S)$ that are multiplicative in the sense that $m(fg) = m(f)m(g)$ whenever $f, g \in C_b(S)$ (such means are commonly called multiplicative means). Functions in LMC(S) are called *left multiplicatively continuous functions* on S.
- LUC(S) is the subspace of $C_b(S)$ of those f for which θ_f is norm continuous. Members of LUC(S) are called *left uniformly continuous functions* on S.
- WAP(S) is the subspace of $C_b(S)$ consisting of all f with the property that the set $Ł(f) = \{\ell_s f \ : \ s \in S\}$ has compact closure in the weak topology of $C_b(S)$. Such functions are called *weakly almost periodic*.
- AP(S) is the space of all $f \in C_b(S)$ such that $Ł(f)$ is relatively compact with respect to the norm topology of $C_b(S)$. Functions in AP(S) are usually called *strongly almost periodic* or simply *almost periodic*.

For the interested reader, relationships between those spaces and their properties are known and may be found in [1]. From now on, X will denote a completely regular topological space. An *action* of S on X is a map $\sigma : S \times X \to X, (s, x) \mapsto s.x$ with the property that $st.x = s.(t.x)$ for every pair of points s, t of S and $x \in X$. For each $s \in S$, we shall let $\sigma_s : X \to X$ denote the partial map $x \mapsto s.x$. Then σ is termed *separately continuous* if it is continuous with respect to each variable separately (that is, if for all $s \in S$ and $x \in X$, the mappings $y \mapsto s.y : X \to X$ and $t \mapsto t.x : S \to X$ are continuous) and *jointly continuous* if $(s, x) \mapsto s.x$ is continuous when $S \times X$ is given the product topology. Now, given a translation-invariant subspace Φ of $C_b(S)$, let

$$X_\Phi := \{x \in X : f_x : S \to \mathbb{R}, s \mapsto f(s.x) \in \Phi \text{ for all } f \in C_b(X)\}.$$

Then we shall say that the action is an *A-action* (respectively *E-action*) of (S, Φ) on X if $X_\Phi = X$ (respectively $X_\Phi \neq \emptyset$). The symbol A stands for 'any', E for 'there exists' and

$C_b(X)$ denotes the Banach algebra of all bounded continuous real-valued functions on X equipped with the sup norm topology. Then it is trivial that an A-action is an E-action.

3. Main results

In this section we shall establish our main results. Given a completely regular space X, let βX denote its Stone–Čech compactification and let $\delta : X \to C_b(X)^*$ be the map defined by $\delta(x)(f) = f(x)$ for every point x of X. And, given a subset F of X, let the symbol $co(F)$ stand for the convex hull of $\delta(F)$ in the dual space $C_b(X)^*$.

Theorem 3.1. *Let S be a semitopological semigroup. Let Φ be a translation-invariant closed subspace of $C_b(S)$ containing the real-valued constant functions on S. If Φ has a left invariant mean, then S possesses the following property:*

(P): Whenever $\sigma : S \times X \to X$ is a separately continuous E-action of (S, Φ) on a completely regular topological space X, then there exist a net $(F_j)_{j\in J}$ of finite subsets of X and $\Lambda_j \in co(F_j)$ for all $j \in J$, and $\Lambda \in C_b(X)^*$ such that:

(1) $\Lambda_j(f) \underset{j}{\to} \Lambda(f)$ *for all* $f \in C_b(X)$*;*

(2) $\Lambda(f \circ \sigma_s) = \Lambda(f)$ *for all* $f \in C_b(X)$ *and for all* $s \in S$.

Conversely, if S has property (P) and Φ is any of the spaces $AP(S)$, $WAP(S)$, $LUC(S)$, $LMC(S)$, $WLUC(S)$ or $C_b(S)$, then Φ has a LIM.

Proof. We start by extending the action on X to its compactification βX. Let

$$\mathfrak{S} : S \times \beta X \to \beta X$$
$$(s, \phi) \mapsto \mathfrak{S}_s(\phi) : f \mapsto \phi(f \circ \sigma_s).$$

In particular, $\mathfrak{S}_s(\delta(x)) := \delta(\sigma_s(x))$ for all $s \in S$ and $x \in X$. We need to justify that $\mathfrak{S}$ is well defined.

Claim: $\mathfrak{S}_s(\phi) \in \beta X$ for all $s \in S$ and $\phi \in \beta X$. In fact, let $(x_j)_j$ be a net in X such that $\delta(x_j) \to \phi$ pointwise. Then, due to the separate continuity of σ given $f \in C_b(X)$, we have $\delta(\sigma_s(x_j))(f) = \delta(x_j)(f \circ \sigma_s) \to \phi(f \circ \sigma_s) = \mathfrak{S}_s(\phi)(f)$. Therefore, $\beta X \ni \mathfrak{S}_s(\delta(x_j)) \to \mathfrak{S}_s(\phi)$, which implies that $\mathfrak{S}_s(\phi) \in \beta X$. So, our claim is true. On the other hand, as readily checked, $\mathfrak{S}$ is an action that is continuous with respect to the second variable. Now let us fix $x \in X_\Phi$. Then, given $\Psi \in C(\beta X)$, the map $\Psi_x : s \mapsto \Psi(\mathfrak{S}_s(\delta(x))$ lies in Φ because $\Psi \circ \delta \in C_b(X)$ and $\Psi_x = (\Psi \circ \delta)_x \in \Phi$ as σ is an E-action. Now let m be a left invariant mean on Φ. Pick a net $(m_t)_{t\in T}$ of finite means on Φ converging pointwise to m; see [1, Theorem 1.8]. Since finite means are convex combinations of point masses, then one can put $m_t := \sum_{i=1}^{n_t} \lambda_i^t \delta_{s_i^t}$ with $\lambda_i^t \geq 0, \sum_{i=1}^{n_t} \lambda_i^t = 1$ and $s_i^t \in S$. Then it follows that $co(\beta X) \ni \Lambda_t := \sum_{i=1}^{n_t} \lambda_i^t \delta(\sigma_{s_i^t}(x))$ for all $t \in T$. As βX is compact in the topology of pointwise convergence, then one can extract a convergent subnet $(\Lambda_{t_j})_{j\in J}$ converging to some limit Λ lying in the pointwise closure of the convex hull of βX. For all $j \in J$, put

$$F_j := \{\sigma_{s_1^{t_j}}(x), \sigma_{s_2^{t_j}}(x), \ldots, \sigma_{s_{n_{t_j}-1}^{t_j}}(x), \sigma_{s_{n_{t_j}}^{t_j}}(x)\}.$$

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Then it is obvious that $\Lambda_{t_j} \in co(F_j)$ for all $j \in J$. Furthermore, given $f \in C_b(X)$, if we let $\widetilde{f \circ \sigma_s} \in C(\beta X)$ denote the continuous extension of $f \circ \sigma_s$ to βX, then it is readily checked that $f \circ \sigma_s = \widetilde{f \circ \sigma_s} \circ \delta$. So,

$$\begin{aligned}
\Lambda(f \circ \sigma_s) &= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \delta(\sigma_{s_i^{t_j}}(x))(f \circ \sigma_s) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} f \circ \sigma_s(\sigma_{s_i^{t_j}}(x)) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \widetilde{f \circ \sigma_s}(\delta(\sigma_{s_i^{t_j}}(x))) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \widetilde{f \circ \sigma_s} \circ \delta(\sigma_{s_i^{t_j}}(x)) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} f \circ \sigma_s(\sigma_{s_i^{t_j}}(x)) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \delta_{s_i^{t_j}}((f \circ \sigma_s)_x) \\
&= \lim_j m_{t_j}(\ell_s f_x) = m(\ell_s f_x) = m(f_x) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \delta_{s_i^{t_j}}(f_x) \\
&= \lim_j \sum_{i=1}^{n_{t_j}} \lambda_i^{t_j} \delta(\sigma_{s_i^{t_j}}(x))(f) \\
&= \lim_j \Lambda_{t_j}(f) \\
&= \Lambda(f).
\end{aligned}$$

Conversely, let us now assume that S possesses the property (P). We consider three cases.

Case 1: $\Phi = \mathrm{AP}(S)$. Fix $f \in \mathrm{AP}(S)$. Let X be the norm closure of the convex hull of the right orbit $\mathcal{R}(f) = \{r_s f \ ; \ s \in S\}$. Then X is norm compact since $\mathcal{R}(f)$ is relatively compact in the norm topology. Define $\sigma : S \times X \to X$ by letting $\sigma_s(h) = r_s h$. Then as readily checked σ is separately continuous; moreover, for all $s \in S$, the mapping $\sigma_s : h \mapsto r_s h$ is norm nonexpansive. So, it follows that $\{\sigma_s \ : \ s \in S\}$ is an equicontinuous family. Hence, it defines an A-action of $(S, \mathrm{AP}(S))$ on X by [8]. Hence, by assumption, there are $\Lambda \in C(X)^*$ and $\Lambda_j \in co(\delta(X))$ such that $\Lambda_j \to \Lambda$ pointwise and $\Lambda(f \circ \sigma_s) = \Lambda(f)$ for all $s \in S$ and $f \in C(X)$. Let $\Lambda_j = \sum_{i=1}^{n_j} t_i^j \delta(h_i^j)$.

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Note that as X is a compact space, we have $\beta X \simeq X$. Moreover, since X is convex and norm compact, one can assume without loss of generality that $X \ni g_j := \sum_{i=1}^{n_j} t_i^j h_i^j \to g$ in norm to some g (because, if not, we just consider a norm-convergent subnet). Then, given $s \in S$ and $\phi \in \mathrm{AP}(S)^*$,

$$\begin{aligned}\phi(r_s g) &= \lim_j \sum_{i=1}^{n_j} t_i^j \phi(r_s h_i^j)\\ &= \lim_j \sum_{i=1}^{n_j} t_i^j \delta(h_i^j)(\phi \circ \sigma_s)\\ &= \lim_j \Lambda_j(\phi \circ \sigma_s)\\ &= \Lambda(\phi \circ \sigma_s) = \Lambda(\phi)\\ &= \lim_j \phi(g_j)\\ &= \phi(g).\end{aligned}$$

Thus, $\phi(r_s g - g) = 0$ for all $\phi \in \mathrm{AP}(S)^*$. Hence, $r_s g = g$ for every $s \in S$.

If S has an identity, say e, then for all $s \in S$ we have $g(s) = r_s g(e) = g(e)$, which shows that $g \equiv g(e)$ is a constant map.

If not, then we adjoin an identity e to S by simply letting $se = s = es$, $ee = e$. Let S^e be the new semigroup obtained. We topologise S^e by adjoining $\{e\}$ to the open sets of S and extend σ to an action of σ^e on X by letting $(e, h) \mapsto h$ ($h \in X$). The extended action σ^e is equicontinuous (because nonexpansive) and separately continuous; it therefore defines an A-action of $(S, \mathrm{AP}(S^e))$ on X, which shows that S^e has property (P). So, by virtue of the previous case, there is a constant function in $X \subset \overline{X}^p$ (the closure of X in the topology of pointwise convergence). Hence, it follows from [5] that $\mathrm{AP}(S)$ has a left invariant mean.

Case 2: $\Phi = \mathrm{WAP}(S)$. Given $f \in \mathrm{WAP}(S)$, let X be as in Case 1. From the relative compactness of $\mathcal{R}(f)$ in the weak topology of $C_b(S)$, it follows that X is weakly compact by virtue of the Krein–Šmulian theorem. Further, we assert that σ is quasi-equicontinuous (that is, the closure $\overline{S}^p$ of S in the product X^X consists entirely of continuous functions; see [1] for more details). Indeed, let $\theta \in \overline{S}^p$ and let us assume by contradiction that θ is not continuous. Then there exist $x \in X, \phi \in \mathrm{WAP}(S)^*, \epsilon > 0$ and a weakly convergent net $h_j \to h$ ($j \in J$) in X such that

$$\phi(\theta(h_j) - \theta(h)) \geq \epsilon \quad \text{for all } j \in J. \tag{3.1}$$

Since $h_j \to h$ weakly, h lies in the closed convex hull of $\{h_j : j \in J\}$. Therefore, there exists a sequence $(y_n)_n$ such that: $y_n = \sum_{i\in J_n} t_i^n h_i^n$, J_n finite, $t_i^n \geq 0, \sum_{i\in J_n} t_i^n = 1$, $h_i^n \in \{h_j : j \in J\}$ and $\|y_n - h\| \to 0$. On the other hand, as $\theta \in \overline{S}^p$, pick a net $(s_t)_t$ in S such that $\sigma_{s_t} \to \theta$ weakly pointwise. Then, by using (3.1), we get for all n,

$$\phi(\theta(y_n) - \theta(h)) = \sum_{i\in J_n} t_i^n \phi(\theta(h_i^n) - \theta(h)) \geq \epsilon.$$

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Thus, it follows that $0 < \epsilon \leq \phi(\theta(y_n) - \theta(h)) = \lim_t \phi(\sigma_{s_t}(y_n - h)) \leq \|\phi\|.\|y_n - h\| \to 0$, leading to a contradiction. Hence, θ must be continuous and our claim is true. Consequently, σ as a separately continuous and quasi-equicontinuous action on the compact space X is by [9] an A-action of $(S, \mathrm{WAP}(S))$ on X. So, by assumption, there is a $\Lambda \in C(X)^*$ such that $\Lambda(f \circ \sigma_s) = \Lambda(f)$ for all $f \in C(X)$. Thus, a similar argument as in Case 1 yields together with [5] the existence of a left invariant mean on $\mathrm{WAP}(S)$. Note that in the case where S has no identity, the extended action on its unitisation S^e is also quasi-equicontinuous because $\overline{S^e}^p = \overline{S}^p \cup \{\mathrm{id}_X\}$.

Case 3: $\Phi = \mathrm{LUC}(S)$, $\mathrm{LM}(S)$, $\mathrm{WLUC}(S)$ or $C_b(S)$. Let X be the collection of all means on Φ. It is known that X is a compact convex subset of Φ^* in the weak* topology. Define $\sigma : S \times X \to X$ by letting $\sigma_s(m) = Ł_s(m)$ with $Ł_s(m)(f) = m(\ell_s f)$ if $f \in \Phi$. Then, from [10], this map is an A-action of (S, Φ) on X whenever $\Phi = \mathrm{LUC}(S)$, $\mathrm{LM}(S)$ or $\mathrm{WLUC}(S)$. If $\Phi = C_b(S)$, then, due to the separate continuity of σ (which is actually jointly continuous), it is automatically an A-action of $(S, C_b(S))$ on X. Hence, by assumption, for each case there is a convergent net $(\Lambda_j^\Phi)_{j \in J_\Phi}$ in the convex hull of $\beta X \simeq X$ such that $\Lambda^\Phi := \lim_j \Lambda_j^\Phi$ has the property $\Lambda^\Phi(f \circ \sigma_s) = \Lambda^\Phi(f)$ for all $f \in C(X)$. Then let us put $\Lambda_j^\Phi := \sum_{i=1}^{n_{\Phi,j}} t_i^{\Phi,j} \delta(m_i^{\Phi,j})$ for some of the $m_i^{\Phi,j} \in X$ and, for all $j \in J$, define

$$m_{\Phi,j} := \sum_{i=1}^{n_{\Phi,j}} t_i^{\Phi,j} m_i^{\Phi,j}$$

and

$$m_\Phi(f) := \Lambda^\Phi(ev_f)$$

for all $f \in \Phi$. Here, given $f \in \Phi$, the symbol ev_f stands for the evaluation map at f on Φ^*, which is also an element of $C(X)$. Then

$$\begin{aligned} m_{\Phi,j}(f) &= \sum_{i=1}^{n_{\Phi,j}} t_i^{\Phi,j} m_i^{\Phi,j}(f) \\ &= \sum_{i=1}^{n_{\Phi,j}} t_i^{\Phi,j} ev_f(m_i^{\Phi,j}) \\ &= \Lambda^{\Phi,j}(ev_f) \\ &\xrightarrow[j]{} \Lambda^\Phi(ev_f) = m_\Phi(f). \end{aligned}$$

So, m_Φ as a pointwise limit of a net of means is then a mean. On the other hand, given $s \in S$,

$$\begin{aligned} m_\Phi(\ell_s f) &= \lim_j m_{\Phi,j}(\ell_s f) \\ &= \lim_j \sum_{i=1}^{n_{\Phi,j}} t_i^{\Phi,j} m_i^{\Phi,j}(\ell_s f) \end{aligned}$$

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$$= \lim_j \Lambda_j^{\Phi}(ev_f \circ \sigma_s)$$
$$= \Lambda^{\Phi}(ev_f \circ \sigma_s) = \Lambda^{\Phi}(ev_f)$$
$$= m_{\Phi}(f).$$

Hence, it follows that m_Φ is a left invariant mean on Φ. □

Remark 3.2. We point out that Theorem 3.1 is a consequence of [14]. On the other hand, if in property (P) the space X is assumed to be a compact convex subset of a Hausdorff convex space and the action is affine, then property (P) yields a common fixed point for S as established in Corollary 4.1.

Example 3.3. Let S be a semitopological semigroup and $\sigma : S \times X \to X$ be a separately continuous action on a compact Hausdorff topological space X. Then it is known that (see, for example, [8–10]):

(1) if S is discrete, then σ is automatically an A-action of $(S, \ell^\infty(S))$ on X (even without any continuity assumption);
(2) if σ is separately continuous, then it is an A-action of $(S, \mathrm{WLUC}(S))$ on X;
(3) if, for all $x \in X$, the mapping $s \mapsto \sigma_s(x)$ is continuous, then σ defines an A-action of $(S, \mathrm{LMC}(S))$ on X;
(4) if σ is jointly continuous, then it defines an A-action of $(S, \mathrm{LUC}(S))$ on X;
(5) if σ is separately continuous and quasi-equicontinuous (that is, the closure $\overline{S}^p$ of S in the product space X^X consists only of continuous functions), then it defines an A-action of $(S, \mathrm{WAP}(S))$ on X;
(6) if, in particular, X is a subset of a topological vector space, and σ is separately continuous and equicontinuous (that is, for each neighbourhood V of 0, there is a neighbourhood W of 0 such that for all $x, y \in X$ we have: $x - y \in W$ implies that $\sigma_s(x) - \sigma_s(y) \in V$ for all $s \in S$), then σ is an A-action of $(S, \mathrm{AP}(S))$ on X.

4. Applications to fixed point theory

In this section, given a semitopological semigroup S, we provide a characterisation of the existence of a left invariant mean on translation-invariant closed subspaces of $C_b(S)$ that contains the constant functions by a fixed point property generalising some common fixed point theorems in the literature.

Corollary 4.1. *Let S be a semitopological semigroup and Φ be a translation-invariant closed subspace of $C_b(S)$ containing constant functions on S. If Φ has a LIM, then S possesses the following fixed point property:*

(F): Whenever $S \times X \to X$ is a separately continuous affine E-action of (S, Φ) on a nonempty compact convex subset X of a Hausdorff locally convex space E, then there exists in X a common fixed point for S.

Conversely, if S has the fixed point property (F) and Φ is any of the spaces: $AP(S)$, $WAP(S)$, $LUC(S)$, $LMC(S)$, $WLUC(S)$ or $C_b(S)$, then Φ has a LIM.

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PROOF. Assume that Φ is left amenable. Since $X \simeq \beta X$, then, by Theorem 3.1, there is a convergent net $\sum_{i=1}^{n_j} t_i^j \delta(x_i^j) \to \Lambda \in C(X)^*$ such that $\Lambda(f \circ \sigma_s) = \Lambda(f)$ for all $f \in C(X)$. Put $x_j := \sum_{i=1}^{n_j} t_i^j x_i^j \in X$ for all $j \in J$. From the compactness of X, by taking a convergent subnet if necessary, we may assume without loss of generality that $(x_j)_{j\in J}$ is convergent. Let $\widetilde{x} \in X$ denote its limit. Fix $\phi \in X^*$ (the continuous dual of X) and $s \in S$. Then, by affineness of σ,

$$\begin{aligned}
\phi(\widetilde{x}) &= \lim_j \phi\Big(\sum_{i=1}^{n_j} t_i^j x_i^j\Big) \\
&= \lim_j \sum_{i=1}^{n_j} t_i^j \delta(x_i^j)(\phi) \\
&= \Lambda(\phi) = \Lambda(\phi \circ \sigma_s) \\
&= \lim_j \sum_{i=1}^{n_j} t_i^j \delta(x_i^j)(\phi \circ \sigma_s) \\
&= \lim_j \sum_{i=1}^{n_j} t_i^j \phi \circ \sigma_s(x_i^j) \\
&= \lim_j \phi \circ \sigma_s\Big(\sum_{i=1}^{n_j} t_i^j x_i^j\Big) \\
&= \phi \circ \sigma_s(\widetilde{x}).
\end{aligned}$$

Thus, we have $\phi(\sigma_s(\widetilde{x}) - \widetilde{x}) = 0$ for all ordered pairs $(s, \phi) \in S \times X^*$. Hence, $\sigma_s(\widetilde{x}) = \widetilde{x}$ (since X^* separates points of X). By arbitrariness of s, it follows that $\widetilde{x}$ is a common fixed point for S. Conversely, if S has fixed point property (F), then the conclusion follows from a similar argument as in the proof of Theorem 3.1. □

For nonlinear actions we also have the following result.

COROLLARY 4.2. *Let S be a semitopological semigroup. Let Φ be a translation-invariant closed subspace of $C_b(S)$ containing constant functions on S. If Φ possesses a left invariant mean, then S possesses the following property:*

(P'): Whenever $S \times X \to X$ is a separately continuous E-action of (S, Φ) on a nonempty compact convex subset X of a Hausdorff locally convex space E, then there exist a net $(\sum_{j=1}^{n_\alpha} t_j^\alpha x_j^\alpha)_{\alpha\in J}$ of convex combinations of elements of X and an element $\widetilde{x} \in X$ such that:

(1) $\sum_{j=1}^{n_\alpha} t_j^\alpha x_j^\alpha \underset{\alpha}{\to} \widetilde{x}$ *strongly;*

(2) $\sum_{j=1}^{n_\alpha} t_j^\alpha \sigma_s(x_j^\alpha) \underset{\alpha}{\to} \widetilde{x}$ *weakly for all $s \in S$;*

(3) $q(\sum_{j=1}^{n_\alpha} t_j^\alpha \sigma_s(x_j^\alpha)) \underset{\alpha}{\to} q(\widetilde{x})$ *for all continuous seminorms q on E, $s \in S$;*

(4) *if in addition the action is affine, then $\widetilde{x}$ is a common fixed point for S.*

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Proof. If Φ possesses a left invariant mean, then, by Theorem 3.1, there exists $\phi \in \overline{co}^{\tau_{wk*}}(\beta X)$ (the weak* closed convex hull of the Stone–Čech compactification βX ($\simeq X$ since X is compact) of X) such that $\phi(f \circ \sigma_s) = \phi(f)$ for all $s \in S$ and for all $f \in C(X)$. Put

$$\phi := \tau_{wk*}\text{-}\lim_j \sum_{i=1}^{n_j} t_i^j \delta(x_i^j)$$

with $t_i^j \in [0,1]$, $\sum_{i=1}^{n_j} t_i^j = 1$ and $x_i^j \in X$. Since K is convex, we have $\sum_{i=1}^{n_j} t_i^j x_i^j \in X$ for all $j \in J$. So, by compactness (of X), from the net $(\sum_{i=1}^{n_j} t_i^j x_i^j)_j$ one can extract a convergent subnet, say $(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} x_i^{j_\alpha})_{\alpha \in J}$; let $\widetilde{x}$ denote its limit. Then, given $\Gamma \in X^*$ (the continuous dual of X), $s \in S$ and a continuous seminorm q on E,

$$\begin{aligned}\Gamma(\widetilde{x}) &= \lim_\alpha \Gamma(\theta_{j_\alpha}) = \phi(\Gamma) = \phi(\Gamma \circ \sigma_s)\\ &= \lim_\alpha \sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \delta(x_i^{j_\alpha})(\Gamma \circ \sigma_s)\\ &= \lim_\alpha \sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \Gamma \circ \sigma_s(x_i^{j_\alpha})\\ &= \lim_\alpha \Gamma\Big(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \sigma_s(x_i^{j_\alpha})\Big).\end{aligned}$$

Therefore, $\Gamma(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \sigma_s(x_i^{j_\alpha})) \to \Gamma(\widetilde{x})$ for all $\Gamma \in X^*$ and $s \in S$, which means that $\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \sigma_s(x_i^{j_\alpha}) \to \widetilde{x}$ weakly whenever $s \in S$. On the other hand, by replacing Γ with q, then, by (weak) lower semicontinuity,

$$\limsup_\alpha q\Big(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \sigma_s(x_i^{j_\alpha})\Big) \le \lim_\alpha \sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} q \circ \sigma_s(x_i^{j_\alpha}) = q(\widetilde{x}) \le \liminf_\alpha q\Big(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} \sigma_s(x_i^{j_\alpha})\Big).$$

Hence, equality holds throughout. The last part follows by affineness and (2). Therefore, $(\sum_{i=1}^{n_{j_\alpha}} t_i^{j_\alpha} x_i^{j_\alpha})_{\alpha \in J}$ is the required net we are looking for. □

Let S be a semitopological semigroup. Then Example 3.3 and Corollary 4.1 or Corollary 4.2 yield the following result.

Corollary 4.3 [10, Theorem 4]. *WLUC(S) possesses a LIM if and only if:*

(F1): Whenever $S \times X \to X$ is a separately continuous affine action on a compact convex subset X of a separated locally convex space, then there is a common fixed point for S.

Corollary 4.4 [10, Theorem 2]. *LUC(S) has a LIM if and only if:*

(F2): Whenever $S \times X \to X$ is a jointly continuous affine action on a compact convex subset X of a separated locally convex space, then there is a common fixed point for S.

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COROLLARY 4.5. *WAPS*(S) *has a LIM if and only if:*

(F3): Whenever $S \times X \to X$ *is a separately continuous and quasi-equicontinuous affine action on a compact convex subset* X *of a separated locally convex space, then there is a common fixed point for* S.

COROLLARY 4.6 [8, Theorem 3.2]. *AP*(S) *has a LIM if and only if:*

(F4): Whenever $S \times X \to X$ *is a separately continuous and equicontinuous affine action on a compact convex subset* X *of a separated locally convex space, then there is a common fixed point for* S.

5. Application to locally compact groups

A *locally compact group* is a group G together with a locally compact topology and such that the mappings $(g.h) \mapsto gh$ from $G \times G$ into G and $g \mapsto g^{-1}$ from G into itself are continuous. Throughout, G denotes a locally compact group with identity e. The purpose of this section is to show that an application of the results established in the previous section yields the existence of a left Haar measure for amenable G; that is, when $C_b(G)$ possesses an invariant mean. Note that for a locally compact group it is a known fact, see [12], that the existence of a mean on the space of left uniformly continuous functions is equivalent to the algebra of bounded continuous functions being amenable. Amenable locally compact groups include compact groups, commutative groups and solvable (discrete) groups [3]. Note that fixed point proofs of existence of a left Haar measure are known for the class of compact groups [11] through the Kakutani fixed point theorem, abelian groups [6] through the Markov–Kakutani fixed point theorem and amenable hypergroups (satisfying a certain positivity property for translations) in [15].

Let $\mathscr{C}$ denote the collection of all nonempty compact subsets of G and let $C_c(G)$ stand for the subspace of $C_b(G)$ consisting of those mappings having a compact support. On $C_c(G)$, let us say that a net $(f_t)_t$ is *τ-convergent* with limit f if there exist a fixed compact set C_f and some t_f such that:

(1) support$(f) \subset C_f$;
(2) support$(f_t) \subset C_f$ for all $t \geq t_f$;
(3) $f_t \to f$ uniformly on C_f.

EXAMPLE 5.1. Let $G = (\mathbb{R}, +)$. Given $x_o \in G, t \in T = (0, 1]$, define

$$f_t(x) := \begin{cases} \dfrac{-1}{1+t}(x - x_o) + 1 & \text{if } x \in [x_o, x_o + 1 + t], \\ \dfrac{1}{1+t}(x - x_o) + 1 & \text{if } x \in [x_o - 1 - t, x_o], \\ 0 & \text{if } |x - x_o| \geq 1 + t, \end{cases}$$

and

$$f(x) := \begin{cases} -(x - x_o) + 1 & \text{if } x \in [x_o, x_o + 1], \\ (x - x_o) + 1 & \text{if } x \in [x_o - 1, x_o], \\ 0 & \text{if } |x - x_o| \geq 1. \end{cases}$$

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Put $C_f = [x_o - 2, x_o + 2]$. Then $\text{support}(f_t) = [x_o - 1 - t, x_o + 1 + t] \subset C_f$ and $\text{support}(f) = [x_o - 1, x_o + 1] \subset C_f$ for all $t \in T$. Furthermore, it is straightforward that $\sup_{x \in C_f} |f_t(x) - f(x)| = (t/(1 + t)) \xrightarrow[t \to 0]{} 0$. Therefore, f is the τ-limit of $(f_t)_{t \in T}$.

Let us say that a linear form ϕ on $C_c(G)$ is *τ-continuous* if we have $\phi(f_t) \to 0$ whenever $(f_t)_t$ is a τ-convergent net to 0. Define

$$\mathcal{D}(G)^* := \{\phi : C_c(G) \to \mathbb{R} : \phi \text{ is linear, } \tau\text{-continuous}\}$$

and fix a relatively compact symmetric neighbourhood N of the identity e of G. Then the author has established, see [13, Ch. 4, Lemma 4.5], the following result.

Lemma 5.2. *Let K be the set*

$$\left[\phi \in \mathcal{D}(G)^* : \phi \geq 0, \begin{cases} \phi(f) \in [0,1] & \text{if } f \in [0,1], \text{support}(f) \subset g.N \text{ for some } g, \\ \phi(f) \geq 1 & \text{if } f \geq 0, f \equiv 1 \text{ on } g.N.N \text{ for some } g \in G. \end{cases}\right]$$

Then K is a nonempty compact convex subset of $\mathcal{D}(G)^*$ with respect to the topology of pointwise convergence.

Then, as an application of Corollary 4.1 and 5.2, we have the following result.

Theorem 5.3. *Every amenable locally compact group G possesses a left Haar measure, that is, a nonnegative nonzero regular Borel measure μ on G such that:*

(1) $\mu(C) < \infty$ *for all* $C \in \mathscr{C}$;
(2) $\mu(g.E) = \mu(E)$ *for all Borel sets E and $g \in G$.*

Proof. Let K be as in the lemma. Define $\sigma : G \times K \to K$, an action of G on K, by the equation $g.\phi(f) := \phi(\ell_g f), (f \in C_c(G))$. Then it is readily checked that σ is a separately continuous A-action of $(G, C_b(G))$ on K. So, if G is amenable, then, as an application of Corollary 4.1 with $\Phi = C_b(G)$ and $X = K$ (equipped with the topology of pointwise convergence), there is $\phi_o \in K$ such that $\phi_o(\ell_g f) = \phi_o(f)$ for all $g \in G$ and $f \in C_b(G)$. Moreover, from the definition of K, ϕ_o is a nonnegative linear form on $C_c(G)$. We claim that ϕ_o is nonzero. Indeed, since $\overline{N.N}$ is compact, then using Urysohn's lemma let us fix $f \in C_c(G)$ such that $f \equiv 1$ on $\overline{N.N}, 0 \leq f \leq 1$. Then $\phi_o(f) \geq 1$, which shows that $\phi_o(f) \neq 0$. Hence, the existence of the measure with the desired properties follows by invoking the Riesz representation theorem to ϕ_o and the properties of ϕ_o. □

Remark 5.4. Actually, see [4, Theorem 5.10], the measure is unique up to a multiplicative positive constant. We also point out that another proof for the amenable case has been established independently in [7].

Remark 5.5. It is a well-known fact that for any topological group G, its space $\text{WAP}(G)$ of weakly almost periodic functions possesses an invariant mean (following from an application of the Ryll–Nardzewsky fixed point theorem [2]). Therefore, it is relevant at this point to raise the following question.

Open problem 1: Can we prove the existence of a left Haar measure for general locally compact groups through a fixed point theorem?

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REMARK 5.6. In Theorem 3.1, we have shown that the convergence $\Lambda_j \to \Lambda$ is pointwise. And, Corollary 4.1 shows that property (P) in Theorem 3.1 yields a common fixed point when the action is affine and the underlying space is assumed to be a compact convex subset of a locally convex space. So, one may ask the following questions.

Open problem 2: Does the convergence in Theorem 3.1 still hold when $C_b(X)^*$ is equipped with the weak topology? If not, under what condition does it hold with respect to the weak topology?

Open problem 3: Is the converse of Theorem 3.1 true for any translation-invariant closed algebra of functions containing the constants?

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References

[1] J. F. Berglund, H. D. Junghenn and P. Milnes, *Analysis on Semigroups* (John Wiley, New York, 1989).
[2] R. B. Burckel, *Weakly Almost Periodic Functions on Semigroups* (Gordon and Breach, New York, 1970).
[3] M. M. Day, 'Amenable semigroups', *Illinois J. Math.* **1** (1957), 509–544.
[4] M. D. De Chiffre, *The Haar Measure*, BSc Thesis, University of Copenhagen, 2011.
[5] E. Granirer and A. T. Lau, 'Invariant means on locally compact groups', *Illinois J. Math.* **15** (1971), 249–257.
[6] A. J. Izzo, 'A functional analysis proof of the existence of Haar measure on locally compact abelian groups', *Proc. Amer. Math. Soc.* **115** (1992), 581–583.
[7] A. J. Izzo, 'A simple proof of the existence of Haar measure on amenable groups', *Math. Scand.* **120** (2017), 317–319.
[8] A. T.-M. Lau, 'Invariant means on almost periodic functions and fixed point properties', *Rocky Mountain J. Math.* **3** (1973), 69–76.
[9] A. T.-M. Lau and Y. Zhang, 'Fixed point properties of semigroups of non-expansive mappings', *J. Funct. Anal.* **254** (2008), 2534–2554.
[10] T. Mitchell, 'Topological semigroups and fixed points', *Illinois J. Math.* **14** (1970), 630–641.
[11] W. Rudin, *Functional Analysis*, 2nd edn (McGraw-Hill, New York, 1991).
[12] V. Runde, *Lectures on Amenability* (Springer, Berlin, 2002).
[13] K. Salame, *Amenability and Fixed Point Properties of Semi-Topological Semigroups of Non-Expansive Mappings in Banach Spaces*, PhD Thesis, University of Alberta, 2016.
[14] K. Salame, 'A characterization of σ-extremely amenable semitopological semigroups', *J. Nonlinear Convex Anal.* **19** (2018), 1443–1458.
[15] B. Wilson, 'A fixed point theorem and the existence of a Haar measure for hypergroups satisfying conditions related to amenability', *Canad. Math. Bull.* **58** (2015), 415–422.

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