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RESEARCH ARTICLE

Slope equality of non-hyperelliptic Eisenbud–Harris special fibrations of genus 4

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Abstract
The Horikawa index and the local signature are introduced for relatively minimal fibered surfaces whose general fiber is a non-hyperelliptic curve of genus 4 with unique trigonal structure.

1. Introduction

Let S (resp. B) be a non-singular projective surface (resp. curve) defined over $\mathbb{C}$ and $f: S \to B$ a relatively minimal fibration whose general fiber F is a non-hyperelliptic curve of genus 4. According to [2], we say that f is *Eisenbud–Harris special* or *E-H special* for short (resp. *Eisenbud–Harris general*) if F has a unique $\mathfrak{g}_3^1$ (resp. two distinct $\mathfrak{g}_3^1$'s), or equivalently, the canonical image of F lies on a quadric surface of rank 3 (resp. rank 4) in $\mathbb{P}^3$.

For E-H general fibrations of genus 4, two important local invariants, the local signature and the Horikawa index, are introduced in the appendix in [2]. The purpose of this short note is to show that an analogous result also holds for E-H special fibrations of genus 4, that is, to show the following:

Theorem 1.1. *Let $\mathcal{A}$ be the set of fiber germs of relatively minimal E-H special fibrations of genus 4. Then, the Horikawa index* $\mathrm{Ind}: \mathcal{A} \to \mathbb{Q}_{\geq 0}$ *and the local signature* $\sigma: \mathcal{A} \to \mathbb{Q}$ *are defined so that for any relatively minimal E-H special fibration* $f: S \to B$ *of genus 4, the slope equality*

$$K_f^2 = \frac{24}{7}\chi_f + \sum_{p \in B} \mathrm{Ind}\left(f^{-1}(p)\right),$$

and the localization of the signature

$$\mathrm{Sign}(S) = \sum_{p \in B} \sigma\left(f^{-1}(p)\right),$$

hold.

Note that the above slope equality was established in [7] under the assumption that the multiplicative map $\mathrm{Sym}^2 f_*\omega_f \to f_*\omega_f^{\otimes 2}$ is surjective, and that for non-hyperelliptic fibrations of genus 4, the slope inequality

$$K_f^2 \geq \frac{24}{7}\chi_f,$$

was shown independently in [3] and [6].

2. Proof of theorem

In this section, we prove Theorem 1.1. Let $f: S \to B$ be a relatively minimal E-H special fibration of genus 4. Since the general fiber F of f is non-hyperelliptic, the multiplicative map $\mathrm{Sym}^2 f_*\omega_f \to f_*\omega_f^{\otimes 2}$ is generically surjective from Noether's theorem. Thus, we have the following exact sequences of sheaves of $\mathcal{O}_B$-modules:

$$0 \to \mathcal{L} \to \mathrm{Sym}^2 f_*\omega_f \to f_*\omega_f^{\otimes 2} \to \mathcal{T} \to 0, \tag{2.1}$$

where the kernel $\mathcal{L}$ is a line bundle on B and the cokernel $\mathcal{T}$ is a torsion sheaf on B. Then, the first injection defines a section $q \in H^0(B, \mathrm{Sym}^2 f_*\omega_f \otimes \mathcal{L}^{-1}) = H^0(\mathbb{P}_B(f_*\omega_f), 2T - \pi^*\mathcal{L})$, where $\pi: \mathbb{P}_B(f_*\omega_f) \to B$ is the projection and $T = \mathcal{O}_{\mathbb{P}_B(f_*\omega_f)}(1)$ is the tautological line bundle on $\mathbb{P}_B(f_*\omega_f)$. The section q can be regarded as a relative quadratic form $q: (f_*\omega_f)^* \to f_*\omega_f \otimes \mathcal{L}^{-1}$, which defines the determinant $\det(q): \det(f_*\omega_f)^{-1} \to \det(f_*\omega_f) \otimes \mathcal{L}^{-4}$. Note that for a non-hyperelliptic fibration f of genus 4, $\det(q) = 0$ if and only if f is E-H special. On the other hand, $Q = (q) \in |2T - \pi^*\mathcal{L}|$ is regarded as the unique relative quadric on $\mathbb{P}_B(f_*\omega_f)$ containing the image of the relative canonical map $\Phi_f: S \dashrightarrow \mathbb{P}_B(f_*\omega_f)$. Since f is E-H special, the general fiber of $\pi|_Q: Q \to B$ is a quadric of rank 3 on $\mathbb{P}(H^0(F, K_F)) = \mathbb{P}^3$. The closure of the set of vertexes of general fibers of $\pi|_Q$ defines a section $v: B \to Q$, which corresponds to some quotient line bundle $\mathcal{F}$ of $f_*\omega_f$. Let $\mathcal{E}$ be the kernel of the surjection $f_*\omega_f \to \mathcal{F}$ and put $P = \mathbb{P}_B(f_*\omega_f)$ and $P' = \mathbb{P}_B(\mathcal{E})$. Let $\tau: \widetilde{P} \to P$ be the blow-up of P along the section $v(B)$. Then, the relative projection $P \dashrightarrow P'$ from the section $v(B)$ extends to the morphism $\tau': \widetilde{P} \to P'$ with

$$\tau'^*T' = \tau^*T - E,$$

where $T' = \mathcal{O}_{\mathbb{P}_B(\mathcal{E})}(1)$ is the tautological line bundle of $\mathbb{P}_B(\mathcal{E})$ and E is the exceptional divisor of τ. Let $\widetilde{Q}$ denote the proper transform of Q on $\widetilde{P}$. It follows that in $\mathrm{Pic}(\widetilde{P})$,

$$\widetilde{Q} = \tau^*Q - 2E = \tau'^*(2T' - \pi'^*\mathcal{L}),$$

where $\pi': P' \to B$ is the projection. Let $Q' = \tau'(\widetilde{Q})$ be the image of $\widetilde{Q}$ via τ'. It follows that $Q' \in |2T' - \pi'^*\mathcal{L}|$ and $\widetilde{Q} = \tau'^*Q'$. The general fiber of $\pi'|_{Q'}: Q' \to B$ is a conic on $\mathbb{P}(H^0(F, \mathcal{E}|_F)) = \mathbb{P}^2$ of rank 3, which is isomorphic to $\mathbb{P}^1$. Note that the composite $\tau' \circ \Phi_f: S \dashrightarrow Q' \subset P'$ of the relative canonical map $\Phi_f: S \dashrightarrow P$ and the projection $\tau': P \dashrightarrow P'$ determines the unique trigonal structure of the general fiber F of f. Let $q' \in H^0(P', 2T' - \pi'^*\mathcal{L}) = H^0\big(B, \mathrm{Sym}^2\mathcal{E} \otimes \mathcal{L}^{-1}\big)$ be a section which defines $Q' = (q')$. Then q' can be regarded as a relative quadratic form $q': \mathcal{E}^* \to \mathcal{E} \otimes \mathcal{L}^{-1}$, which has non-zero determinant $\det(q'): \det(\mathcal{E})^{-1} \to \det(\mathcal{E}) \otimes \mathcal{L}^{-3}$ since Q' is of rank 3. Thus, $\det(q') \in H^0(B, \det(\mathcal{E})^{\otimes 2} \otimes \mathcal{L}^{-3})$ defines an effective divisor $\Delta_{Q'} = (\det(q'))$ on B. The degree of $\Delta_{Q'}$ is

$$\deg \Delta_{Q'} = 2\deg\mathcal{E} - 3\deg\mathcal{L}. \tag{2.2}$$

Let $\rho: \widetilde{S} \to S$ be the minimal desingularization of the rational map $\tau^{-1} \circ \Phi_f: S \dashrightarrow \widetilde{P}$ and $\widetilde{\Phi}: \widetilde{S} \to \widetilde{P}$ the induced morphism. Put $\Phi = \tau \circ \widetilde{\Phi}: \widetilde{S} \to P$, $\Phi' = \tau' \circ \widetilde{\Phi}: \widetilde{S} \to P'$, $M = \Phi^*T$ and $M' = \Phi'^*T'$. Then we can write $\rho^*K_f = M + Z$ for some effective vertical divisor Z on $\widetilde{S}$. Since $M' = M - \widetilde{\Phi}^*E$, we can also write $\rho^*K_f = M' + Z'$, where $Z' = Z + \widetilde{\Phi}^*E$ is also an effective vertical divisor on $\widetilde{S}$. Since Φ' is of degree 3 onto the image Q', we have $\Phi'_*\widetilde{S} = 3Q'$ as cycles. It follows that

$$\begin{aligned} M'^2 &= (\Phi'^*T')^2\widetilde{S} = T'^2\Phi'_*\widetilde{S} \\ &= 3T'^2Q' = 3T'^2(2T' - \pi'^*\mathcal{L}) \\ &= 6\deg\mathcal{E} - 3\deg\mathcal{L}, \end{aligned}$$

while we have

$$M'^2 = \big(\rho^*K_f - Z'\big)^2 = K_f^2 - \big(\rho^*K_f + M'\big)Z'.$$

Hence, we get

$$K_f^2 = 6\deg\mathcal{E} - 3\deg\mathcal{L} + \big(\rho^*K_f + M'\big)Z'. \tag{2.3}$$

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From (2.2) and (2.3), we can delete the term $\deg\mathcal{E}$ and then we have

$$\deg\mathcal{L} = \frac{1}{6}K_f^2 - \frac{1}{6}(\rho^*K_f + M')Z' - \frac{1}{2}\deg\Delta_{Q'}. \tag{2.4}$$

On the other hand, taking the degree of (2.1), we get

$$K_f^2 = 4\chi_f - \deg\mathcal{L} + \operatorname{length}\mathcal{T}. \tag{2.5}$$

Substituting (2.4) in the equation (2.5), we get

$$K_f^2 = \frac{24}{7}\chi_f + \frac{1}{7}\left(\rho^*K_f + M'\right)Z' + \frac{3}{7}\deg\Delta_{Q'} + \frac{6}{7}\operatorname{length}\mathcal{T}.$$

For a fiber germ $f^{-1}(p)$, we define $\operatorname{Ind}\left(f^{-1}(p)\right)$ by

$$\operatorname{Ind}\left(f^{-1}(p)\right) = \frac{1}{7}\left(\rho^*K_f + M'\right)Z'_p + \frac{3}{7}\operatorname{mult}_p\Delta_{Q'} + \frac{6}{7}\operatorname{length}_p\mathcal{T},$$

where $Z = \sum_{p\in B} Z_p$ is the natural decomposition with $(f\circ\rho)(Z_p) = \{p\}$ for any $p \in B$. For the definitions of M', Z', etc., we do not use the completeness of the base B. Thus, we can modify the definition of Ind for any fiber germ of relatively minimal E-H special fibrations of genus 4 which is invariant under holomorphically equivalence. Thus, we can define the Horikawa index $\operatorname{Ind}\colon \mathcal{A} \to \mathbb{Q}_{\geq 0}$ such that

$$K_f^2 = \frac{24}{7}\chi_f + \sum_{p\in B}\operatorname{Ind}\left(f^{-1}(p)\right).$$

The non-negativity of $\operatorname{Ind}\left(f^{-1}(p)\right)$ is as follows. From the nefness of K_f, we have $\rho^*K_fZ'_p \geq 0$. For a sufficiently ample divisor $\mathfrak{a}$ on B, the linear system $|M' + (f\circ\rho)^*\mathfrak{a}|$ is free from base points. Thus, by Bertini's theorem, there is a smooth horizontal member $C \in |M' + (f\circ\rho)^*\mathfrak{a}|$ and then $M'Z'_p = (M' + (f\circ\rho)^*\mathfrak{a})Z'_p = CZ'_p \geq 0$.

Once the Horikawa index is introduced, we can define the local signature since $\operatorname{Sign}(S) = K_f^2 - 8\chi_f$ and $e_f = 12\chi_f - K_f^2$ is localized by using the topological Euler numbers of the singular fibers (cf. [1, Section 2]). Indeed, we put

$$\sigma\left(f^{-1}(p)\right) = \frac{7}{15}\operatorname{Ind}\left(f^{-1}(p)\right) - \frac{8}{15}e_f\left(f^{-1}(p)\right),$$

where $e_f\left(f^{-1}(p)\right) = e_{\mathrm{top}}\left(f^{-1}(p)\right) + 6$ is the Euler contribution at $p \in B$. Then we have $\operatorname{Sign}(S) = \sum_{p\in B}\sigma\left(f^{-1}(p)\right)$.

Remark 2.1. *In [5], we define a Horikawa index* $\operatorname{Ind}_{g,n}$ *for fibered surfaces of genus g admitting a cyclic covering of degree n over a ruled surface (called primitive cyclic covering fibrations of type (g, 0, n)). For $g = 4$ and $n = 3$, these fibrations are non-hyperelliptic E-H special fibrations of genus 4. One can check the Horikawa index* $\operatorname{Ind}_{4,3}\left(f^{-1}(p)\right)$ *in [5, (4.5)] and* $\operatorname{Ind}\left(f^{-1}(p)\right)$ *in Theorem 1.1 are coincide by using the technique of [4, Appendix] which we left to the reader.*

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