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SCHWARZ LEMMA FOR HARMONIC FUNCTIONS IN THE UNIT BALL

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Abstract Recently, it is proven that positive harmonic functions defined in the unit disc or the upper half-plane in $\mathbb{C}$ are contractions in hyperbolic metrics [14]. Furthermore, the same result does not hold in higher dimensions as shown by given counterexamples [16]. In this paper, we shall show that positive (or bounded) harmonic functions defined in the unit ball in $\mathbb{R}^n$ are Lipschitz in hyperbolic metrics. The involved method in main results allows to establish essential improvements of Schwarz type inequalities for monogenic functions in Clifford analysis [24, 25] and octonionic analysis [21] in a unified approach.

Keywords: harmonic function; Schwarz–Pick lemma; hyperbolic metric

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1. Introduction

Let $\mathbb{B}_n$ be the open unit ball and $\mathbb{H}_n$ be the upper half-space in $\mathbb{R}^n$, respectively. Specially, $\mathbb{B}_2$ and $\mathbb{H}_2$ are denoted as $\mathbb{D}$ and $\mathbb{H}$, identified with the open unit disc and the upper half-plane of $\mathbb{C}$. The classical Schwarz–Pick lemma states that holomorphic functions $f : \mathbb{D} \to \mathbb{D}$ satisfy

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}, \tag{1.1}$$

and

$$|\varphi_{f(w)}(f(z))| \leq |\varphi_w(z)|, \quad z, w \in \mathbb{D}, \tag{1.2}$$

where $\varphi_w(z) = (w - z)(1 - \overline{w}z)^{-1}$ is the Möbius transformation of $\mathbb{D}$ onto itself.

Recall that the hyperbolic metric on $\mathbb{D}$ is given by

$$d_{\mathbb{D}}(z, w) = \log \frac{1 + |\varphi_w(z)|}{1 - |\varphi_w(z)|} = 2 \tanh^{-1}(|\varphi_w(z)|), \quad z, w \in \mathbb{D}.$$

Note that $\tanh^{-1}$ is monotone increasing, then (1.2) can be rewritten as

$$d_{\mathbb{D}}(f(z), f(w)) \leq d_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D}.$$

That is to say every holomorphic function $f : \mathbb{D} \to \mathbb{D}$ is a contraction with respect to the hyperbolic metric on $\mathbb{D}$.

In 2012, Kalaj and Vuorinen in [7, Theorem 1.12] proved that, for harmonic functions $f : \mathbb{D} \to (-1, 1)$,

$$|\nabla f(z)| \leq \frac{4}{\pi} \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}, \tag{1.3}$$

where the constant $4/\pi$ is sharp.

Equivalently, harmonic functions $f : \mathbb{D} \to (-1, 1)$ are Lipschitz in the hyperbolic metric, i.e.

$$d_{\mathbb{D}}(f(z), f(w)) \leq \frac{4}{\pi} d_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D},$$

which holds also for harmonic functions defined in hyperbolic plane domains (see [15, Theorem 4]).

In 2013, Chen in [3, Theorem 1.2] obtained a sharper version of (1.3)

$$|\nabla f(z)| \leq \frac{4}{\pi} \frac{\cos \frac{\pi |f(z)|}{2}}{1 - |z|^2}, \quad z \in \mathbb{D},$$

which was generalized into pluriharmonic functions (see [26, Theorem 1.5]).

Motivated by the result of Kalaj and Vuorinen, Marković in 2015 showed [14, Theorem 1.1] that harmonic functions $f : \mathbb{H} \to \mathbb{R}^+ = (0, +\infty)$ are contractible in the hyperbolic metric, i.e.

$$d_{\mathbb{R}^+}(f(z), f(w)) \leq d_{\mathbb{H}}(z, w), \quad z, w \in \mathbb{H}, \tag{1.4}$$

where the hyperbolic metric $d_{\mathbb{H}}$ on the upper half-plane $\mathbb{H}$ is given by

$$d_{\mathbb{H}}(z, w) = 2 \tanh^{-1} \left| \frac{z - w}{\overline{z} - w} \right|, \quad z, w \in \mathbb{H}.$$

In particular, the hyperbolic distance $d_{\mathbb{R}^+}$ on $\mathbb{R}^+$ is

$$d_{\mathbb{R}^+}(x, y) = d_{\mathbb{H}}(ix, iy) = |\log \frac{x}{y}|, \quad x, y \in \mathbb{R}^+.$$

In [16], Melentijević established some refinements of Schwarz's lemma for holomorphic functions with the invariant gradient and gave another proof of (1.4) based on Harnack inequality. By using the same strategy, one can show that harmonic functions $f : \mathbb{D} \to \mathbb{R}^+$ are also contractible in the hyperbolic metric,

$$d_{\mathbb{R}^+}(f(z), f(w)) \leq d_{\mathbb{D}}(z, w), \quad z, w \in \mathbb{D}. \tag{1.5}$$

Furthermore, Melentijević provide counterexamples to show that these results in (1.4) and (1.5) do not hold in higher dimensions for positive harmonic functions defined in $\mathbb{B}_n$ or $\mathbb{H}_n$ when $n \geq 3$; see [16, Example 1 and Example 2].

In fact, up to multiplying a constant depending on the dimension, these results in (1.4) and (1.5) still hold for positive harmonic functions defined in higher dimensions. To be more precise, we shall establish the following result in this paper.

Theorem 1.1. *Let $n \geq 2$ be integer and $f : \mathbb{B}_n \to \mathbb{R}^+$ be a harmonic function. Then*

$$d_{\mathbb{R}^+}(f(x), f(y)) \leq (n-1) d_{\mathbb{B}_n}(x, y), \quad x, y \in \mathbb{B}_n, \tag{1.6}$$

where $d_{\mathbb{B}_n}$ is the hyperbolic metric on $\mathbb{B}_n$ given by $d_{\mathbb{B}_n}(x, y) = 2\tanh^{-1}(|\varphi_y(x)|)$, and $\varphi_y(x)$ is the Möbius transformation of $\mathbb{B}_n$ defined by (2.1).

The proof of Theorem 1.1 is built on the following estimate. Moreover, this estimate is sharp.

Theorem 1.2. *Let $n \geq 2$ be integer and $f : \mathbb{B}_n \to \mathbb{R}^+$ be a harmonic function. Then*

$$|(|x|^2 - 1)\nabla f(x) + (n-2)x f(x)| \leq n f(x), \quad x \in \mathbb{B}_n. \tag{1.7}$$

If the equality in (1.7) is attained for some $a \in \mathbb{B}_n$, then there is a point $\xi \in \partial\mathbb{B}_n$ such that

$$f(x) = f(a)|1 - \varphi_a(x)\overline{a}|^{n-2} P_\xi \circ \varphi_a(x), \quad x \in \mathbb{B}_n, \tag{1.8}$$

where P_ξ is the Poisson kernel given by

$$P_\xi(x) = P(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}.$$

Moreover, every positive and harmonic function f defined by (1.8) satisfies the equality in (1.7) for all $x \in \mathbb{B}_n$ and $\xi \in \partial\mathbb{B}_n$.

The natural question is to ask: what is the analogue of Theorem 1.1 for bounded harmonic functions $f : \mathbb{B}_n \to (-1, 1)$? Based on the proved Khavinson conjecture in [11], Liu very recently has given an answer to this question by established the following Schwarz–Pick type inequality [12, Theorem 1], which can be viewed as a counterpart of Theorem 1.2 for bounded harmonic functions.

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Theorem 1.3. *Let $n \geq 4$ be integer and $f : \mathbb{B}_n \to (-1, 1)$ be a harmonic function. Then*

$$|\nabla f(x)| \leq \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} \frac{2}{1-|x|^2}, \quad x \in \mathbb{B}_n, \tag{1.9}$$

where $|\mathbb{B}_n|$ denotes the volume of the unit ball $\mathbb{B}_n$. The equality in (1.9) holds if and only if $\mathrm{x} = 0$ and $f = U \circ T$ for some $T \in O(n)$, where U is the Poisson integral of the function that equals 1 on a hemisphere and -1 on the remaining hemisphere and $O(n)$ denotes the set of orthogonal transformations of $\mathbb{R}^n$.

Note that, for $n = 2$, (1.9) can be obtained directly from (1.3). Curiously, for $n = 3$, (1.9) should be replaced by

$$|\nabla f(x)| < \frac{8}{3\sqrt{3}} \frac{1}{1-|x|^2}, \quad x \in \mathbb{B}_3,$$

where $\frac{8}{3\sqrt{3}} (> 2\frac{|\mathbb{B}_2|}{|\mathbb{B}_3|} = 1.5)$ is the best possible; see [10, Note] and [11, Remark 1]. Until 2019, Melentijević in [17, Theorem 2] established the following sharp inequality:

$$|\nabla f(x)| \leq \frac{1}{|x|^2} \Big(\frac{(1+\frac{1}{3}|x|^2)^{\frac{3}{2}}}{1-|x|^2} - 1 \Big), \quad x \in \mathbb{B}_3,$$

for every harmonic function $f : \mathbb{B}_3 \to (-1, 1)$.

Remark 1.4. Factually, (1.9) at $x = 0$ holds for all $n \geq 2$ and the constant $2|\mathbb{B}_{n-1}|/|\mathbb{B}_n|$ is optimal in this case; see [2, Theorem 6.26] or [6, Corollary 2.2]. Furthermore, the requirement that f is real-valued is crucial in the validity of (1.9). In fact, (1.9) fails even at $x = 0$ for complex-valued harmonic functions [2, p. 126]. In this paper, for vector-valued harmonic functions $f : \mathbb{B}_n \to \mathbb{R}^m$, we find that (1.9) still hold by using the matrix (operator) norm of the Jacobian matrix $\nabla f(x) \in \mathbb{R}^{m\times n}$, that is the square root of the biggest eigenvalue of $(\nabla f(x))^T \nabla f(x)$.

Theorem 1.5. *Let $f : \mathbb{B}_n \to \mathbb{B}_m$ be harmonic functions with $n = 2$, or $n \geq 4$. Then*

$$\|\nabla f(x)\| \leq \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} \frac{2}{1-|x|^2}, \quad x \in \mathbb{B}_n, \tag{1.10}$$

where $\|\nabla f(x)\|$ denotes the matrix norm of $\nabla f(x) \in \mathbb{R}^{m\times n}$.

We restate (1.10) in the terms of the hyperbolic metric as follows. The proof is standard and omitted here.

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Theorem 1.6. *Let $f : \mathbb{B}_n \to \mathbb{B}_m$ be harmonic functions with $n = 2$, or $n \geq 4$. Then*

$$|f(x) - f(y)| \leq \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} d_{\mathbb{B}_n}(x, y), \quad x, y \in \mathbb{B}_n. \tag{1.11}$$

Remark 1.7. The distance in the left side of (1.11) is Euclidean but not hyperbolic. Based on the inequality (1.3) and Theorem 1.1, one would conjecture a sharper version of (1.9) that, for harmonic functions $f : \mathbb{B}_n \to (-1, 1)$ with $n \geq 4$,

$$\frac{|\nabla f(x)|}{1 - |f(x)|^2} \leq \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} \frac{2}{1 - |x|^2}, \quad x \in \mathbb{B}_n.$$

However, it is not the case as shown by a counter-example [9, Theorem 2.1].

Remark 1.8. Let $f : \mathbb{B}_n \to \mathbb{R}^m$. When $m = 1$, the matrix norm concises with the Euclidean norm of $\nabla f(x) \in \mathbb{R}^n$, i.e. $\|\nabla f(x)\| = |\nabla f(x)|$. Furthermore, it holds that

$$|\triangledown |f|(x)| \leq \|\nabla f(x)\|, \quad x \in \mathbb{B}_n,$$

where $\triangledown |f|(x) = (\frac{\partial |f|}{\partial x_1}, \frac{\partial |f|}{\partial x_2}, \dots \frac{\partial |f|}{\partial x_n})$ denotes the gradient of the Euclidean norm of $f(x)$.

In the study of the Schwarz–Pick inequality for holomorphic functions, the quantity $|\triangledown |f||$ was first adopted by Pavlović [18] due to that the classical form in (1.1) does not hold generally for vector-valued holomorphic functions. To obtain analogous form of (1.1), Pavlović gave that, for holomorphic mappings $f = (f_1, \dots, f_n) : \mathbb{D} \to \mathbb{C}^n$ with $|f| = (|f_1|^2 + \cdots + |f_n|^2)^{1/2} < 1$,

$$|\triangledown |f|(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{D}. \tag{1.12}$$

Following the idea of Pavlović, Chen and Hamada established the vector-valued version of the Khavinson conjecture for the norm of harmonic functions from the Euclidean unit ball $\mathbb{B}_n$ into the unit ball of the real Minkowski space by complicated calculations [4]. By the same technique, they gave several sharp Schwarz–Pick type inequalities for pluriharmonic functions from the Euclidean unit ball (or the unit polydisc) in $\mathbb{C}^n$ into the unit ball of the Minkowski space. Very recently, using the technique of the present author [27], Chen *et al.* have provided some improvements and generalizations of the corresponding results in [4] into Banach spaces by a relatively simple proof [5].

As a large subclass of the harmonic functions, the concept of monogenic functions appears in Clifford analysis, which is also a natural generalization of complex analysis into higher dimensions over non-commutative algebras. For monogenic functions, the Schwarz lemma does not hold at least in the original form, observed by Yang and Qian [23, Remark 2] and they established a Schwarz lemma outside of the unit ball in $\mathbb{R}^{n+1}$. Recently, some analogues of Schwarz lemma inside the unit ball were obtained in Clifford analysis [24, 25], quaternionic analysis [13, 22] and octonionic analysis [21]. For example, by integral representations of harmonic functions and Möbius transformations with Clifford coefficients, Zhang established the following Schwarz type lemma.

Theorem 1.9. [25, Theorem 3.2]. *Let $f : \mathbb{B}_{n+1} \to \mathbb{R}_{0,n}$ be a Clifford algebra valued monogenic function with $|f(x)| \leq 1, x \in \mathbb{B}_{n+1}$. If $f(a) = 0$ for some $a \in \mathbb{B}_{n+1}$, then*

$$|f(x)| \leq \frac{(1+|a|)^n}{\sqrt[n+1]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n+1}}, \quad x \in \mathbb{B}_{n+1}, \tag{1.13}$$

where $|\cdot|$ is the norm and $\bar{\cdot}$ is the conjugate in $\mathbb{R}_{0,n}$.

By the same technique as in the prove of Theorem 1.1, we shall offer a unify method to establish Schwarz type inequalities for harmonic functions in Clifford analysis and octonionic analysis as follows, instead of monogenic functions.

Theorem 1.10. *Let $f : \mathbb{B}_{n+1} \to \mathbb{R}_{0,n}$ be a Clifford algebra valued harmonic function with $|f(x)| \leq 1, x \in \mathbb{B}_{n+1}$. If $f(a) = 0$ for some $a \in \mathbb{B}_{n+1}$, then*

$$|f(x)| \leq \frac{(1+|a|)^{n-1}}{\sqrt[n+1]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^{n}}, \quad x \in \mathbb{B}_{n+1}, \tag{1.14}$$

and as a corollary

$$|\nabla f(a)| \leq \frac{1}{\sqrt[n+1]{2}-1} \frac{1}{(1+|a|)(1-|a|)^n}.$$

Theorem 1.11. *Let $f : \mathbb{B}_8 \to \mathbb{O}$ be an octonion valued harmonic function with $|f(x)| \leq 1, x \in \mathbb{B}_8$. If $f(a) = 0$ for some $a \in \mathbb{B}_8$, then*

$$|f(x)| \leq \frac{(1+|a|)^6}{\sqrt[8]{2}-1} \frac{|x-a|}{|1-\overline{a}x|^7}, \quad x \in \mathbb{B}_8.$$

where $|\cdot|$ is the norm and $\bar{\cdot}$ is the conjugate in $\mathbb{O}$.

Note that $|1-\overline{a}x| < 1+|a|$ for $a, x \in \mathbb{B}_{n+1}$. Hence, in a broader function class being harmonic, the obtained results in Theorems 1.10 and 1.11 are essential improvements of monogenic versions in [24, Theorem 4.8], [25, Theorem 3.2] and [21, Theorem 4], respectively.

The remaining part of the paper is organized as follows. The next section shall recall preliminaries on Clifford algebras and use it to rewrite some known properties of Möbius transformations of $\mathbb{B}_n$, which shall be used in the proof of main results. The § 3 is devoted to the proof of Theorems 1.1, 1.2 and 1.5. In § 4, we recall the concepts of monogenic functions in Clifford analysis and octonionic analysis and show that they are subclasses of harmonic functions. Finally, we give the proof of Theorems 1.10 and 1.11.

2. Preliminaries

In this section, we first recall preliminaries on Clifford algebras; see e.g. [19].

Denote by $\mathbb{R}_{0,n}$ the real Clifford algebra over imaginary units $\{e_1, e_2, \ldots, e_n\}$ which satisfy

$$e_i e_j + e_j e_i = -2\delta_{ij} e_0, \quad 1 \leq i \leq j \leq n,$$

where e_0 is identify with 1, δ_{ij} is Kronecker function.

Each element $a \in \mathbb{R}_{0,n}$ has the form of

$$a = \sum_A a_A e_A, \quad a_A \in \mathbb{R},$$

where $A = h_1 h_2 \cdots h_r$ with $1 \leq h_1 < h_2 < \cdots < h_r \leq n, e_A = e_{h_1} e_{h_2} \cdots e_{h_r}$ and $e_\emptyset = e_0 = 1$. The *real* part of $a \in \mathbb{R}_{0,n}$ is $\operatorname{Re} a = a_\emptyset = a_0$. The norm of a is defined by $|a| = (\sum_A |a_A|^2)^{\frac{1}{2}}$. As a real vector space, the dimension of Clifford algebra $\mathbb{R}_{0,n}$ is 2^n. The paravector x in $\mathbb{R}_{0,n}$ is given by

$$x = \sum_{i=0}^{n} x_i e_i, \quad x_i \in \mathbb{R}.$$

Hence, the space $\mathbb{R}^{n+1}$ can be identified as the set of all paravector in Clifford algebra $\mathbb{R}_{0,n}$. For paravector $x \neq 0$, it inverse is given by

$$x^{-1} = \frac{\overline{x}}{|x|^2},$$

where $\overline{x}$ denotes the conjugate of x, that is $\overline{x} = \sum_{i=0}^{n} x_i \overline{e_i} = x_0 - \sum_{i=1}^{n} x_i e_i$. Note that Clifford algebra is associative and non-commutative, but not divisible generally. The equality $|ab| = |a||b|$ does not hold generally for $a, b \in \mathbb{R}_{0,n}$ when $n \geq 3$. However, it is holds for in the following special case in Clifford algebras (see [19, Theorem 3.14]).

Lemma 2.1. *Let $a \in \mathbb{R}_{0,n}$ and $x \in \mathbb{R}^{n+1}$. Then*

$$|ax| = |xa| = |a||x|.$$

Now we introduce some known properties of Möbius transformations of $\mathbb{B}_n$ by using the language of Clifford algebras, which shall be used in the sequel. These results can be founded in [1, 20].

It is known that any Möbius transformation ψ of $\mathbb{B}_n$ onto itself has the form $\psi = T\varphi_a$, where $T \in O(n)$ and φ_a is Möbius transformations of $\mathbb{B}_n$ with $\varphi_a(0) = a \in \mathbb{B}_n$, given by

$$\varphi_a(x) = \frac{(1 - |a|^2)(a - x) + |a - x|^2 a}{[x, a]^2}, \quad x \in \mathbb{B}_n, \tag{2.1}$$

where

$$[x, a] = \sqrt{1 + |a|^2 |x|^2 - 2\langle a, x \rangle}.$$

Here $\langle \cdot, \cdot \rangle$ is real inner product in $\mathbb{R}^n$.

The vector space $\mathbb{R}^n$ can be viewed as the paravector in $\mathbb{R}_{0,n-1}$, and then

$$\langle a, x\rangle = \mathrm{Re}\,(x\overline{a}) = \mathrm{Re}\,(\overline{a}x).$$

Hence,

$$[x, a] = \sqrt{1 + |a|^2|x|^2 - 2\mathrm{Re}\,(x\overline{a})} = |1 - x\overline{a}|.$$

Consequently, the mapping φ_a can be expressed as

$$\varphi_a(x) = \frac{(1 - |a|^2 + a\overline{(a - x)})(a - x)}{|1 - x\overline{a}|^2} = \frac{(1 - a\overline{x})(a - x)}{|1 - x\overline{a}|^2} = (1 - x\overline{a})^{-1}(a - x).$$

Furthermore, it holds that

$$\varphi_a^{-1} = \varphi_a. \tag{2.2}$$

To see this, we should notice that

$$\varphi_a(x) = (1 - x\overline{a})^{-1}(a - x) = (a - x)(1 - \overline{a}x)^{-1}.$$

Let $y = \varphi_a(x)$. Then

$$a - x = (1 - x\overline{a})y = y - x\overline{a}y,$$

which implies that

$$a - y = x(1 - \overline{a}y) \Rightarrow x = (a - y)(1 - \overline{a}y)^{-1} = \varphi_a(y).$$

Denote by $\mathcal{M}(\mathbb{B}_n)$ Möbius transformations of $\mathbb{B}_n$ onto $\mathbb{B}_n$. Recall a useful identity [20, Theorem 2.1.3]

$$\frac{|\varphi_{(x)} - \varphi_{(y)}|^2}{(1 - |\varphi_{(x)}|^2)(1 - |\varphi_{(y)}|^2)} = \frac{|x - y|^2}{(1 - |x|^2)(1 - |y|^2)}, \quad \varphi \in \mathcal{M}(\mathbb{B}_n).$$

This formula implies that

$$\|\nabla\varphi_a(x)\| = \overline{\lim}_{y\to x}\frac{|\varphi_a(x) - \varphi_a(y)|}{|x - y|} = \frac{1 - |\varphi_a(x)|^2}{1 - |x|^2}. \tag{2.3}$$

From the identity

$$1 - |\varphi_a(x)|^2 = 1 - |a - x|^2|1 - x\overline{a}|^{-2} = \frac{(1 - |a|^2)(1 - |x|^2)}{|1 - x\overline{a}|^2}, \tag{2.4}$$

it is easy to see that

$$\|\nabla\varphi_a(x)\| = \frac{1 - |a|^2}{|1 - x\overline{a}|^2} \in (\frac{1 - |a|}{1 + |a|}, \frac{1 + |a|}{1 - |a|}). \tag{2.5}$$

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3. Proof of Theorems 1.1, 1.2 and 1.5

To prove main results, we recall the invariance of the Laplace equation [8, Chapter 1.10, p. 22].

Lemma 3.1. *Let $a \in \mathbb{B}_n$. If an independent variable undergoes the transformation $y = \varphi_a(x), x \in \mathbb{B}_n$, and the function is transformed by*

$$Y(y) = \Big(\frac{|1-x\overline{a}|^2}{1-|a|^2}\Big)^{\frac{n}{2}-1} X(x), \tag{3.1}$$

then

$$(1-|x|^2)^{\frac{n}{2}+1} \sum_{i=0}^{n-1} \frac{\partial^2 X}{\partial x_i^2} = (1-|y|^2)^{\frac{n}{2}+1} \sum_{i=0}^{n-1} \frac{\partial^2 Y}{\partial y_i^2}.$$

Let $y = \varphi_a(x)$. From the identity

$$|1-x\overline{a}||1-y\overline{a}| = 1-|a|^2, \tag{3.2}$$

and (2.2), the transformation in (3.1) can be rewritten as

$$Y(y) = \Big(\frac{1-|a|^2}{|1-y\overline{a}|^2}\Big)^{\frac{n}{2}-1} X(\varphi_a^{-1}(y)) = \Big(\frac{1-|a|^2}{|1-y\overline{a}|^2}\Big)^{\frac{n}{2}-1} X(\varphi_a(y)).$$

Hence, Lemma 3.1 gives directly the following result.

Lemma 3.2. *Let $a \in \mathbb{B}_n$ and* f *be a harmonic function in $\mathbb{B}_n$. Then the function*

$$f[\varphi_a](x) := \Big(\frac{1-|a|^2}{|1-x\overline{a}|^2}\Big)^{\frac{n}{2}-1} f \circ \varphi_a(x)$$

is still harmonic in $\mathbb{B}_n$.

Now we are in a position to prove Theorem 1.2.

Proof of Theorem 1.2. Firstly, let us prove the estimate (1.7) in the special case $x = 0$, i.e.

$$|\nabla f(0)| \leq n f(0), \tag{3.3}$$

for the positive harmonic function f defined in $\mathbb{B}_n$.

From the Poisson–Herglotz representation, it holds that

$$f(x) = \int_{\partial\mathbb{B}_n} P(x,\xi)\,\mathrm{d}\mu(\xi), \quad x \in \mathbb{B}_n,$$

where μ is a positive Borel measure on $\partial\mathbb{B}_n$ such that $\int_{\partial\mathbb{B}_n} \mathrm{d}\mu(\xi) = f(0)$.

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Note that

$$\nabla P_\xi(x) = -\frac{2x}{|x-\xi|^n} - \frac{n(1-|x|^2)}{|x-\xi|^{n+2}}(x-\xi), \quad (x,\xi) \in \mathbb{B}_n \times \partial\mathbb{B}_n. \tag{3.4}$$

In particular,

$$\nabla P_\xi(0) = n\xi, \quad \xi \in \partial\mathbb{B}_n.$$

Hence,

$$|\nabla f(0)| = n\Big|\int_{\partial\mathbb{B}_n} \xi \, \mathrm{d}\mu(\xi)\Big| \le n \int_{\partial\mathbb{B}_n} \mathrm{d}\mu(\xi) = nf(0),$$

where the equality is attained if and only if the measure μ is a singleton, that is to say, there exits some $\xi \in \partial\mathbb{B}_n$ such that

$$\mu(\{\xi\}) = f(0), \ \mu(\partial\mathbb{B}_n \setminus \{\xi\}) = 0.$$

This shows (3.3) and that the equality in (3.3) is attained if and only if $f(x) = f(0)P_\xi(x)$ for some $\xi \in \partial\mathbb{B}_n$, and in this case, we have, by (3.4),

$$\begin{aligned}
&(|x|^2-1)\nabla f(x) + (n-2)xf(x) \\
&= f(0)(|x|^2-1)\Big(-\frac{2x}{|x-\xi|^n} - \frac{n(1-|x|^2)}{|x-\xi|^{n+2}}(x-\xi) - \frac{(n-2)x}{|x-\xi|^n}\Big) \\
&= f(0)\frac{n(|x|^2-1)}{|x-\xi|^{n+2}}((|x|^2-1)(x-\xi) - x|x-\xi|^2) \\
&= f(0)\frac{n(|x|^2-1)}{|x-\xi|^{n+2}}(|x|^2-1-x\overline{(x-\xi)})(x-\xi) \\
&= f(0)\frac{n(|x|^2-1)}{|x-\xi|^{n+2}}(x\overline{\xi}-1)(x-\xi).
\end{aligned}$$

By Lemma 2.1, it follows that

$$|(x\overline{\xi}-1)(x-\xi)| = |x\overline{\xi}-1||x-\xi| = |x-\xi|^2.$$

Therefore, the harmonic function $f(x) = f(0)P_\xi(x)$ satisfies the following identity

$$|(|x|^2-1)\nabla f(x) + (n-2)xf(x)| = nf(0)\frac{1-|x|^2}{|x-\xi|^n} = nf(x), \quad (x,\xi) \in \mathbb{B}_n \times \partial\mathbb{B}_n. \tag{3.5}$$

Secondly, we prove the conclusion in the general case $x = a$. Fix $a \in \mathbb{B}_n$. By Lemma 3.2, the function $f[\varphi_a](x)$ is harmonic in $\mathbb{B}_n$. Hence, by applying the inequality (3.3) to the

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positive and harmonic function $f[\varphi_a](x)$, we have

$$|\nabla f[\varphi_a](0)| \leq n f[\varphi_a](0). \tag{3.6}$$

Direct calculations gives that

$$\nabla \frac{1}{|1-x\overline{a}|^{n-2}} = \frac{(2-n)(|a|^2x-a)}{|1-x\overline{a}|^{n-2}}, \quad x \in \mathbb{B}_n,$$

and

$$\nabla(\frac{1}{|1-x\overline{a}|^{n-2}} f \circ \varphi_a(x))|_{x=0} = (|a|^2-1)\nabla f(a) + (n-2)af(a),$$

then (3.6) reduces into

$$|(|a|^2-1)\nabla f(a) + (n-2)af(a)| \leq nf(a).$$

Let us consider the case of the equality in (1.7) is attained at $x = a$, that is

$$|\nabla f[\varphi_a](0)| = nf[\varphi_a](0).$$

Therefore, the previously obtained result of $x = 0$ gives that $f[\varphi_a](x) = f[\varphi_a](0)P_\xi(x)$ for some $\xi \in \partial\mathbb{B}_n$. More precisely,

$$\Big(\frac{1-|a|^2}{|1-x\overline{a}|^2}\Big)^{\frac{n}{2}-1} f \circ \varphi_a(x) = (1-|a|^2)^{\frac{n}{2}-1} f(a) P_\xi(x), \quad x \in \mathbb{B}_n.$$

Thus

$$f \circ \varphi_a(x) = f(a)|1-x\overline{a}|^{n-2} P_\xi(x), \quad x \in \mathbb{B}_n.$$

Replacing x with $\varphi_a(x)$ in the above formula and noticing (2.2), we obtain

$$f(x) = f(a)|1-\varphi_a(x)\overline{a}|^{n-2} P_\xi \circ \varphi_a(x), \quad x \in \mathbb{B}_n.$$

Finally, we verify that every positive harmonic function f defined by (1.8) satisfies the equality in (1.7) for all $x \in \mathbb{B}_n$ and $\xi \in \partial\mathbb{B}_n$. Observing (3.2) and (2.4), straightforward

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calculations give

$$\begin{aligned}
&\nabla \log((1-|x|^2)^{\frac{n}{2}-1}f(x))\\
&\quad= \nabla \log \Big(f(a)(1-|x|^2)^{\frac{n}{2}-1}\Big(\frac{1-|a|^2}{|1-x\overline{a}|}\Big)^{n-2}P_\xi\circ\varphi_a(x)\Big)\\
&\quad= \nabla \log \Big(\Big(\frac{(1-|a|^2)(1-|x|^2)}{|1-x\overline{a}|^2}\Big)^{\frac{n}{2}-1}P_\xi\circ\varphi_a(x)\Big)\\
&\quad= \nabla \log((1-|\varphi_a(x)|^2)^{\frac{n}{2}-1}P_\xi\circ\varphi_a(x))\\
&\quad= \frac{n}{2}\nabla \log \frac{1-|\varphi_a(x)|^2}{|\varphi_a(x)-\xi|^2}\\
&\quad= (\nabla \log((1-|\cdot|^2)^{\frac{n}{2}-1}P_\xi(\cdot)))(\varphi_a(x))\nabla\varphi_a(x).
\end{aligned}$$

Observe that (3.5) has an equivalent representation

$$|\nabla \log((1-|x|^2)^{\frac{n}{2}-1}P_\xi(x))| = \frac{n}{1-|x|^2}, \quad (x,\xi)\in\mathbb{B}_n\times\partial\mathbb{B}_n.$$

Combining this with $\nabla\varphi_a(x)/\|\nabla\varphi_a(x)\|\in O(n)$ and (2.3), we infer that, for all $(x,\xi)\in\mathbb{B}_n\times\partial\mathbb{B}_n$,

$$\begin{aligned}
&|\nabla \log((1-|x|^2)^{\frac{n}{2}-1}f(x))|\\
&\quad= |\nabla \log((1-|\cdot|^2)^{\frac{n}{2}-1}P_\xi(\cdot)))(\varphi_a(x))|\|\nabla\varphi_a(x)\|\\
&\quad= \frac{n}{1-|\varphi_a(x)|^2}\|\nabla\varphi_a(x)\|\\
&\quad= \frac{n}{1-|x|^2},
\end{aligned}$$

which completes the proof. □

With the help of Theorem 1.2, we turn back to prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 1.2, it holds that, for the harmonic function $f:\mathbb{B}_n\to\mathbb{R}^+$,

$$|(|x|^2-1)\nabla f(x)+(n-2)xf(x)|\le nf(x), \quad x\in\mathbb{B}_n,$$

which has an equivalent representation

$$|\nabla \log((1-|x|^2)^{\frac{n}{2}-1}f(x))|\le\frac{n}{1-|x|^2}, \quad x\in\mathbb{B}_n.$$

Hence, by Cauchy–Schwarz inequality,

$$|d(\log((1-|x|^2)^{\frac{n}{2}-1}f(x)))|\le|\nabla \log((1-|x|^2)^{\frac{n}{2}-1}f(x))||dx|\le\frac{n|dx|}{1-|x|^2}.$$

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Recall that the hyperbolic metric on $\mathbb{B}_n$ is given by

$$d_{\mathbb{B}_n}(x,y) = 2\tanh^{-1}(|\varphi_y(x)|) = \log\frac{1+|\varphi_y(x)|}{1-|\varphi_y(x)|},$$

and its responding element of arclength is

$$ds = \frac{2|dx|}{1-|x|^2}.$$

Integrating both sides of the above inequality along geodesics for the hyperbolic metric from x to y, we have

$$|\log\frac{(1-|x|^2)^{\frac{n}{2}-1}f(x)}{(1-|y|^2)^{\frac{n}{2}-1}f(y)}| \le \frac{n}{2}d_{\mathbb{B}_n}(x,y),$$

which implies that

$$d_{\mathbb{R}^+}(f(x),f(y)) = |\log\frac{f(x)}{f(y)}| \le \frac{n}{2}d_{\mathbb{B}_n}(x,y) + (\frac{n}{2}-1)|\log\frac{1-|y|^2}{1-|x|^2}|.$$

Combining this with the inequality

$$\frac{1+|\varphi_y(x)|}{1-|\varphi_y(x)|} \ge \frac{1-|y|^2}{1-|x|^2}, \quad x,y\in\mathbb{B}_n, \tag{3.7}$$

we get

$$d_{\mathbb{R}^+}(f(x),f(y)) \le (n-1)d_{\mathbb{B}_n}(x,y).$$

To see (3.7), using the identity (2.4), we first obtain

$$\frac{1+|\varphi_y(x)|}{1-|\varphi_y(x)|} = \frac{(1+|\varphi_y(x)|)^2}{1-|\varphi_y(x)|^2} = \frac{(1+|\varphi_y(x)|)^2|1-x\overline{y}|^2}{(1-|x|^2)(1-|y|^2)},$$

then the question reduces into proving that, for $x,y\in\mathbb{B}_n$ with $|x|>|y|$,

$$|1-x\overline{y}| + |x-y| \ge 1-|y|^2. \tag{3.8}$$

If $|x-y|\ge 1-|y|^2$, it is a trivial assertion. Otherwise, for $|x-y|<1-|y|^2$,

$$\begin{aligned}
&\Leftrightarrow |1-x\overline{y}|^2 \ge (1-|y|^2-|x-y|)^2 = (1-|y|^2)^2 + |x-y|^2 - 2|x-y|(1-|y|^2)\\
&\Leftrightarrow (1-|x|^2)(1-|y|^2) \ge (1-|y|^2)^2 - 2|x-y|(1-|y|^2)\\
&\Leftrightarrow 1-|x|^2 \ge 1-|y|^2 - 2|x-y|\\
&\Leftrightarrow |x|^2-|y|^2 \le 2|x-y|.
\end{aligned}$$

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Under the assumption that $x, y \in \mathbb{B}_n$ with $|x| > |y|$, it holds naturally that

$$|x|^2 - |y|^2 \le 2(|x| - |y|) \le 2|x - y|.$$

Now the proof is complete. □

Remark 3.3. In the proof of Theorem 1.1, we give a direct and basic proof of inequality (3.7). As pointed out by an anonymous reader, (3.7) is a consequence of the known inequality

$$\frac{|\rho(x,z) - \rho(z,y)|}{1 - \rho(x,z)\rho(z,y)} \le \rho(x,y) \le \frac{\rho(x,z) + \rho(z,y)}{1 + \rho(x,z)\rho(z,y)}, \quad x, y, z \in \mathbb{B}_n, \tag{3.9}$$

where $\rho(x, y) = |\varphi_y(x)|$ is the so-called pseudo-hyperbolic metric on $\mathbb{B}_n$.

To see this, we need only to consider $x, y \in \mathbb{B}_n$ with $|x| \ge |y|$. In this case, it follows that

$$\frac{1 - |y|^2}{1 - |x|^2} \le \frac{(1 + |x|)(1 - |y|)}{(1 - |x|)(1 + |y|)} = \frac{1 + \frac{|x| - |y|}{1 - |x||y|}}{1 - \frac{|x| - |y|}{1 - |x||y|}}. \tag{3.10}$$

According to (3.9), it holds that

$$\frac{|x| - |y|}{1 - |x||y|} = \frac{|\rho(x,0) - \rho(0,y)|}{1 - \rho(x,0)\rho(0,y)} \le \rho(x,y). \tag{3.11}$$

Note that $(1 + t)(1 - t)^{-1}$ is an increasing function for $t \in (0, 1)$, then (3.10) and (3.11) give the desired inequality (3.7).

Proof of Theorem 1.5. Let $l \in \partial\mathbb{B}_m$. For the harmonic $f : \mathbb{B}_n \to \mathbb{B}_m$, consider the scalar harmonic function $g = \langle f, l\rangle$, where $\langle\cdot,\cdot\rangle$ is real inner product in $\mathbb{R}^m$. Now the scalar harmonic function $g : \mathbb{B}_n \to (-1, 1)$ satisfies the condition of (1.9). Hence,

$$|\nabla\langle f(x), l\rangle| = |(\nabla f(x))^T \cdot l| \le \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} \frac{2}{1 - |x|^2}, \quad x \in \mathbb{B}_n,$$

where $\cdot$ denotes the matrix product of $(\nabla f(x))^T \in \mathbb{R}^{n\times m}$ with $l \in \mathbb{R}^{m\times 1}$. Due to the arbitrariness of $l \in \partial\mathbb{B}_m$, we obtain

$$\|\nabla f(x)\| \le \frac{|\mathbb{B}_{n-1}|}{|\mathbb{B}_n|} \frac{2}{1 - |x|^2}, \quad x \in \mathbb{B}_n,$$

as desired. □

4. Proof of Theorems 1.10 and 1.11

Before proving the theorem, we recall the concepts of monogenic functions in Clifford analysis and octonionic analysis and show that they are subclasses of harmonic functions.

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First, we give the definition of monogenic functions in Clifford analysis [19].

Definition 4.1. *Let* $\Omega \subset \mathbb{R}^{n+1}$ *and* $f : \Omega \to \mathbb{R}_{0,n}$ *be a Clifford algebra valued* C^1 *function. The function* $f = \sum_A e_A f_A$ *is called (left) monogenic in* Ω *if*

$$Df(x) := \sum_{i=0}^{n} e_i \frac{\partial f}{\partial x_i}(x) = \sum_{i=0}^{n} \sum_{A} e_i e_A \frac{\partial f_A}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

And the function f *is called (left) anti-monogenic in* Ω *if*

$$\overline{D} f(x) := \sum_{i=0}^{n} \overline{e_i} \frac{\partial f}{\partial x_i}(x) = \sum_{i=0}^{n} \sum_{A} \overline{e_i} e_A \frac{\partial f_A}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

Due to the non-commutation of Clifford algebra, the right monogenic functions could be defined similarly. Note that all monogenic functions on $\mathbb{B}_{n+1}$ is real analytic. For the Clifford algebra valued C^2 functions *f*, by the association of Clifford algebra, it holds that

$$D\overline{D} f(x) = \overline{D} D f(x) = \Delta_{n+1} f(x), \tag{4.1}$$

where Δ_{n+1} is Laplace operator in $\mathbb{R}^{n+1}$.

In fact, Definition 4.1 can also be built in octonionic analysis.

Definition 4.2. *Denote by* $\mathbb{O}$ *the non-commutative and non-associative algebra with canonical vector basis* $\{e_0 = 1, e_1, e_2, \ldots, e_7\}$. *Let* $\Omega \subset \mathbb{O}$ *and* $f : \Omega \to \mathbb{O}$ *be a octonionic valued* C*1* *function. The function* $f = \sum_{i=0}^{7} e_i f_i$ *is called (left) monogenic in* Ω *if*

$$\mathcal{D} f(x) := \sum_{i=0}^{7} e_i \frac{\partial f}{\partial x_i}(x) = \sum_{i=0}^{7} \sum_{j=0}^{7} e_i e_j \frac{\partial f_j}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

And the function f *is called (left) anti-monogenic in* Ω *if*

$$\overline{\mathcal{D}} f(x) := \sum_{i=0}^{7} \overline{e_i} \frac{\partial f}{\partial x_i}(x) = \sum_{i=0}^{7} \sum_{j=0}^{7} \overline{e_i} e_j \frac{\partial f_j}{\partial x_i}(x) = 0, \quad x \in \Omega.$$

Even though the algebra of octonions is non-associative, (4.1) still holds in the octonionic setting. Indeed, the Artin theorem shows that the subalgebra generated by two elements ($\mathcal{D}$ and f) in octonions is associative, which implies

$$\Delta f(x) = (\overline{\mathcal{D}} \mathcal{D}) f(x) = \overline{\mathcal{D}}(\mathcal{D} f(x)), \tag{4.2}$$

where Δ is Laplace operator in $\mathbb{R}^8$.

Hence, monogenic functions in Clifford analysis and octonionic analysis belong to harmonic functions from (4.1) and (4.2).

Since the proof of Theorem 1.11 is completely similar to Theorem 1.10, we only show Theorem 1.10 in this section.

Proof of Theorem 1.10. Let f be as described in Theorem 1.10. First, if $a=0$ (that is $f(0)=0$), then [25, Theorem 3.1] gives that

$$|f(x)| \leq \frac{1}{\sqrt[n+1]{2}-1}|x|, \quad x \in \mathbb{B}_{n+1}. \tag{4.3}$$

Now (1.14) at $x=0$ is obtained. Otherwise, as in the prove of Theorem 1.1, consider the Clifford algebra valued harmonic function

$$g_1(x) = \left(\frac{1-|a|^2}{|1-x\overline{a}|^2}\right)^{\frac{n+1}{2}-1} f\circ\varphi_a(x), \quad x \in \mathbb{B}_{n+1}.$$

In view of the estimate (2.5), set

$$g(x) = \left(\frac{1-|a|}{1+|a|}\right)^{\frac{n-1}{2}} g_1(x) = \left(\frac{1-|a|}{|1-x\overline{a}|}\right)^{n-1} f\circ\varphi_a(x)$$

with $|g(x)|<1$ for $x \in \mathbb{B}_{n+1}$. Applying the inequality (4.3) to the harmonic function $g(x)$, we obtain

$$\left(\frac{1-|a|}{|1-x\overline{a}|}\right)^{n-1}|f\circ\varphi_a(x)| \leq \frac{1}{\sqrt[n+1]{2}-1}|x|, \quad x \in \mathbb{B}_{n+1}.$$

Let $y=\varphi_a(x)$. From the identity (3.2), we have

$$|f(y)| \leq \frac{1}{\sqrt[n+1]{2}-1}\left(\frac{1+|a|}{|1-y\overline{a}|}\right)^{n-1}|\varphi_a^{-1}(y)|, \quad y \in \mathbb{B}_{n+1}.$$

Thus the fact $\varphi_a=\varphi_a^{-1}$ in (2.2) gives the desired inequality

$$|f(x)| \leq \frac{(1+|a|)^n}{\sqrt[n+1]{2}-1}\frac{|x-a|}{|1-\overline{a}x|^{n+1}}, \quad x \in \mathbb{B}_{n+1}.$$

The proof is completed. □

Data availability statement. Data sharing is not applicable to this article as no datasets were generated during the current study.

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