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ORIGINAL RESEARCH PAPER

Risk analysis of a multivariate aggregate loss model with dependence

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Abstract

This paper studies a hierarchical risk model where an accident can cause a combination of different types of claims, whose sizes could be dependent. In addition, the frequencies of accidents that cause the different combinations of claims are dependent. We first derive formulas for computing risk measures, such as the Tail Conditional Expectation and Tail Variance of the aggregate losses for a portfolio of businesses. Then, we present formulas for performing the associated capital allocation to different types of claims in the portfolio. The main tool we used is the moment (or size-biased) transform of the multivariate distributions.

Keywords: capital allocation; dependence modeling; moment transform; multivariate aggregate losses; risk measures

1. Introduction

Insurance companies typically write policies in multiple lines of business and each line of business may cause claims of different types. For example, in auto insurance, as illustrated in Frees and Valdez (2008), an accident can lead to any combination of the claims of the following three types: (1) injury to the third party, (2) property damage (PD) to the third party, and (3) injury/PDs to the policyholder. In insurance pricing, actuaries need to ensure that the aggregate premium level for a portfolio of business is fair and adequate and that the premium for each type of risk coverage reflects its contribution to the total risk of the portfolio. Therefore, it is important to accurately evaluate the risk of the aggregate losses and allocate the total capital requirement to different types of risks in the portfolio of business. To this end, effective modeling of the joint distribution of losses from different types of risks is essential.

Multivariate aggregate loss models with different types of dependence structures have been discussed extensively in the literature. Some of them, for example, Hesselager (1996), Vernic (1999), Walhin and Paris (2000), Cossette et al. (2012), and Kim et al. (2019), assumed that claim frequencies are dependent but claim sizes are mutually independent and independent of the claim frequencies. Others, such as Sundt (1999) and Sundt and Vernic (2004), assumed that the claim number is univariate, but each claim can generate several types of losses whose sizes are dependent. Models that allow dependence between claim frequencies and claim sizes have also been developed recently. For example, Gschlößl and Czado (2007), Frees et al. (2011), and Garrido et al. (2016) took a regression approach where the claim frequency is treated as an explanatory variable in the regression model for the claim sizes; Boudreault et al. (2006), Cossette et al. (2008), and Marri and Furman (2012) assumed that the inter-claim times and claim sizes are dependent; Czado et al. (2012), Frees et al. (2016), Cossette et al. (2019), and Oh et al. (2020) employed bivariate copulas to model the dependency relationship between the number of claims and the average claim amount; Shi and Zhao (2020) used a copula to model the relation between the frequency and the individual severity directly; Yang and Shi (2019) proposed a multivariate framework for pricing property insurance contracts with multiperil coverage in the longitudinal context by using copulas to capture the dependence within and between perils.

In this paper, following Cummins and Wiltbank (1983) and Frees and Valdez (2008), we consider a hierarchical risk model where an accident can cause a combination of different types of claims, whose sizes could be dependent. In addition, the frequencies of accidents that cause the different combinations of claims are dependent. As pointed out in Cummins and Wiltbank (1983), this structure of multivariate compound distribution modeling of risk explicitly considers the intrinsic dependencies among the different components of the generating process. It is advantageous to the traditional approach of pooling the data from the entire portfolio of risks to obtain collective estimates of the frequency and severity parameters. Frees et al. (2009) provided statistical tools to apply this hierarchical model to analyze the risk profile of either a single policy or a portfolio of risks. It was argued that the model allows actuaries to "unbundle" insurance contracts and price more primitive elements of insurance coverages.

It is usually challenging to compute the risk measures of compound distributions explicitly. However, some results exist in the actuarial literature. For example, Cossette et al. (2012) used a top-down approach to derive closed-form expressions for Tail Conditional Expectation (TCE) based capital allocation for multivariate compound distributions; Kim et al. (2019) derived recursive algorithms to compute TCE for the sum of dependent compound mixed Poisson variables and to perform the associated capital allocation computation.

In this paper, we first derive formulas for evaluating TCE and Tail Variance (TV) of the aggregate loss amount in the hierarchical multivariate risk model. Then, we provide explicit expressions and computation methods for allocating the required capital to the different types of risks. Note that the multivariate loss model and the corresponding capital allocation studied here pertain to the portfolio of businesses level, not to the individual policyholder level. For the latter, data on the characteristics of individual policyholders are needed, as illustrated in Frees et al. (2009).

In terms of methodology, we apply the method introduced by Furman and Landsman (2005), Furman and Landsman (2006), and Furman and Zitikis (2008), which showed that tail moments risk measures can be analyzed through the moment (size-biased) transform of distributions. The theory of moment transformation has a long history and is widely used in statistics. For details, one is referred to, for example, Patil and Ord (1976), Arratia and Goldstein (2010), and references therein. Based on this method, Denuit (2020) presented explicit expressions for TCE of some univariate compound distributions; Denuit and Robert (2022) illustrated how to apply moment transform to analyze TCE of multivariate random variables. Ren (2022) derived formulas for TCE and TV of multivariate compound models based on Sundt (1999), where claim frequency is one-dimensional, and one claim can yield multiple dependent losses; Jiang and Ren (2022) provided methods for computing the TCE and TV of the multivariate aggregate losses, where the claim frequencies are dependent but the claim sizes are mutually independent and independent of the claim frequencies.

The remainder of the paper is organized as follows. Section 2 provides some preliminary results and definitions needed. Section 3 presents results for computing the hierarchical risk model's TCE and TV and performing the corresponding capital allocations. Section 4 provides numerical examples with details of the computations and then extends the model by considering a case when the distribution of the claim counts and the claim size are dependent. Section 5 concludes.

2. Models and definitions

We first introduce the hierarchical multivariate compound aggregate loss model studied in this paper. Assume that an insurance policy covers K categories of risks, denoted by $\mathcal{K} = \{1, \cdots, K\}$.

An accident can cause different types of claims in combinations, $\mathbf{h} = (h_1, \ldots, h_K)^\top$, where the kth coordinate h_k equals to 1 if type k claim occurs and 0 otherwise.

Let $\mathcal{M} = \{1, \cdots, M\}$, where $M = 2^K - 1$ be the set of indexes of possible combinations that include at least one claim. For $m \in \mathcal{M}$, let N_m denote the number of accidents resulting in the mth combination of claims for a portfolio of policies during a time period. The random variables N_m, $m \in \mathcal{M}$, can be dependent. Let $\mathbf{N} = (N_1, \cdots, N_M)^\top$ and denote its joint probability function by

$$p_{\mathbf{N}}(\mathbf{n}) = \Pr[\mathbf{N} = \mathbf{n}],$$

where $\mathbf{n} = (n_1, \cdots, n_M)^\top \in \mathbb{N}^M$.

For a given claim combination $m \in \mathcal{M}$, let $\mathbf{X}_{(m)} = \left(X_{(m),1}, \ldots, X_{(m),K}\right)^\top$ denote the random vector of claim sizes, where $X_{(m),k}$ for $k = 1, \ldots, K$ represents the claim size of type k risk in this combination. $X_{(m),k} = 0$ if the mth combination does not include a type k claim. Since the claim size vector $\mathbf{X}_{(m)}$ is generated by one accident, its elements are stochastically dependent. However, since the loss size vectors $\mathbf{X}_{(1)}, \ldots, \mathbf{X}_{(M)}$ result from different accidents, we assume that they are mutually independent and are independent of $\mathbf{N}$. A slight extension of the model where $\mathbf{N}$ and $(\mathbf{X}_{(1)}, \ldots, \mathbf{X}_{(M)})$ are dependent is considered in Section 4.2 of the paper.

For the portfolio of policies, let the aggregate amount of the K types of claims resulting from the mth combination be represented by the vector

$$\mathbf{S}_{N_m} = \left(S_{N_m,1}, \ldots, S_{N_m,K}\right)^\top = \sum_{i=1}^{N_m} \mathbf{X}_{(m)i} = \sum_{i=1}^{N_m} \left(X_{(m)i,1}, \ldots, X_{(m)i,K}\right)^\top,$$

where $\mathbf{X}_{(m)i}$, $i \geq 1$ are independent copies of $\mathbf{X}_{(m)}$.

Let

$$\mathbf{S}_{\mathbf{N}} = \left(\mathbf{S}_{N_1}, \ldots, \mathbf{S}_{N_M}\right)$$

be the $K \times M$ dimensional compound loss matrix. Equivalently,

$$\mathbf{S}_{\mathbf{N}} = \begin{bmatrix} S_{N_1,1} & S_{N_2,1} & \cdots & S_{N_M,1} \\ S_{N_1,2} & S_{N_2,2} & \cdots & S_{N_M,2} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N_1,K} & S_{N_2,K} & \cdots & S_{N_M,K} \end{bmatrix}, \tag{2.1}$$

where the element $S_{N_m,k}$ represents the aggregate amount of the type k claims that result from the claim combination m. Then, the total amount of claims of all types for the portfolio of business is given by

$$S_\bullet = \sum_{m=1}^{M} \sum_{k=1}^{K} S_{N_m,k}.$$

Example 2.1. Suppose that an auto insurance policy covers two types of risks: PD and bodily injury (BI). An accident can cause a PD claim only, a BI claim only, or a claim that combines both PD and BI. Then we may denote the type of risk by $\mathcal{K} = \{1, 2\}$; and the possible combinations of claims caused by an accident can be represented by three two-dimensional vectors $\mathbf{h}_1 = (1, 0)^\top$, $\mathbf{h}_2 = (0, 1)^\top$, and $\mathbf{h}_3 = (1, 1)^\top$. Then we have $\mathcal{M} = \{1, 2, 3\}$, and the numbers of claims of the three combinations of risk types, i.e., PD only, BI only, and both PD and BI, incurred in a time period is given by $\mathbf{N} = (N_1, N_2, N_3)^\top$. The claim sizes are given by $\mathbf{X}_{(1)} = (X_{(1),1}, 0)^\top$, $\mathbf{X}_{(2)} = (0, X_{(2),2})^\top$ and $\mathbf{X}_{(3)} = (X_{(3),1}, X_{(3),2})^\top$, respectively.

For this case, the multivariate aggregate loss matrix $\mathbf{S}_{\mathbf{N}}$ is given by

$$\mathbf{S}_{\mathbf{N}} = \begin{bmatrix} S_{N_1,1} & 0 & S_{N_3,1} \\ 0 & S_{N_2,2} & S_{N_3,2} \end{bmatrix} = \begin{bmatrix} \sum_{i=1}^{N_1} X_{(1)i,1} & 0 & \sum_{i=1}^{N_3} X_{(3)i,1} \\ 0 & \sum_{i=1}^{N_2} X_{(2)i,2} & \sum_{i=1}^{N_3} X_{(3)i,2} \end{bmatrix}.$$

For $q \in (0, 1)$, let s_q denote the Value-at-Risk (VaR) of $S_\bullet$ at the $100q\%$ confidence level. The TCE at level q of $S_\bullet$ is defined by

$$\mathrm{TCE}_{S_\bullet}(q) = \mathbb{E}[S_\bullet | S_\bullet > s_q] .$$

According to the TCE-based capital allocation rule (Dhaene et al., 2008), the total capital requirement in a portfolio of business is $\mathrm{TCE}_{S_\bullet}(q)$ for some q and the part allocated to the type k risk in the portfolio is

$$\mathrm{TCE}_{S_{\bullet,k}}(q) = \sum_{m=1}^{M} \mathbb{E}[S_{N_m,k} | S_\bullet > s_q], \quad k \in \mathcal{K} . \tag{2.2}$$

The capital required for the mth combination of risk type is given by

$$\mathrm{TCE}_{S_{N_m,\bullet}}(q) = \sum_{k=1}^{K} \mathbb{E}[S_{N_m,k} | S_\bullet > s_q], \quad m \in \mathcal{M} . \tag{2.3}$$

Note that

$$\sum_{k=1}^{K} \mathrm{TCE}_{S_{\bullet,k}}(q) = \sum_{m=1}^{M} \mathrm{TCE}_{S_{N_m,\bullet}}(q) = \mathrm{TCE}_{S_\bullet}(q) .$$

Likewise, if the total required capital is determined by the TV of $S_\bullet$ at probability level q, which is defined by (Furman and Landsman, 2006)

$$\mathrm{TV}_{S_\bullet}(q) = \mathrm{Var}[S_\bullet | S_\bullet > s_q] ,$$

then, according to the TV-based capital allocation rule, the capital allocated to the type k risk is given by

$$\mathrm{TV}_{S_{\bullet,k}}(q) = \sum_{m=1}^{M} \mathrm{Cov}[(S_{N_m,k}, S_\bullet) | S_\bullet > s_q] , \tag{2.4}$$

and the capital required for the mth combination of risk type is given by

$$\mathrm{TV}_{S_{N_m,\bullet}}(q) = \sum_{k=1}^{K} \mathrm{Cov}[(S_{N_m,k}, S_\bullet) | S_\bullet > s_q] . \tag{2.5}$$

Notably,

$$\sum_{k=1}^{K} \mathrm{TV}_{S_{\bullet,k}}(q) = \sum_{m=1}^{M} \mathrm{TV}_{S_{N_m,\bullet}}(q) = \mathrm{Var}[S_\bullet | S_\bullet > s_q] .$$

For more details about the TCE- and TV-based capital allocation, one can refer to, for example, Cummins (2000), Dhaene et al. (2008), Furman and Zitikis (2008), and references therein. We note that other frameworks of risk measures exist in the literature. As an example, Furman et al. (2017) proposed Gini-type risk measures and developed the corresponding capital allocation rules. Notably, this framework requires only finiteness of the first moment of the underlying random variable.

In the next section, we derive formulas for computing the TCE and TV risk measures of the proposed multivariate compound loss model and performing the associated capital allocation. The main tool we use is the concept of moment (size-biased) transforms, which is widely used in statistics. For detailed studies of the moment transforms, one is referred to, for example, Patil and Ord (1976), Arratia and Goldstein (2010), Furman and Landsman (2005), Denuit (2020), Denuit and Robert (2022), Mohammed et al. (2021), and Furman et al. (2021).

For completeness of this paper, we provide definitions for the size-biased transform of univariate and multivariate random variables in the following.

Definition 2.1. *Consider a non-negative random variable X with the distribution function F_X and moments $\mathbb{E}[X^\alpha]<\infty$ for some positive integer α. A random variable $\widetilde{X^\alpha}$ is said to be a copy of the αth moment transform of X if its cumulative distribution function (c.d.f.) is given by*

$$F_{\widetilde{X^\alpha}}(x)=\frac{\int_0^x t^\alpha\, dF_X(t)}{\mathbb{E}[X^\alpha]}=\frac{\mathbb{E}[X^\alpha I(X\le x)]}{\mathbb{E}[X^\alpha]},\qquad x\ge 0.$$

The first-moment transform of X is commonly referred to as the size-biased transform. It is simply denoted by $\widetilde{X}$.

Definition 2.2. *Consider a random vector $\mathbf{X}=(X_1,\ldots,X_K)^\top$ with the c.d.f. $F_{\mathbf{X}}$ and moments $\mathbb{E}[X_k^\alpha]<\infty$ and $\mathbb{E}[X_{k_1}^{\alpha_1}X_{k_2}^{\alpha_2}]<\infty$ for some $k,k_1,k_2\in\{1,\ldots,K\}$ and non-negative integers, α, α_1, and α_2.*

The kth component αth moment transform of $\mathbf{X}$ is any random vector $\widetilde{\mathbf{X}^\alpha}^{[k]}$ with the c.d.f.

$$F_{\widetilde{\mathbf{X}^\alpha}^{[k]}}(\mathbf{x})=\frac{\int_0^{x_1}\cdots\int_0^{x_K} t_k^\alpha\, dF_{\mathbf{X}}(t_1,\ldots,t_K)}{\mathbb{E}[X_k^\alpha]}=\frac{\mathbb{E}[X_k^\alpha I(\mathbf{X}\le\mathbf{x})]}{\mathbb{E}[X_k^\alpha]},\qquad \mathbf{x}\ge\mathbf{0},$$

where $\mathbf{x}=(x_1,\ldots,x_K)^\top$.

The (k_1,k_2)th component, (α_1,α_2)th moment transform of $\mathbf{X}$ is any random vector $\widetilde{\mathbf{X}^{\alpha_1,\alpha_2}}^{[k_1,k_2]}$ with the c.d.f.

$$F_{\widetilde{\mathbf{X}^{\alpha_1,\alpha_2}}^{[k_1,k_2]}}(\mathbf{x})=\frac{\int_0^{x_1}\cdots\int_0^{x_K} t_{k_1}^{\alpha_1}t_{k_2}^{\alpha_2}\, dF_{\mathbf{X}}(t_1,\ldots,t_K)}{\mathbb{E}[X_{k_1}^{\alpha_1}X_{k_2}^{\alpha_2}]}=\frac{\mathbb{E}[X_{k_1}^{\alpha_1}X_{k_2}^{\alpha_2}I(\mathbf{X}\le\mathbf{x})]}{\mathbb{E}[X_{k_1}^{\alpha_1}X_{k_2}^{\alpha_2}]},\qquad \mathbf{x}\ge\mathbf{0}.$$

The kth component first-moment transform of $\mathbf{X}$ is denoted as $\widetilde{\mathbf{X}}^{[k]}$, and the (k_1,k_2)th component $(1,1)$th moment transform of $\mathbf{X}$ is denoted as $\widetilde{\mathbf{X}}^{[k_1,k_2]}$.

For discrete distributions, following Patil and Ord (1976), we consider the factorial moment transform. For a positive integer I, define

$$I^{(\alpha)}=\begin{cases} I(I-1)\ldots(I-\alpha+1), & \text{if } \alpha\le I\\ 0, & \textit{otherwise}\end{cases}.$$

Then we have

Definition 2.3. *Consider a non-negative discrete random variable N with the probability mass function (p.m.f.) p_N. A random variable $\widetilde{N^{(\alpha)}}$ is said to be a copy of the αth factorial moment transform of N if its p.m.f. is given by*

$$p_{\widetilde{N^{(\alpha)}}}(n)=\frac{\mathbb{E}[N^{(\alpha)}I(N=n)]}{\mathbb{E}[N^{(\alpha)}]}=\frac{n^{(\alpha)}p_N(n)}{\mathbb{E}[N^{(\alpha)}]},\qquad n\ge 0.$$

The first factorial moment transform of N is denoted by $\widetilde{N}$.

Definition 2.4. *Consider a vector of discrete random variables $\mathbf{N}=(N_1,\ldots,N_K)^\top$ having the p.m.f. $p_{\mathbf{N}}(\mathbf{n})$.*

The kth component, αth moment transform of $\mathbf{N}$ is any random vector $\widetilde{\mathbf{N}^{(\alpha)}}^{[k]}$ with p.m.f.

$$p_{\widetilde{\mathbf{N}^{(\alpha)}}^{[k]}}(\mathbf{n})=\frac{n_k^{(\alpha)}p_{\mathbf{N}}(\mathbf{n})}{\mathbb{E}[N_k^\alpha]}=\frac{\mathbb{E}[N_k^{(\alpha)}I(\mathbf{N}=\mathbf{n})]}{\mathbb{E}[N_k^\alpha]},\qquad \mathbf{n}\ge\mathbf{0}.$$

The (k_1, k_2)*th component,* (α_1, α_2)*th moment transform of* $\mathbf{N}$ *is any random vector* $\widetilde{\mathbf{N}^{(\alpha_1),(\alpha_2)}}^{[k_1,k_2]}$ *with the p.m.f.*

$$p_{\widetilde{\mathbf{N}^{(\alpha_1),(\alpha_2)}}^{[k_1,k_2]}}(\mathbf{n}) = \frac{n_{k_1}^{(\alpha_1)} n_{k_2}^{(\alpha_2)} p_{\mathbf{N}}(\mathbf{n})}{\mathbb{E}[N_{k_1}^{(\alpha_1)} N_{k_2}^{(\alpha_2)}]} = \frac{\mathbb{E}[N_{k_1}^{(\alpha_1)} N_{k_2}^{(\alpha_2)} I(\mathbf{N} = \mathbf{n})]}{\mathbb{E}[N_{k_1}^{(\alpha_1)} N_{k_2}^{(\alpha_2)}]}, \qquad \mathbf{n} \geq \mathbf{0}\,.$$

The kth component, first-moment transform of $\mathbf{N}$ is denoted as $\widetilde{\mathbf{N}}^{[k]}$, and the (k_1, k_2)th component $(1, 1)$th moment transform of $\mathbf{N}$ is denoted as $\widetilde{\mathbf{N}}^{[k_1,k_2]}$.

3. Risk measures and capital allocation for the multivariate compound loss model

In this section, we present results for calculating the risk measures and capital allocation for the multivariate compound loss model represented by the matrix $\mathbf{S_N}$ defined in equation (2.1). To this purpose, we define the (k, m)th (kth row, mth column) component, first-moment transform of $\mathbf{S_N}$ to be a matrix of random variables $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$, which has the same size as $\mathbf{S_N}$ and the c.d.f.

$$F_{\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}}(\mathbf{s}) = \frac{\mathbb{E}[S_{N_m,k} I(\mathbf{S_N} \leq \mathbf{s})]}{\mathbb{E}[S_{N_m,k}]}\,,$$

where $\mathbf{s}$ is a matrix of non-negative constants

$$\mathbf{s} = \begin{bmatrix} s_{1,1} & \cdots & s_{M,1} \\ \vdots & \ddots & \vdots \\ s_{1,K} & \cdots & s_{M,K} \end{bmatrix},$$

and the $\leq$ operation is defined piecewisely.

Then, following Proposition 1 of Furman and Landsman (2005), Proposition 3.1 of Denuit and Robert (2022), or Lemma 2.1 in Jiang and Ren (2022), we have, for $m \in \mathcal{M}$ and $k \in \mathcal{K}$,

$$\mathbb{E}[S_{N_m,k} | S_\bullet > s_q] = \mathbb{E}[S_{N_m,k}] \frac{\Pr(\widetilde{S_\bullet}^{m[k]} > s_q)}{\Pr(S_\bullet > s_q)}\,, \tag{3.1}$$

where

$$\widetilde{S_\bullet}^{m[k]} = \sum_{i=1}^{M} \sum_{j=1}^{K} \widetilde{S_{\mathbf{N}^{[m]}}}^{[k]}_{i,j}$$

and $\widetilde{S_{\mathbf{N}^{[m]}}}^{[k]}_{i,j}$ is the i, jth element of $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$.

In addition, define the $[(k_1, m_1), (k_2, m_2)]$th components joint moment transform $\mathbf{S_N}$ to be a random matrix $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$ with c.d.f.

$$F_{\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}}(\mathbf{s}) = \frac{\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2} I(\mathbf{S_N} \leq \mathbf{s})]}{\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2}]}\,.$$

Then, for $m_1, m_2 \in \mathcal{M}$ and $k_1, k_2 \in \mathcal{K}$,

$$\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2} | S_\bullet > s_q] = \mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2}] \frac{\Pr(\widetilde{S_\bullet}^{m_1,m_2[k_1,k_2]} > s_q)}{\Pr(S_\bullet > s_q)}\,, \tag{3.2}$$

where

$$\widetilde{S}_{\bullet}^{m_1,m_2[k_1,k_2]} = \sum_{i=1}^{M}\sum_{j=1}^{K} \widetilde{S_{\mathbf{N}^{[m_1,m_2]}}}_{i,j}^{[k_1,k_2]}$$

and $\widetilde{S_{\mathbf{N}^{[m_1,m_2]}}}_{i,j}^{[k_1,k_2]}$ is the (i,j)th element of $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$.

Therefore, together with Equations (2.2) to (2.5), it is seen that if we can compute the distribution function of $S_\bullet$, $\widetilde{S}_{\bullet}^{m[k]}$ and $\widetilde{S}_{\bullet}^{m_1,m_2[k_1,k_2]}$, then the TCE and TV of $S_\bullet$ can be determined and the associated capital allocation can be performed.

To determine the distribution of $S_\bullet$, $\widetilde{S}_{\bullet}^{m[k]}$, and $\widetilde{S}_{\bullet}^{m_1,m_2[k_1,k_2]}$, we need the distribution functions of $\mathbf{S}_{\mathbf{N}}$, $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$, and $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$, for which we have the following results.

Theorem 3.1. *For $m \in \mathcal{M}$, let $\mathbf{1}^{[m]}$ denote an M dimensional vector with the mth element being one and all others zero. Let*

$$\mathbf{L}^{[m]} = \widetilde{\mathbf{N}}^{[m]} - \mathbf{1}^{[m]},$$

and

$$\mathbf{S}_{\mathbf{L}^{[m]}} = \left(\mathbf{S}_{L_1^{[m]}}, \ldots, \mathbf{S}_{L_M^{[m]}}\right),$$

where

$$\mathbf{S}_{L_i^{[m]}} = \sum_{j=1}^{L_i^{[m]}} \mathbf{X}_{(m)j} = \sum_{j=1}^{L_i^{[m]}} \left(X_{(m)j,1}, \ldots, X_{(m)j,K}\right)^\top = \left(S_{L_i^{[m]},1}, \ldots, S_{L_i^{[m]},K}\right)^\top$$

and $L_i^{[m]}$, $i \in \mathcal{M}$, is the ith element of $\mathbf{L}^{[m]}$. Then, for $k \in \mathcal{K}$,

$$\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]} \overset{d}{=} \mathbf{S}_{\mathbf{L}^{[m]}} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \times \mathbf{1}^{[m]\top}, \tag{3.3}$$

where $\widetilde{\mathbf{X}_{(m)1}}^{[k]}$ is an independent copy of the kth component, first-moment transform of $\mathbf{X}_{(m)}$.

Further, let

$$\mathbf{L}^{(2)[m]} = \widetilde{\mathbf{N}^{(2)}}^{[m]} - 2 \times \mathbf{1}^{[m]},$$

then, for $k_1, k_2 \in \mathcal{K}$,

$$\begin{aligned}
&\Pr\left(\widetilde{\mathbf{S}_{\mathbf{N}^{[m,m]}}}^{[k_1,k_2]} \le \mathbf{s}\right) \\
&= \frac{\mathbb{E}[N_m]\mathbb{E}[X_{(m)1,k_1}X_{(m)1,k_2}]}{\mathbb{E}[S_{N_m,k_1}S_{N_m,k_2}]}\Pr\left(\mathbf{S}_{\mathbf{L}^{[m]}} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \times \mathbf{1}^{[m]\top} \le \mathbf{s}\right) \\
&\quad + \frac{\mathbb{E}[N_m^{(2)}]\mathbb{E}[X_{(m)1,k_1}]\mathbb{E}[X_{(m)2,k_2}]}{\mathbb{E}[S_{N_m,k_1}S_{N_m,k_2}]}\Pr\left(\mathbf{S}_{\mathbf{L}^{(2)[m]}} + (\widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]}) \times \mathbf{1}^{[m]\top} \le \mathbf{s}\right),
\end{aligned} \tag{3.4}$$

where $\widetilde{\mathbf{X}_{(m)1}}^{[k_1]}$ and $\widetilde{\mathbf{X}_{(m)2}}^{[k_2]}$ are copies of the first-moment transform of $\mathbf{X}_{(m)}$, and $\widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]}$ is a copy of the (k_1,k_2)th component (1,1)th moment transform of $\mathbf{X}_{(m)}$. The random variables $\widetilde{\mathbf{X}_{(m)1}}^{[k_1]}$, $\widetilde{\mathbf{X}_{(m)2}}^{[k_2]}$, $\widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]}$, and $\mathbf{X}_{(m)}$ are mutually independent.

In addition, for $m_1, m_2 \in \mathcal{M}$ and $m_1 \neq m_2$, let

$$\mathbf{L}^{[m_1,m_2]} = \widetilde{\mathbf{N}}^{[m_1,m_2]} - \mathbf{1}^{[m_1]} - \mathbf{1}^{[m_2]},$$

then

$$\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]} \stackrel{d}{=} \mathbf{S}_{\mathbf{L}^{[m_1,m_2]}} + \widetilde{\mathbf{X}_{(m_1)1}}^{[k_1]} \times \mathbf{1}^{[m_1]\top} + \widetilde{\mathbf{X}_{(m_2)1}}^{[k_2]} \times \mathbf{1}^{[m_2]\top}. \tag{3.5}$$

The proof of this theorem is provided in the appendix of the paper.

Remark 3.1. Equation (3.4) shows that the distribution of $\widetilde{\mathbf{S}_{\mathbf{N}^{[m,m]}}}^{[k_1,k_2]}$ is a mixture of

$$\mathbf{S}_{\mathbf{L}^{[m]}} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \times \mathbf{1}^{[m]\top}$$

and

$$\mathbf{S}_{\mathbf{L}^{(2)[m]}} + (\widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]}) \times \mathbf{1}^{[m]\top},$$

with weights

$$\frac{\mathbb{E}[N_m]\mathbb{E}[X_{(m)1,k_1}X_{(m)1,k_2}]}{\mathbb{E}[S_{N_m,k_1}S_{N_m,k_2}]}$$

and

$$\frac{\mathbb{E}[N_m^{(2)}]\mathbb{E}[X_{(m)1,k_1}]\mathbb{E}[X_{(m)2,k_2}]}{\mathbb{E}[S_{N_m,k_1}S_{N_m,k_2}]},$$

respectively.

Remark 3.2. If one accident can only give rise to one type of claim, then $\mathcal{M} = \mathcal{K}$ and the claim size variables are univariate. In this case, Theorem 3.1 was reduced to Theorem 3.1 by Jiang and Ren (2022). On the other hand, if the claim number random vectors $\mathbf{N}$ is univariate and the claim size random vector contains all risk types, Theorem 3.1 reduces to Theorem 3 in Ren (2022).

We summarize the procedures for performing the capital allocation computation as follows:

Computation Procedure 3.1.

Step 1. *Determine the distributions of* $\mathbf{N}$, $\mathbf{L}^{[m]}$, *and* $\mathbf{L}^{[m_1,m_2]}$ *for* $m, m_1, m_2 \in \mathcal{M}$. *Some commonly used distribution functions of* $\mathbf{N}$, *such as multinomial, additive common shock, and common Poisson mixture, were studied in the literature by, for example, Hesselager (1996) and Kim et al. (2019). In these cases, as shown by Jiang and Ren (2022), the distributions of* $\mathbf{L}^{[m]}$ *and* $\mathbf{L}^{[m_1,m_2]}$ *are in fact mixture of some distributions in the same family as* $\mathbf{N}$ *and can be conveniently computed.*

Step 2. *Determine the distributions of* $\mathbf{S}_{\mathbf{N}}$, $\mathbf{S}_{\mathbf{L}^{[m]}}$, *and* $\mathbf{S}_{\mathbf{L}^{[m_1,m_2]}}$ *for* $m, m_1, m_2 \in \mathcal{M}$. *When the distribution of* $\mathbf{N}$ *is as described in Step 1, this can be implemented by using the recursive methods introduced by Hesselager (1996) and Kim et al. (2019). Alternatively, the Fast Fourier transform (FFT) method, as discussed by Robertson (1992) and Wang (1998), can be applied if the characteristic function of* $\mathbf{S}_{\mathbf{N}}$, $\mathbf{S}_{\mathbf{L}^{[m]}}$, *and* $\mathbf{S}_{\mathbf{L}^{[m_1,m_2]}}$ *for* $m, m_1, m_2 \in \mathcal{M}$ *can be determined. This is possible if the characteristic functions of* $\mathbf{N}$ *(therefore* $\mathbf{L}$*'s) and* $\mathbf{X}_{(m)}$ *are known. For the cases discussed in this paper, both methods can be applied. We choose to use the FFT method since it can be implemented conveniently using software such as R and Matlab.*

Step 3. *Determine the distributions of* $\mathbf{X}_{(m)}$, $\widetilde{\mathbf{X}_{(m)1}}^{[k]}$, *and* $\widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]}$ *for* $m \in \mathcal{M}$ *and* $k, k_1, k_2 \in \mathcal{K}$. *The probability density function (p.d.f.) of* $\widetilde{\mathbf{X}_{(m)1}}^{[k]}$ *and* $\widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]}$ *for* $m \in \mathcal{M}$ *and* $k, k_1, k_2 \in \mathcal{K}$ *can be determined by applying Definition 2.2 for the continuous case and Definition 2.4 for the discrete case.*

Step 4. *Determine the distributions of* $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$ *and* $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$ *for* $m, m_1, m_2 \in \mathcal{M}$ *and* $k, k_1, k_2 \in \mathcal{K}$*. This can be done by applying Theorem 3.1. The required convolutions can be computed using the FFT method.*

Step 5. *Determine the distributions of* $S_\bullet$, $\widetilde{S}_\bullet^{m[k]}$, *and* $\widetilde{S}_\bullet^{m_1,m_2[k_1,k_2]}$ *for* $m, m_1, m_2 \in \mathcal{M}$ *and* $k, k_1, k_2 \in \mathcal{K}$*. The FFT of the distribution of* $S_\bullet$, $\widetilde{S}_\bullet^{m[k]}$, *and* $\widetilde{S}_\bullet^{m_1,m_2[k_1,k_2]}$ *is given by the diagonal term of the FFT of distributions of* $S_{\mathbf{N}}$, $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$, *and* $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$*. Their distributions can be obtained using the one-dimensional inverse Fast Fourier transformation (IFFT).*

Step 6. *Determine the TCE- and TV-based capital allocations. The TCE- and TV-based capital requirement and the capital allocation can be determined by applying Equations (3.1) and (3.2).*

Remark 3.3. Notice from equations (3.1) and (3.2) that in calculating TCE, TV, and the associate capital allocation, we only need the distribution of univariate random variables $S_\bullet$, $\widetilde{S}_\bullet^{m[k]}$, and $\widetilde{S}_\bullet^{m_1,m_2[k_1,k_2]}$, not the whole joint distributions. Consequently, when computing their distributions by applying the FFT method, we do not need to apply multi-dimensional IFFT to the array of the FFT of $\mathbf{S}_{\mathbf{N}}$, $\widetilde{\mathbf{S}_{\mathbf{N}^{[m]}}}^{[k]}$, and $\widetilde{\mathbf{S}_{\mathbf{N}^{[m_1,m_2]}}}^{[k_1,k_2]}$. Instead, we only need to apply one-dimensional IFFT to the diagonal terms to obtain the distribution of the sums.

4. Numerical examples

In this section, we illustrate how to apply the formulas derived in last section to compute the risk measures and to perform the capital allocations for the proposed multivariate aggregate loss models.

4.1 The multivariate aggregate claim model

In this subsection, we provide the general structure of the model that will be used in the numerical examples. Let W be a counting random variable that follows NB(r, β) distribution with probability mass function

$$p_W(w) = \binom{r+w-1}{w}\left(\frac{\beta}{1+\beta}\right)^w \left(\frac{1}{1+\beta}\right)^r, \quad w \geq 0\,.$$

Conditional on $W = w$, let the claim number vector $\mathbf{N} = (N_1, \ldots, N_M)^\top$ follow a multinomial distribution with parameters $(w, q_1, \ldots, q_M)$. That is,

$$\Pr\left(\mathbf{N} = \mathbf{n} | W = w\right) = \frac{w!}{n_1! n_2! \ldots n_M!} q_1^{n_1} q_2^{n_2} \ldots q_M^{n_M}, \quad n_1 + n_2 + \ldots + n_M = w\,.$$

This model was introduced by Hesselager (1996), and Jiang and Ren (2022) denoted the unconditional distribution of $\mathbf{N}$ by HMN$(W, q_1, \ldots, q_M)$. Regression analysis of this model was provided by Frees et al. (2009).

We could easily obtain that

$$\mathbb{E}[N_m] = \mathbb{E}[W] q_m = r\beta q_m, \quad m \in \mathcal{M}\,,$$

$$\mathbb{E}[N_m^{(2)}] = \mathbb{E}[W^{(2)}] q_m^2 = r(r+1)\beta^2 q_m^2, \quad m \in \mathcal{M}\,,$$

and

$$\mathbb{E}[N_{m_1} N_{m_2}] = \mathbb{E}[W^{(2)}] q_{m_1} q_{m_2} = r(r+1)\beta^2 q_{m_1} q_{m_2}, \quad m_1, m_2 \in \mathcal{M}, \; m_1 \neq m_2\,.$$

The joint p.g.f. of $\mathbf{N}$ is

$$\mathcal{P}_{\mathbf{N}}(z_1,\ldots,z_M)=\left[1-\beta(q_1z_1+\ldots+q_Mz_M-1)\right]^{-r}.$$

As shown in Theorem 4.1 of Jiang and Ren (2022), for $m,m_1,m_2\in\mathcal{M}$ and $m_1\neq m_2$, we have

$$\mathbf{L}^{[m]}=\widetilde{\mathbf{N}}^{[m]}-\mathbf{1}^{[m]}\sim\mathrm{HMN}(\widetilde{W}-1,q_1,\ldots,q_M),$$

$$\mathbf{L}^{(2)^{[m]}}=\widetilde{\mathbf{N}^{(2)}}^{[m]}-2\times\mathbf{1}^{[m]}\sim\mathrm{HMN}(\widetilde{W}^{(2)}-2,q_1,\ldots,q_M),$$

and

$$\mathbf{L}^{[m_1,m_2]}=\widetilde{\mathbf{N}}^{[m_1,m_2]}-\mathbf{1}^{[m_1]}-\mathbf{1}^{[m_2]}\sim\mathrm{HMN}(\widetilde{W}^{(2)}-2,q_1,\ldots,q_M).$$

In addition, $\widetilde{W}-1$ follows $\mathrm{NB}(r+1,\beta)$ distribution, and $\widetilde{W}^{(2)}-2$ follows $\mathrm{NB}(r+2,\beta)$ distribution. Therefore, the distribution of $\mathbf{L}^{[m]}$, $\mathbf{L}^{(2)^{[m]}}$, and $\mathbf{L}^{[m_1,m_2]}$ is all in the same family as $\mathbf{N}$.

For claim severity, we assume that if a claim combination m only consists of a type $k\in\mathcal{K}$ claim, then its size follows a Poisson distribution with mean a_k. Then we have

$$\mathbb{E}[X_{(m),k}]=a_k,$$

and

$$\mathbb{E}[X^2_{(m),k}]=a_k+a_k^2.$$

If a claim combination m consists of non-zero claims of types $\{k_1,\ldots,k_h\}$, then the joint distribution of the claim sizes is assumed to be a common Poisson mixture. That is, conditional on a mixing variable $\Lambda=\lambda$, for $j=1,\ldots,h$, the size of type k_j claim follows a Poisson distribution with mean $b_{k_j}\lambda$. Further, we assume Λ follows a gamma distribution with shape parameter α and p.d.f.

$$f_\Lambda(\lambda)=\frac{\alpha^\alpha\lambda^{\alpha-1}e^{-\alpha\lambda}}{\Gamma(\alpha)}.$$

Consequently, $\mathbb{E}[\Lambda]=1$ and for $i,j\in\{1,\ldots,h\}$ and $i\neq j$, we have

$$\mathbb{E}[X_{(m),k_j}]=b_{k_j},$$

$$\mathbb{E}[X^2_{(m),k_j}]=b_{k_j}+\frac{\alpha+1}{\alpha}b^2_{k_j},$$

$$\mathbb{E}[X_{(m),k_i}X_{(m),k_j}]=\frac{\alpha+1}{\alpha}b_{k_i}b_{k_j}.$$

In addition, for claim combinations $m,m_1,m_2\in\mathcal{M}$, $m_1\neq m_2$, and $k_1,k_2\in\mathcal{K}$, we have

$$\mathbb{E}[S_{N_m,k_1}S_{N_m,k_2}]=\mathbb{E}[N_m]\mathbb{E}[X_{(m),k_1}X_{(m),k_2}]+\mathbb{E}[N_m^{(2)}]\mathbb{E}[X_{(m),k_1}]\mathbb{E}[X_{(m),k_2}],$$

and

$$\mathbb{E}[S_{N_{m_1},k_1}S_{N_{m_2},k_2}]=\mathbb{E}[N_{m_1}N_{m_2}]\mathbb{E}[X_{(m_1),k_1}]\mathbb{E}[X_{(m_2),k_2}].$$

Let

$$\psi_{\mathcal{S}_{\mathbf{N}}}(\mathbf{t})=\mathbb{E}\left[exp(i\cdot tr(\mathbf{t}^\top\mathbf{S}_{\mathbf{N}}))\right],$$

where

$$\mathbf{t} = \left((t_{1,1}, \ldots, t_{K,1})^\top, \ldots, (t_{1,M}, \ldots, t_{K,M})^\top \right),$$

denote the characteristic function (c.f.) of $\mathbf{S_N}$. Let $\mathcal{P}_\mathbf{N}(\cdot)$ denote the probability generating function (p.g.f.) of $\mathbf{N}$ and $\phi_{\mathbf{X_{(m)}}}(\cdot)$ the c.f. of $\mathbf{X_{(m)}}$. Then

$$\begin{aligned}
\psi_{\mathcal{S}_\mathbf{N}}(\mathbf{t}) &= \mathbb{E}\left[exp\left(i \sum_{m=1}^{M} \sum_{k=1}^{K} t_{k,m} \sum_{j=1}^{N_m} X_{(m)j,k} \right) \right] \\
&= \mathbb{E}\left[\prod_{m=1}^{M} \prod_{j=1}^{N_m} exp\left(it_{k,m} \sum_{k=1}^{K} X_{(m)j,k} \right) \right] = \mathbb{E}\left[\mathbb{E}\left[\prod_{m=1}^{M} \prod_{j=1}^{N_m} exp\left(it_{k,m} \sum_{k=1}^{K} X_{(m)j,k} \right) \middle| \mathbf{N} \right] \right] \\
&= \mathbb{E}\left[\prod_{m=1}^{M} \prod_{j=1}^{N_m} \mathbb{E}\left[exp\left(it_{k,m} \sum_{k=1}^{K} X_{(m)j,k} \right) \right] \right] = \mathbb{E}\left[\prod_{m=1}^{M} \left(\mathbb{E}\left[exp\left(it_{k,m} \sum_{k=1}^{K} X_{(m),k} \right) \right] \right)^{N_m} \right] \\
&= \mathcal{P}_\mathbf{N}\left(\phi_{\mathbf{X_{(1)}}}(t_{1,1}, \ldots, t_{K,1}), \ldots, \phi_{\mathbf{X_{(M)}}}(t_{1,M}, \ldots, t_{K,M}) \right) \\
&= \left[1 - \beta\left(q_1 \phi_{\mathbf{X_{(1)}}}(t_{1,1}, \ldots, t_{K,1}) + \ldots + q_M \phi_{\mathbf{X_{(M)}}}(t_{1,M}, \ldots, t_{K,M}) - 1 \right) \right]^{-r}.
\end{aligned}$$

The c.f. of $\mathbf{S}_{\mathbf{L}^{[m]}}$, $\mathcal{S}_{\mathbf{L}^{(2)[m]}}$, and $\mathcal{S}_{\mathbf{L}^{[m_1,m_2]}}$ can be derived similarly. Specifically,

$$\psi_{\mathcal{S}_{\mathbf{L}^{[m]}}}(\mathbf{t}) = \left[1 - \beta\left(q_1 \phi_{\mathbf{X_{(1)}}}(t_{1,1}, \ldots, t_{K,1}) + \ldots + q_M \phi_{\mathbf{X_{(M)}}}(t_{1,M}, \ldots, t_{K,M}) - 1 \right) \right]^{-(r+1)},$$

and

$$\begin{aligned}
&\psi_{\mathcal{S}_{\mathbf{L}^{(2)[m]}}}(\mathbf{t}) = \psi_{\mathcal{S}_{\mathbf{L}^{[m_1,m_2]}}}(\mathbf{t}) \\
&= \left[1 - \beta\left(q_1 \phi_{\mathbf{X_{(1)}}}(t_{1,1}, \ldots, t_{K,1}) + \ldots + q_M \phi_{\mathbf{X_{(M)}}}(t_{1,M}, \ldots, t_{K,M}) - 1 \right) \right]^{-(r+2)}.
\end{aligned}$$

With the above, all steps in computation procedure 3.1 can be carried out, and the risk analysis of $\mathbf{S_N}$ can be performed.

We remark that the selection of the distribution of $\mathbf{N}$ and $\mathbf{X}_{(m)}$ is arbitrary in this section. Other distributions of $\mathbf{N}$ and $\mathbf{X}_{(m)}$ can be used as long as their moments, characteristic function, and moment transforms can be evaluated.

4.1.1 An example with two types of risks

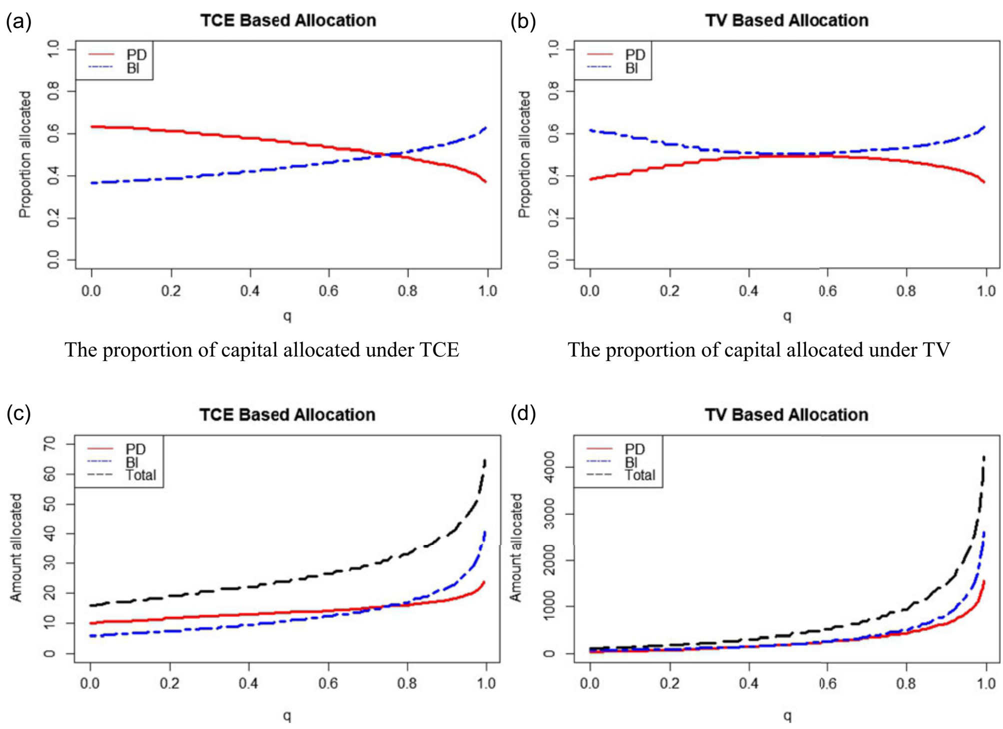
We apply the setting in Example 2.1 where two types of claims, PD and BI, are considered. Recall that the claim number vector is $\mathbf{N} = (N_1, N_2, N_3)^\top$ and the claim sizes be $\mathbf{X}_{(1)} = (X_{(1),1}, 0)^\top$, $\mathbf{X}_{(2)} = (0, X_{(2),2})^\top$, and $\mathbf{X}_{(3)} = (X_{(3),1}, X_{(3),2})^\top$, respectively.

We assume that $\mathbf{N} \sim \text{HMN}(W, q_1 = 0.9, q_2 = 0.02, q_3 = 0.08)$ and $W \sim \text{NB}(r = 10, \beta = 1)$.

Let $X_{(1),1} \sim \text{Poi}(a_1 = 1)$, $X_{(2),2} \sim \text{Poi}(a_2 = 5)$, and $\mathbf{X}_{(3)}$ follow a common Poisson mixture, where conditional on $\Lambda = \lambda$, $X_{(3),1} \sim \text{Poi}(1.2\lambda)$ and $X_{(3),2} \sim \text{Poi}(6\lambda)$, and Λ follows a gamma distribution with shape parameter $\alpha = 2$ and mean one.

These parameter values are selected hypothetically to reflect the fact that accidents that cause only PDs usually have high frequency and low severity; it is unlikely ($q_2 = 0.02$) that an accident causes BI but no PDs; accidents that cause both BI and PDs have low frequency and high severity. Note that we assume discrete distributions for the claim sizes for simplicity. If continuous distributions are assumed, they need to be discretized to apply the FFT or recursive methods.

The proportions of capital allocated to the two types of risks according to TCE with selected values of q in $(0, 1)$ are plotted in Figure 1, panel (a). It shows that the proportion of risk capital



The proportion of capital allocated under TCE

The proportion of capital allocated under TV

The amount of capital allocated under TCE

The amount of capital allocated under TV

Figure 1. The proportions and amounts of capital allocated to the two types of risks according to TCE and TV criteria.

allocated to PD (BI) claims decreases (increases) with q. When q is small, more capital is allocated to PD claims, whereas more risk capital is allocated to BI claims when q is large.

The proportions of capital allocated according to TV are shown in panel (b) of Figure 1. We observe that when q is small, the proportion allocated to BI claims is a decreasing function of q, and when q is large, the proportion allocated to BI claims increases with q. The opposite pattern is observed for PD claims.

The amounts of capital allocated to the two types of risks according to both TCE and TV criteria increase with q, as shown in panels (c) and (d) of Figure 1.

Figure 2 compares the proportions of capital allocated to the two types of risks according to TCE and TV criteria obtained by using the moment transform method proposed in this paper and those by using the Monte Carlo simulation (with 10^7 runs). It shows that the results based on the moment transform are accurate. We note that the moment transform method takes much less computation time than the Monte Carlo simulation.

The capital allocation to the three combinations of risk types according to TCE and TV criteria can be performed following the same procedure. To avoid redundancy, we omit the analysis here.

Tables 1 and 2 show the numerical values of the amounts and proportions of capital allocated to the two types of risk according to TCE and TV criteria for the HMN models under different values of α, which is the parameter of the mixing random variable Λ for $\mathbf{X}_{(3)} = (X_{(3),1}, X_{(3),2})$. Since $\mathrm{Var}(X_{(3),1}) = b_1 + b_1^2/\alpha$, $\mathrm{Var}(X_{(3),2}) = b_2 + b_2^2/\alpha$, and

$$r(X_3, X_4) = \frac{\mathrm{Cov}(X_3, X_4)}{\sqrt{\mathrm{Var}(X_3)}\sqrt{\mathrm{Var}(X_4)}} = \frac{b_1 b_2/\alpha}{\sqrt{b_1 + b_1^2/\alpha}\sqrt{b_2 + b_2^2/\alpha}} = \sqrt{\frac{b_1 b_2}{\alpha^2 + (b_1 + b_2)\alpha + b_1 b_2}},$$

Table 1. Comparison of the amounts of capital allocated to the two risk types under Tail Conditional Expectation (TCE) criterion

$q = 0.995$	s_q	$TCE_{S_{\bullet,1}}(q)$	$TCE_{S_{\bullet,2}}(q)$	$TCE_{S_\bullet}(q)$	$\frac{TCE_{S_{\bullet,1}}(q)}{TCE_{S_\bullet}(q)}$	$\frac{TCE_{S_{\bullet,2}}(q)}{TCE_{S_\bullet}(q)}$
α=0.1	151	41.6	157.1	198.7	21.0%	79.0%
α=1	64	24.0	49.8	73.8	32.5%	67.5%
α=10	51	24.2	32.6	56.8	42.7%	57.3%
$\alpha \to \infty$	49	24.2	30.3	54.5	44.5%	55.5%

Table 2. Comparison of the amounts and proportions of capital allocated to the two risk types under Tail Variance (TV) criterion

	$TV_{S_{\bullet,1}}(q)$	$TV_{S_{\bullet,2}}(q)$	$TV_{S_\bullet}(q)$	$\frac{TV_{S_{\bullet,1}}(q)}{TV_{S_\bullet}(q)}$	$\frac{TV_{S_{\bullet,2}}(q)}{TV_{S_\bullet}(q)}$
α=0.1	8794.7	33920.4	42715.0	20.6%	79.4%
α=1	1793.8	3794.1	5587.9	32.1%	67.9%
α=10	1386.3	1877.0	3263.3	42.5%	57.5%
$\alpha \to \infty$	1328.7	1667.6	2996.3	44.3%	55.7%

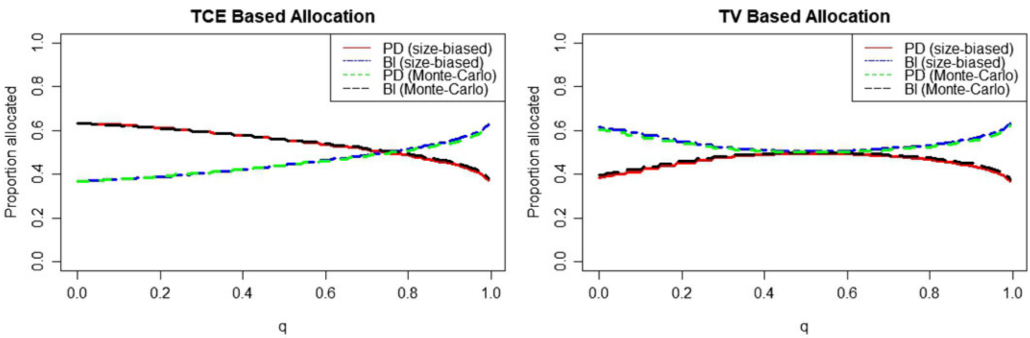


Figure 2. Comparison of the simulated and theoretical results.

we see that the variance of $X_{(3),1}$ and $X_{(3),2}$ and their correlation increases as α decreases. In particular, when $\alpha \to \infty$, $X_{(3),1}$, and $X_{(3),2}$ are uncorrelated; when $\alpha \to 0$, the corelation coefficient approaches to one.

From Tables 1 and 2, we observe that larger variance of and stronger dependence between $X_{(3),1}$ and $X_{(3),2}$ lead to greater values of VaR and TCE of the total losses and higher proportion of capital allocated to BI risks.

4.1.2 An example with three types of risks

In this subsection, we consider the automobile insurance claim model discussed by Frees and Valdez (2008), in which three types of claims, own damage (OD), third-party property (TPP), and third-party injury (TPI), are considered. An accident can cause any combination of the three types of claims with proportions shown in Table 3. Frees and Valdez (2008) proposed a hierarchical, three-component (loss frequency, severity, and dependence) regression model to analyze this highly complex data structure.

Table 3. Possible combinations and their occurrence frequencies

Combination	m	Frequency
TPI	1	0.4%
OD	2	73.2%
TPP	3	12.3%
TPI & OD	4	0.3%
TPI & TPP	5	0.1%
OD & TPP	6	13.5%
TPI & OD & TPP	7	0.2%

In this example, we study the risk measure and capital allocation problem for the model. We assume that the vector of the number of claim combinations is given by

$$\mathbf{N} = (N_1, N_2, \ldots, N_7)^\top \sim \mathrm{HMN}(W, q_1, \ldots, q_7),$$

where $W \sim \mathrm{NB}(r = 10, \beta = 1)$, with $q_1 = 0.004$, $q_2 = 0.732$, $q_3 = 0.123$, $q_4 = 0.003$, $q_5 = 0.001$, $q_6 = 0.135$, $q_7 = 0.002$. The claim size vectors are denoted by $\mathbf{X}_{(1)} = (X_{(1),1}, 0, 0)^\top$, $\mathbf{X}_{(2)} = (0, X_{(2),2}, 0)^\top$, $\mathbf{X}_{(3)} = (0, 0, X_{(3),3})^\top$, $\mathbf{X}_{(4)} = (X_{(4),1}, X_{(4),2}, 0)^\top$, $\mathbf{X}_{(5)} = (X_{(5),1}, 0, X_{(5),3})^\top$, $\mathbf{X}_{(6)} = (0, X_{(6),2}, X_{(6),3})^\top$, and $\mathbf{X}_{(7)} = (X_{(7),1}, X_{(7),2}, X_{(7),3})^\top$.

Frees and Valdez (2008) fitted the claim sizes by the generalized beta of the second kind (GB2) distribution and modelled their dependence by multivariate t-copula. Here, for illustration of our method, we simply assume that $X_{(1),1} \sim \mathrm{Poi}(a_1)$, $X_{(2),2} \sim \mathrm{Poi}(a_2)$, $X_{(3),1} \sim \mathrm{Poi}(a_3)$, and the non-zero elements of $\mathbf{X}_{(4)}$, $\mathbf{X}_{(5)}$, $\mathbf{X}_{(6)}$, $\mathbf{X}_{(7)}$ follow common Poisson mixtures. Specifically, let Λ_i for $i = 1, 2, 3, 4$ assumed to be independent; all follow a gamma distribution with shape parameter $\alpha = 2$ and mean one. Conditional on $\Lambda_1 = \lambda_1$, $X_{(4),1}$ and $X_{(4),2}$ are independent Poisson variables $\mathrm{Poi}(b_1\lambda_1)$ and $\mathrm{Poi}(b_2\lambda_1)$; conditional on $\Lambda_2 = \lambda_2$, $X_{(5),1}$ and $X_{(5),3}$ are independent Poisson variables $\mathrm{Poi}(b_1\lambda_2)$ and $\mathrm{Poi}(b_3\lambda_2)$; conditional on $\Lambda_3 = \lambda_3$, $X_{(6),2}$ and $X_{(6),3}$ are independent Poisson variables $\mathrm{Poi}(b_2\lambda_3)$ and $\mathrm{Poi}(b_3\lambda_3)$; and conditional on $\Lambda_4 = \lambda_4$, $X_{(7),1}$, $X_{(7),2}$, and $X_{(7),3}$ are independent Poisson variables $\mathrm{Poi}(b_1\lambda_4)$, $\mathrm{Poi}(b_2\lambda_4)$, and $\mathrm{Poi}(b_3\lambda_4)$. The parameter values are set to be $a_1 = 5$, $a_2 = 1$, $a_3 = 0.8$, $b_1 = 6$, $b_2 = 1.2$, and $b_3 = 0.96$.

The proportions and amounts of capital allocated to the three types of risks according to TCE and TV with selected values of q are plotted in Figure 3. As we can see, the pattern for TPI (OD) is similar to BI (PD) in the last example, and the pattern for TPP is somewhat in the middle.

Remark 4.1. Applying simulation methods to estimate risk measures and compute capital allocations for this complex hierarchical risk model can be time-consuming and/or inaccurate. Our proposed method, based on moment transform and FFT, can solve the problem efficiently. This is especially true because, as pointed out in Remark 3.3, we do not need to apply multivariate IFFT to obtain the joint distribution of the seven possible combinations of the three types of losses. Instead, we only need to perform one-dimensional IFFT to the diagonal terms to get the distribution of the total.

4.2 A model with dependent claim frequency and size

In this subsection, we study a model in which the claim frequency and size are dependent through a common mixing variable, Ξ defined on $(0, \infty)$. Similar to the example in Section 4.1.1, we suppose that insurance policies cover two types of claims, PD and BI. The claim frequency vector $\mathbf{N} = (N_1, N_2, N_3)^\top$ follows the $\mathrm{HMN}(W, q_1, q_2, q_3)$ distribution defined in Section 4.1, where $W \sim \mathrm{NB}(r\xi, \beta)$. The claim sizes are denoted by $\mathbf{X}_{(1)} = (X_{(1),1}, 0)^\top$, $\mathbf{X}_{(2)} =$

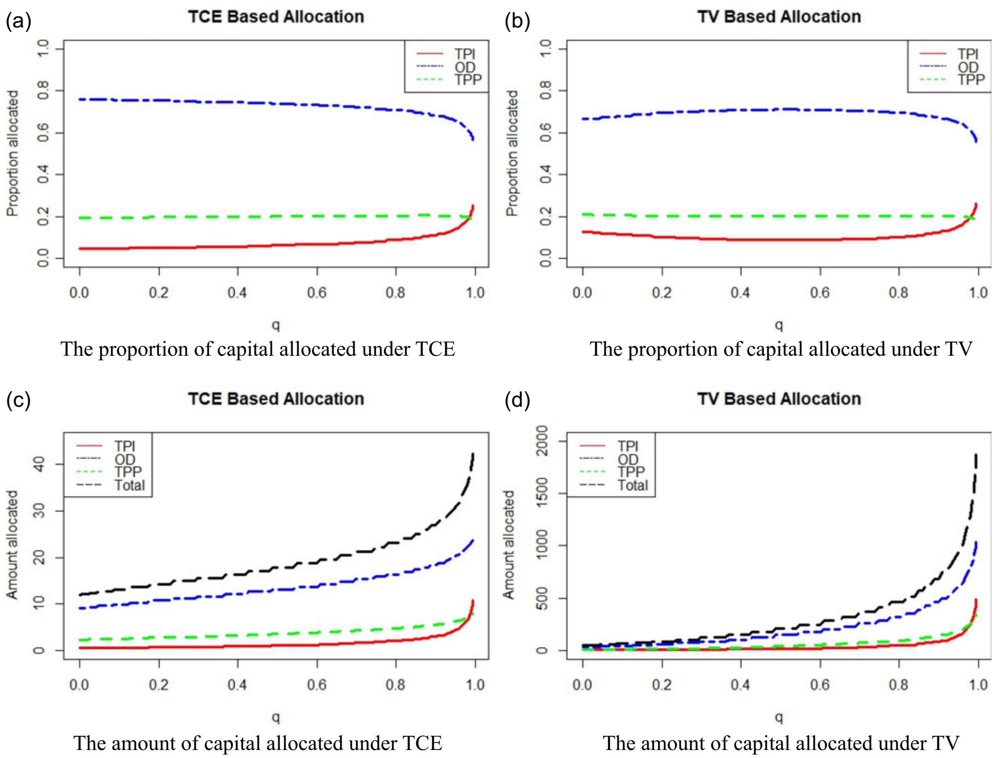


Figure 3. The proportions and amounts of capital allocated to the three types of risks under TCE and TV criteria.

$(0, X_{(2),2})^\top$, and $\mathbf{X}_{(3)} = (X_{(3),1}, X_{(3),2})^\top$, respectively. We assume that, conditional on $\Xi = \xi$, $X_{(1),1} \sim \text{Poi}(a_1\xi)$, $X_{(2),2} \sim \text{Poi}(a_2\xi)$, and $\mathbf{X}_3$ follows a common Poisson mixture, where conditional on $\Lambda = \lambda$, $X_{(3),1} \sim \text{Poi}(b_1\lambda\xi)$ and $X_{(3),2} \sim \text{Poi}(b_2\lambda\xi)$. Finally, we assume that Ξ and Λ are independent and follow gamma distribution with shape parameters α_1 and α_2, respectively. Both have unit mean.

The characteristic function of $\mathbf{S_N}$ is given by

$$\begin{aligned}
\psi_{\mathcal{S}_\mathbf{N}}(\mathbf{t}) = &\mathbb{E}\left[\mathbb{E}\left[exp\left(i\sum_{m=1}^{3}\sum_{k=1}^{2} t_{k,l}\sum_{j=1}^{N_m(\Xi)} X_{(m)j,k}(\Xi)\right)\middle|\Xi\right]\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[\prod_{m=1}^{3}\left(\mathbb{E}\left[exp\left(it_{k,m}\sum_{k=1}^{2} X_{(m),k}(\Xi)\right)\right]\right)^{N_m(\Xi)}\middle|\Xi\right]\right] \\
=&\mathbb{E}\left[\mathbb{E}\left[\mathcal{P}_{\mathbf{N}(\Xi)}\left(\phi_{\mathbf{X_{(1)}}(\Xi)}(t_{1,1}, t_{2,1}), \phi_{\mathbf{X_{(2)}}(\Xi)}(t_{1,2}, t_{2,2}), \phi_{\mathbf{X_{(3)}}(\Xi)}(t_{1,3}, t_{2,3})\right)\middle|\Xi\right]\right] \\
=&\mathbb{E}\left[\mathcal{P}_{\mathbf{N}(\Xi)}\left(\phi_{\mathbf{X_{(1)}}(\Xi)}(t_{1,1}, t_{2,1}), \phi_{\mathbf{X_{(2)}}(\Xi)}(t_{1,2}, t_{2,2}), \phi_{\mathbf{X_{(3)}}(\Xi)}(t_{1,3}, t_{2,3})\right)\right]. \qquad (4.1)
\end{aligned}$$

Since it is difficult to calculate the explicate expression of integral in Equation (4.1), even with the simple assumptions for the distributions of claim frequency and severity, in computation, we discretize the distribution of Ξ and compute the expectation in Equation (4.1) numerically.

The computation for capital allocation can be performed by using the following equations. For $m, m_1, m_2 = 1, 2, 3$ and $k, k_1, k_2 = 1, 2$

$$\mathbb{E}[S_{N_m,k}|S_\bullet > s_q] = \frac{\mathbb{E}[S_{N_m,k} I(S_\bullet > s_q)]}{\Pr(S_\bullet > s_q)} = \frac{\mathbb{E}\left[\mathbb{E}[S_{N_m,k} I(S_\bullet > s_q)|\Xi]\right]}{\Pr(S_\bullet > s_q)}$$
$$= \frac{\mathbb{E}\left[\mathbb{E}[S_{N_m,k}(\Xi)|\Xi]\Pr\left(\widetilde{S}_\bullet^{m[k]}(\Xi) > s_q|\Xi\right)\right]}{\Pr(S_\bullet > s_q)},$$

and

$$\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2}|S_\bullet > s_q] = \frac{\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2} I(S_\bullet > s_q)]}{\Pr(S_\bullet > s_q)}$$
$$= \frac{\mathbb{E}\left[\mathbb{E}[S_{N_{m_1},k_1}(\Xi) S_{N_{m_2},k_2}(\Xi)|\Xi]\Pr\left(\widetilde{S}_\bullet^{m_1,m_2[k_1,k_2]}(\Xi) > s_q|\Xi\right)\right]}{\Pr(S_\bullet > s_q)}.$$

Conditional on $\Xi = \xi$, the above quantities can be calculated by applying Theorem 3.1. Specifically, for $m = 1, 2, 3$ and $k_1, k_2 = 1, 2$, we have the following:

$$\mathbb{E}[S_{N_m,k_1}(\xi) S_{N_m,k_2}(\xi)] = \mathbb{E}[N_m(\xi)]\mathbb{E}[X_{(m),k_1}(\xi) X_{(m),k_2}(\xi)] + \mathbb{E}[N_m^{(2)}(\xi)]\mathbb{E}[X_{(m),k_1}(\xi)]\mathbb{E}[X_{(m),k_2}(\xi)],$$

where

$$\mathbb{E}[N_m(\xi)] = r\xi\beta q_m, \quad m = 1, 2, 3,$$

$$\mathbb{E}[N_m^{(2)}(\xi)] = r\xi(r\xi + 1)\beta^2 q_m^2, \quad m = 1, 2, 3,$$

$$\mathbb{E}[X_{(m),k}(\xi)] = a_k\xi, \quad m = 1, 2, 3, \ k = 1, 2, \text{ and } X_{(m),k} \neq 0,$$

$$\mathbb{E}[X^2_{(m),k}(\xi)] = a_k\xi + a_k^2\xi^2, \quad m = 1, 2, \ k = 1, 2, \text{ and } X_{(m),k} \neq 0,$$

$$\mathbb{E}[X^2_{(3),k}(\xi)] = b_k\xi + \frac{\alpha_2 + 1}{\alpha_2} b_k^2\xi^2, \quad k = 1, 2,$$

$$\mathbb{E}[X_{(3),1}(\xi) X_{(3),2}(\xi)] = \frac{\alpha_2 + 1}{\alpha_2} b_1 b_2 \xi^2.$$

For $m_1, m_2 = 1, 2, 3$, $k_1, k_2 = 1, 2$, and $m_1 \neq m_2$, we have

$$\mathbb{E}[S_{N_{m_1},k_1}(\xi) S_{N_{m_2},k_2}(\xi)] = \mathbb{E}[N_{m_1}(\xi) N_{m_2}(\xi)]\mathbb{E}[X_{(m_1),k_1}(\xi)]\mathbb{E}[X_{(m_2),k_2}(\xi)],$$

where

$$\mathbb{E}[N_{m_1}(\xi) N_{m_2}(\xi)] = r\xi(r\xi + 1)\beta^2 q_{m_1} q_{m_2}, \quad m = 1, 2, 3.$$

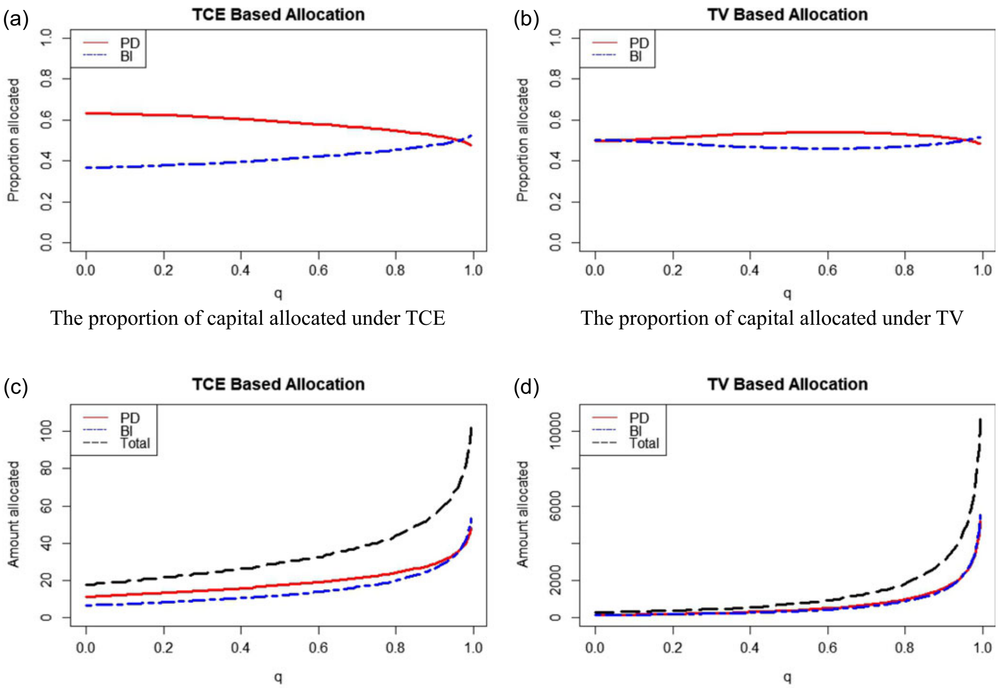
Then, the expectation with regard to Ξ can be computed numerically by discretizing the distribution of Ξ.

In the following, we set the value of the shape parameter of the gamma distributed variable Ξ to $\alpha_1 = 10$ and assume that all other parameters are the same as those in Section 4.1.1.

The proportions and amounts of capital allocated to the two types of risks according to TCE and TV are shown in Figure 4. It shows a similar pattern to that in Figure 1 in Section 4.1.1. That is, according to TCE, the proportion of risk capital allocated to PD (BI) claims decreases (increases)

Table 4. Comparison of the amounts and proportions of capital allocated to the two types of risks according to Tail Conditional Expectation (TCE) criterion when the dependence between loss frequency and sizes changes

$q = 0.995$	s_q	$\mathrm{TCE}_{S_{\bullet,1}}(q)$	$\mathrm{TCE}_{S_{\bullet,2}}(q)$	$\mathrm{TCE}_{S_\bullet}(q)$	$\frac{\mathrm{TCE}_{S_{\bullet,1}}(q)}{\mathrm{TCE}_{S_\bullet}(q)}$	$\frac{\mathrm{TCE}_{S_{\bullet,2}}(q)}{\mathrm{TCE}_{S_\bullet}(q)}$
α_1=0.1	147	1550.8	917.0	2467.8	62.8%	37.2%
α_1=1	122	130.3	83.7	214.0	60.9%	39.1%
α_1=10	88	48.1	53.0	101.2	47.5%	52.5%
$\alpha_1 \to \infty$	56	23.6	39.9	63.5	37.2%	62.8%



The proportion of capital allocated under TCE

The proportion of capital allocated under TV

The amount of capital allocated under TCE

The amount of capital allocated under TV

Figure 4. The proportions and amounts of capital allocated to the two types of risks according to TCE and TV criteria for the model with dependent frequency and severity.

with q; and according to TV, the proportion allocated to BI(PD) claims decreases (increases) with q when q is small and increases (decreases) when q is large. Also, the amounts of capital allocated to the two types of risks increase with q in all cases.

Tables 4 and 5 show the numerical values of the amounts of capital allocated to the two types of risk according to TCE and TV criteria for different values of α_1, the coefficient of the mixing random variable Ξ. Recall that a smaller value of α_1 indicates a larger variance of Ξ, and a stronger dependence between the claim frequency and severity. When $\alpha_1 \to \infty$, the loss frequency and severities are independent.

From Tables 4 and 5, we conclude that larger variance of the claim frequency and size, and stronger dependence between them, leads to greater values of VaR, TCE and TV of the total losses. The total capital allocated to each type of risk also increases.

Table 5. Comparison of the amounts and proportions of capital allocated to the two types of risks according to Tail Variance (TV) criterion when the dependence between loss frequency and sizes changes

	$TV_{S_{\bullet,1}}(q)$	$TV_{S_{\bullet,2}}(q)$	$TV_{S_\bullet}(q)$	$\frac{TV_{S_{\bullet,1}}(q)}{TV_{S_\bullet}(q)}$	$\frac{TV_{S_{\bullet,2}}(q)}{TV_{S_\bullet}(q)}$
α_1=0.1	594980.2	448985.5	1043965.7	57.0%	43.0%
α_1=1	34731.6	30199.2	64930.8	53.5%	46.5%
α_1=10	5107.9	5427.7	10535.6	48.5%	51.5%
$\alpha_1 \to \infty$	1513.6	2590.5	4104.1	36.9%	63.1%

5. Conclusions

This paper presents formulas for computing TCE and TV and performing corresponding capital allocation for a hierarchical multivariate compound model introduced by Cummins and Wiltbank (1983) and Frees and Valdez (2008), where both the claim frequencies and the claim sizes are dependent. The main methodology we used is the multivariate moment transform.

Future research will study the risk measures and capital allocation problems for multivariate compound models with more complicated dependence structures between claim frequencies and claim sizes.

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Appendix: Proof of Theorem 3.1

Proof. We firstly assume that $\mathbf{N}$ takes fixed values $\mathbf{N} = \mathbf{n} = (n_1, \ldots, n_M)^\top$. For $m \in \mathcal{M}$, let

$$\mathbf{S}_{n_m} = \sum_{i=1}^{n_m} \mathbf{X}_{(m)i} = \sum_{i=1}^{n_m} \left(X_{(m)i,1}, \ldots, X_{(m)i,K}\right)^\top = \left(S_{n_m,1}, \ldots, S_{n_m,K}\right)^\top,$$

and

$$\mathbf{S}_{\mathbf{n}} = \left(\mathbf{S}_{n_1}, \ldots, \mathbf{S}_{n_M}\right).$$

Then, similar to the results in Denuit and Robert (2022) and Ren (2022), for $m \in \mathcal{M}$, $i \in \{1, \ldots, n_m\}$, and $k \in \mathcal{K}$, we have

$$\mathbb{E}[X_{(m)i,k} I(\mathbf{S}_{n_m} \leq \mathbf{s}_m)] = \mathbb{E}[X_{(m)i,k}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)i} + \widetilde{\mathbf{X}_{(m)i}}^{[k]} \leq \mathbf{s}_m\right).$$

Since $\mathbf{X}_{(m)i}$'s are assumed to be i.i.d.,

$$\mathbb{E}[S_{n_m,k} I(\mathbf{S}_{n_m} \leq \mathbf{s}_m)] = n_m \mathbb{E}[X_{(m)1,k}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \leq \mathbf{s}_m\right).$$

Further, for $i, j \in \{1, \ldots, n_m\}$, $i \neq j$, and $k_1, k_2 \in \mathcal{K}$, we have

$$\mathbb{E}[X_{(m)i,k_1} X_{(m)i,k_2} I(\mathbf{S}_{n_m} \leq \mathbf{s}_m)] = \mathbb{E}[X_{(m)i,k_1} X_{(m)i,k_2}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)i} + \widetilde{\mathbf{X}_{(m)i}}^{[k_1,k_2]} \leq \mathbf{s}_m\right).$$

and

$$\begin{aligned}\mathbb{E}[X_{(m)i,k_1} X_{(m)j,k_2} I(\mathbf{S}_{n_m} \leq \mathbf{s}_m)] =& \mathbb{E}[X_{(m)i,k_1}] \mathbb{E}[X_{(m)j,k_2}] \times \\ & \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)i} - \mathbf{X}_{(m)j} + \widetilde{\mathbf{X}_{(m)i}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)j}}^{[k_2]} \leq \mathbf{s}_m\right),.\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}[S_{n_m,k_1} S_{n_m,k_2} I(\mathbf{S}_{n_m} \leq \mathbf{s}_m)] \\ =& n_m \mathbb{E}[X_{(m)1,k_1} X_{(m)1,k_2}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \leq \mathbf{s}_m\right) \\ &+ n_m(n_m - 1) \mathbb{E}[X_{(m)1,k_1}] \mathbb{E}[X_{(m)2,k_2}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} - \mathbf{X}_{(m)2} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]} \leq \mathbf{s}_m\right).\end{aligned}$$

Since $\mathbf{S}_{n_1}, \ldots, \mathbf{S}_{n_M}$ are mutually independent, we have

$$\mathbb{E}[S_{n_m,k} I(\mathbf{S}_{\mathbf{n}} \leq \mathbf{s})] = n_m \mathbb{E}[X_{(m)1,k}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \leq \mathbf{s}_m\right) \prod_{\xi \in \mathcal{M} - \{m\}} \Pr(\mathbf{S}_{n_\xi} \leq \mathbf{s}_\xi), \tag{A.1}$$

and

$$\begin{aligned}&\mathbb{E}[S_{n_m,k_1} S_{n_m,k_2} I(\mathbf{S}_{\mathbf{n}} \leq \mathbf{s})] \\ =& \left\{ n_m \mathbb{E}[X_{(m)1,k_1} X_{(m)1,k_2}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \leq \mathbf{s}_m\right) \right. \\ &+ n_m(n_m - 1) \mathbb{E}[X_{(m)1,k_1}] \mathbb{E}[X_{(m)2,k_2}] \times \\ & \left. \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} - \mathbf{X}_{(m)2} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]} \leq \mathbf{s}_m\right) \right\} \prod_{\xi \in \mathcal{M} - \{m\}} \Pr(\mathbf{S}_{n_\xi} \leq \mathbf{s}_\xi).\end{aligned} \tag{A.2}$$

In addition, for $m_1, m_2 \in \mathcal{M}$, $m_1 \neq m_2$,

$$\mathbb{E}[S_{n_{m_1},k_1} S_{n_{m_2},k_2} I(\mathbf{S}_{\mathbf{n}} \leq \mathbf{s})] \tag{A.3}$$

$$= n_{m_1} n_{m_2} \mathbb{E}[X_{(m_1)1,k_1}] \mathbb{E}[X_{(m_2)1,k_2}] \times \prod_{i \in \{1,2\}} \Pr\left(\mathbf{S}_{n_{m_i}} - \mathbf{X}_{(m_i)1} + \widetilde{\mathbf{X}_{(m_i)1}}^{[k_i]} \leq \mathbf{s}_{m_i}\right) \prod_{\xi \in \mathcal{M} - \{m_1, m_2\}} \Pr(\mathbf{S}_{n_\xi} \leq \mathbf{s}_\xi). \tag{A.4}$$

Therefore, applying the law of total probability to Equation (A.1) leads to

$$\begin{aligned}
&\mathbb{E}[S_{N_m,k} I(\mathbf{S}_{\mathbf{N}} \le \mathbf{s})] \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} p_{\mathbf{N}}(\mathbf{n}) n_m \mathbb{E}[X_{(m)1,k}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} p_{\widetilde{\mathbf{N}}^{[m]}}(\mathbf{n}) \mathbb{E}[N_m] \mathbb{E}[X_{(m)1,k}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \\
&= \mathbb{E}[N_m] \mathbb{E}[X_{(m)1,k}] \Pr\left(\mathbf{S}_{\widetilde{N_m}^{[m]}} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k]} \le \mathbf{s}_m,\ \mathbf{S}_{\widetilde{N_\xi}^{[m]}} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right),
\end{aligned}$$

which leads to Equation (3.3).

Similarly, applying the law of total probability to Equations (A.2) and (A.3), respectively, yields

$$\begin{aligned}
&\mathbb{E}[S_{N_m,k_1} S_{N_m,k_2} I(\mathbf{S}_{\mathbf{N}} \le \mathbf{s})] \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} p_{\mathbf{N}}(\mathbf{n}) \times \\
&\quad \Bigg\{ n_m \mathbb{E}[X_{(m)1,k_1} X_{(m)1,k_2}] \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \\
&\qquad + n_m(n_m - 1) \mathbb{E}[X_{(m)1,k_1}] \mathbb{E}[X_{(m)2,k_2}] \times \\
&\qquad \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} - \mathbf{X}_{(m)2} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \Bigg\} \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} \Bigg\{ p_{\widetilde{\mathbf{N}}^{[m]}}(\mathbf{n}) \mathbb{E}[N_m] \mathbb{E}[X_{(m)1,k_1} X_{(m)1,k_2}] \times \\
&\qquad\qquad \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \\
&\qquad\qquad + p_{\widetilde{\mathbf{N}^{(2)}}^{[m]}}(\mathbf{n}) \mathbb{E}[N_m^{(2)}] \mathbb{E}[X_{(m)1,k_1}] \mathbb{E}[X_{(m)2,k_2}] \times \\
&\qquad\qquad \Pr\left(\mathbf{S}_{n_m} - \mathbf{X}_{(m)1} - \mathbf{X}_{(m)2} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{n_\xi} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \Bigg\} \\
&= \mathbb{E}[N_m] \mathbb{E}[X_{(m)1,k_1} X_{(m)1,k_2}] \Pr\left(\mathbf{S}_{\widetilde{N_m}^{[m]}} - \mathbf{X}_{(m)1} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1,k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{\widetilde{N_\xi}^{[m]}} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right) \\
&\quad + \mathbb{E}[N_m^{(2)}] \mathbb{E}[X_{(m)1,k_1}] \mathbb{E}[X_{(m)2,k_2}] \times \\
&\quad \Pr\left(\mathbf{S}_{\widetilde{N_m^{(2)}}^{[m]}} - \mathbf{X}_{(m)1} - \mathbf{X}_{(m)2} + \widetilde{\mathbf{X}_{(m)1}}^{[k_1]} + \widetilde{\mathbf{X}_{(m)2}}^{[k_2]} \le \mathbf{s}_m,\ \mathbf{S}_{\widetilde{N_\xi^{(2)}}^{[m]}} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m\}\right),
\end{aligned}$$

which leads to Equation (3.4).

In addition,

$$
\begin{aligned}
&\mathbb{E}[S_{N_{m_1},k_1} S_{N_{m_2},k_2} I(\mathbf{S}_{\mathbf{N}} \le \mathbf{s})] \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} p_{\mathbf{N}}(\mathbf{n}) \Bigg\{ n_{m_1} n_{m_2} \mathbb{E}[X_{(m_1)1,k_1}] \mathbb{E}[X_{(m_2)1,k_2}] \prod_{i \in \{1,2\}} \Pr\left(\mathbf{S}_{n_{m_i}} - \mathbf{X}_{(m_i)1} + \widetilde{\mathbf{X}_{(m_i)1}}^{[k_i]} \le \mathbf{s}_{m_i} \right) \\
&\qquad \prod_{\xi \in \mathcal{M} - \{m_1, m_2\}} \Pr(\mathbf{S}_{n_\xi} \le \mathbf{s}_\xi) \Bigg\} \\
&= \sum_{n_1=0}^{\infty} \cdots \sum_{n_M=0}^{\infty} p_{\widetilde{\mathbf{N}}^{[m_1,m_2]}}(\mathbf{n}) \Bigg\{ \mathbb{E}[N_{m_1} N_{m_2}] \mathbb{E}[X_{(m_1)1,k_1}] \mathbb{E}[X_{(m_2)1,k_2}] \times \\
&\quad \Pr\Bigg(\mathbf{S}_{n_{m_1}} - \mathbf{X}_{(m_1)1} + \widetilde{\mathbf{X}_{(m_1)1}}^{[k_1]} \le \mathbf{s}_{m_1}, \mathbf{S}_{n_{m_2}} - \mathbf{X}_{(m_2)1} + \widetilde{\mathbf{X}_{(m_2)1}}^{[k_2]} \le \mathbf{s}_{m_2}, \\
&\qquad \mathbf{S}_{n_{m_\xi}} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m_1, m_2\} \Bigg) \Bigg\} \\
&= \mathbb{E}[N_{m_1} N_{m_2}] \mathbb{E}[X_{(m_1)1,k_1}] \mathbb{E}[X_{(m_2)1,k_2}] \times \\
&\quad \Pr\Bigg(\mathbf{S}_{\widetilde{N_{m_1}}^{[m_1,m_2]}} - \mathbf{X}_{(m_1)1} + \widetilde{\mathbf{X}_{(m_1)1}}^{[k_1]} \le \mathbf{s}_{m_1},\ \mathbf{S}_{\widetilde{N_{m_2}}^{[m_1,m_2]}} - \mathbf{X}_{(m_2)1} + \widetilde{\mathbf{X}_{(m_2)1}}^{[k_2]} \le \mathbf{s}_{m_2}, \\
&\qquad \mathbf{S}_{\widetilde{N_\xi}^{[m_1,m_2]}} \le \mathbf{s}_\xi,\ \xi \in \mathcal{M} - \{m_1, m_2\} \Bigg),
\end{aligned}
$$

which leads to Equation (3.5). This ends the proof. □

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