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Ergodic cocycles of IDPFT systems and non-singular Gaussian actions

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To the memory of Sergiy Sinel'shchikov, our colleague and friend

Abstract. It is proved that each Gaussian cocycle over a mildly mixing Gaussian transformation is either a Gaussian coboundary or sharply weak mixing. The class of non-singular infinite direct products T of transformations T_n, $n \in \mathbb{N}$, of finite type is studied. It is shown that if T_n is mildly mixing, $n \in \mathbb{N}$, the sequence of Radon–Nikodym derivatives of T_n is asymptotically translation quasi-invariant and T is conservative then the Maharam extension of T is sharply weak mixing. This technique provides a new approach to the non-singular Gaussian transformations studied recently by Arano, Isono and Marrakchi.

Key words: ergodic transformation, Gaussian cocycle, Maharam extension
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1. *Introduction*

The original motivation of this paper was to tackle a problem (stated in [**LeLeSk**]) that is related to the theory of Gaussian dynamical systems. Let T be an ergodic (equivalently, weakly mixing) Gaussian transformation on a standard probability space $(X, \mathfrak{B}, \mu)$ and let H be the corresponding invariant Gaussian subspace of the real Hilbert space $L^2_0(X, \mu)$.

Conjecture. For each function $f \in H$, either f is a T-coboundary (equivalently, a Gaussian coboundary) or the skew product transformation T_f acting on $X \times \mathbb{R}$ is ergodic.

In this paper we obtain an affirmative answer under an assumption which is slightly stronger than weak mixing. We say that an ergodic conservative non-singular transformation R is *sharply weak mixing* if the direct product of R with each ergodic conservative transformation is either totally dissipative or ergodic. We also recall that T_f is conservative for each $f \in H$ [**At**].

THEOREM 1.1. *If T is mildly mixing and f is not a coboundary for T then T_f is sharply weak mixing.*

To prove Theorem 1.1 we note that there exists a decomposition of T into direct product of mildly mixing transformations T_n in a such a way that f splits into a sum of coboundaries $f_n := a_n - a_n \circ T_n$ for T_n, $n \in \mathbb{N}$. Moreover, the sequence of distributions of the transfer functions $(a_n)_{n=1}^{\infty}$ satisfies a certain property that we call *asymptotic translation invariance* (ATI) in Definition 2.2. Then Theorem 1.1 follows from the following result.

THEOREM 1.2. *Given a locally compact second countable abelian group G, a sequence of mildly mixing dynamical systems (X_n, ν_n, T_n) and a sequence of functions $f_n : X_n \to G$, $n \in \mathbb{N}$, consider the infinite direct product $(X, \nu, T) := \bigotimes_{n=1}^{\infty}(X_n, \nu_n, T_n)$. Suppose that a function $f(x) := \sum_{n=1}^{\infty}(f_n(T_n x_n) - f_n(x_n)) \in G$ is well defined for ν-almost every (a.e.) $x = (x_n)_{n=1}^{\infty} \in X$. If the sequence of distributions $(\nu_n \circ f_n^{-1})_{n\in\mathbb{N}}$ on G is asymptotically translation invariant and the f-skew product extension $T_f : X \times G \to X \times G$ of T is conservative then T_f is sharply weak mixing.*

The proof of Theorem 1.2 is based on the two ideas.

- The mild mixing and the product structure of T_f yield that each T_f-invariant subset is also invariant under a large group of 'finitary' transformations, that is, transformations that 'move' finitely many coordinates only.
- The ATI property implies that this finitary group is ergodic via techniques related to computation of the essential values of cocycles.

The former idea was inspired by the proof [**ArIsMa**, Theorem D] of ergodic properties of some non-singular Gaussian group actions.

We then turn to classical problems of non-singular ergodic theory. We mention recent progress in providing natural examples for non-singular ergodic theory: non-singular Bernoulli and Markov shiftwise actions (see [**DaLe**, **BjKoVa**, **KoSo**, **Av**, **MaVa**] and references therein), non-singular Gaussian systems [**ArIsMa**] and non-singular Poisson systems [**DaKoRo1**, **DaKoRo2**]. In the present work we introduce one more natural family of non-singular transformations. We say that a non-singular transformation T on a standard probability space (X, μ) is an *infinite direct product of finite types* (IDPFT) if there is a sequence of ergodic probability-preserving dynamical systems (X_n, ν_n, T_n) and a sequence of probability measures μ_n on X_n, $n \in \mathbb{N}$, such that $\mu_n \sim \nu_n$ for each n and $(X, \mu, T) = \bigotimes_{n=1}^{\infty}(X_n, \mu_n, T_n)$. Kakutani's theorem [**Ka**] provides a criterion where μ is quasi-invariant under T. We are interested in the case where $\mu \perp \nu$ and μ does not admit an equivalent T-invariant probability. It is possible that (X, μ, T) is totally dissipative.

THEOREM 1.3. *Let (X_n, ν_n, T_n) be mildly mixing for each $n > 0$. If T is μ-conservative and the sequence of distributions of the random variables $\log(d\mu_n/d\nu_n)$, $n \in \mathbb{N}$, is asymptotically translation quasi-invariant then T is ergodic of stable type Krieger's type III_1. Moreover, the Maharam extension of T is sharply weak mixing.*

The asymptotic translation quasi-invariance (ATQI) property (see Definition 3.8) in the statement of Theorem 1.3 is an analogue of ATI though neither ATI implies ATQI nor

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vice versa. The scheme of the proof of Theorem 1.3 is similar to that of Theorem 1.2 and we again use the aforementioned two ideas. However, there is a 'non-singular' nuance. Namely, a formal repetition of the proof of Theorem 1.2 yields that the group of finitary transformations is ergodic with respect to the 'wrong' measure. Hence, it does not work. We recall that there are two different (mutually singular) natural measures associated with an IDPFT system: ν (invariant) and μ (quasi-invariant). Therefore a certain additional argument and the ATQI property rather than ATI are needed to prove ergodicity for the 'right' measure. We also provide examples of rigid IDPFT systems T of Krieger's type III_λ for an arbitrary $\lambda \in (0, 1)$.

We have already mentioned that non-singular Gaussian systems were recently studied in [**ArIsMa**]. However, the exposition there is based heavily on affine geometry and often uses a non-standard (from the dynamical viewpoint) terminology. Therefore, we decided to provide here an alternative exposition of this important topic. We define the non-singular Gaussian systems as transformations on Hilbert spaces $\mathcal{H}$ furnished with Gaussian measures stressing the fact that the systems are compositions of classical Gaussian automorphisms and totally dissipative transformations (given by non-singular rotations). Connections with the underlying Fock space, the first chaos and the exponential map are explicitly made. We also explain interrelation between non-singular Gaussian systems and non-singular Poisson systems. Our main observation is that Gaussian transformations (except for a 'small' family of degenerate ones) are a subclass of IDPFT systems. Hence we deduce [**ArIsMa**, Theorem D] (we consider only the case of $\mathbb{Z}$-actions) from Theorem 1.2. $\mathcal{H}_0$ below is a linear subspace of $\mathcal{H}$ endowed with a new inner product; see §4.

THEOREM 1.4. *Let an orthogonal operator V of a real Hilbert space $\mathcal{H}_0$ be mildly mixing. Let $f \in \mathcal{H}_0$ not be a V-coboundary (that is, $f \neq Va - a$ for any $a \in \mathcal{H}_0$). If the non-singular Gaussian transformation $T_{(f,V)}$ associated with the pair (f, V) is conservative then the Maharam extension of $T_{(f,V)}$ is sharply weak mixing. In particular, $T_{(f,V)}$ is of type III_1.*

The outline of the paper is as follows. In §2 we introduce important definitions: Hellinger distance, weak mixing properties for non-singular actions, the ATI property, skew product extension, essential value of a cocycle, etc. Then we prove Theorem 1.2 (see Theorem 2.5) and deduce Theorem 1.1 from it (see Theorem 2.6). We also provide a generalization of Theorem 1.1 (see Conjecture II and discussion preceding it). In §3 we consider non-singular versions of the problems studied in §2. IDPFT systems are introduced in Definition 3.2. Radon–Nikodym cocycles, Maharam extensions and Krieger's types III_λ, $0 \le \lambda \le 1$, are discussed there. We show that each IDPFT system is either conservative or totally dissipative (Corollary 3.7), introduce the ATQI property (Definition 3.8) and prove Theorem 1.3 (Theorem 3.10). Type III_λ rigid IDPFT systems are also constructed there for each $\lambda \in (0.1)$ (Proposition 3.12). The final §4 is devoted to non-singular Gaussian systems. We first recall the definition of Gaussian measure in a separable Hilbert space. Then we discuss the main properties of the related Fock space and exponential map. Given an orthogonal operator V in a Hilbert space $\mathcal{H}_0$ and a vector $f \in \mathcal{H}_0$, we associate a non-singular transformation $T_{(f,V)}$ acting on the corresponding Hilbert space $\mathcal{H} \supset \mathcal{H}_0$ equipped with a Gaussian measure μ. We show that $T_{(f,V)}$ is the composition of the classic Gaussian μ-preserving transformation associated to V with the (totally dissipative) rotation by f. It is well known that the non-singular transformation group $\{T_{(f,0)} \mid f \in \mathcal{H}_0\}$ generated by the rotations is ergodic (see, for example, [**Gu**]) but Krieger's type has not been specified so far. We prove that it is III_1 (Theorem 4.7). We show that the Koopman operator generated by $T_{(f,V)}$ is the Weyl operator associated to the pair $(f/2, V)$. A criterion for the existence of an invariant equivalent probability measure for $T_{(f,V)}$ is established in Theorem 4.9 (cf. [**DaKoRo1**, Proposition 6.4] and [**ArIsMa**, Theorem 6.3(i)]). Theorem 1.4 in proved in this section (Theorem 4.12).

After completion of this paper we learnt of a work [**MaVa**] devoted to non-singular Gaussian actions of arbitrary groups. It was written independently of but at the same time as our work†. Some of our results overlap with theirs. For example, Theorem 4.7 is [**MaVa**, Theorem 3.1] and Theorem 1.1, though stated in a more general form, is, in fact, equivalent to [**MaVa**, Theorem 9.1(3)] in the case of $\mathbb{Z}$-actions. Our proofs are different. They are based solely on elementary techniques of the non-singular ergodic (measurable orbit) theory. We do not use affine geometry, representation theory or harmonic analysis.

2. *Weak mixing cocycles of product type*

2.1. *Hellinger distance and Kakutani's theorem.* Let γ and δ be two equivalent probability measures on a standard Borel space $(Y, \mathfrak{C})$. The square of the Hellinger distance between γ and δ is

$$H^2(\gamma,\delta) := \frac{1}{2}\int_Y \left(1-\sqrt{\frac{d\gamma}{d\delta}}\right)^2 d\delta = 1-\int_Y \sqrt{\frac{d\gamma}{d\delta}}\,d\delta.$$

By the Cauchy–Schwarz inequality, $0 \le H(\gamma,\delta) < 1$. We also recall [**Ni**] the following inequalities between the Hellinger distance and the total variation:

$$H^2(\gamma,\delta) \le \|\gamma-\delta\|_1 := \sup_{C\in\mathfrak{C}} |\gamma(C)-\delta(C)| \le \sqrt{2}H(\gamma,\delta). \tag{2.1}$$

We now state the Kakutani theorem on equivalence of infinite products of probability measures [**Ka**].

THEOREM A. *Let μ_n and ν_n be two equivalent probability measures on a standard Borel space $(X_n, \mathfrak{B}_n)$ for each $n \in \mathbb{N}$. Let μ and ν denote the infinite product measures $\bigotimes_{n\in\mathbb{N}} \mu_n$ and $\bigotimes_{n\in\mathbb{N}} \nu_n$ respectively on the standard Borel space $(X, \mathfrak{B}) := \bigotimes_{n\in\mathbb{N}}(X_n, \mathfrak{B}_n)$. If*

$$\prod_{n=1}^{\infty}(1-H^2(\mu_n,\nu_n)) > 0 \quad \textit{or, equivalently,} \quad \sum_{n=1}^{\infty} H^2(\mu_n,\nu_n) < \infty \tag{2.2}$$

then $\mu \sim \nu$, $\prod_{n=1}^{\infty}(1-H^2(\mu_n,\nu_n)) = 1-H^2(\mu,\nu)$ and $(d\mu/d\nu)(x) = \prod_{n\in\mathbb{N}}(d\mu_n/d\nu_n)(x_n)$ at a.e. $x = (x_n)_{n\in\mathbb{N}} \in X$. If (2.2) does not hold then $\mu \perp \nu$.

2.2. *Weak mixing properties of non-singular actions.* We recall that, given a non-singular transformation R of a standard Borel probability space $(Y, \mathfrak{C}, \nu)$, there is a unique decomposition $Y = \mathcal{D}(R) \sqcup \mathcal{C}(R)$ (called *Hopf's decomposition*) of Y into two

† The two papers appeared on arXiv on two successive days. We thank S. Vaes for informing us about [**MaVa**].

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Borel sets such that $\mathcal{D}(R)$ is the disjoint union of the orbit of a *wandering set* W, that is, $\mathcal{D}(R) = \bigsqcup_{n\in\mathbb{Z}} R^n W$ and $\mathcal{C}(R) = Y \setminus \mathcal{D}(R)$ contains no non-trivial wandering set. If $\mathcal{C}(R) = Y$ then R is called *conservative* and if $\mathcal{D}(R) = Y$ then R is called *totally dissipative*. As both parts $\mathcal{C}(R)$ and $\mathcal{D}(R)$ are R-invariant, each ergodic R is either conservative or totally dissipative. An ergodic conservative non-singular transformation R is called *weakly mixing* if, for each ergodic probability-preserving transformation S, the Cartesian product $R \times S$ is ergodic. We now introduce a stronger concept of weak mixing.

Definition 2.1. An ergodic conservative non-singular transformation R is called *sharply weak mixing* if, for each ergodic conservative non-singular transformation S, the direct product $R \times S$ is either totally dissipative or ergodic.

If S in the above definition admits an equivalent invariant probability measure (that is, S is of type II_1) then $T \times S$ is conservative (see [**Aa**, Proposition 1.1.6, part 2]). Hence $R \times S$ is ergodic according to Definition 2.1. Thus, every sharply weak mixing transformation is weakly mixing. It follows from [**SiTh**] that every conservative non-singular transformation with property K is sharply weak mixing (see also [**AaLiWe**, Theorem 6.7] for other examples). In [**AdFrSi**, **Da**] examples of weakly mixing infinite measure-preserving rank-one transformations R were constructed such that $R \times R$ is conservative but not ergodic. Hence R is not sharply weak mixing. We recall that an ergodic probability-preserving transformation R defined on a space $(Y, \mathfrak{C}, \nu)$ is called *mildly mixing* ([**FuWe**], see also [**AaLiWe**] and [**ScWa**]) if every function $f \in L^\infty(\nu)$ such that $\|f \circ T^{n_i} - f\|_1 \to 0$ for some sequence $n_i \to \infty$ is constant.

We will utilize the following result from [**ScWa**].

THEOREM B. *Let R be a mildly mixing transformation of a standard probability space $(Y, \mathfrak{C}, \nu)$ and let C be a conservative non-singular transformation of a standard probability space $(Z, \mathfrak{F}, \tau)$. If a function $F \in L^\infty(Y \times Z, \nu \otimes \tau)$ is invariant under $R \times C$ then there is $f \in L^\infty(Z, \tau)$ such that $F(y, z) = f(z)$ almost everywhere.*

We note that Theorem B was proved in [**ScWa**] for the ergodic conservative C only, but the proof remains valid for an arbitrary conservative C as well. Direct products of finitely (and countably) many mildly mixing transformations are mildly mixing.

It follows from Theorem B that an ergodic finite measure-preserving transformation is sharply weak mixing if and only if it is mildly mixing. In Theorems 2.5 and 2.6 below we provide examples of mildly mixing transformations (including the zero-entropy case) which have locally compact group extensions that are sharply weak mixing infinite measure-preserving (and hence not mildly mixing).

2.3. *ATI property.* Fix a locally compact second countable abelian group G. Denote by λ_G a Haar measure on G.

Definition 2.2. A sequence $(\xi_n)_{n=1}^\infty$ of probability Borel measures on G is called *asymptotically translation invariant* if

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$$\lim_{m\to\infty} \|\xi_n * \xi_{n+1} * \cdots * \xi_{n+m} * \delta_a - \xi_n * \xi_{n+1} * \cdots * \xi_{n+m}\|_1 = 0$$

for each $n \in \mathbb{N}$ and $a \in G$.

Example 2.3. Let $\mathcal{N}_{a,\sigma^2}$ denote the normal distribution on $\mathbb{R}$ with parameters a and σ^2, that is, $\widehat{\mathcal{N}_{a,\sigma^2}}(t) = e^{iat-(1/2)\sigma^2 t^2}$ for all $t \in \mathbb{R}$. We leave verification of the formula

$$H^2(\mathcal{N}_{a,\sigma^2}, \mathcal{N}_{b,\tau^2}) = 1 - \sqrt{\frac{2\sigma\tau}{\sigma^2+\tau^2}}\, e^{-(1/4)((a-b)^2/(\sigma^2+\tau^2))} \quad \text{for all } a, b, \sigma, \tau \in \mathbb{R},$$

as an exercise for the reader. Given two sequences $(a_n)_{n\in\mathbb{Z}}$ and $(\sigma_n)_{n=1}^{\infty}$ of reals such that $\sum_{n=1}^{\infty} \sigma_n^2 = +\infty$, the sequence of probabilities $(\mathcal{N}_{a_n,\sigma_n^2})_{n=1}^{\infty}$ is asymptotically translation invariant. Indeed,

$$\mathcal{N}_{a_n,\sigma_n^2} * \cdots * \mathcal{N}_{a_{n+m},\sigma_{n+m}^2} = \mathcal{N}_{\sum_{k=n}^{n+m} a_k,\, \sum_{k=n}^{n+m} \sigma_k^2},$$

$$\mathcal{N}_{a_n,\sigma_n^2} * \cdots * \mathcal{N}_{a_{n+m},\sigma_{n+m}^2} * \delta_a = \mathcal{N}_{a+\sum_{k=n}^{n+m} a_k,\, \sum_{k=n}^{n+m} \sigma_k^2} \quad \text{and}$$

$$H^2\Big(\mathcal{N}_{a+\sum_{k=n}^{n+m} a_k,\, \sum_{k=n}^{n+m} \sigma_k^2},\ \mathcal{N}_{\sum_{k=n}^{n+m} a_k,\, \sum_{k=n}^{n+m} \sigma_k^2}\Big) = 1 - e^{-(1/8)\cdot a^2/(\sum_{k=n}^{n+m} \sigma_k^2)} \to 0$$

as $m \to \infty$. Hence $(\xi_n)_{n=1}^{\infty}$ is asymptotically translation invariant in view of (2.1).

2.4. *Ergodic cocycles of ergodic transformation groups.* Given a standard Borel σ-finite measure space $(Y, \mathfrak{C}, \nu)$, we denote by $\mathrm{Aut}(Y, \nu)$ the group of all ν-non-singular invertible Borel transformations on Y. Let $\mathrm{Aut}_0(Y, \nu)$ denote the subgroup of ν-preserving transformations from $\mathrm{Aut}(Y, \nu)$. Let Γ be an ergodic countable subgroup in $\mathrm{Aut}(Y, \nu)$. The *full group* $[\Gamma]$ of Γ is defined by

$$[\Gamma] := \{\theta \in \mathrm{Aut}(Y, \nu) \mid \theta y \in \{\gamma y \mid \gamma \in \Gamma\} \text{ at a.e. } y \in Y\}.$$

Let G be a locally compact second countable abelian group and let λ_G be a Haar measure on G. A measurable map $\alpha : \Gamma \times Y \to G$ is called a *cocycle* of Γ if

$$\alpha(\gamma_1\gamma_2, y) = \alpha(\gamma_1, \gamma_2 y) + \alpha(\gamma_2, y) \quad \text{at a.e. } y \in Y \tag{2.3}$$

for all $\gamma_1, \gamma_2 \in \Gamma$. From now on we assume that Γ is free, that is, if $\gamma \in \Gamma \setminus \{I\}$ then $\gamma y \neq y$ for a.e. y. Then α can be 'extended' to $[\Gamma]$ if we set

$$\alpha(\theta, y) := \alpha(\gamma, y) \quad \text{where } \gamma \text{ is defined uniquely by } \theta y = \gamma y.$$

It is straightforward to verify that (2.3) holds if we replace γ_1 and γ_2 with arbitrary elements from $[\Gamma]$. A cocycle α is a *coboundary* if there is a measurable map $a : Y \to G$ such that

$$\alpha(\gamma, y) = a(\gamma y) - a(y) \quad \text{at a.e. } y \in Y$$

for all $\gamma \in \Gamma$. Given a pair (Γ, α), we can construct a transformation group $\Gamma_\alpha := \{\gamma_\alpha \mid \gamma \in \Gamma\} \subset \mathrm{Aut}(Y \times G, \nu \times \lambda_G)$, where

$$\gamma_\alpha(y, g) := (\gamma y, \alpha(\gamma, y) + g) \quad \text{for all } y \in Y, g \in G.$$

The group Γ_α is called the *α-skew product extension* of Γ. If Γ preserves ν then Γ_α preserves the product measure $\nu \otimes \lambda_G$. If Γ_α is ergodic then α is called *ergodic*. A

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coboundary is never ergodic (unless G is a singleton). It is easy to verify that if $\Gamma = \{R^n \mid n \in \mathbb{Z}\}$ for a transformation $R \in \mathrm{Aut}(Y, \nu)$ then each measurable function $f : Y \to G$ uniquely defines a cocycle α_f of Γ via the condition

$$\alpha_f(R, y) := f(y) \quad \text{for each } y \in Y.$$

For brevity we will write R_f for the α_f-skew product extension R_{α_f} of R.

We now recall an important concept of essential value for a cocycle.

Definition 2.4. Suppose that Γ preserves ν. An element $g \in G$ is called an *essential value of* α if, for each subset $A \subset Y$ of positive measure and a neighborhood U of g, there are a Borel subset $B \subset A$ and an element $\gamma \in \Gamma$ such that $\nu(B) > 0$, $\gamma B \subset A$ and $\alpha(\gamma, y) \in U$ for all $y \in B$.

It appears that the set $r(\alpha)$ of all essential values of a cocycle is a closed subgroup of G. Our interest in the essential values of α is explained by the fact that α is ergodic if and only if $r(\alpha) = G$ [Sc]. It is often easier to check the aforementioned condition on essential values not for each subset $A \in \mathfrak{C}$ of positive measure but only for a dense subfamily of subsets in $\mathfrak{C}$. However, in this case we have to strengthen this condition. More precisely, we will use the following folklore lemma.

LEMMA C. *Let $(Y, \mathfrak{C}, \nu)$ be a standard probability space, $\mathfrak{A}$ a dense subset in $\mathfrak{C}$, Γ an ergodic countable subgroup of $\mathrm{Aut}_0(Y, \nu)$ and $\alpha : \Gamma \times Y \to G$ a Borel cocycle of Γ. If, for some $a \in G$ and each subset $B \in \mathfrak{A}$ and each neighborhood U of 0 in G, there are a measurable subset $D \subset B$ and an element $\theta \in [\Gamma]$ such that $\theta D \subset B$, $\nu(D) > 0.5\nu(B)$ and $\alpha(\theta, x) \in a + U$ for all $x \in D$ then a is an essential value of α.*

2.5. *Sharp weak mixing of skew products for cocycles of product type.* In this subsection we prove the following theorem.

THEOREM 2.5. *Let T_n be a mildly mixing transformation of a standard probability space $(X_n, \mathfrak{B}_n, \nu_n)$ for each $n \in \mathbb{N}$. Let*

$$(X, \mathfrak{B}, \nu, T) := \bigotimes_{n \in \mathbb{Z}} (X_n, \mathfrak{B}_n, \nu_n, T).$$

Suppose that, for a measurable function $f : X \to G$, there are functions $f_n : X_n \to G$ such that $f(x) = \sum_{n \in \mathbb{N}} (f_n(T_n x_n) - f_n(x_n))$ at ν-a.e. $x = (x_n)_{n=1}^\infty \in X$ and the sequence of measures $(\nu_n \circ f_n^{-1})_{n \in \mathbb{N}}$ is asymptotically translation invariant. If the skew product extension $T_f : X \times G \to X \times G$ of T is conservative then T_f is sharply weak mixing.

Proof. Let C be an ergodic conservative transformation of a standard probability space $(Z, \mathfrak{Z}, \kappa)$. Suppose that $T_f \times C$ is not totally dissipative. Then it follows from [Aa, Proposition 1.2.4] that $T_f \times C$ is conservative. All that remains is to show that $T_f \times C$ is ergodic.

Let a function $F \in L^\infty(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa)$ be invariant under $T_f \times C$. We first show that F is also invariant under a huge group of transformations. Fix $n > 0$. For each $x \in X$, we write $x_1^n := (x_1, \dots, x_n) \in X_1 \times \cdots \times X_n$ and $x_{n+1}^\infty := (x_{n+1}, x_{n+2}, \dots) \in$

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$X_{n+1} \times X_{n+2} \times \cdots$. Then $x = (x_1^n, x_{n+1}^\infty)$. We define a measure-preserving automorphism E_n of $(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa)$ and a non-singular automorphism V_n of $(\bigotimes_{k=n+1}^\infty (X_k, \nu_k)) \otimes (G, \lambda_G) \otimes (Z, \kappa)$ respectively by setting

$$E_n(x, g, z) := \left(x, g + \sum_{k=1}^{n} f_k(x_k), z\right) \quad \text{and}$$

$$V_n((x_k)_{k=n+1}^\infty, g, z) := \left((T_k x_k)_{k=n+1}^\infty, g + \sum_{k>n} (f_k(T_k x_k) - f_k(x_k)), Cz\right).$$

A straightforward verification shows that

$$E_n^{-1}(T_f \times C)E_n = (T_1 \times \cdots \times T_n) \times V_n.$$

Since V_n is a factor of the transformation $E_n^{-1}(T_f \times C)E_n$ and the latter transformation is conservative, it follows that V_n is conservative. On the other hand, the function $F \circ E_n$ is invariant under $E_n^{-1}(T_f \times C)E_n$. Utilizing these two facts, we deduce from Theorem B that $F \circ E_n$ does not depend on the coordinates $x_1, \ldots, x_n$. Hence, for each transformation $S \in \mathrm{Aut}_0(X_1 \times \cdots \times X_n, \bigotimes_{k=1}^n \nu_k)$, we have that $F \circ E_n \circ (S \times I) = F \circ E_n$. Therefore F is invariant under the transformation $E_n(S \times I)E_n^{-1} \in \mathrm{Aut}_0(X \times G \times Z, \mu \otimes \lambda_G \otimes \kappa)$ and

$$E_n(S \times I)E_n^{-1}(x, g, z) = (Sx_1^n, x_{n+1}^\infty, g - A_n(x_1^n) + A_n(Sx_1^n), z), \tag{2.4}$$

where A_n stands for the mapping

$$X_1 \times \cdots \times X_n \ni (x_1, \ldots, x_n) \mapsto A_n(x_1, \ldots, x_n) := \sum_{k=1}^{n} f_k(x_k).$$

Thus, we have shown that F is invariant under each transformation from the set

$$\mathcal{G} := \bigcup_{n>0} E_n\left(\mathrm{Aut}_0\left(X_1 \times \cdots \times X_n, \bigotimes_{k=1}^{n} \nu_k\right) \times \{I\}\right)E_n^{-1}.$$

We now consider a new dynamical system. The space of this system is the product $(X, \mathfrak{B}, \mu)$. Denote by Γ the group of transformations of this space generated by mutually commuting measure-preserving transformations $\widehat{T}_1, \widehat{T}_2, \ldots$, where

$$\widehat{T}_n x = (x_1^{n-1}, T_n x_n, x_{n+1}^\infty), \quad n \in \mathbb{N}.$$

Then Γ is countable, abelian† and ergodic. For each $n > 0$, we consider a coboundary

$$\alpha_n : X \ni x \mapsto \alpha_n(x) := f_n(T_n x_n) - f_n(x_n) \in G$$

of $\widehat{T}_n$. It is straightforward to verify‡ that the α_n-skew product extensions $(\widehat{T}_n)_{\alpha_n}$ of $\widehat{T}_n$, $n \in \mathbb{N}$, mutually commute. It follows that a cocycle $\alpha : \Gamma \times X \to G$ of Γ with values in G is well defined by the formula

$$\alpha(\widehat{T}_n, x) := \alpha_n(x), \quad n \in \mathbb{N}.$$

† It is isomorphic to $\bigoplus_{n=1}^\infty \mathbb{Z}$.

‡ This follows from the fact that each function α_n depends only on a single coordinate x_n, $n = 1, 2, \ldots$.

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Since $\alpha_n(x) = A_n((I \times T_n)x_1^n) - A_n(x_1^n)$, it follows from (2.4) that

$$E_n(I \times T_n \times I)E_n^{-1} = (\widehat{T}_n)_{\alpha_n} \times I_Z.$$

Hence $(\widehat{T}_n)_{\alpha_n} \times I_Z \in \mathcal{G}$. Although, each α_n is a coboundary for the $\mathbb{Z}$-action given by T_n, the cocycle α is not a coboundary for Γ. In fact, we will now show the following claim.

CLAIM I. *The cocycle α of Γ is ergodic.*

Proof. For this purpose we show that each element $a \in G$ is an essential value of α. Given $n > 0$ and a subset $B \subset X_1 \times \cdots \times X_n$, denote by $[B]_1^n \subset X$ the corresponding cylinder with 'head' B, that is, $[B]_1^n := \{x \in X \mid x_1^n \in B\}$. Let U be a symmetric neighborhood of 0 in G. Choose a countable partition $\mathcal{P}$ of G into Borel subsets Δ such that $g - h \in U$ for all $g, h \in \Delta$ and each $\Delta \in \mathcal{P}$. Let $\psi_k := \nu_k \circ f_k^{-1}$ for each $k > 0$. Using the ATI assumption, we can find $m > n$ such that

$$\|\psi_{n+1} * \cdots * \psi_m * \delta_a - \psi_{n+1} * \cdots * \psi_m\|_1 < \epsilon. \tag{2.5}$$

For each $\Delta \in \mathcal{P}$, we let

$$\begin{aligned} A_\Delta &:= \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \;\middle|\; \sum_{k=n+1}^m f_k(y_k) \in \Delta \right\} \quad \text{and} \\ B_\Delta &:= \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \;\middle|\; a + \sum_{k=n+1}^m f_k(y_k) \in \Delta \right\}. \end{aligned} \tag{2.6}$$

Then $\{A_\Delta\}_{\Delta\in\mathcal{P}}$ and $\{B_\Delta\}_{\Delta\in\mathcal{P}}$ are two measurable partitions of $X_{n+1} \times \cdots \times X_m$. It follows from (2.5) that

$$\begin{aligned} \sum_{\Delta\in\mathcal{P}} |\nu_{n+1}^m(A_\Delta) - \nu_{n+1}^m(B_\Delta)| &= \sum_{\Delta\in\mathcal{P}} |\psi_{n+1} * \cdots * \psi_m * \delta_a(\Delta) - \psi_{n+1} * \cdots * \psi_m(\Delta)| \\ &\le \|\psi_{n+1} * \cdots \psi_m * \delta_a - \psi_{n+1} * \cdots * \psi_m\|_1 \\ &< \epsilon, \end{aligned}$$

where ν_{n+1}^m denotes the direct product $\bigotimes_{k=n+1}^m \nu_k$. We can find subsets $A'_\Delta \subset A_\Delta$ and $B'_\Delta \subset B_\Delta$ such that

$$\nu_{n+1}^m(A'_\Delta) = \nu_{n+1}^m(B'_\Delta) = \min(\nu_{n+1}^m(A_\Delta), \nu_{n+1}^m(B_\Delta)). \tag{2.7}$$

Note that the group $\Gamma_{n+1,m}$ generated by $m - n$ mutually commuting transformations $T_{n+1} \times I \times \cdots \times I,\quad I \times T_{n+2} \times I \times \cdots \times I, \ldots, I \times \cdots \times I \times T_m \in \mathrm{Aut}_0(X_{n+1} \times \cdots \times X_m, \nu_{n+1}^m)$ is ergodic. Hence, in view of (2.7), Hopf's lemma [**HaOs**, Lemma 10] yields that there is a transformation $S_0 \in [\Gamma_{n+1,m}]$ such that $S_0 A'_\Delta = B'_\Delta$ for each $\Delta \in \mathcal{P}$. (We recall that Hopf's equivalence lemma claims that, given an ergodic conservative measure-preserving countable transformation group Σ of a standard σ-finite measure space $(Y, \mathfrak{Y}, \omega)$ and two subsets $A, B \in \mathfrak{Y}$ with $\omega(A) = \omega(B)$, there is a transformation $\gamma \in [\Sigma]$ such that $\gamma A = B \bmod \omega$. The lemma is proved via the standard exhaustion argument.) We note that

$$\sum_{\Delta\in\mathcal{P}} \nu_{n+1}^m(A_\Delta \setminus A'_\Delta) \le \sum_{\Delta\in\mathcal{P}} |\nu_{n+1}^m(A_\Delta) - \nu_{n+1}^m(B_\Delta)| < \epsilon.$$

It follows that $\nu_{n+1}^m(\bigsqcup_{\Delta\in\mathcal{P}} A'_\Delta) > 1-\epsilon$. On the other hand, in view of (2.6), for each $y \in A^+ := \bigsqcup_{\Delta\in\mathcal{P}} A'_\Delta$,

$$\bigg(\sum_{k=n+1}^{m} f_k\bigg)(y) - \bigg(\sum_{k=n+1}^{m} f_k\bigg)(S_0 y) \in a + U.$$

We now 'extend' S_0 to a transformation $S \in \mathrm{Aut}_0(X,\mu)$ by setting

$$Sx := (x_1^n, S_0 x_{n+1}^m, x_{m+1}^\infty) \in X \quad \text{for all } x \in X.$$

Then $S \in [\Gamma]$ and

$$\alpha(S,x) \in a + U \quad \text{whenever } x_{n+1}^m \in A^+. \tag{2.8}$$

Then we have that $[B\times A^+]_1^m \subset [B]_1^n$, $S[B\times A^+]_1^m \subset [B]_1^n$, $\mu([B\times A^+]_1^m) > \frac{1}{2}\mu([B]_1^n)$ and (2.8) holds for all $x \in [B\times A^+]_1^m$. Since the set of all cylinders is dense in $\mathfrak{B}$, it follows from Lemma C that a is an essential value of α. Thus, Claim I is proved. □

To complete the proof of the theorem, we have already noticed that $(\widehat{T_n})_{\alpha_n} \times I_Z \in \mathcal{G}$ for each $n \in \mathbb{N}$. Hence $F(\gamma_\alpha(x,g),z) = F(x,g,z)$ at a.e. $(x,g,z) \in X\times G\times Z$ for each $\gamma \in \Gamma$. Claim I yields that there is a function $M : Z \to \mathbb{R}$ such that $F(x,g,z) = M(z)$ at a.e. $(x,g,z) \in X\times G\times Z$. Since F is invariant under $T_f \times C$, we obtain that M is invariant under C. Since C is ergodic, M is constant almost everywhere and hence F is constant almost everywhere, that is, $T_f \times C$ is ergodic. □

We call the cocycle f in the statement of Theorem 2.5 *a cocycle of product type.*

2.6. *Application to Gaussian cocycles.* Let $(X,\mathfrak{B},\mu,T)$ be an ergodic Gaussian dynamical system. It is completely determined by a restriction of the corresponding Koopman unitary operator U_T to a closed (real) Gaussian subspace $H \subset L_0^2(X,\mu)$, called *the first chaos* (see, for example, [**LePaTh**] for the definitions). Let κ denote the maximal spectral type of $U_T \restriction H$. It is known that T is ergodic if and only if T is weakly mixing if and only if κ is non-atomic [**Mar**]. Take $f \in H$. Then the measurable map $f : X \to \mathbb{R}$ considered as a cocycle of T is called a *Gaussian* cocycle. It was shown in [**LeLeSk**] that if f is a T-coboundary, that is, $f = h\circ T - h$ for a measurable function $h : X \to \mathbb{R}$, then $h \in H$. We now recall a conjecture from [**LeLeSk**].

Conjecture I. If a Gaussian cocycle f is not a coboundary then f is ergodic.

We now prove this conjecture (in fact, we prove a stronger result) under an additional assumption that T is mildly mixing.

THEOREM 2.6. *If T is a mildly mixing Gaussian transformation and f is a Gaussian cocycle of T which is not a coboundary then T_f is sharply weak mixing.*

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Proof. Since $f \in H$, it follows that $\int_X f\, d\mu = 0$. Hence, by Atkinson's theorem [**At**], T_f is conservative. Consider now the spectral decomposition for the pair (H, U_T):

$$H = \int_{\mathbb{T}}^{\oplus} \mathcal{H}_z\, d\kappa(z) \quad \text{and} \quad U_T = \int_{\mathbb{T}}^{\oplus} zI_z\, d\kappa(z), \tag{2.9}$$

where $\mathbb{T} \ni z \mapsto \mathcal{H}_z$ is the corresponding measurable field of Hilbert spaces and I_z is the identity operator in $\mathcal{H}_z$. In other words, we can consider an element h of H as a measurable map $\mathbb{T} \ni z \mapsto h(z) \in \mathcal{H}_z$ such that $\|h\|^2 = \int_{\mathbb{T}} \|h(z)\|^2 d\kappa(z) < \infty$. We now let $\Delta_n := \{z \in \mathbb{T} \mid (1/(n+1) < |z-1| \le (1/n)\}$. Then we obtain a countable partition $\bigsqcup_{n=1}^{\infty} \Delta_n$ of $\mathbb{T} \setminus \{1\}$. Since $\kappa(\{1\}) = 0$, this countable partition generates a decomposition of H into a direct sum $\bigoplus_{n\in\mathbb{N}} H_n$ of closed U_T-invariant subspaces H_n consisting of the measurable maps $h : \mathbb{T} \ni z \mapsto h(z) \in \mathcal{H}_z$ such that $\|h\| < \infty$ and $h(z) = 0$ whenever $z \notin \Delta_n$. This decomposition induces a decomposition of (X, μ, T) into the infinite direct product $(X, \mu, T) = \bigotimes_{n=1}^{\infty} (X_n, \mu_n, T_n)$, where (X_n, μ_n, T_n) is the Gaussian dynamical system associated with the pair $(H_n, U_T \restriction H_n)$ for each $n \in \mathbb{N}$. Now we can expand f into an orthogonal sum $f = \bigoplus_{n=1}^{\infty} f_n$ with $f_n \in H_n$ for each $n \in \mathbb{N}$. We claim that, for each $n > 0$, there is $a_n \in H_n$ such that $f_n = U_T a_n - a_n$. Indeed, it follows from this equation and (2.9) that if we represent f_n as a measurable map $\Delta_n \ni z \mapsto f_n(z) \in \mathcal{H}_z$ then $f_n(z) = za_n(z) - a_n(z)$ for a.e. $z \in \Delta_n$. Solving this equation, we obtain that $a_n(z) = (z-1)^{-1} f_n(z)$ for a.e. $z \in \Delta_n$. Since $|z-1|^{-1} < n+1$ for all $z \in \Delta_n$, we obtain that $a_n \in H_n$. This yields an expansion

$$f = \bigoplus_{n=1}^{\infty} (U_T a_n - a_n) = \bigoplus_{n=1}^{\infty} (a_n \circ T_n^{-1} - a_n) \tag{2.10}$$

of f into an infinite sum of T_n-coboundaries. Of course, $\sum_{n\in\mathbb{N}} \|a_n\|^2 = +\infty$. Otherwise, the series $\sum_{n\in\mathbb{N}} a_n$ converges in H and hence f would be a coboundary, which contradicts the assumption of the theorem. We have that $\mu_n \circ a_n^{-1} = \mathcal{N}_{0,\|a_n\|^2}$ for each $n \in \mathbb{N}$. Passing, if necessary, to a subsequence, we may assume without loss of generality that the convergence in (2.10) is almost everywhere. Example 2.3 yields that the sequence $(\mu_n \circ a_n^{-1})_{n=1}^{\infty}$ is asymptotically translation invariant. It now follows from Theorem 2.5 that T_f is sharply weak mixing. □

Consider now the general case. Then there is a maximal (with respect to κ) subset A of $\mathbb{T}$ such that U_T restricted to the closed subspace $\int_A^{\oplus} \mathcal{H}_z d\kappa(z)$ of H is mildly mixing. We note that A is symmetric. Then κ decomposes into a sum of two orthogonal measures: $\kappa_{mm} := \kappa \restriction A$ (the mildly mixing part of κ) and $\kappa_r := \kappa \restriction (\mathbb{T} \setminus A)$ (the rigid part of κ). This decomposition defines a decomposition of (X, μ, T) into a direct product $(X_1, \mu_{mm}, M) \times (X_2, \mu_r, R)$, where (X_1, μ_{mm}, M) is the Gaussian dynamical system corresponding to the pair $(\int_A^{\oplus} \mathcal{H}_z d\kappa_{mm}(z), U_T)$ and (X_2, μ_r, R) is the Gaussian dynamical system corresponding to the pair $(\int_{\mathbb{T}\setminus A}^{\oplus} \mathcal{H}_z d\kappa_r(z), U_T)$. Also, we obtain a decomposition of f into a sum $f_{mm} + f_r$, where $f_{mm} := f 1_A$ and $f_r = f 1_{\mathbb{T}\setminus A}$. There are two possible cases: either f_{mm} is a coboundary or f_{mm} is not a coboundary. In the first case T_f is isomorphic to $Q \times R_{f_r}$. Moreover, f_r is not a coboundary because otherwise f would be a coboundary. Since Q is mildly mixing and T_f is conservative, T_f is ergodic if

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and only if R_{f_r} is ergodic. In the second case, T_f is isomorphic to $Q_{f_{mm}} \times R_{f_r}$ and $Q_{f_{mm}}$ is sharply weak mixing by Theorem 2.5. Since T_f is conservative, it follows that T_f is ergodic if and only if R_{f_r} is ergodic. Thus, we have reduced the conjecture from [**LeLeSk**] as follows.

Conjecture II. If a Gaussian cocycle f is not a coboundary and κ has only rigid part then f is ergodic.

In [**LeLeSk**], there were constructed some concrete rigid Gaussian transformations admitting ergodic Gaussian cocycles. In [**MaVa**] this result was extended to arbitrary rigid Gaussian transformations which have at least one Gaussian non-coboundary. However, it is unknown whether ergodicity holds for each Gaussian non-coboundary in those examples.

3. *Krieger's type of infinite direct products of dynamical systems of finite type*

3.1. *IDPFT systems.* Let T_n be a non-singular invertible transformation of a standard probability space $(X_n, \mathfrak{B}_n, \mu_n)$ for each $n \in \mathbb{N}$. Denote by T the infinite direct product of T_n, $n \in \mathbb{N}$, acting on the infinite product space $(X, \mathfrak{B}, \mu) := \bigotimes_{n\in\mathbb{Z}}(X_n, \mathfrak{B}_n, \mu_n)$. By Theorem A, T is μ-non-singular if and only if

$$\prod_{n=1}^{\infty}(1 - H^2(\mu_n \circ T_n^{-1}, \mu_n)) > 0 \quad \text{or} \quad \sum_{n=1}^{\infty} H^2(\mu_n \circ T_n^{-1}, \mu_n) < \infty. \tag{3.1}$$

If (3.1) does not hold then $\mu \circ T^{-1} \perp \mu$. If T is μ-non-singular then

$$\frac{d\mu \circ T^{-1}}{d\mu}(x) = \prod_{n=1}^{\infty} \frac{d\mu_n \circ T_n^{-1}}{d\mu_n}(x_n) \quad \text{at a.e. } x \in X.$$

Suppose now that T_n is *of finite type*, that is, that there exists a μ_n-equivalent probability measure ν_n which is invariant under T_n for each $n \in \mathbb{N}$. We then put $\phi_n := d\mu_n/d\nu_n$. Since $1 - H^2(\mu_n \circ T_n^{-1}, \mu_n) = \int_{X_n} \sqrt{((\phi_n \circ T_n^{-1})/\phi_n)}\phi_n d\nu_n$, formula (3.1) and Theorem A yield the following corollary.

COROLLARY 3.1. *T is μ-non-singular if and only if*

$$\prod_{n=1}^{\infty} \int_{X_n} \sqrt{\phi_n \cdot \phi_n \circ T_n^{-1}} d\nu_n > 0. \tag{3.2}$$

$\mu \perp \nu$ if and only if

$$\prod_{n=1}^{\infty} \int_{X_n} \sqrt{\phi}_n d\nu_n = 0. \tag{3.3}$$

Definition 3.2. If T is μ-non-singular and T_n is of finite type for all $n > 0$ then we say that the dynamical system $(X, \mathfrak{B}, \mu, T)$ is *an infinite direct product of finite types*.

Our purpose in this section is to investigate dynamical properties of IDPFT systems. The first result is about ergodicity of conservative IDPFT systems under the mild mixing assumption on the factors.

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PROPOSITION 3.3. *Let (X_n, ν_n, T_n) be mildly mixing for each $n > 0$ and (3.2) and (3.3) hold. Suppose that T is μ-conservative. Then T is μ-sharply weak mixing and $\mu \perp \nu$.*

Proof. Let S be an ergodic conservative transformation of a standard probability space $(Y, \mathfrak{C}, \nu)$. By [**Aa**, Proposition 1.2.4], $T \times S$ is either totally dissipative or conservative. Suppose that $T \times S$ is conservative. We have to prove that it is ergodic. Let a subset $A \in \mathfrak{B} \otimes \mathfrak{C}$ be invariant under $T \times S$. It follows from Theorem B that, for each $n > 0$, A belongs to the σ-algebra $\{\emptyset, X_1 \times \cdots \times X_n\} \otimes \mathfrak{B}_{n+1} \otimes \mathfrak{B}_{n+2} \otimes \cdots \otimes \mathfrak{C}$ (When applying Theorem B, we consider the measure $(\bigotimes_{k=1}^n \nu_k) \otimes (\bigotimes_{k>n} \mu_k) \otimes \nu$ on $X \times Y$. This measure is equivalent to $\mu \otimes \nu$.). By the Kolmogorov 0–1 law, the intersection of these σ-algebras is $\mathfrak{N} \otimes \mathfrak{C}$, where $\mathfrak{N}$ is the trivial σ-algebra on X. Thus $A = X \times D$ for some subset $D \in \mathfrak{C}$. Since A in invariant under $T \times S$, it follows that D is invariant under S. Since S is ergodic, we obtain that either $\mu \otimes \nu(A) = 0$ or $\mu \otimes \nu(A) = 1$. □

Remark 3.4. In §4 we give examples of (X, ν, T) and μ such that (X, μ, T) is of type III_1. In particular, there is no μ-equivalent invariant probability measure. On the other hand, we do not know of examples where (X, μ, T) is of type II_1, that is, T is mildly mixing with respect to a μ-equivalent invariant probability measure.

3.2. *Radon–Nikodym cocycle and type III_1.* Let Γ be an ergodic countable subgroup of $\mathrm{Aut}(Y, \nu)$. Denote by $\rho_\nu : \Gamma \times Y \to \mathbb{R}$ the logarithm of the *Radon–Nikodym* cocycle of Γ, that is,

$$\rho_\nu(\gamma, y) := \log \frac{d\nu \circ \gamma}{d\nu}(y).$$

The ρ_ν-skew product extension Γ_{ρ_ν} of Γ is called the *Maharam extension* of R. We note that Γ_{ρ_ν} preserves an equivalent σ-finite measure $\nu \otimes \kappa$, where κ is a Lebesgue absolutely continuous σ-finite measure on $\mathbb{R}$ such that $d\kappa(t) = e^{-t} dt$ for all $t \in \mathbb{R}$. Similar to the finite measure-preserving case, ρ_ν 'extends' to the full group $[\Gamma]$ in such a way that the cocycle identity holds. Moreover, we do not need the freeness condition for Γ to define this extension.

We note that ρ_ν is a coboundary if and only if there is a Γ-invariant ν-equivalent σ-finite measure on $(Y, \mathfrak{C})$.

By the Maharam theorem (see [**Sc**]), Γ_{ρ_ν} is conservative if and only if Γ is conservative. However, if Γ is ergodic then Γ_{ρ_ν} is not necessarily ergodic. If the Maharam extension of Γ is ergodic then Γ is called *of Krieger's type III_1*. If, for each homomorphism $\vartheta : \Gamma \to \mathrm{Aut}_0(Y, \nu)$ such that the image $\{\vartheta(\gamma) \mid \gamma \in \Gamma\}$ is ergodic, the direct product $\{\gamma \times \vartheta(\gamma) \mid \gamma \in \Gamma\}$ is ergodic and of type III_1 then Γ is said *to be of stable Krieger's type III_1*.

It is possible to define essential values of ρ_ν in the same way as in the finite measure-preserving case.

Definition 3.5. An element $g \in \mathbb{R}$ is called an *essential value of* ρ_ν if, for each subset $A \subset Y$ of positive measure and a neighborhood U of g, there are a Borel subset $B \subset A$ and an element $\gamma \in \Gamma$ such that $\nu(B) > 0$, $\gamma B \subset A$ and $\rho_\nu(\gamma, y) \in U$ for all $y \in B$.

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We refer to [**Sc**, **HaOs**] for the proof of the following results:

- The set $r(\rho_\nu)$ is a closed subgroup in $\mathbb{R}$.
- Γ is of type III_1 if and only if $r(\rho_\nu) = \mathbb{R}$.

If there is $\lambda \in (0, 1)$ such that $r(\rho_\nu) = \{n \log \lambda \mid n \in \mathbb{Z}\}$ then Γ is said to be *of Krieger's type III*$_\lambda$.

We will need the following analog of Lemma C. We do not provide a proof of the lemma because it is routine.

LEMMA D. *An element $a \in \mathbb{R}$ is an essential value of ρ_ν if there exists $\delta > 0$ such that for each $\epsilon > 0$ and each subset B from a dense collection $\mathfrak{C}_0$ of subsets in $\mathfrak{C}$, there are a subset $B_0 \subset B$ and a transformation $\theta \in [\Gamma]$ such that $\nu(B_0) > \delta\nu(B)$, $\theta B_0 \subset B$ and either $|\rho_\nu(\theta, y) - a| \le \epsilon$ for all $y \in B_0$ or $|\rho_\nu(\theta, y) + a| \le \epsilon$ for all $y \in B_0$.*

3.3. *On conservativeness of IDPFT systems.* In this subsection we first establish a general result on conservativeness of infinite direct product systems. We note that the argument used below to prove conservativeness of IDPFT systems is similar to the argument used in [**Ko**, **Da**] to prove ergodicity of non-singular Bernoulli and Markov systems.

PROPOSITION 3.6. *Let $(X_n, \mathfrak{B}_n, \mu_n, T_n)$ be an ergodic non-singular dynamical system on a standard probability space for each $n \in \mathbb{N}$ and let (3.1) hold. Let $(X, \mathfrak{B}, \mu, T) := \bigotimes_{n=1}^\infty (X_n, \mathfrak{B}_n, \mu_n, T_n)$. If, for each $n \in \mathbb{N}$, there is a function $\alpha_n : X_n \to [1, +\infty)$ such that, for each $k \in \mathbb{N}$,*

$$\alpha_n(x)^{-1} \le \frac{d\mu_n \circ T^k}{d\mu_n}(x) \le \alpha_n(x) \quad \text{at a.e. } \mu_n\text{-a.e. } x \in X_n$$

then the dynamical system $(X, \mathfrak{B}, \mu, T)$ is either conservative or totally dissipative. Moreover, if $(Y, \mathfrak{C}, \nu, S)$ is an ergodic conservative non-singular dynamical system then the direct product $T \times S$ is either conservative or totally dissipative.

Proof. We will prove the second claim only. By the Hopf criterion [**Aa**, Proposition 1.3.1],

$$\mathcal{D}(T \times S) = \left\{(x, y) \in X \times Y \;\middle|\; \sum_{k=1}^{\infty} \frac{d(\mu \otimes \nu) \circ (T \times S)^k}{d(\mu \otimes \nu)}(x, y) < \infty\right\}.$$

For each $r > 0$, we consider a transformation γ_r of X by setting $\gamma_r(x_1, x_2, \ldots) := (x_1, \ldots, x_{r-1}, T_r x_r, x_{r+1}, \ldots)$. Of course, $\gamma_r \in \mathrm{Aut}(X, \mu)$. Denote by Γ the transformation group generated by γ_r, $r \in \mathbb{N}$. It follows from the Kolmogorov 0–1 law that Γ is ergodic. We claim that $\mathcal{D}(T \times S)$ is invariant under $\gamma_r \times I$ for each r. Let $(x, y) \in \mathcal{D}(T \times S)$. Since, for each $k > 0$,

$$\begin{aligned}\frac{d\mu \circ T^k}{d\mu}(\gamma_r x) &= \frac{d\mu_r \circ T_r^k}{d\mu_r}(T_r x_r)\left(\frac{d\mu_r \circ T_r^k}{d\mu_r}(x_r)\right)^{-1} \prod_{n=1}^{\infty} \frac{d\mu_n \circ T_n^k}{d\mu_n}(x_n) \\ &\le \alpha_r(T_r x_r)\alpha_r(x_r)^{-1} \frac{d\mu \circ T^k}{d\mu}(x),\end{aligned}$$

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it follows that

$$\sum_{k=1}^{\infty}\frac{d(\mu\otimes\nu)\circ(T\times S)^k}{d(\mu\otimes\nu)}(\gamma_r x,y)=\sum_{k=1}^{\infty}\frac{d\mu\circ T^k}{d\mu}(\gamma_r x)\frac{d\nu\circ S^k}{d\nu}(y)$$
$$\leq\frac{\alpha_r(T_r x_r)}{\alpha_r(x_r)}\sum_{k=1}^{\infty}\frac{d\mu\circ T^k}{d\mu}(x)\frac{d\nu\circ S^k}{d\nu}(y)$$
$$=\frac{\alpha_r(T_r x_r)}{\alpha_r(x_r)}\sum_{k=1}^{\infty}\frac{d(\mu\otimes\nu)\circ(T\times S)^k}{d(\mu\otimes\nu)}(x,y)<\infty.$$

Thus, $(\gamma_r x, y)\in\mathcal{D}(T\times S)$. Since $\mathcal{D}(T\times S)$ is invariant under $I\times S$, we obtain that $\mathcal{D}(T\times S)$ is invariant under an ergodic transformation group on $X\times Y$ generated by $I\times S$ and $\gamma\times I$, $\gamma\in\Gamma$. Hence, either $(\mu\otimes\nu)(\mathcal{D}(T\times S))=0$ or $(\mu\otimes\nu)(\mathcal{D}(T\times S))=1$, as desired. □

We now apply Proposition 3.6 to IDPFT systems.

COROLLARY 3.7. *Let $(X_n,\mathfrak{B}_n,\mu_n,T_n)$ be an ergodic non-singular dynamical system on a standard probability space for each $n\in\mathbb{N}$ and let (3.1) hold. Suppose that, for each $n\in\mathbb{N}$, there is a μ_n-equivalent T_n-invariant probability measure on X_n. Let $(X,\mathfrak{B},\mu,T):=\bigotimes_{n=1}^{\infty}(X_n,\mathfrak{B}_n,\mu_n,T_n)$. Then the dynamical system $(X,\mathfrak{B},\mu,T)$ is either conservative or totally dissipative. Moreover, if $(Y,\mathfrak{C},\nu,S)$ is an ergodic conservative non-singular dynamical system then the direct product $T\times S$ is either conservative or totally dissipative.*

Proof. Let $\phi_n:=d\mu_n/d\nu_n$ for each $n>0$. If, for each $n>0$, there is a real $\alpha_n\geq 1$ such that $\alpha_n^{-1}\leq\phi_n\leq\alpha_n$ almost everywhere then the claim of the corollary follows directly from Proposition 3.6. We now show that the general case can be reduced to the 'bounded' case. Indeed, for each $n>0$, we can find a probability measure $\widetilde{\mu}_n\sim\mu_n$ such that $H^2(\widetilde{\mu}_n,\mu_n)\leq 2^{-n}$ and the Radon–Nikodym derivative $d\widetilde{\mu}_n/d\nu_n$ is bounded from above and separated from 0 from below (For that purpose, take the Radon–Nikodym derivative $d\mu_n/d\nu_n$ and change it on a subset of very small measure to get the boundedness. The 'modified' function will be the Radon–Nikodym derivative $d\widetilde{\mu}_n/d\mu_n$.). Since $\sum_{n=1}^{\infty}H^2(\widetilde{\mu}_n,\mu_n)<\infty$, it follows from Theorem A that $\mu\sim\widetilde{\mu}:=\bigotimes_{n=1}^{\infty}\widetilde{\mu}_n$. All that remains is to note that the conservativeness of a dynamical system does not depend on the choice of quasi-invariant measure within its equivalence class. □

3.4. *Sharp weak mixing for Maharam extensions of IDPFT systems.* We first introduce a 'non-singular analogue' of the ATI property (cf. Definition 2.2).

Definition 3.8. A sequence $(\xi_n)_{n=1}^{\infty}$ of probability non-atomic Borel measures on G is called asymptotically translation quasi-invariant if, for each $a\in G$, there exists $\zeta_a>0$ such that, for every $n\in\mathbb{N}$, there are $m>n$ and a Borel subset $W_{n,m}\subset G$

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such that

$$\zeta_a \le (\xi_n * \xi_{n+1} * \cdots * \xi_{n+m})(W_{n,m}),$$
$$\xi_n * \xi_{n+1} * \cdots * \xi_{n+m} * \delta_a \prec \xi_n * \xi_{n+1} * \cdots * \xi_{n+m} \quad \text{and}$$
$$\zeta_a \le \frac{d(\xi_n * \xi_{n+1} * \cdots * \xi_{n+m} * \delta_a)}{d(\xi_n * \xi_{n+1} * \cdots * \xi_{n+m})}(t) \quad \text{for each } t \in W_{n,m}.$$

We will need the following lemma on continuous measures.

LEMMA 3.9. *Given a standard non-atomic probability space $(Y, \mathfrak{C}, \nu)$, a non-negative function $\phi \in L^1(Y, \nu)$ and $\delta \in (0, 1)$, then*

$$\max\left\{ \int_A \phi \, d\nu \;\middle|\; \nu(A) = \delta \right\} \ge \frac{\delta}{2} \int_Y \phi \, d\nu.$$

Proof. Let $\alpha := \max\{\int_A \phi d\nu \mid \nu(A) = \delta\}$. Find $n \ge 1$ such that $\delta \le (1/n)$. Then there is a partition $Y = Y_1 \sqcup \cdots \sqcup Y_{n+1}$ of Y into subsets Y_k such that $\nu(Y_k) = \delta$ for each $k = 1, \ldots, n$ and $\mu(Y_{n+1}) \le \delta$. We now have

$$\int_Y \phi \, d\nu = (n+1)\alpha \le (\delta^{-1} + 1)\alpha < \frac{2}{\delta}\alpha. \qquad \square$$

The next theorem is a non-singular analogue of Theorem 2.5. The skeleton of the proof is similar to that of Theorem 2.5.

THEOREM 3.10. *Let a dynamical system $(X_n, \mathfrak{B}_n, \nu_n, T_n)$ be mildly mixing for each $n > 0$. Let μ_n be a probability on X_n such that $\mu_n \sim \nu_n$ for each $n \in \mathbb{N}$. Let $\phi_n := d\mu_n/d\nu_n$ and (3.2) hold. We set*

$$(X, \mathfrak{B}, \nu, T) := \bigotimes_{n \in \mathbb{N}} (X_n, \mathfrak{B}_n, \nu_n, T_n)$$

and $\mu := \bigotimes_{n=1}^{\infty} \mu_n$. If T is μ-conservative and the sequence of probability measures $(\nu_n \circ (\log \phi_n)^{-1})_{n=1}^{\infty}$ is asymptotically translation quasi-invariant then $T \in Aut(X, \mu)$ is ergodic of stable type III_1. Moreover, the Maharam extension of T is sharply weak mixing.

Proof. By the Maharam theorem, the Maharam extension T_{ρ_μ} is conservative. Let C be an ergodic conservative transformation of a standard probability space $(Z, \mathfrak{Z}, \eta)$. As in the proof of Theorem 2.5, we see that $T_{\rho_\mu} \times C$ is either totally dissipative or conservative. Suppose that it is conservative and prove that it is ergodic.

Let a function $F \in L^\infty(X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta)$ be invariant under $T_{\rho_\mu} \times C$. We first show that F is also invariant under a huge group of transformations. Fix $n > 0$. We define a non-singular automorphism $T^{(n)}$ of $(\bigotimes_{k>n} X_k, \mu^{(n)})$, where $\mu^{(n)} := \bigotimes_{k=n+1}^{\infty} \mu_k$, and a measure-preserving isomorphism E_n of $(X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta)$ onto the product space $(X \times \mathbb{R} \times Z, (\bigotimes_{k=1}^{n} \nu_k) \otimes \mu^{(n)} \otimes \kappa \otimes \eta)$ by setting

$$T^{(n)}(x_k)_{k=n+1}^{\infty} := (T_k x_k)_{k=n+1}^{\infty} \quad \text{and}$$
$$E_n(x, t, z) := \left(x, t + \sum_{k=1}^{n} \log \phi_k(x_k), z \right).$$

Since

$$T_{\rho_\mu}(x,t) = \left(T_1x_1, T_2x_2, \ldots, t + \sum_{k=1}^{\infty} \log \frac{d\mu_k \circ T_k}{d\mu_k}(x_k)\right)$$
$$= \left(T_1x_1, T_2x_2, \ldots, t + \sum_{k=1}^{\infty} (\log \phi_k(T_kx_k) - \log \phi_k(x_k))\right),$$

it follows that

$$E_n^{-1}(T_{\rho_\mu} \times C)E_n = (T_1 \times \cdots \times T_n) \times (T^{(n)})_{\rho_{\mu(n)}} \times C. \tag{3.4}$$

Since $T_{\rho_\mu} \times C$ is conservative, it follows from (3.4) that the product $(T^{(n)})_{\rho_{\mu(n)}} \times C$ is also conservative. On the other hand, the function $F \circ E_n$ is invariant under $E_n^{-1}(T_f \times C)E_n$. Utilizing these two facts plus the mild mixing of the transformation $T_1 \times \cdots \times T_n$, we deduce from Theorem B that $F \circ E_n$ does not depend on the coordinates $x_1, \ldots, x_n$. Hence, for each transformation $S \in \mathrm{Aut}_0(X_1 \times \cdots \times X_n, \bigotimes_{k=1}^n \nu_k)$, we have that $F \circ E_n \circ (S \times I \times I_Z) = F \circ E_n$. Therefore F is invariant under the transformation $E_n(S \times I \times I_Z)E_n^{-1} \in \mathrm{Aut}_0(X \times \mathbb{R} \times Z, \mu \otimes \kappa \otimes \eta)$ and

$$E_n(S \times I \times I_Z)E_n^{-1} = (S \times I)_{\rho_\mu} \times I_Z. \tag{3.5}$$

Denote by Γ the group of non-singular transformations of $(X, \mathfrak{B}, \mu)$ generated by $I \times T_n \times I$, $n \in \mathbb{N}$. Then Γ is an ergodic abelian countable subgroup of $\mathrm{Aut}(X, \mu)$ and F is invariant under $\{\gamma_{\rho_\mu} \times I_Z \mid \gamma \in \Gamma\}$ by (3.5).

CLAIM II. *We claim that Γ is of type III_1.*

Proof. Equivalently, we will show that each $a \in \mathbb{R}$ is an essential value for the cocycle ρ_μ of Γ. For that purpose, fix $n > 0$, $\epsilon > 0$ and a Borel subset $B \subset X_1 \times \cdots \times X_n$. Denote by ψ_k the pushforward of ν_k under $\log \phi_k$ for each $k > 0$. By ATQI, there is $\zeta_a > 0$ (which does not depend on n), $m > n$ and a subset $W_{n+1,m} \subset \mathbb{R}$ such that

$$\begin{aligned} \zeta_a &\le (\psi_{n+1} * \cdots * \psi_m)(W_{n+1,m}), \\ \psi_{n+1} * \cdots * \psi_m * \delta_a &\prec \psi_{n+1} * \cdots * \psi_m \quad \text{and} \\ \zeta_a &\le \frac{d(\psi_{n+1} * \cdots * \psi_{n+m} * \delta_a)}{d(\psi_{n+1} * \cdots * \psi_{n+m})}(t) \quad \text{for each } t \in W_{n+1,m}. \end{aligned} \tag{3.6}$$

Choose a countable partition $\mathcal{P}$ of $W_{n+1,m}$ into subsets of diameter no more than ϵ. For each $\Delta \in \mathcal{P}$, we let

$$A_\Delta := \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \;\middle|\; \sum_{k=n+1}^m \log \phi_k(y_k) \in \Delta \right\} \quad \text{and}$$

$$B_\Delta := \left\{ y = (y_k)_{k=n+1}^m \in X_{n+1} \times \cdots \times X_m \;\middle|\; a + \sum_{k=n+1}^m \log \phi_k(y_k) \in \Delta \right\}.$$

Let $\mu_{n+1}^m := \bigotimes_{k=n+1}^m \mu_k$, $\nu_{n+1}^m := \bigotimes_{k=n+1}^m \nu_k$ and $\phi_{n+1}^m := d\mu_{n+1}^m / d\nu_{n+1}^m$. Dropping off some atoms of $\mathcal{P}$ if necessary, we may assume without loss of generality that

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$\nu_{n+1}^m(A_\Delta) > 0$ (and hence $\nu_{n+1}^m(B_\Delta) > 0$ in view of (3.6)) for each $\Delta \in \mathcal{P}$. Note that the group $\Gamma_{n+1,m}$ generated by $m-n$ mutually commuting transformations $T_{n+1} \times I \times \cdots \times I,\ I \times T_{n+2} \times I \times \cdots \times I, \ldots, I \times \cdots \times I \times T_m \in \mathrm{Aut}_0(X_{n+1} \times \cdots \times X_m, \nu_{n+1}^m)$ is ergodic. Suppose that $\nu_{n+1}^m(A_\Delta) > \nu_{n+1}^m(B_\Delta)$ for some $\Delta \in \mathcal{P}$. We now apply Lemma 3.9 to the space A_Δ equipped with the conditional measure $\nu_{n+1}^m(\cdot)/\nu_{n+1}^m(A_\Delta)$, the function $\nu_{n+1}^m(A_\Delta)\phi_{n+1}^m$ and $\delta := \nu_{n+1}^m(B_\Delta)/\nu_{n+1}^m(A_\Delta)$. Then there is a Borel subset $A'_\Delta \subset A_\Delta$ such that $\nu_{n+1}^m(A'_\Delta)/\nu_{n+1}^m(A_\Delta) = \delta$ and

$$\int_{A'_\Delta} \phi_{n+1}^m d\nu_{n+1}^m \geq \frac{\delta}{2} \int_{A_\Delta} \phi_{n+1}^m d\nu_{n+1}^m.$$

In other words, $\mu_{n+1}^m(A'_\Delta) \geq (\delta/2)\mu_{n+1}^m(A_\Delta)$. It follows from (3.6) that

$$\delta = \frac{\nu_{n+1}^m(B_\Delta)}{\nu_{n+1}^m(A_\Delta)} = \frac{(\psi_n * \psi_{n+1} * \cdots * \psi_{n+m} * \delta_a)(\Delta)}{(\psi_n * \psi_{n+1} * \cdots * \psi_{n+m})(\Delta)} \geq \zeta_a.$$

Therefore $\mu_{n+1}^m(A'_\Delta) \geq (\zeta_a/2)\mu_{n+1}^m(A'_\Delta)$. By Hopf's lemma, there is a transformation $S_0 \in [\Gamma_{n+1,m}]$ such that

- $S_0 A_\Delta \subset B_\Delta$ if $\nu_{n+1}^m(A_\Delta) \leq \nu_{n+1}^m(B_\Delta)$ and
- $S_0 A'_\Delta = B_\Delta$ if $\nu_{n+1}^m(A_\Delta) > \nu_{n+1}^m(B_\Delta)$, because in this case we have that $\nu_{n+1}^m(A'_\Delta) = \delta\nu_{n+1}^m(A_\Delta) = \nu_{n+1}^m(B_\Delta)$.

Let

$$A^+ := \bigsqcup_{\nu_{n+1}^m(A_\Delta) \leq \nu_{n+1}^m(B_\Delta)} A_\Delta \sqcup \bigsqcup_{\nu_{n+1}^m(A_\Delta) > \nu_{n+1}^m(B_\Delta)} A'_\Delta.$$

Then $\mu_{n+1}^m(A^+) \geq (\zeta_a/2)\mu_{n+1}^m(\bigsqcup_{\Delta\in\mathcal{P}} A_\Delta) = (\zeta_a/2)(\psi_{n+1} * \cdots * \psi_m)(W_{n+1,m}) \geq (\zeta_a^2/2)$. Of course, for each $y \in A^+$,

$$\left(\sum_{k=n+1}^m \log \phi_k\right)(y) - \left(\sum_{k=n+1}^m \log \phi_k\right)(S_0 y) = a \pm \epsilon. \tag{3.7}$$

We now 'extend' S_0 to a transformation $S \in \mathrm{Aut}(X, \mu)$ by setting

$$S := I \times S_0 \times I.$$

Then $S \in [\Gamma]$ and, in view of (3.7),

$$\rho_\mu(S, x) = -a \pm \epsilon \quad \text{whenever } x_{n+1}^m \in A^+. \tag{3.8}$$

We now have that

$$[B \times A^+]_1^m \subset [B]_1^n, \quad S[B \times A^+]_1^m \subset [B]_1^n, \quad \mu([B \times A^+]_1^m) \geq \frac{\zeta_a^2}{2}\mu([B]_1^n)$$

and (3.8) holds for all $x \in [B \times A^+]_1^m$. Since the set of all cylinders is dense in $\mathfrak{B}$, it follows from Lemma D that a is an essential value of α. Thus, Claim II is proved. □

The assertion of the theorem follows from Claim II in the very same way as the assertion of Theorem 2.5 follows from Claim I (in the proof of Theorem 2.5). □

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Remark 3.11. In this remark we clarify some subtle points in the proof of Theorem 3.10. Let $\mathfrak{B}_0$ stand for the collection of all cylinders in X. Then $\mathfrak{B}_0$ is dense in $\mathfrak{B}$ both with respect to μ and with respect to ν. Though $\mu \perp \nu$, the two measures are equivalent on $\mathfrak{B}_0$, that is, $\mu(B) = 0$ if and only if $\nu(B) = 0$ whenever $B \in \mathfrak{B}_0$. Given a transformation $\theta \in \mathrm{Aut}(X, \mu)$, the Radon–Nikodym derivative $(d\mu \circ \theta)/d\mu$ is defined up to a subset of zero μ-measure. Hence it makes no sense as a function on (X, ν). However, if we consider transformations of a specific product structure, say $\gamma \in \Gamma$, then $(d\mu \circ \gamma)/d\mu$ is defined, in fact, up a subset of zero μ-measure from $\mathfrak{B}_0$. Therefore, $(d\mu \circ \gamma)/d\mu$ is well defined as a measurable function on (X, ν) as well. Thus, the cocycle $\rho_\mu : \Gamma \times X \to \mathbb{R}$ is well defined simultaneously on (X, μ) and on (X, ν). Another observation is that given a transformation $S_0 \in [\Gamma_{n+1,m}]$, the extension $S := I \times S_0 \times I$ of S_0 to X is a well-defined transformation from $\mathrm{Aut}(X, \mu)$ as well as from $\mathrm{Aut}_0(X, \nu)$. Thus, although an element of the full group $[\Gamma]$ is defined up to subset of zero measure, the relation $S \in [\Gamma]$ is well defined with respect to μ as well as with respect to ν.

3.5. *On type III_λ for rigid IDPFT systems.* We would like to emphasize that the conclusion of Theorem 3.10 does not hold if we drop the mild mixing condition on T_n and the ATQI property. We illustrate this on a family of IDPFT systems consisting of infinite products of periodic transformations. Let $(p_n)_{n=1}^\infty$ be a sequence of mutually coprime positive integers such that $p_n > 2p_1 \cdots p_{n-1}$ for each $n \in \mathbb{N}$. Below we will specify more conditions on the growth of $(p_n)_{n=1}^\infty$. For $n \in \mathbb{N}$, we set $X_n := \{0, 1, \ldots, p_n - 1\}$ and identify X_n with the cyclic group $\mathbb{Z}/p_n\mathbb{Z}$. Then $T_n : X_n \to X_n$, given by $T_n x = x + 1 \pmod{p_n}$, is a bijection of X_n. The infinite product $T = \bigotimes_{n=1}^\infty T_n$ is a minimal rotation on the compact totally disconnected abelian group $X := \bigotimes_{n=1}^\infty X_n$. Of course, the Haar measure ν on X is the only T-invariant Borel probability measure on X. This measure is the infinite direct product of the equidistributions on X_n, $n \in \mathbb{N}$.

Fix $\lambda \in (0, 1)$. For each $n > 0$, let μ_n denote the unique probability measure on X_n such that:

- $\mu_n(j) = \mu_n(0)$ for each $j \leq p_n/2$;
- $\mu_n(j) = \mu_n(p_n - 1)$ for each $j > p_n/2$;
- $\mu_n(p_n - 1)/\mu_n(0) = \lambda$.

It is straightforward to verify that $H^2(\mu_n, \mu_n \circ T_n^{-1}) \to 0$ as $n \to \infty$. Hence, passing to a subsequence in $(p_n)_{n=1}^\infty$, we may assume without loss of generality that

(∘) $\sum_{n=1}^\infty H^2(\mu_n, \mu_n \circ T_n^{-1}) < \infty$.

We now let

$$Y_n := \left\{x_n \in X_n \mid x_n < \frac{p_n}{2} - \prod_{k=1}^{n-1} p_k\right\} \cup \left\{x_n \in X_n \mid \frac{p_n}{2} < x_n < p_n - \prod_{k=1}^{n-1} p_k\right\},$$
$$Z_n := \{x_n \in X_n \mid p_1 \cdots p_{n-1} < x_n < p_n/2\}.$$

Passing to a further subsequence in $(p_n)_{n=1}^\infty$, we will assume that the following two conditions are satisfied:

(•) $\mu_n(Y_n) > 1 - 2^{-n-1}$ for each $n > 0$;

(⋆) $\mu_n(Z_n) > (1/(2(\lambda + 1)))$ for each $n > 0$.

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Of course, ($\circ$) is satisfied for this subsequence as well. We now let $\mu = \bigotimes_{n=1}^{\infty} \mu_n$. By Theorem A, in view of ($\circ$), T is μ-non-singular (Although the topological system (X, T) is a topological odometer, that is, a minimal rotation on a monothetic compact totally disconnected abelian group, the non-singular system (X, μ, T) should not be confused with the non-singular product odometers which have been well studied in the literature (see, for example, [**DaSi**, **HaOs**, **Sc**]) because μ does not split into an infinite product when X is written in the product form suitable for the odometer 'addition with carry'.). It follows from the Kolmogorov 0–1 law that T is μ-ergodic.

PROPOSITION 3.12. *(X, μ, T) is of Krieger's type III_λ.*

Proof. Since $\log(d\mu \circ T^{-1}/d\mu)(x) \in \{n \log \lambda \mid n \in \mathbb{Z}\}$ at a.e. $x \in X$, it suffices to show that $\log \lambda$ is an essential value of the Radon–Nikodym cocycle ρ_μ of T. For each $n > 0$, denote by l_n the positive integer such that $l_n p_1 \cdots p_{n-1} \le p_n/2 < (l_n + 1) p_1 \cdots p_{n-1}$. For a Borel subset $B \subset X_1 \times \cdots \times X_n$, we set $A := B \times Z_{n+1} \times Y_{n+2} \times Y_{n+3} \times \cdots \subset X$. Then A is a Borel subset of the cylinder $[B]_1^n$. Of course, $T^{p_1 \cdots p_n}[B]_1^n = [B]_1^n$ and hence $T^{p_1 \cdots p_n l_{n+1}} A \subset [B]_1^n$. Since

- $T_m^{p_1 \cdots p_n} = I$ for each $m = 1, \ldots, n$,
- $(d\mu_{n+1} \circ T^{p_1 \cdots p_n l_{n+1}})/(d\mu_{n+1})(x_{n+1}) = \lambda$ if $x_{n+1} \in Z_{n+1}$ and
- $((d\mu_m \circ T^k)/d\mu_m)(x_m) = 1$ if $x_m \in Y_m$ and $0 \le k \le p_1 \cdots p_{m-1}$ and every $m > n + 1$,

it follows that, for each $x = (x_m)_{m=1}^{\infty} \in A$,

$$\frac{d\mu \circ T^{p_1 \cdots p_n l_{n+1}}}{d\mu}(x) = \prod_{m=1}^{\infty} \frac{d\mu_m \circ T_m^{p_1 \cdots p_n l_{n+1}}}{d\mu_m}(x_m) = \lambda.$$

We also note that $\mu(A) > (\mu([B]_1^n)/2(\lambda + 1)) \prod_{m=1}^{\infty} (1 - 2^{-m-1})$ in view of ($\bullet$) and ($\star$). Since the family of Borel cylinders $\{[B]_1^n \mid B \subset X_1 \times \cdots \times X_n, n \in \mathbb{N}\}$ generates a dense subring of the entire Borel σ-algebra on X, it follows from Lemma D that $\log \lambda$ is an essential value of ρ_μ. □

4. *Gaussian dynamical systems*

4.1. *Integration in Hilbert spaces.* Let $\mathcal{H}$ denote a separable infinite-dimensional real Hilbert space. Given a Borel probability measure μ on $\mathcal{H}$, we denote by $\widehat{\mu}$ the *characteristic functional* of μ, that is,

$$\widehat{\mu}(y) := \int_{\mathcal{H}} e^{i\langle x, y \rangle} d\mu(x), \quad y \in \mathcal{H}.$$

We note that each Borel probability measure on $\mathcal{H}$ is defined completely by its characteristic functional. If there is a vector $h \in \mathcal{H}$ and a bounded linear operator $B > 0$ in $\mathcal{H}$ such that $\widehat{\mu}(y) = e^{i\langle h, y \rangle - (1/2)\langle By, y \rangle}$ for all $y \in \mathcal{H}$ then μ is called the *(non-degenerate) Gaussian measure with covariance operator B and mean h*. Then, for each $t \in \mathbb{R}$ and $y \in \mathcal{H}$,

$$\int_{\mathbb{R}} e^{its}\, d(\mu \circ \langle \cdot, y \rangle^{-1})(s) = \int_{\mathcal{H}} e^{i\langle x, ty \rangle}\, d\mu(x) = e^{it\langle h, y \rangle - (1/2)t^2 \langle By, y \rangle}.$$

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Therefore the continuous linear functional $\mathcal{H} \ni x \mapsto \langle x, y\rangle$ has normal distribution $\mathcal{N}_{\langle h,y\rangle,\langle By,y\rangle}$. In particular, each continuous linear functional belongs to $L^2(\mathcal{H}, \mu_B)$. By the Minlos–Sazonov theorem, B is a nuclear operator, that is, $\mathrm{tr}(B) < \infty$ [**Sk**, Theorem 1 and Example from §4]. Conversely, each strictly positive nuclear operator B in $\mathcal{H}$ determines a unique Gaussian measure on $\mathcal{H}$ with zero mean and covariance operator B. We denote this measure by μ_B. Thus $\widehat{\mu_B}(y) = e^{-(1/2)\langle By,y\rangle}$ for all $y \in \mathcal{H}$. We note that $\int_{\mathcal{H}} \langle x, y\rangle d\mu_B(y) = 0$ for each $h \in \mathcal{H}$. It is well known (see, for instance, [**DalFo**, Ch. II, §1.2]) that

$$\int_{\mathcal{H}} \langle x, y\rangle\langle z, y\rangle \, d\mu_B(y) = \langle Bx, z\rangle$$

and hence $\mathrm{tr}(B) = \int_{\mathcal{H}} \|y\|^2 \, d\mu_B(y)$. We now let $\mathcal{H}_0 := B^{1/2}\mathcal{H} \subset \mathcal{H}$ and define an inner product and the corresponding norm on $\mathcal{H}_0$ by setting

$$\langle x, y\rangle_0 := \langle B^{-(1/2)}x, B^{-(1/2)}y\rangle \text{ and } \|x\|_0^2 := \langle x, x\rangle_0 \quad \text{for } x, y \in \mathcal{H}_0.$$

Then $(\mathcal{H}_0, \langle ., .\rangle_0)$ is a Hilbert space. We now show that there is a canonical isometric embedding of $\mathcal{H}_0$ into $L^2(\mathcal{H}, \mu_B)$. To do so, we first take $\theta \in B\mathcal{H} \subset \mathcal{H}_0$. Then the mapping

$$l_\theta : \mathcal{H} \ni y \mapsto \langle B^{-1}\theta, y\rangle$$

is a continuous linear functional on $\mathcal{H}$. Moreover, for all $\theta, \eta \in B\mathcal{H}$,

$$\langle l_\theta, l_\eta\rangle_{L^2(\mathcal{H},\mu_B)} = \int_{\mathcal{H}} \langle B^{-1}\theta, y\rangle\langle B^{-1}\eta, y\rangle \, d\mu_B(y) = \langle \theta, B^{-1}\eta\rangle = \langle \theta, \eta\rangle_0.$$

In particular, the linear mapping

$$l : B\mathcal{H} \ni \theta \mapsto l_\theta \in L^2(\mathcal{H}, \mu_B)$$

is isometric†. We note that $B\mathcal{H}$ is dense in $\mathcal{H}_0$. Indeed, since the linear span $\mathcal{L}$ of the orthonormal basis in $\mathcal{H}$ consisting of eigenvectors for B is dense in $\mathcal{H}$, it follows that $B^{1/2}\mathcal{L}$ is dense in $\mathcal{H}_0$ because $B^{1/2}$ is an isometric isomorphism of $(\mathcal{H}, \langle, .\rangle)$ onto $(\mathcal{H}_0, \langle, .\rangle_0)$. All that remains is to observe that $B^{1/2}\mathcal{L} = B\mathcal{L} = \mathcal{L}$. Since $B\mathcal{H}$ is dense in $\mathcal{H}_0$, the isometry l extends by continuity to an isometry from $\mathcal{H}_0$ to $L^2(\mathcal{H}, \mu_B)$. Thus, for each $y \in \mathcal{H}_0$, there is a sequence $(\theta_n)_{n=1}^\infty$ of elements from $B\mathcal{H}$ such that $\|y - \theta_n\|_0 \to 0$ and the sequence $(l_{\theta_n})_{n=1}^\infty$ converges to some element $l_y \in L^2(\mathcal{H}, \mu_B)$. Hence a subsequence $(l_{\theta_{n_k}})_{k=1}^\infty$ converges to l_y almost everywhere. Let D_y denote the set of all $x \in \mathcal{H}$ such that the sequence $(l_{\theta_{n_k}}(x))_{k=1}^\infty$ converges. It is easy to verify that D_y is a (Borel) linear subspace of $\mathcal{H}$, $\mu_B(D_y) = 1$ and l_y is linear on D_y. That is why l_y is often called a *measurable linear functional on* $\mathcal{H}$. Moreover, $D_y \supset \mathcal{H}_0$ and l_y is defined uniquely by the restriction to $\mathcal{H}_0$ even though $\mu_B(\mathcal{H}_0) = 0$. It is often convenient to write $\langle B^{-1}y, x\rangle$ instead of $l_y(x)$ for μ_B-a.e. $x \in \mathcal{H}$. We note that the distribution of l_θ is $\mathcal{N}_{0,\langle\theta,B^{-1}\theta\rangle} = \mathcal{N}_{0,\|\theta\|_0^2}$ for each $\theta \in B\mathcal{H}$. Passing to a limit, we obtain that the distribution of l_y is $\mathcal{N}_{0,\|y\|_0^2}$ for each $y \in \mathcal{H}_0$.

† If $B\mathcal{H}$ is furnished with $\|.\|_0$.

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For each $y \in \mathcal{H}$, we denote by L_y the rotation by y, that is, $L_y x = x + y$ for all $x \in \mathcal{H}$. By the Cameron–Martin theorem (see [**Gu**, Corollary 7.4], [**Sk**]),

$$\begin{aligned} &\mathcal{H}_0 = \{y \in \mathcal{H} \mid \mu_B \sim \mu_B \circ L_y^{-1}\} \quad \text{and, for each } y \in \mathcal{H}_0, \\ &\frac{d\mu_B \circ L_y^{-1}}{d\mu_B}(x) = e^{\langle B^{-1}y,x\rangle - (1/2)\|y\|_0^2} \quad \text{at a.e. } x \in \mathcal{H}. \end{aligned} \tag{4.1}$$

4.2. *Fock space and exponential map.* Given a separable Hilbert space $\mathcal{K}$, the *(bosonic) Fock space $\mathcal{F}(\mathcal{K})$ built over $\mathcal{K}$* is the Hilbert space $\bigoplus_{n=0}^{\infty} \mathcal{K}^{\odot n}$. The subspace $\mathcal{K}^{\odot n}$ of $\mathcal{F}(\mathcal{K})$ is called the *n-chaos* in $\mathcal{F}(\mathcal{K})$, $n \in \mathbb{Z}_+$. Given $h \in \mathcal{K}$, we let $\exp_h := \bigoplus_{n=0}^{\infty}(h^{\otimes n}/\sqrt{n!}) \in \mathcal{F}(\mathcal{K})$. In particular, $\exp_0 = (1, 0, 0, \dots,)$ is called the *vacuum vector* in $\mathcal{F}(\mathcal{H})$. The map $\exp : \mathcal{K} \ni h \mapsto \exp_h \in \mathcal{F}(\mathcal{K})$ is called the *exponential map*. It satisfies the following properties [**Gu**]:

(i) exp is continuous;
(ii) $\langle \exp_h, \exp_k\rangle_{\mathcal{F}(\mathcal{K})} = e^{\langle h,k\rangle_{\mathcal{K}}}$ for all $h, k \in \mathcal{K}$;
(iii) the set $\{\exp_h \mid h \in \mathcal{K}\}$ is linearly independent and total in $\mathcal{F}(\mathcal{K})$.

Given an orthogonal operator V in $\mathcal{K}$, we can define a linear operator $\exp V$ of $\mathcal{F}(\mathcal{K})$, called *the second quantization of* V, by setting

$$(\exp V)h^{\otimes n} := (Vh)^{\otimes n} \quad \text{for all } n \geq 0 \text{ and } h \in \mathcal{K}.$$

Then $\exp V$ preserves each chaos in $\mathcal{F}(\mathcal{K})$ and the restriction of $\exp V$ to the first chaos is V. Of course, $(\exp V)\exp_h = \exp_{Vh}$ for each $h \in \mathcal{K}$. The most important property of the Fock spaces is the following: given a decomposition $\mathcal{K} = \bigoplus_{j=1}^{\infty} \mathcal{K}_j$ of $\mathcal{K}$ into an orthogonal sum of subspaces $\mathcal{K}_j$, there is a unique unitary isomorphism Φ of $(\mathcal{F}(\mathcal{K}), \exp_0)$ onto $\bigotimes_{j=1}^{\infty}(\mathcal{F}(K_j), \exp_0)$ such that

$$\Phi(\exp_{\bigoplus_{j=1}^{\infty} h_j}) = \bigotimes_{j=1}^{\infty} \Phi(\exp_{h_j})$$

for each vector $\bigoplus_{j=1}^{\infty} h_j \in \mathcal{K}$ such that $h_j = 0$ for all but finitely many j [**Gu**, Proposition 2.3] (We consider the infinite tensor product in the category of Hilbert spaces furnished with unit vectors (see [**Gu**, Appendix A]). It is assumed that the unitary isomorphism in this category intertwines the corresponding unit vectors.).

Denote the orthogonal group of $\mathcal{K}$ by $\mathcal{O}(\mathcal{K})$. Let $\mathrm{Aff}(\mathcal{K}) := \mathcal{K} \rtimes \mathcal{O}(\mathcal{K})$ stand for the group of affine operators in $\mathcal{K}$. We recall that an operator $A = (f, V) \in \mathrm{Aff}(\mathcal{K})$ acts on $\mathcal{K}$ by the formula $Ah := f + Vh$. One can verify that the multiplication law in $\mathrm{Aff}(\mathcal{K})$ is given by

$$(f, V)(f', V') := (f + Vf', VV').$$

We note that $\mathrm{Aff}(\mathcal{K})$ is a Polish group if endowed with the product of the norm topology on $\mathcal{K}$ and the weak operator topology on $\mathcal{O}(\mathcal{K})$. We recall the well-known *Weyl unitary representation* $W = (W_{(f,V)})_{(f,V)\in\mathrm{Aff}(\mathcal{K})}$ of $\mathrm{Aff}(\mathcal{K})$ in $\mathcal{F}(\mathcal{K})$ [**Gu**, §2.2]:

$$W_{(f,V)} \exp_h := e^{-\langle f,Vh\rangle_{\mathcal{K}} - (1/2)\|f\|_{\mathcal{K}}^2} \exp_{f+Vh}, \quad h \in \mathcal{K}. \tag{4.2}$$

It is well defined due to (ii) and (iii). Of course, $W_{(0,V)} = \exp V$ for each $V \in \mathcal{O}(\mathcal{K})$.

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By [Gu, Theorem 7.1], there is a unique (canonical) unitary isomorphism of $L^2(\mathcal{H}, \mu_B)$ with $\mathcal{F}(\mathcal{H}_0)$ such that (For simplicity, we write $L^2(\mathcal{H}, \mu_B) = \mathcal{F}(\mathcal{H}_0)$ and hence identify $\exp_h$ with an L^2-function on $(\mathcal{H}, \mu_B)$, $h \in \mathcal{H}_0$.)

$$\exp_h(x) := e^{\langle B^{-1}h, x\rangle - (1/2)\|h\|_0^2}, \quad \text{for a.e. } x \in \mathcal{H}. \tag{4.3}$$

Moreover, the map $\mathcal{H}_0 \ni h \mapsto l_h \in L^2(\mathcal{H}, \mu_B)$ identifies (isometrically) $\mathcal{H}_0$ with the first chaos in $L^2(\mathcal{H}, \mu_B)$. It follows from (4.1) and (4.3) that

$$\exp_h = \frac{d\mu_B \circ L_h^{-1}}{d\mu_B} \quad \text{for each } h \in \mathcal{H}_0. \tag{4.4}$$

It is straightforward to verify that the following additional properties for exp hold:

(iv) $\exp_h > 0$ for each $h \in \mathcal{H}_0$;

(v) $\exp_h \in \bigcap_{p=1}^{\infty} L^p(\mathcal{H}, \mu_B)$ because the map $\mathcal{H} \ni x \mapsto \langle B^{-1}h, x\rangle - (1/2)\|h\|_0^2$ has normal distribution $\mathcal{N}_{-(1/2)\|h\|_0^2, \|h\|_0^2}$ and (4.3) holds;

(vi) $\|\exp_h\|_1 = 1$ for each $h \in \mathcal{H}_0$;

(vii) the cone $\{\sum_{k=1}^{n} a_k \exp_{h_k} \mid a_1, \ldots, a_n > 0, h_1, \ldots, h_n \in \mathcal{H}_0, n \in \mathbb{N}\}$ is dense in the cone $L^2_+(\mathcal{H}, \mu_B)$ of non-negative functions from $L^2(\mathcal{H}, \mu_B)$;

(viii) $\exp_h \cdot \exp_k = e^{\langle h,k\rangle_0} \exp_{h+k}$ for all $h, k \in \mathcal{H}_0$, and hence

(ix) $\sqrt{\exp_h} = e^{-(1/8)\|h\|_0^2} \exp_{h/2}$ for each $h \in \mathcal{H}_0$;

(x) $\exp_h \circ L_f^{-1} = e^{-\langle B^{-1}h, f\rangle} \exp_h = e^{-\langle h, f\rangle_0} \exp_h$ for all $h, f \in \mathcal{H}_0$.

Remark 4.1.

(i) We recall that $\mathcal{H}_0$ is determined by the pair $(\mathcal{H}, B)$ (see §4.1). Conversely, if $\mathcal{H}_0$ is given beforehand as an abstract Hilbert space, then it uniquely determines the probability space $(\mathcal{H}, \mu_B)$ for some pair $(\mathcal{H}, B)$ such that $\mathcal{H}_0 = B^{1/2}\mathcal{H}$. Indeed, if there is another Hilbert space $\mathcal{K}$ and a non-degenerated nuclear operator $C > 0$ on $\mathcal{K}$ such that the space $\mathcal{K}_0 := C^{1/2}\mathcal{K}$ furnished with the corresponding Hilbert norm is unitarily isomorphic to $\mathcal{H}_0$ via some unitary isomorphism Ψ then, according to [Gu, Theorem 7.1] and (4.3), there is a unique unitary isomorphism Φ of $L^2(\mathcal{H}, \mu_B)$ with $L^2(\mathcal{K}, \mu_C)$ which maps $\exp_h$ onto $\exp_{\Psi^{-1}h}$ for each $h \in \mathcal{H}_0$. Hence in view of (vii), Φ maps $L^2_+(\mathcal{K}, \mu_C)$ onto $L^2_+(\mathcal{K}, \mu_C)$. Moreover, $\Phi 1 = 1$. Therefore Φ is *spacial*, that is, there is a measure-preserving isomorphism $\theta : (\mathcal{H}, \mu_B) \to (\mathcal{K}, \mu_C)$ such that $\Phi h = h \circ \theta^{-1}$ for each $h \in \mathcal{H}$.

(ii) Another useful observation is that, given a Hilbert space $\mathcal{K}_0$, there is another Hilbert space $\mathcal{K} \supset \mathcal{K}_0$ and a nuclear operator C of $\mathcal{K}$ such that $C^{1/2}$ is a unitary isomorphism of $\mathcal{K}$ onto $\mathcal{K}_0$.

Remark 4.2. Given a decomposition $\mathcal{H}_0 = \bigoplus_{j=1}^{\infty} \mathcal{H}_{0,j}$ of $\mathcal{H}_0$ into an orthogonal sum of subspaces $\mathcal{H}_{0,j}$, consider the corresponding decomposition $\mathcal{H} = \bigoplus_{j=1}^{\infty} \mathcal{H}_j$ of $\mathcal{H}$ into an orthogonal sum of subspaces $\mathcal{H}_j := B^{-(1/2)}\mathcal{H}_{j,0}$, $j \in \mathbb{N}$. Let $P_j : \mathcal{H} \to \mathcal{H}_j$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_j$ and let $B_j := P_j B P_j^*$. Then $B_j : H_j \to H_j$ is a nuclear operator and $B_j^{1/2}\mathcal{H}_j = \mathcal{H}_{0,j}$ for each $j \in \mathbb{N}$. Moreover, $(\mathcal{H}, \mu_B)$ splits into the direct product $(\mathcal{H}, \mu_B) = \bigotimes_{j=1}^{\infty} (\mathcal{H}_j, \mu_{B_j})$ of Gaussian probability spaces $(\mathcal{H}_j, \mu_{B_j})$ in

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such a way that $\{\exp_h \mid h \in \mathcal{H}_{0,j}\}$ is total in $L^2(H_j, \mu_{B_j})$ and $\mu_{B_j} = \mu_B \circ P_j^*$ for each $j \in \mathbb{N}$.

4.3. *Non-singular Gaussian action of* Aff $\mathcal{H}_0$. Let $(Y, \mathfrak{C}, \nu)$ be a standard non-atomic probability space. Denote by $\mathcal{U}(L^2(Y, \nu))$ the group of unitary operators in $L^2(Y, \nu)$ and by $\mathcal{U}_{\mathbb{R}}(L^2(Y, \nu))$ the subgroup of unitaries that preserve the subspace $L^2_{\mathbb{R}}(Y, \nu)$ of real-valued functions in $L^2(Y, \nu)$. Let

$$U : \mathrm{Aut}(Y, \nu) \ni T \mapsto U_T \in \mathcal{U}_{\mathbb{R}}(L^2(Y, \nu))$$

stand for the unitary Koopman representation of $\mathrm{Aut}(Y, \nu)$ in $L^2(Y, \nu)$. We recall that $U_T f := f \circ T^{-1}\sqrt{((d\mu \circ T^{-1})/d\mu)}$ for all $f \in L^2(Y, \nu)$. The following results are well known:

(•) $\{U_T \mid T \in \mathrm{Aut}(Y, \nu)\} = \{V \in \mathcal{U}_{\mathbb{R}}(L^2(Y, \nu)) \mid V L^2_+(Y, \nu) = L^2_+(Y, \nu)\}$;
(∘) $\{U_T \mid T \in \mathrm{Aut}_0(Y, \nu)\} = \{V \in \mathcal{U}_{\mathbb{R}}(L^2(Y, \nu)) \mid V L^2_+(Y, \nu) = L^2_+(Y, \nu), V1 = 1\}$.

We also note that U is one-to-one and the image of U is closed in $\mathcal{U}_{\mathbb{R}}(L^2(Y, \nu))$ in the weak (and the strong) operator topology.

Let $\mathbb{R}^*$ denote the multiplicative group of reals. It is straightforward to verify that, for each $t \in \mathbb{R}^*$, the map $\alpha_t : \mathrm{Aff}(\mathcal{H}_0) \to \mathrm{Aff}(\mathcal{H}_0)$ given by

$$(f, V) \mapsto \alpha_t(f, V) := (tf, V) \tag{4.5}$$

is a continuous automorphism of $\mathrm{Aff}(\mathcal{H}_0)$. Moreover, $\alpha_{t_1}\alpha_{t_2} = \alpha_{t_1t_2}$ for all $t_1, t_2 \in \mathbb{R}^*$.

It is straightforward to verify that, for each $A \in \mathrm{Aff}(\mathcal{H}_0)$, the corresponding Weyl unitary operator W_A (see (4.2)) preserves the cone

$$\left\{ \sum_{k=1}^{n} a_k \exp_{h_k} \;\middle|\; a_k > 0, h_k \in \mathcal{H}_0, \text{for each } k = 1, \ldots, n \text{ and } n \in \mathbb{N} \right\}.$$

Hence it preserves $L^2_+(\mathcal{H}, \mu_B)$ in view of (vii) from §4.2. Therefore, by (•), there is a (unique) transformation $T_A \in \mathrm{Aut}(\mathcal{H}, \mu_B)$ such that $U_{T_A} = W_{\alpha_{1/2}(A)}$.

Definition 4.3. T_A is called the *non-singular Gaussian transformation generated by* $A \in \mathrm{Aff}(\mathcal{H}_0)$.

Since the image of $\mathrm{Aff}(\mathcal{H}_0)$ under the unitary Weyl representation is closed in the unitary group of the space $L^2(\mathcal{H}, \mu_B)$ [**Gu**, Theorem 2.1], it follows that the group $\{T_A \mid A \in \mathrm{Aff}(\mathcal{H}_0)\}$ of non-singular Gaussian transformations is closed in $\mathrm{Aut}(\mathcal{H}, \mu_B)$.

PROPOSITION 4.4.

(i) *If* $V \in \mathcal{O}(\mathcal{H}_0)$ *then* $T_{(0,V)}$ *is the usual (classic) measure-preserving Gaussian transformation generated by the orthogonal operator* V*, that is,* $U_{T_{(0,V)}} = \exp V$ *(see [**LePaTh**, Lemma 2]).*
(ii) *If* $f \in \mathcal{H}_0$ *then* $T_{(f,I)} = L_f$.

Proof. (i) We note that

$$U_{T_{(0,V)}} \exp_h = W_{(0,V)} \exp_h = \exp_{Vh} = (\exp V) \exp_h .$$

Hence $U_{T_{(0,V)}} = \exp V$.

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(ii) Using (4.4) and (viii)–(x) from 4.2, we obtain that

$$\begin{aligned} U_{L_f}\exp_h &= \sqrt{\frac{d\mu_B\circ L_f^{-1}}{d\mu_B}}\,\exp_h\circ L_f^{-1} \\ &= \sqrt{\exp_f}\,e^{-\langle f,h\rangle_0}\,\exp_h \\ &= e^{-(1/8)\|f\|_0^2}\exp_{f/2}\,e^{-\langle h,f\rangle_0}\,\exp_h \\ &= e^{-(1/8)\|f\|_0^2-\langle f,h\rangle_0+(1/2)\langle f,h\rangle_0}\exp_{f/2+h}. \end{aligned}$$

Hence $U_{L_f}\exp_h = e^{-(1/8)\|f\|_0^2-(1/2)\langle f,h\rangle_0}\exp_{f/2+h} = W_{(f/2,I)}\exp_h = U_{T_{(f,I)}}\exp_h$. It follows that $T_{(f,I)} = L_f$. □

COROLLARY 4.5. *Every non-singular Gaussian transformation $T_{(f,V)}$ is the composition of the classic μ_B-preserving Gaussian transformation $T_{(0,V)}$ and a μ_B-non-singular translation $L_f = T_{(f,I)}$ which is totally dissipative (Let $\mathcal{K}$ stand for the orthogonal complement in $\mathcal{H}$ to the one-dimensional subspace generated by f. Then the set $\{sf + k \mid 0 \le s < 1,\ k \in \mathcal{K}\} \subset \mathcal{H}$ is a Borel fundamental domain for L_f.). These two transformations commute if and only if $Vf = f$.*

Remark 4.6. Let $(X, \mathfrak{B}, \mu)$ be a standard σ-finite non-atomic measure space. Let a transformation $S \in \mathrm{Aut}(X,\mu)$ be such that $\sqrt{(d\mu\circ S^{-1})/d\mu} - 1 \in L^2(X,\mu)$. Then a non-singular Poisson suspension S_* of S is well defined on a standard probability space $(X^*, \mathfrak{B}^*, \mu^*)$ [**DaKoRo1**]. Let $A := (U_S, \sqrt{((d\mu\circ S^{-1})/d\mu)} - 1) \in \mathrm{Aff}(L^2(X,\mu))$. It was shown in [**DaKoRo1**] that U_{S_*} is unitarily equivalent to W_A. It follows that each non-singular Poisson transformation is unitarily equivalent to a non-singular Gaussian transformation: S_* is unitarily equivalent to $T_{\alpha_2(A)}$ (see (4.5)). The converse is not true even in the classic (finite measure preserving) case: there is no II_∞ automorphism which is spectrally isomorphic to a Gaussian–Kronecker automorphism (see [**Ro**, Theorem 4.13]).

It is well known that the transformation group $\{T_{(f,I)} \mid f \in \mathcal{H}_0\} \subset \mathrm{Aut}(\mathcal{H}, \mu_B)$ is ergodic (see [**Gu**, **Sk**]). However, its Krieger's type has not been determined so far. We will show that it is type III_1, that is, a dense countable subgroup of it is of type III_1 (hence every dense countable subgroup is of type III_1).

THEOREM 4.7. *$\{T_{(f,I)} \mid f \in \mathcal{H}_0\}$ is of type III_1.*

Proof. Let $\{e_n \mid n \in \mathbb{N}\}$ be an orthonormal basis of $\mathcal{H}$ consisting of the eigenvectors of B. As B is positive and nuclear, $Be_n = \lambda_n e_n$, $\lambda_n > 0$ for each $n \in \mathbb{N}$ and $\sum_{n=1}^\infty \lambda_n < \infty$. Denote by Γ the group generated by translations $L_{\sqrt{\lambda_k}e_k}$ for all $k \in \mathbb{N}$. Then Γ is an ergodic countable abelian subgroup of $\mathrm{Aut}(\mathcal{H}, \mu_B)$. We will show that Γ is of type III_1.

Denote by $\mathfrak{B}_n$ the smallest Borel σ-algebra on $\mathcal{H}$ such that the map $\mathcal{H} \ni x \mapsto \langle x, e_k\rangle \in \mathbb{R}$ is $\mathfrak{B}_n$-measurable for each $k = 1, \ldots, n$. Then $\mathfrak{B}_1 \subset \mathfrak{B}_2 \subset \cdots$ and the union $\bigcup_{n>0}\mathfrak{B}_n$ is dense in $\mathfrak{B}$. We deduce from (4.3) and (4.4) that, for each $n > 0$,

$$\log\frac{d\mu_B\circ L_{\sqrt{\lambda_{n+1}}e_{n+1}}}{d\mu_B}(x) = \frac{\langle x, e_{n+1}\rangle}{\sqrt{\lambda_{n+1}}} - \frac{1}{2}.$$

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Take $a \in \mathbb{R}$ and $\epsilon > 0$. We now let

$$D_n := \left\{ x \in \mathcal{H} \,\middle|\, a + \frac{1}{2} - \epsilon < \frac{\langle x, e_{n+1}\rangle}{\sqrt{\lambda_{n+1}}} < a + \frac{1}{2} + \epsilon \right\}.$$

Since $e_{n+1} \perp e_k$ for each $k = 1, \ldots, n$ and the random variable $\mathcal{H} \ni x \mapsto \langle x, e_k\rangle \in \mathbb{R}$ is Gaussian for all $k = 1, \ldots, n+1$ (and the joint distribution is also Gaussian), it follows that D_n is independent of $\mathfrak{B}_n$. Moreover, the measure

$$\mu_B(D_n) = \frac{1}{\sqrt{2\pi\lambda_{n+1}}} \int_{(a+(1/2)-\epsilon, a+(1/2)+\epsilon)\cdot\sqrt{\lambda_{n+1}}} e^{-(t^2/2\lambda_{n+1})}\,dt$$
$$= \frac{1}{\sqrt{2\pi}} \int_{(a+(1/2)-\epsilon, a+(1/2)+\epsilon)} e^{-(t^2/2)}\,dt$$

of D_n does not depend on n (We use here the fact that the random variable $\langle \cdot, e_{n+1}\rangle$ has normal distribution $\mathcal{N}_{0,\lambda_{n+1}}$.). We denote it by $\delta > 0$. Since each subset $A \in \mathfrak{B}_n$ depends only on the 'first n coordinates' $x_1, \ldots, x_n$ while the translation $L_{e_{n+1}}$ changes only the $(n+1)$th coordinate, we have that $L_{e_{n+1}} A = A$ and hence

- $(A \cap D_n) \cup L_{e_{n+1}}(A \cap D_n) \subset A$,
- $\mu_B(A \cap D_n) = \mu_B(A)\mu_B(D_n) = \delta\mu_B(A)$ and
- $\log((d\mu_B \circ L_{e_{n+1}})/d\mu_B)(x) = a \pm \epsilon$ for each $x \in A \cap D_n$.

It follows from Lemma D that a is an essential value of the logarithm of the Radon–Nikodym cocycle of Γ. Since a is an arbitrary element of $\mathbb{R}$, the Radon–Nikodym cocycle is ergodic, that is, Γ is of type III_1. □

4.4. *When non-singular Gaussian systems are of type II_1.* We recall a standard definition.

Definition 4.8. Given $V \in \mathcal{O}(\mathcal{H}_0)$, we say that a vector $f \in \mathcal{H}_0$ is a *V-coboundary* if there is $a \in \mathcal{H}_0$ such that $f = a - Va$.

In this subsection we prove the following statement (cf. [**DaKoRo1**, Proposition 6.4] and [**ArIsMa**]).

THEOREM 4.9. *Let $(f, V) \in \mathrm{Aff}(\mathcal{H}_0)$. For $n \in \mathbb{Z}$, we define $f^{(n)} \in \mathcal{H}_0$ by setting $(f, V)^n = (f^{(n)}, V^n)$. The following statements are equivalent.*

(i) *$T_{(f,V)}$ admits an equivalent invariant probability measure.*
(ii) *f is a V-coboundary.*
(iii) *The affine operator (f, V) has a fixed point.*
(iv) *The sequence $(f^{(n)})_{n\in\mathbb{Z}}$ is bounded in $\mathcal{H}_0$.*

Proof. (ii) $\Longleftrightarrow$ (iv) is classic; see [**BeKaVal**, Proposition 2.2.9], for a proof.

(ii) $\Longleftrightarrow$ (iii) is obvious because the equality $(f, V)a = a$ for some $a \in \mathcal{H}_0$ means $f + Va = a$, that is, f is a V-coboundary.

(ii) $\Longrightarrow$ (i). In view of Proposition 4.4 and Corollary 4.5,

$$\frac{d\mu_B \circ T^{-1}_{(f,V)}}{d\mu_B} = \frac{d(\mu_B \circ T^{-1}_{(0,V)}) \circ T^{-1}_{(f,I)}}{d\mu_B} = \frac{d\mu_B \circ T^{-1}_{(f,I)}}{d\mu_B} = \frac{d\mu_B \circ L^{-1}_f}{d\mu_B}.$$

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Therefore, by (4.4), we obtain that

$$\frac{d\mu_B \circ T_{(f,V)}^{-1}}{d\mu_B} = \exp_f.$$

Let $f = a - Va$ for some $a \in \mathcal{H}_0$. We claim that

$$\exp_f = \frac{\exp_a}{\exp_a \circ T_{(f,V)}^{-1}}. \tag{4.6}$$

Indeed, applying Proposition 4.4 and (viii) and (x) from 4.2, we obtain that

$$\begin{aligned}\exp_f \, \exp_a \circ T_{(f,V)}^{-1} &= \exp_f \, \exp_a \circ T_{(0,V)^{-1}} \circ L_f^{-1} \\ &= \exp_f \, ((\exp V) \exp_a) \circ L_f^{-1} \\ &= \exp_f \ \exp_{Va} \ e^{-\langle Va, f\rangle} \\ &= \exp_{f+Va} \\ &= \exp_a.\end{aligned}$$

Since $\exp_a \in L^1(\mathcal{H}, \mu_B)$, (i) follows from (4.6).

(i) $\Longrightarrow$ (iv). We first note that, for each $h \in \mathcal{H}_0$,

$$\|\sqrt{\exp_h}\|_1 = e^{-(\|h\|_0^2/8)}.$$

We now have

$$\begin{aligned}\langle (U_{T_{(f,V)}})^n 1, 1\rangle &= \langle U_{T_{(f^{(n)},V^n)}} 1, 1\rangle \\ &= \left\langle \sqrt{\frac{d\mu_B \circ T_{(f^{(n)},V^n)}^{-1}}{d\mu_B}}, 1 \right\rangle \\ &= \|\sqrt{\exp_{f^{(n)}}}\|_1 \\ &= e^{-(\|f^{(n)}\|_0^2/8)}.\end{aligned}$$

The rest of the argument is almost a repetition of the proof that (4) implies (1) in [**DaKoRo1**, Proposition 6.4]. Suppose that the sequence $(f^{(n)})_{n=1}^{\infty}$ is unbounded. Then there is an increasing sequence $n_1 < n_2 < \cdots$ such that $\|f^{(n_k)}\|_0^2 \to +\infty$ as $k \to \infty$. Hence $\langle (U_{T_{(f,V)}})^{n_k} 1, 1\rangle \to 0$ as $k \to \infty$. Since the operator $U_{T_{(f,V)}}$ is positive with respect to the cone $L^2_+(\mathcal{H}, \mu_B)$, it follows that $U^{n_k}_{T_{(f,V)}} \to 0$ weakly as $k \to \infty$. Since $T_{(f,V)}$ admits an equivalent invariant probability measure, $U_{T_{(f,V)}}$ is unitarily equivalent to the Koopman operator of a probability-preserving transformation. The latter does not have subsequences weakly converging to zero because 1 is a fixed point of this operator. □

Remark 4.10. In fact, we showed more: if $f = a - Va$ and ν is a μ_B-equivalent $T_{(f,V)}$-invariant measure then $d\nu/d\mu_B = \exp_a$.

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4.5. *Gaussian transformations as IDPFT systems.* Suppose that we are given an affine operator $(f, V) \in \mathrm{Aff}(\mathcal{H}_0)$. Suppose also that V has no non-trivial invariant vectors†. Using the spectral decomposition of V, as in the proof of Theorem 2.6, we can choose an orthogonal decomposition $\mathcal{H}_0 = \bigoplus_{r=1}^{\infty} \mathcal{H}_{0,r}$ of $\mathcal{H}_0$ in such a way that $V\mathcal{H}_{0,r} = \mathcal{H}_{0,r}$ and the orthogonal projection f_r of f onto $\mathcal{H}_{0,r}$ is a V-coboundary for each $r \in \mathbb{N}$. Let $V_r := V \restriction \mathcal{H}_r$. Then $(f_r, V_r) \in \mathrm{Aff}(\mathcal{H}_{0,r})$ for each $r \in \mathbb{N}$ and $(f, V) = \bigoplus_{r=1}^{\infty}(f_r, V_r)$. Let $\mathcal{H}_r$ and μ_r stand for the Hilbert space and a Gaussian measure on $\mathcal{H}_r$ respectively such that $\mathcal{F}(\mathcal{H}_{0,r})$ is canonically isomorphic to $L^2(\mathcal{H}_r, \mu_r)$ (see Remark 4.1(ii)). Then the standard probability space $(\mathcal{H}, \mu_B)$ is isomorphic to the infinite product $\bigotimes_{r=1}^{\infty}(\mathcal{H}_r, \mu_r)$ according to Remark 4.2. It follows that

$$(\mathcal{H}, \mu_B, T_{(f,V)}) = \bigotimes_{r=1}^{\infty}(\mathcal{H}_r, \mu_r, T_{(f_r,V_r)}). \tag{4.7}$$

Since f_r is a V_r-coboundary, there is $a_r \in \mathcal{H}_{0,r}$ such that $f_r = a_r - V_r a_r$ for each $r \in \mathbb{N}$. By Theorem 4.9, the system $(\mathcal{H}_r, \mu_r, T_{(f_r,V_r)})$ admits an equivalent invariant probability measure ν_r. Moreover, $d\mu_r/d\nu_r = \exp_{-a_r}$ for each $r \in \mathbb{N}$ in view of Remark 4.10. Thus, we have shown that each non-singular Gaussian dynamical system $(\mathcal{H}, \mu_B, T_{(f,V)})$ such that V has no non-trivial invariant vectors is IDPFT (see (4.7)). Therefore, Corollary 3.7 yields the following result.

COROLLARY 4.11. *If V has no non-trivial invariant vectors then the non-singular Gaussian dynamical system $(\mathcal{H}, \mu_B, T_{(f,V)})$ is either conservative or totally dissipative. In fact, if $(Y, \mathfrak{C}, \nu, S)$ is an ergodic conservative non-singular dynamical system then the direct product $T_{(f,V)} \times S$ is either conservative or totally dissipative.*

The following theorem was first proved in [**ArIsMa**] in the case of mixing V. We extend it to the mildly mixing case with a different proof.

THEOREM 4.12. *Let $T_{(0,V)}$ be mildly mixing and let f not be a V-coboundary. If $T_{(f,V)}$ is conservative then the Maharam extension of $T_{(f,V)}$ is sharply weak mixing. In particular, $T_{(f,V)}$ is of type III_1.*

Proof. Since $T_{(f,V)}$ is conservative, it follows from (4.7) and Proposition 3.3 that $T_{(f,V)}$ is sharply weak mixing. Let a_r, μ_r and ν_r be as above in this subsection. Since $d\mu_r/d\nu_r = \exp_{-a_r}$ for each $r \in \mathbb{N}$, it follows from (4.1) and (4.4) that the distribution ψ_r of $\log(d\mu_r/d\nu_r)$ defined on $(\mathcal{H}_{0,r}, \nu_r)$ is $\mathcal{N}_{-\|a_r\|_0^2/2, \|a_r\|_0^2}$. Hence, for all $m > n$, we have

$$\psi_{n+1} * \cdots * \psi_m = \mathcal{N}_{-0.5\sum_{r=n+1}^{m}\|a_r\|_0^2,\, \sum_{r=n+1}^{m}\|a_r\|_0^2} \quad \text{and}$$
$$\psi_{n+1} * \cdots * \psi_m * \delta_a = \mathcal{N}_{a-0.5\sum_{r=n+1}^{m}\|a_r\|_0^2,\, \sum_{r=n+1}^{m}\|a_r\|_0^2}$$

† Equivalently, the measure of maximal spectral type of V has no atom at 1.

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for each $a \in \mathbb{R}$. We will show that the sequence $(\psi_r)_{r=1}^{\infty}$ is asymptotically translation quasi-invariant. First, it is straightforward to verify that, for each $\sigma \in \mathbb{R}$ and $b > 0$,

$$\log\left(\frac{d\mathcal{N}_{b-\sigma^2/2,\sigma^2}}{d\mathcal{N}_{-\sigma^2/2,\sigma^2}}(t)\right) = \frac{b(2t+\sigma^2-b)}{2\sigma^2} = \frac{bt}{\sigma^2} + \frac{b(\sigma^2-b)}{2\sigma^2}, \quad t \in \mathbb{R}.$$

Hence, if $t \geq -\sigma^2$ and $\sigma^2 \geq 2b$ then

$$\frac{d\mathcal{N}_{b-\sigma^2/2,\sigma^2}}{d\mathcal{N}_{-\sigma^2/2,\sigma^2}}(t) \geq e^{-b+(b(\sigma^2-b)/2\sigma^2)} > e^{-(3b/4)}. \tag{4.8}$$

Moreover,

$$\begin{aligned}\int_{-\sigma^2}^{+\infty} d\mathcal{N}_{-\sigma^2/2,\sigma^2}(t) &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\sigma^2}^{+\infty} e^{-(1/2)((t+\sigma^2/2)/\sigma)^2}\,dt\\ &= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\sigma^2/2}^{+\infty} e^{-(t^2)/2\sigma^2}\,dt\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\sigma/2}^{+\infty} e^{-(t^2/2)}\,dt,\end{aligned}$$

that is, $\mathcal{N}_{-\sigma^2/2,\sigma^2}((-\sigma^2,+\infty)) = \mathcal{N}_{0,1}((-\sigma/2,+\infty))$. Obviously, we have

$$\psi_{n+1} * \cdots * \psi_m * \delta_a \sim \psi_{n+1} * \cdots * \psi_m \quad \text{for all } n < m.$$

We now set $\zeta_a := e^{-(3a/4)}$. Next, we note that f is not a V-coboundary if and only if $\sum_{r=1}^{\infty}\|a_r\|_0^2 = \infty$. Hence for each $n > 0$, there is $m > n$ such that $\sum_{r=n+1}^{m}\|a_r\|_0^2 > 2a$. Let $W_{n+1,m} := [-\sum_{r=n+1}^{m}\|a_r\|_0^2, +\infty) \subset \mathbb{R}$. Then (4.8) yields that

$$\frac{d(\psi_{n+1} * \cdots * \psi_m * \delta_a)}{d(\psi_{n+1} * \cdots * \psi_m)}(t) \geq \zeta_a \quad \text{for all } t \in W_{n+1,m}.$$

Moreover, $(\psi_{n+1} * \cdots * \psi_m)(W_{n+1,m}) = \mathcal{N}_{0,1}((-0.5\sqrt{\sum_{r=n+1}^{m}\|a_r\|_0^2}, +\infty)) \approx 1$ if m is large. Hence $(\psi_r)_{r=1}^{\infty}$ is asymptotically translation quasi-invariant. It follows now from Theorem 3.10 that the Maharam extension of $T_{(f,V)}$ is sharply weak mixing. □

4.6. *One-parametric family of non-singular Gaussian systems.* We note that (4.5) determines a one-to-one homomorphism $\mathbb{R}^* \ni t \mapsto \alpha_t$ from the multiplicative group $\mathbb{R}^*$ to the group of continuous automorphisms of $\text{Aff}(\mathcal{H}_0)$. Therefore, for each $A \in \text{Aff}(\mathcal{H}_0)$, one can consider a one-parametric family of non-singular Gaussian transformations $T_{\alpha_t(A)} \in \text{Aut}(\mathcal{H}, \mu_B)$, $t \in \mathbb{R}^*$†. Our purpose is this section is to investigate how the dynamical properties of $T_{\alpha_t(A)}$ depend on t. It is straightforward to verify that the linear operator $-I$ of $\mathcal{H}$ preserves μ_B and conjugates $T_{\alpha_t(A)}$ with $T_{\alpha_{-t}(A)}$. Therefore it suffices to consider only the transformations $T_{\alpha_t(A)}$ with $t \in \mathbb{R}^*_+$.

PROPOSITION 4.13. [**ArIsMa**] *Given $A = (f, V) \in \text{Aff}(\mathcal{H}_0)$ such that V has no non-zero invariant vectors, there is $t_{\text{diss}}(A) \in [0, +\infty]$ such that the transformation $T_{\alpha_t(A)}$ is conservative if $0 < t < t_{\text{diss}}(A)$ and totally dissipative if $t > t_{\text{diss}}(A)$.*

† We note that the map $\mathbb{R}^* \ni t \mapsto T_{\alpha_t(A)}$ is not a group homomorphism

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Proof. Let $A = (f, V)$ with $f \in \mathcal{H}_0$ and $V \in \mathcal{O}(\mathcal{H}_0)$. It is sufficient to show that if T_A is totally dissipative then, for each $t > 1$, the Gaussian transformation $T_{\alpha_t(A)}$ is totally dissipative. Since T_A is totally dissipative, the Hopf criterion yields that $\sum_{n=0}^{\infty}((d\mu_B \circ T_A^n)/d\mu_B)(x) = \sum_{n=0}^{\infty} e^{\langle B^{-1} f^{(n)}, x\rangle - (1/2)\|f^{(n)}\|_0^2} < \infty$ for μ_B-a.e. $x \in \mathcal{H}$. Hence, there is $N_x > 0$ such that $\langle B^{-1} f^{(n)}, x\rangle - \frac{1}{2}\|f^{(n)}\|_0^2 < 0$ for all $n > N_x$. It follows that

$$\langle tB^{-1} f^{(n)}, x\rangle - \frac{t^2\|f^{(n)}\|_0^2}{2} < t\bigg(\langle B^{-1} f^{(n)}, x\rangle - \frac{\|f^{(n)}\|_0^2}{2}\bigg) < \langle B^{-1} f^{(n)}, x\rangle - \frac{\|f^{(n)}\|_0^2}{2}$$

for all $n > N_x$. Hence $\sum_{n=0}^{\infty} e^{\langle tB^{-1} f^{(n)}, x\rangle - \frac{1}{2}\|tf^{(n)}\|_0^2} < \infty$ for μ_B-a.e. $x \in \mathcal{H}$. Since $tf^{(n)} = (tf)^{(n)}$, we deduce from the Hopf criterion that $T_{\alpha_t(A)}$ is dissipative, as desired. □

We recall that the *Poincaré exponent* of $A = (f, V) \in \mathrm{Aff}(\mathcal{H}_0)$ [**ArIsMa**] is

$$\delta_A := \inf\bigg\{\alpha > 0 \mid \sum_{n=1}^{\infty} e^{-\alpha\|f^{(n)}\|_0^2} < +\infty\bigg\} \in [0, +\infty].$$

For completeness of our argument we give a proof of the following proposition.

PROPOSITION 4.14. [**ArIsMa**] $\sqrt{2\delta_A} \le t_{\mathrm{diss}}(A) \le 2\sqrt{2\delta_A}$.

Proof [**ArIsMa**]. Let $t > t_{\mathrm{diss}}(A)$. Since $T_{\alpha_t(A)}$ is isomorphic to $T_{\alpha_{-t}(A)}$, the two transformations are dissipative. Therefore, by the Hopf criterion,

$$\sum_{n=0}^{\infty} e^{t\langle B^{-1} f^{(n)}, x\rangle - (t^2/2)\|f^{(n)}\|_0^2} < \infty \quad \text{and} \quad \sum_{n=0}^{\infty} e^{-t\langle B^{-1} f^{(n)}, x\rangle - (t^2/2)\|f^{(n)}\|_0^2} < \infty$$

at a.e. x. Since $e^{t\langle B^{-1} f^{(n)}, x\rangle} + e^{-t\langle B^{-1} f^{(n)}, x\rangle} \ge 2$ for each $x \in X$, it follows that $\sum_{n=0}^{\infty} e^{-(t^2/2)\|f^{(n)}\|_0^2} < \infty$, that is, $\delta_A \le t^2/2$ and hence $\delta_A \le t_{\mathrm{diss}}(A)^2/2$.

On the other hand, if $t < t_{\mathrm{diss}}(A)$ then $T_{\alpha_t(A)}$ is conservative and hence

$$\sum_{n=0}^{\infty} e^{t\langle B^{-1} f^{(n)}, x\rangle - (t^2/2)\|f^{(n)}\|_0^2} = +\infty.$$

Therefore,

$$+\infty = \sum_{n=0}^{\infty} \int_{\mathcal{H}} e^{(1/2)t\langle B^{-1} f^{(n)}, x\rangle - (t^2/4)\|f^{(n)}\|_0^2} d\mu_B(x) = \sum_{n=0}^{\infty} e^{-(t^2/8)\|f^{(n)}\|_0^2}.$$

Hence $\delta_A \ge (t^2/8)$ and therefore $\delta_A \ge (t_{\mathrm{diss}}(A)^2/8)$.

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