

POWER SERIES PROOFS FOR LOCAL STABILITIES OF KÄHLER AND BALANCED STRUCTURES WITH MILD $\partial\bar{\partial}$ -LEMMA

SHENG RAO*, XUEYUAN WAN† AND QUANTING ZHAO‡

Abstract. By use of a natural map introduced recently by the first and third authors from the space of pure-type complex differential forms on a complex manifold to the corresponding one on the small differentiable deformation of this manifold, we will give a power series proof for Kodaira–Spencer’s local stability theorem of Kähler structures. We also obtain two new local stability theorems, one of balanced structures on an n -dimensional balanced manifold with the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma by power series method and the other one on p -Kähler structures with the deformation invariance of (p, p) -Bott–Chern numbers.

Contents

1	Introduction	305
2	Stability of Kähler structures	310
2.1	Obstruction equations	312
2.2	Construction of power series	317
2.3	Hölder convergence	319
2.4	Regularity argument	322
3	Stability of balanced structures with $(n-1, n)$th mild $\partial\bar{\partial}$-lemma	322
3.1	The $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma and examples	323
3.2	Proof for stability of balanced structures with mild $\partial\bar{\partial}$ -lemma	329
4	Stability of p-Kähler structures	338
4.1	Obstruction of extension for d -closed and $\partial\bar{\partial}$ -closed forms	338
4.2	Un-obstruction of extension for transverse forms	340

§1. Introduction

The local stability of some special complex structure is an interesting topic in deformation theory of complex structures and the power series method, initiated by Kodaira–Nirenberg–Spencer and Kuranishi, plays a prominent role there. One main goal of this paper is to present a power series proof for the classical Kodaira–Spencer’s local stability of Kähler structures, which is a problem at latest dated back to [39, Remark 1 on Page 180]: “A good

Received August 5, 2020. Revised February 16, 2021.

2020 Mathematics Subject Classification: Primary 32G05; Secondary 13D10, 14D15, 53C55.

*Rao is partially supported by the National Natural Science Foundations of China No. 11671305, 11771339, 11922115 and China Scholarship Council/University of California, Los Angeles Joint Scholarship Program.

†Wan is partially supported by Scientific Research Foundation of Chongqing University of Technology.

‡Zhao is partially supported by the National Natural Science Foundations of China No. 11801205.

© (2021) The Authors. The publishing rights in this article are licenced to Foundation Nagoya Mathematical Journal under an exclusive license.

problem would be to find an elementary proof (e.g., using power series methods). Our proof uses nontrivial results from partial differential equations.”

THEOREM 1.1 ([29, Theorem 15]). *Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact complex manifolds. If a fiber $X_0 := \pi^{-1}(t_0)$ admits a Kähler metric, then, for a sufficiently small neighborhood U of t_0 on B , the fiber $X_t := \pi^{-1}(t)$ over any point $t \in U$ still admits a Kähler metric, which depends smoothly on t and coincides for $t = t_0$ with the given Kähler metric on X_0 .*

The other goal is to prove a new local stability theorem of balanced structures when the reference fiber satisfies the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma, a new-type $\partial\bar{\partial}$ -lemma, using the power series method developed above, and also one of p -Kähler structures with the deformation invariance of (p, p) -Bott–Chern numbers by two different proofs. Recall that a *balanced metric* ω on an n -dimensional complex manifold is a real positive $(1, 1)$ -form satisfying $d(\omega^{n-1}) = 0$, and a complex manifold is called *balanced* if there exists such a metric on it.

This paper is a sequel to [36, 43], whose notions are adopted here. All manifolds in this paper are assumed to be compact complex n -dimensional manifolds. The symbol $A^{p,q}(X, E)$ stands for the space of the holomorphic vector bundle E -valued (p, q) -forms on a complex manifold X . A *Beltrami differential* on X , generally denoted by ϕ , is an element in $A^{0,1}(X, T_X^{1,0})$, where $T_X^{1,0}$ denotes the holomorphic tangent bundle of X . Then ι_ϕ or $\phi \lrcorner$ denotes the contraction operator with respect to $\phi \in A^{0,1}(X, T_X^{1,0})$ or other analogous vector-valued complex differential forms alternatively if there is no confusion. We also follow the convention

$$e^\spadesuit = \sum_{k=0}^\infty \frac{1}{k!} \spadesuit^k, \tag{1}$$

where \spadesuit^k denotes k -time action of the operator \spadesuit . Since the dimension of X is finite, the summation in the above formulation is always finite.

We will always consider the differentiable family $\pi : \mathcal{X} \rightarrow B$ of compact complex n -dimensional manifolds over a sufficiently small domain in \mathbb{R}^k with the reference fiber $X_0 := \pi^{-1}(t_0)$ and the general fibers $X_t := \pi^{-1}(t)$. For simplicity, we set $k = 1$. Denote by $\zeta := (\zeta_j^\alpha(z, t))$ the holomorphic coordinates of X_t induced by the family with the holomorphic coordinates $z := (z^i)$ of X_0 , under a coordinate covering $\{\mathcal{U}_j\}$ of \mathcal{X} , when t is assumed to be fixed, as the standard notions in deformation theory described at the beginning of [39, Chapter 4]. This family induces a canonical differentiable family of integrable Beltrami differentials on X_0 , denoted by $\varphi(z, t)$, $\varphi(t)$, and φ interchangeably, to be explained at the beginning of Section 2.

In [43], the first and third authors introduce an extension map

$$e^{\iota_{\varphi(t)} | \iota_{\overline{\varphi(t)}}} : A^{p,q}(X_0) \rightarrow A^{p,q}(X_t),$$

which plays an important role in this paper.

DEFINITION 1.2. For $s \in A^{p,q}(X_0)$, we define

$$e^{\iota_{\varphi(t)} | \iota_{\overline{\varphi(t)}}}(s) = s_{i_1 \dots i_p j_1 \dots j_q}(z(\zeta)) (e^{\iota_{\varphi(t)}}(dz^{i_1} \wedge \dots \wedge dz^{i_p})) \wedge (e^{\iota_{\overline{\varphi(t)}}}(d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q})),$$

where s is locally written as

$$s = s_{i_1 \dots i_p j_1 \dots j_q}(z) dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \dots \wedge d\bar{z}^{j_q}$$

and the operators $e^{\iota_{\varphi(t)}}$, $e^{\iota_{\overline{\varphi(t)}}}$ follow the convention (1). It is easy to check that this map is a real linear isomorphism as in [43, Lemma 2.8].

Now, let us describe our approach to reprove Kodaira–Spencer’s local stability of Kähler structures. We will use Kuranishi’s completeness theorem [31] to reduce the proof to Kuranishi family $\varpi : \mathcal{K} \rightarrow T$ and a power series method to construct a natural Kähler extension $\tilde{\omega}_t$ of the Kähler form ω_0 on X_0 , such that $\tilde{\omega}_t$ is a Kähler form on the general fiber $\varpi^{-1}(t) = X_t$. More precisely, the extension is given by

$$e^{\iota_{\varphi}|_{\iota_{\overline{\varphi}}}} : A^{1,1}(X_0) \rightarrow A^{1,1}(X_t), \quad \omega_0 \rightarrow \tilde{\omega}_t := e^{\iota_{\varphi}|_{\iota_{\overline{\varphi}}}}(\omega(t)),$$

where $\omega(t)$ is a family of smooth $(1, 1)$ -forms on X_0 , depending smoothly on t , and $\omega(0) = \omega_0$. This method is developed in [36, 37, 43, 48, 49, 56, 57]. The following proposition will be used many times in this paper:

PROPOSITION 1.3 ([36, Theorem 3.4], [43, Proposition 2.2]). *Let $\phi \in A^{0,1}(X, T_X^{1,0})$. Then on the space $A^{*,*}(X)$,*

$$d \circ e^{\iota_{\phi}} = e^{\iota_{\phi}}(d + \partial \circ \iota_{\phi} - \iota_{\phi} \circ \partial - \iota_{\bar{\partial}\phi - \frac{1}{2}[\phi, \phi]}). \tag{2}$$

By a careful use of Proposition 1.3, we can show that $d(e^{\iota_{\varphi}|_{\iota_{\overline{\varphi}}}}(\omega)) = 0$ with $\omega \in A^{1,1}(X_0)$ is equivalent to

$$\begin{cases} ([\partial, \iota_{\varphi}] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi)\lrcorner\omega = 0, \\ ([\bar{\partial}, \iota_{\overline{\varphi}}] + \partial)(\mathbb{1} - \varphi\bar{\varphi})\lrcorner\omega = 0. \end{cases} \tag{3}$$

In this paper, we always use the notations: $\varphi\bar{\varphi} = \bar{\varphi}\lrcorner\varphi$, $\bar{\varphi}\varphi = \varphi\lrcorner\bar{\varphi}$ and $\mathbb{1}$ is the identity operator defined as:

$$\mathbb{1} = \frac{1}{p+q} \left(\sum_i^n dz^i \otimes \frac{\partial}{\partial z^i} + \sum_i^n dz^{\bar{i}} \otimes \frac{\partial}{\partial \bar{z}^i} \right)$$

when it acts on (p, q) -forms of a complex manifold. Obviously, the identity operator is a real operator. It is worth noticing that this definition is a little different from that in [43], where for a (p, q) -form α on a complex manifold,

$$\mathbb{1}(\alpha) = (p+q) \cdot \alpha$$

since the constant factor $\frac{1}{p+q}$ did not appear in that definition.

Then one has the crucial reduction:

PROPOSITION 1.4(=PROPOSITION 2.7). *If the power series*

$$\omega(t) = \sum_{k=0}^{\infty} \sum_{i+j=k} \omega_{i,j} t^i \bar{t}^j$$

of $(1, 1)$ -forms on X_0 is a real formal solution of the system of equations

$$\begin{cases} \bar{\partial}\omega = \bar{\partial}(\bar{\varphi}\varphi\lrcorner\omega - \varphi\lrcorner\bar{\varphi}\omega) - \partial(\varphi\lrcorner\omega), \\ \partial\omega = \partial(\varphi\bar{\varphi}\lrcorner\omega - \bar{\varphi}\lrcorner\varphi\omega) - \bar{\partial}(\bar{\varphi}\lrcorner\omega), \end{cases} \tag{4}$$

then it is also one of the system (3).

Based on Observation 2.11, one can solve the system (4) easily by the Kählerian condition on the reference fiber inductively. Then the Hölder convergence and regularity argument in Sections 2.3 and 2.4 gives rise to the desired Kähler form on the deformation X_t of X_0 .

In Section 3, we will discuss the local stability problem of balanced structures on the complex manifolds also satisfying various $\partial\bar{\partial}$ -lemmata. The $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma is introduced in Section 3.1: an n -dimensional complex manifold X satisfies the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma, if for any $(n-2, n)$ -complex differential form ξ on X , there exists an $(n-2, n-1)$ -form θ on X such that

$$\partial\bar{\partial}\theta = \partial\xi.$$

Then another main result of this paper can be described as follows:

THEOREM 1.5(=THEOREM 3.12). *Let X_0 be a compact balanced manifold of complex dimension n , satisfying the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma. Then X_t also admits a balanced metric for t small.*

Nilmanifolds with invariant abelian complex structures satisfy the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma as shown in Corollary 3.5. Many examples and results, such as [10, Proposition 4.4, Remark 4.6, Remark 4.7, and Example 4.10] and [23, Corollaries 8 and 9], become consequences of this theorem.

It is an obvious generalization of Wu's result [55, Theorem 5.13] that the balanced structure is preserved under small deformation if the reference fiber satisfies the $\partial\bar{\partial}$ -lemma. Fu–Yau [23, Theorem 6] show that the balanced structure is deformation open, assuming that the $(n-1, n)$ th weak $\partial\bar{\partial}$ -lemma, introduced by them, holds on the general fibers X_t for $t \neq 0$. Recall that the $(n-1, n)$ th weak $\partial\bar{\partial}$ -lemma on a compact complex manifold X says that if for any real $(n-1, n-1)$ -form ψ on X such that $\bar{\partial}\psi$ is ∂ -exact, there exists an $(n-2, n-1)$ -form θ on X such that $\partial\bar{\partial}\theta = \bar{\partial}\psi$. It is well known from [3] that a small deformation of the Iwasawa manifold, which satisfies the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma but does not satisfy the mild one from Example 3.8, may not be balanced. Thus, the condition “ $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma” in Theorem 1.5 can't be replaced by the weak one. In [53, Example 3.7] or Example 3.14, Ugarte–Villacampa construct an explicit family of nilmanifolds I_λ of complex dimension 3 with invariant balanced metric on each fiber, for $\lambda \in [0, 1)$. However, the general fiber X_t does not satisfy the $(n-1, n)$ th weak $\partial\bar{\partial}$ -lemma for $t \neq 0$. Fortunately, the mild one holds on the reference fiber and thus, Fu–Yau's theorem is not applicable to this example, while ours is applicable.

Based on Fu–Yau's theorem, Angella–Ugarte [10, Theorem 4.9] prove that if X_0 admits a locally conformal balanced metric and satisfies the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma, then X_t is balanced for t small. They define the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma of a complex manifold X as: for any ∂ -closed $(n-1, n)$ -form Γ on X of the type $\Gamma = \partial\xi + \bar{\partial}\psi$, there exists a suitable θ on X with $\partial\bar{\partial}\theta = \Gamma$. Interestingly, a complex manifold satisfies the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma if and only if both of the mild one and the dual mild one hold on it, and the $(n-1, n)$ th dual mild $\partial\bar{\partial}$ -lemma guarantees that a locally conformal balanced metric is also a global one. The $(n-1, n)$ th dual mild $\partial\bar{\partial}$ -lemma refers to that the induced mapping $\iota_{BC, \bar{\partial}}^{n-1, n} : H_{BC}^{n-1, n}(X) \rightarrow H_{\bar{\partial}}^{n-1, n}(X)$ from the $(n-1, n)$ th Bott–Chern cohomology group by the identity map is injective. From this point of view, one can understand Angella–Ugarte's theorem more intrinsically. Besides, there indeed exist examples that satisfy the $(n-1, n)$ th

mild $\partial\bar{\partial}$ -lemma but not the strong one, such as a nilmanifold endowed with an invariant abelian complex structure from Corollary 3.5 and [9, Proposition 2.9].

Similarly to the Kähler case, we will prove Theorem 1.5 in Section 3.2 by reducing the proof to Kuranishi family and constructing a power series $\Omega(t) \in A^{n-1,n-1}(X_0)$ such that

$$\begin{cases} d(e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Omega(t))) = 0, \\ \Omega(t) = \overline{\Omega(t)}, \\ \Omega(0) = \omega^{n-1}, \end{cases} \tag{5}$$

where ω is the original balanced metric on X_0 . By Proposition 1.3 again and setting

$$\tilde{\Omega}(t) = e^{-\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}} \circ e^{-\iota_\varphi} \circ e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Omega(t)),$$

one reduces the obstruction system (5) of equations to:

$$\begin{cases} (\bar{\partial} + \partial \circ \iota_\varphi + \bar{\partial} \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) = 0, \\ (\partial + \bar{\partial} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \partial \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) = 0, \\ \tilde{\Omega}(0) = \omega^{n-1}, \end{cases} \tag{6}$$

and solves this system formally also by the power series method, when the “ $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma” just comes from the strategy of using Observation 2.11 to resolve (6). Inspired by the Hölder convergence and regularity argument for the integrable Beltrami differential $\varphi(t)$ in the deformation theory of complex structures, we complete that of $\tilde{\Omega}(t)$. The key point is to deal with the Green’s operator in the explicit canonical solution of $\tilde{\Omega}(t)$ there. It is worth noticing that this analogous proof is different in the resolution of the obstruction equation (6) by use of the Bott–Chern Green’s operator and thus undergoes more difficult regularity argument. It is also applicable to the $(1, 1)$ -case without the Kähler or deformation invariance of $(1, 1)$ -Bott–Chern numbers assumption on the reference fiber X_0 essentially in Kodaira–Spencer’s original proof:

PROPOSITION 1.6. *Assume that the reference fiber X_0 satisfies the $(1, 2)$ th mild $\partial\bar{\partial}$ -lemma, that is, any d -closed ∂ -exact $(1, 2)$ -form on X_0 is also $\partial\bar{\partial}$ -exact. Then any d -closed $(1, 1)$ -form Ω_0 on X_0 can be extended as a d -closed $(1, 1)$ -form Ω_t varying smoothly at t on its small differentiable deformation X_t .*

Notice that the $(1, 2)$ -mild $\partial\bar{\partial}$ -lemma is different from the $\partial\bar{\partial}$ -lemma on a complex manifold. It is easy to see that the $(1, 2)$ -mild $\partial\bar{\partial}$ -lemma amounts to the injectivity of the mapping

$$\iota_{BC,\partial}^{1,2} : H_{BC}^{1,2}(X) \rightarrow H_{\partial}^{1,2}(X),$$

or equivalently, that of

$$\iota_{BC,\bar{\partial}}^{2,1} : H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X).$$

See Example 3.2 for an example of a non- $\partial\bar{\partial}$ -manifold which satisfies the $(1, 2)$ -mild $\partial\bar{\partial}$ -lemma.

Section 4 is devoted to the local stabilities of p -Kähler structures. In Section 4.1, by means of Wu’s result [55, Theorem 5.13], we will use the cohomological method, originally from [29], to get:

PROPOSITION 1.7(=PROPOSITION 4.1). *Let r and s be non-negative integers. Assume that the reference fiber X_0 satisfies the $\partial\bar{\partial}$ -lemma. Then any d -closed (r, s) -form Ω_0 and $\partial_0\bar{\partial}_0$ -closed (r, s) -form Ψ_0 on X_0 can be extended unobstructed to a d -closed (r, s) -form Ω_t and a $\partial_t\bar{\partial}_t$ -closed (r, s) -form Ψ_t on its small differentiable deformation X_t , respectively.*

We can also prove this proposition in the $(n-1, n-1)$ -case by another way inspired by the results of [10, 23, 55]. It is impossible to prove Theorem 1.5 by this method since the proof would rely on the deformation openness of $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma, which contradicts with Ugarte–Villacampa’s Example 3.14.

Finally, in Section 4.2, we study some basic properties of p -Kähler structures, a possibly more intrinsic notion for the local stabilities of complex structures. Based on un-obstruction of extension for transverse forms and (the proof of) Proposition 1.7, we use two different approaches to obtain:

THEOREM 1.8(= THEOREM 4.9+ REMARK 4.13). *For any positive integer $p \leq n-1$, any small differentiable deformation X_t of a compact p -Kähler manifold X_0 satisfying the deformation invariance of (p, p) -Bott–Chern numbers is still p -Kählerian.*

Notation Without specially mentioned, the hermitian metrics will be identified with their fundamental forms. We only consider the small differentiable deformations in this paper, that is, the parameter t is always assumed to be small. All subindices in the power series, such as i, j, \dots , are set no less than zero, while Einstein sum convention is adopted in the local settings and calculations. In many places, we fix a Kähler metric or a balanced one on the reference fiber of the differentiable family to induce the dual operators and the associated Hodge decomposition with respect to $\bar{\partial}$ and ∂ on it. A complex differential form, linear operator or current is called *real* if it is invariant under conjugation.

§2. Stability of Kähler structures

We introduce some basics on deformation theory of complex structures to be used throughout this paper. For holomorphic family of compact complex manifolds, we adopt the definition [28, Definition 2.8]; while for differentiable one, we follow:

DEFINITION 2.1 ([28, Definition 4.1]). Let \mathcal{X} be a differentiable manifold, B a domain of \mathbb{R}^k and π a smooth map of \mathcal{X} onto B . By a *differentiable family of n -dimensional compact complex manifolds* we mean the triple $\pi : \mathcal{X} \rightarrow B$ satisfying the following conditions:

- (i) The rank of the Jacobian matrix of π is equal to k at every point of \mathcal{X} .
- (ii) For each point $t \in B$, $\pi^{-1}(t)$ is a compact connected subset of \mathcal{X} .
- (iii) $\pi^{-1}(t)$ is the underlying differentiable manifold of the n -dimensional compact complex manifold X_t associated to each $t \in B$.
- (iv) There is a locally finite open covering $\{\mathcal{U}_j \mid j = 1, 2, \dots\}$ of \mathcal{X} and complex-valued smooth functions $\zeta_j^1(p), \dots, \zeta_j^n(p)$, defined on \mathcal{U}_j such that for each t ,

$$\{p \rightarrow (\zeta_j^1(p), \dots, \zeta_j^n(p)) \mid \mathcal{U}_j \cap \pi^{-1}(t) \neq \emptyset\}$$

form a system of local holomorphic coordinates of X_t .

Let us sketch Kodaira–Spencer’s proof of local stability theorem [29]. Let F_t be the orthogonal projection to the kernel \mathbb{F}_t of the *first fourth order Kodaira–Spencer operator*

(also often called *Bott–Chern Laplacian*)

$$\square_{BC,t} = \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t + \bar{\partial}_t^* \partial_t \partial_t^* \bar{\partial}_t + \partial_t^* \bar{\partial}_t \bar{\partial}_t^* \partial_t + \bar{\partial}_t^* \bar{\partial}_t + \partial_t^* \partial_t \tag{7}$$

and \mathbb{G}_t the corresponding Green’s operator with respect to α_t on X_t . Here

$$\alpha_t = \sqrt{-1} g_{i\bar{j}}(\zeta, t) d\zeta^i \wedge d\bar{\zeta}^{\bar{j}}$$

is a hermitian metric on X_t depending differentiably on t and α_0 is a Kähler metric on X_0 . By a cohomological argument with the upper semi-continuity theorem, they prove that F_t and \mathbb{G}_t depend differentiably on t . Then they can construct the desired Kähler metric on X_t as

$$\tilde{\alpha}_t = \frac{1}{2} (F_t \alpha_t + \overline{F_t \alpha_t}).$$

See also [54, Section 9.3].

Now let us describe our basic philosophy to reprove the Kodaira–Spencer’s local stability of Kähler structures. By (the proof of) Kuranishi’s completeness theorem [31], for any compact complex manifold X_0 , there exists a complete holomorphic family $\varpi : \mathcal{K} \rightarrow T$ of complex manifolds at the reference point $0 \in T$ in the sense that for any differentiable family $\pi : \mathcal{X} \rightarrow B$ with $\pi^{-1}(s_0) = \varpi^{-1}(0) = X_0$, there is a sufficiently small neighborhood $E \subseteq B$ of s_0 , and smooth maps $\Phi : \mathcal{X}_E \rightarrow \mathcal{K}$, $\tau : E \rightarrow T$ with $\tau(s_0) = 0$ such that the diagram commutes

$$\begin{array}{ccc} \mathcal{X}_E & \xrightarrow{\Phi} & \mathcal{K} \\ \pi \downarrow & & \downarrow \varpi \\ (E, s_0) & \xrightarrow{\tau} & (T, 0), \end{array}$$

Φ maps $\pi^{-1}(s)$ biholomorphically onto $\varpi^{-1}(\tau(s))$ for each $s \in E$, and

$$\Phi : \pi^{-1}(s_0) = X_0 \rightarrow \varpi^{-1}(0) = X_0$$

is the identity map. This family is called *Kuranishi family* and constructed as follows. Let $\{\eta_\nu\}_{\nu=1}^m$ be a base for $\mathbb{H}^{0,1}(X_0, T_{X_0}^{1,0})$, where some suitable hermitian metric is fixed on X_0 and $m \geq 1$; Otherwise the complex manifold X_0 would be *rigid*, that is, for any differentiable family $\kappa : \mathcal{M} \rightarrow P$ with $s_0 \in P$ and $\kappa^{-1}(s_0) = X_0$, there is a neighborhood $V \subseteq P$ of s_0 such that $\kappa : \kappa^{-1}(V) \rightarrow V$ is trivial. Then one can construct a holomorphic family

$$\varphi(t) = \sum_{|I|=1}^{\infty} \varphi_I t^I := \sum_{j=1}^{\infty} \varphi_j(t), \quad I = (i_1, \dots, i_m), \quad t = (t_1, \dots, t_m) \in \mathbb{C}^m, \tag{8}$$

for $|t| < \rho$ a small positive constant, of Beltrami differentials as follows:

$$\varphi_1(t) = \sum_{\nu=1}^m t_\nu \eta_\nu \tag{9}$$

and for $|I| \geq 2$,

$$\varphi_I = \frac{1}{2} \bar{\partial}^* \mathbb{G} \sum_{J+L=I} [\varphi_J, \varphi_L]. \tag{10}$$

It is obvious that $\varphi(t)$ satisfies the equation

$$\varphi(t) = \varphi_1 + \frac{1}{2} \bar{\partial}^* \mathbb{G}[\varphi(t), \varphi(t)].$$

Let

$$T = \{t \mid \mathbb{H}[\varphi(t), \varphi(t)] = 0\}.$$

Thus, for each $t \in T$, $\varphi(t)$ satisfies

$$\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)], \tag{11}$$

and determines a complex structure X_t on the underlying differentiable manifold of X_0 . More importantly, $\varphi(t)$ represents the complete holomorphic family $\varpi : \mathcal{K} \rightarrow T$ of complex manifolds. Roughly speaking, Kuranishi family $\varpi : \mathcal{K} \rightarrow T$ contains all sufficiently small differentiable deformations of X_0 .

By means of these, one can reduce the local stability Theorem 1.1 to the Kuranishi family by shrinking E if necessary, that is, it suffices to construct a Kähler metric on each X_t . From now on, one uses $\varphi(t)$ and φ interchangeably to denote this holomorphic family of integrable Beltrami differentials, and assumes $m = 1$ for simplicity.

Using this reduction, we should construct a natural Kähler extension $\tilde{\omega}_t$ of a given Kähler metric ω_0 on X_0 , such that $\tilde{\omega}_t$ is a Kähler metric on the general fiber $\varpi^{-1}(t) = X_t$. More precisely, the extension is given by

$$e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}} : A^{1,1}(X_0) \rightarrow A^{1,1}(X_t), \quad \omega_0 \rightarrow \tilde{\omega}_t := e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\omega(t)),$$

where $\omega(t)$ is a family of smooth $(1, 1)$ -forms on X_0 , depending smoothly on t , and $\omega(0) = \omega_0$. As we need to construct a Kähler extension, the following conditions appear:

- (1) $d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\omega(t))) = 0$ and
- (2) $\omega(t) = \overline{\omega(t)}$.

As t is sufficiently small, $\omega(t)$ is positive by the convergence argument and thus $e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\omega(t))$ is a Kähler form on X_t . Here, we will use an elementary power series method to complete the construction.

2.1 Obstruction equations

We will discuss the obstruction equation to extend the d -closed pure-type complex differential forms on a complex manifold to the ones on its small differentiable deformation in this subsection. The argument in this and next subsections is applicable to a general differentiable family of complex manifolds.

For a general $\alpha \in A^{p,q}(X_0)$, by Proposition 1.3 and the integrability condition (11), one has

$$\begin{aligned} d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\alpha)) &= d \circ e^{\iota_\varphi} \circ e^{-\iota_\varphi} \circ e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\alpha) \\ &= e^{\iota_\varphi} \circ ([\partial, \iota_\varphi] + \bar{\partial} + \partial) \circ e^{-\iota_\varphi} \circ e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\alpha) \\ &= e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}} \circ \left(e^{-\iota_\varphi|_{-\iota_{\bar{\varphi}}}} \circ e^{\iota_\varphi} \circ ([\partial, \iota_\varphi] + \bar{\partial} + \partial) \circ e^{-\iota_\varphi} \circ e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\alpha) \right). \end{aligned} \tag{12}$$

Here,

$$e^{-\iota_\varphi(t)|_{-\iota_{\bar{\varphi}(t)}}} : A^{p,q}(X_t) \rightarrow A^{p,q}(X_0)$$

is the inverse map of $e^{\iota_{\varphi(t)}|_{\overline{\varphi(t)}}$, defined by

$$\begin{aligned} & e^{-\iota_{\varphi(t)}|_{\overline{\varphi(t)}}(s) \\ &= s_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(\zeta) \left(e^{-\iota_{\varphi(t)} \left((dz^{i_1} + \varphi(t) \lrcorner dz^{i_1}) \wedge \dots \wedge (dz^{i_p} + \varphi(t) \lrcorner dz^{i_p}) \right) \wedge \right. \\ & \quad \left. e^{-\iota_{\overline{\varphi(t)}} \left((d\bar{z}^{j_1} + \overline{\varphi(t)} \lrcorner d\bar{z}^{j_1}) \wedge \dots \wedge (d\bar{z}^{j_q} + \overline{\varphi(t)} \lrcorner d\bar{z}^{j_q}) \right) \right), \end{aligned} \tag{13}$$

where $s \in A^{p,q}(X_t)$ is locally written as

$$\begin{aligned} s &= s_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(\zeta) (dz^{i_1} + \varphi(t) \lrcorner dz^{i_1}) \wedge \dots \wedge (dz^{i_p} + \varphi(t) \lrcorner dz^{i_p}) \\ & \quad \wedge (d\bar{z}^{j_1} + \overline{\varphi(t)} \lrcorner d\bar{z}^{j_1}) \wedge \dots \wedge (d\bar{z}^{j_q} + \overline{\varphi(t)} \lrcorner d\bar{z}^{j_q}), \end{aligned}$$

and the operators $e^{-\iota_{\varphi(t)}}$, $e^{-\iota_{\overline{\varphi(t)}}}$ also follow the convention (1) as in the proof of [43, Lemma 2.8]. We introduce one more new notation \lrcorner to denote the *simultaneous contraction* on each component of a complex differential form. For example, $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner \alpha$ means that the operator $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi})$ acts on α simultaneously as:

$$\begin{aligned} & (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner (f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}) \\ &= f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_p} \\ & \quad \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_q}, \end{aligned} \tag{14}$$

if α is locally expressed by:

$$\alpha = f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}.$$

This new simultaneous contraction is well-defined since $\varphi(t)$ is a global $(1,0)$ -vector valued $(0,1)$ -form on X_0 (See [39, pp. 150–151]) as reasoned in [43, Proof of Lemma 2.8]. Notice that $(\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner \alpha \neq \mathbb{1} \lrcorner \alpha - \bar{\varphi}\varphi \lrcorner \alpha + \bar{\varphi} \lrcorner \alpha$ in general. Using this notation, one can rewrite the extension map $e^{\iota_{\varphi}|_{\overline{\varphi}}}$ in Definition 1.2:

$$e^{\iota_{\varphi}|_{\overline{\varphi}}} = (\mathbb{1} + \varphi + \bar{\varphi}) \lrcorner.$$

Then one has:

LEMMA 2.2. For any $\alpha \in A^{p,q}(X_0)$,

$$e^{-\iota_{\varphi}} \circ e^{\iota_{\varphi}|_{\overline{\varphi}}}(\alpha) = (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner \alpha. \tag{15}$$

Proof. Following the above notations and the definition of $e^{\iota_{\varphi}|_{\overline{\varphi}}}$, we have

$$\begin{aligned} e^{\iota_{\varphi}|_{\overline{\varphi}}}(\alpha) &= f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} + \varphi) \lrcorner dz^{i_1} \wedge \dots \wedge (\mathbb{1} + \varphi) \lrcorner dz^{i_p} \\ & \quad \wedge (\mathbb{1} + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} + \bar{\varphi}) \lrcorner d\bar{z}^{j_q}. \end{aligned} \tag{16}$$

On the other hand,

$$\begin{aligned} & e^{\iota_{\varphi}} \circ (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner \alpha \\ &= e^{\iota_{\varphi}} \left(f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_p} \right. \\ & \quad \left. \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_q} \right) \\ &= f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_1} \wedge \dots \wedge (\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_p} \end{aligned}$$

$$\begin{aligned} & \wedge (\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \cdots \wedge (\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_q} \\ & = f_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} (\mathbb{1} + \varphi) \lrcorner dz^{i_1} \wedge \cdots \wedge (\mathbb{1} + \varphi) \lrcorner dz^{i_p} \wedge (\mathbb{1} + \bar{\varphi}) \lrcorner d\bar{z}^{j_1} \wedge \cdots \wedge (\mathbb{1} + \bar{\varphi}) \lrcorner d\bar{z}^{j_q}, \end{aligned} \tag{17}$$

where the last equality holds by

$$(\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner dz^{i_k} = (\mathbb{1} + \varphi) \lrcorner dz^{i_k}$$

and

$$(\mathbb{1} + \varphi) \lrcorner (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_k} = (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner d\bar{z}^{j_k} + (\bar{\varphi}\varphi) \lrcorner d\bar{z}^{j_k} = (\mathbb{1} + \bar{\varphi}) \lrcorner d\bar{z}^{j_k}.$$

Therefore, (15) is proved by (16) and (17). □

Similarly:

LEMMA 2.3. For any $\alpha \in A^{p,q}(X_0)$,

$$e^{-\iota_\varphi | -\iota_{\bar{\varphi}}} \circ e^{\iota_\varphi}(\alpha) = ((\mathbb{1} - \bar{\varphi}\varphi)^{-1} - (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi}) \lrcorner \alpha, \tag{18}$$

where $((\mathbb{1} - \bar{\varphi}\varphi)^{-1} - (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi})$ acts on α just as (14).

Notice that the more intrinsic proofs of (15) and (18) can be found in the proof of [43, Proposition 2.12]. Substituting (15) and (18) into (12), one has

PROPOSITION 2.4. For any $\alpha \in A^{p,q}(X_0)$,

$$\begin{aligned} & d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) \\ & = e^{\iota_\varphi | \iota_{\bar{\varphi}}} \left(((\mathbb{1} - \bar{\varphi}\varphi)^{-1} - (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi}) \lrcorner ([\partial, \iota_\varphi] + \bar{\partial}) (\mathbb{1} - \bar{\varphi}\varphi + \bar{\varphi}) \lrcorner \alpha \right). \end{aligned} \tag{19}$$

From (18), we know that

$$e^{-\iota_\varphi | -\iota_{\bar{\varphi}}} \circ e^{\iota_\varphi} : A^{p,q}(X_0) \rightarrow \bigoplus_{i=0}^{\min\{q, n-p\}} A^{p+i, q-i}(X_0).$$

Thus, by carefully comparing the form types in both sides of (19), we have

$$\bar{\partial}_t(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) = e^{\iota_\varphi | \iota_{\bar{\varphi}}} \left((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner ([\partial, \iota_\varphi] + \bar{\partial}) (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \alpha \right). \tag{20}$$

The d -closed condition $d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) = 0$ amounts to

$$\begin{cases} \bar{\partial}_t(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) = 0, \\ \partial_t(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) = \bar{\partial}_t \circ e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha) = 0, \end{cases}$$

which, together with (20), implies that

$$\begin{cases} ([\partial, \iota_\varphi] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \alpha = 0, \\ ([\bar{\partial}, \iota_{\bar{\varphi}}] + \partial)(\mathbb{1} - \varphi\bar{\varphi}) \lrcorner \alpha = 0 \end{cases} \tag{21}$$

by the invertibility of the operators $e^{\iota_\varphi | \iota_{\bar{\varphi}}}$, $(\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner$ and their conjugations. Remark that the operator \lrcorner in (21) is just the ordinary contraction operator \lrcorner when acting on $A^{1,1}(X)$ of a complex manifold X . Actually, one can obtain an equivalent expression of (19)

$$\begin{aligned} & d(e^{\iota_\varphi | \iota_{\bar{\varphi}}}(\alpha)) \\ & = e^{\iota_\varphi | \iota_{\bar{\varphi}}} \left((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner ([\partial, \iota_\varphi] + \bar{\partial})(\mathbb{1} - \bar{\varphi}\varphi) \lrcorner \alpha + (\mathbb{1} - \varphi\bar{\varphi})^{-1} \lrcorner ([\bar{\partial}, \iota_{\bar{\varphi}}] + \partial)(\mathbb{1} - \varphi\bar{\varphi}) \lrcorner \alpha \right). \end{aligned}$$

From the original proof of the stability theorem sketched at the beginning of this section, one knows that the system (21) of obstruction equations indeed has a real solution $\alpha(t) \in A^{1,1}(X_t)$ with $\alpha(0) = \omega(0)$. To get this solution by a power series method, we will formulate an effective obstruction system (23) of equations in Proposition 2.7 for (21).

We will use the commutator formula repeatedly, which is originated from [50, 51] and whose various versions appeared in [11, 13, 21, 34, 37] and also [35, 36] for vector bundle valued forms.

LEMMA 2.5. For $\phi, \psi \in A^{0,1}(X, T_X^{1,0})$ and $\alpha \in A^{*,*}(X)$ on a complex manifold X ,

$$[\phi, \psi] \lrcorner \alpha = -\partial(\psi \lrcorner (\phi \lrcorner \alpha)) - \psi \lrcorner (\phi \lrcorner \partial \alpha) + \phi \lrcorner \partial(\psi \lrcorner \alpha) + \psi \lrcorner \partial(\phi \lrcorner \alpha). \tag{22}$$

There are several formulae to be established, whose proofs are given in Appendix 4.2.

PROPOSITION 2.6. Let $\phi \in A^{0,1}(T_X^{1,0})$ and $\alpha \in A^{p,q}(X)$ on a complex manifold X . Then we have:

- (1) $\phi \lrcorner \bar{\phi} \lrcorner \alpha - (\phi \lrcorner \bar{\phi}) \lrcorner \alpha = \bar{\phi} \lrcorner \phi \lrcorner \alpha - (\bar{\phi} \lrcorner \phi) \lrcorner \alpha.$
- (2) $[\phi, \phi] \lrcorner \bar{\phi} \lrcorner \alpha = 2\phi \lrcorner \partial(\phi \lrcorner \bar{\phi} \lrcorner \alpha) - \phi \lrcorner \phi \lrcorner \partial(\bar{\phi} \lrcorner \alpha).$
- (3) For $\psi \in A^{r,s}(T_X^{1,0})$, $\bar{\partial}(\psi \lrcorner \alpha) = (\bar{\partial}\psi) \lrcorner \alpha + (-1)^{r+s+1} \psi \lrcorner \bar{\partial}\alpha.$
In particular, if $\psi \in A^{1,0}(T_X^{1,0})$ or $A^{0,1}(T_X^{1,0})$, then

$$\bar{\partial}(\psi \lrcorner \alpha) = (\bar{\partial}\psi) \lrcorner \alpha + \psi \lrcorner \bar{\partial}\alpha.$$

- (4) $\bar{\phi} \lrcorner \bar{\phi} \lrcorner \phi \lrcorner \alpha - \phi \lrcorner \bar{\phi} \lrcorner \bar{\phi} \lrcorner \alpha = 2(\bar{\phi} \lrcorner \phi \bar{\phi} \lrcorner \alpha - \bar{\phi} \lrcorner \phi \bar{\phi} \lrcorner \alpha).$
In particular, if $\alpha \in A^{1,1}(X)$, then one has

$$\bar{\phi} \lrcorner \bar{\phi} \lrcorner \phi \lrcorner \alpha = 2(\bar{\phi} \lrcorner \phi \bar{\phi} \lrcorner \alpha),$$

since $\phi \lrcorner \bar{\phi} \lrcorner \bar{\phi} \lrcorner \alpha = 0$ and $\bar{\phi} \lrcorner \phi \bar{\phi} \lrcorner \alpha = 0$.

We first explain the homogenous notation for a power series to be used here and henceforth. Assuming that $\alpha(t)$ is a power series of (bundle-valued) (p, q) -forms, expanded as

$$\alpha(t) = \sum_{k=0}^{\infty} \sum_{i+j=k} \alpha_{i,j} t^i \bar{t}^j,$$

one uses the notation

$$\begin{cases} \alpha(t) = \sum_{k=0}^{\infty} \alpha_k, \\ \alpha_k = \sum_{i+j=k} \alpha_{i,j} t^i \bar{t}^j, \end{cases}$$

where α_k is the k -degree homogeneous part in the expansion of $\alpha(t)$ and all $\alpha_{i,j}$ are smooth (bundle-valued) (p, q) -forms on X_0 with $\alpha(0) = \alpha_{0,0}$. Similarly, according to the expansion (8), one will also adopt this notation to other terms related with φ , such as

$$(\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \varphi = \sum_{k=1}^{\infty} ((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \varphi)_k,$$

where $((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \varphi)_k$ stands for the k -degree homogeneous part of the power series $(\mathbb{1} - \bar{\varphi}\varphi)^{-1} \lrcorner \varphi$ in t, \bar{t} . Then we come to the crucial reduction:

PROPOSITION 2.7. *If the power series*

$$\omega(t) = \sum_{k=0}^{\infty} \sum_{i+j=k} \omega_{i,j} t^i \bar{t}^j$$

of $(1,1)$ -forms on X_0 is a real formal solution of the system of equations

$$\begin{cases} \bar{\partial}\omega = \bar{\partial}(\bar{\varphi}\lrcorner\omega - \varphi\lrcorner\bar{\varphi}\lrcorner\omega) - \partial(\varphi\lrcorner\omega), \\ \partial\omega = \partial(\varphi\bar{\varphi}\lrcorner\omega - \bar{\varphi}\lrcorner\varphi\lrcorner\omega) - \bar{\partial}(\bar{\varphi}\lrcorner\omega), \end{cases} \tag{23}$$

up to the degree N , then $\bar{\partial}(\varphi(t)\lrcorner\omega(t))_k = 0$ for each $0 \leq k \leq N + 1$ and thus $\omega(t)$ is also one real formal solution of the system (21) up to the degree N .

We will realize the importance of the $\bar{\partial}$ -closedness $\bar{\partial}(\varphi(t)\lrcorner\omega(t))_{N+1} = 0$, which fulfills (26) in Observation 2.11 under the Kähler condition and guarantees the existence of a real solution of $(23)_{N+1}$, the equation (23) at the $(N + 1)$ th degree.

Proof. Comparing the power series expansion of the equations (21) and (23), we use induction on the degrees to complete the proof.

Denote by ω_k the homogenous k -part of the power series $\omega(t)$, that is, $\omega_k = \sum_{i+j=k} \omega_{i,j} t^i \bar{t}^j$. Without danger of confusion, we will use φ and ω to denote $\varphi(t)$ and $\omega(t)$, respectively.

The case $N = 0$ is trivial. By induction, assuming that the proposition holds for the degrees $\leq N - 1$, we need to show the proposition for the degree N . That is, if $(23)_k$ has a real solution and

$$\bar{\partial}(\varphi\lrcorner\omega)_k = 0$$

for the degrees $k \leq N$, then we will show that this solution is also one for the system $(21)_N$ and satisfies $\bar{\partial}(\varphi\lrcorner\omega)_{N+1} = 0$. Without loss of generality, we always assume that $N \geq 4$ since the lower-degree cases are also obtained by the same formulation as follows.

Here is an important observation to be proved in Appendix 4.2. □

OBSERVATION 2.8.

$$\bar{\partial}(\varphi\lrcorner\omega)_k = 0, \quad k \leq N + 1.$$

So by Lemma 2.5, Proposition 2.6.(1) and the induction assumption, we have

$$\begin{aligned} & \varphi\lrcorner\partial((\mathbf{1} - \bar{\varphi}\varphi)\lrcorner\omega) \\ &= \varphi\lrcorner\partial(\omega - \varphi\bar{\varphi}\lrcorner\omega + \varphi\bar{\varphi}\lrcorner\omega - \bar{\varphi}\varphi\lrcorner\omega) \\ &= \varphi\lrcorner(-\partial(\bar{\varphi}\lrcorner\varphi\lrcorner\omega) - \bar{\partial}(\bar{\varphi}\lrcorner\omega)) + \varphi\lrcorner\partial(\varphi\bar{\varphi}\lrcorner\omega - \bar{\varphi}\varphi\lrcorner\omega) \\ &= -\varphi\lrcorner\partial(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) - \varphi\lrcorner\bar{\partial}(\bar{\varphi}\lrcorner\omega) \\ &= -\varphi\lrcorner\partial(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) - \bar{\partial}(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) + \bar{\partial}\varphi\lrcorner\bar{\varphi}\lrcorner\omega \\ &= -\varphi\lrcorner\partial(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) - \bar{\partial}(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) + \frac{1}{2}[\varphi, \varphi]\lrcorner\bar{\varphi}\lrcorner\omega \\ &= -\bar{\partial}(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) - \frac{1}{2}\varphi\lrcorner\varphi\lrcorner\partial(\bar{\varphi}\lrcorner\omega) - \frac{1}{2}\partial(\varphi\lrcorner\varphi\lrcorner\bar{\varphi}\lrcorner\omega). \end{aligned}$$

Here, the subscript N is omitted. Using Proposition 2.6.(4) and Observation 2.8, one has

$$(\partial(\varphi\lrcorner\bar{\varphi}\lrcorner\omega) + \varphi\lrcorner\partial(\omega - \bar{\varphi}\varphi\lrcorner\omega) + \bar{\partial}(\varphi\lrcorner\bar{\varphi}\lrcorner\omega))_N = 0,$$

whose left-hand side is exactly the difference of the first equations in $(23)_N$ and $(21)_N$. Therefore, this real solution of the first equation in $(23)_N$ is also that in $(21)_N$, and similarly for the second equations in $(21)_N$ and $(23)_N$.

2.2 Construction of power series

For the resolution of the system (23), we need several more lemmas. As usual, the $\partial\bar{\partial}$ -lemma refers to: for every pure-type d -closed form on X_0 , the properties of d -exactness, ∂ -exactness, $\bar{\partial}$ -exactness and $\partial\bar{\partial}$ -exactness are equivalent.

LEMMA 2.9. *Let X be a complex manifold satisfying the $\partial\bar{\partial}$ -lemma. Consider the system of equations:*

$$\begin{cases} \partial x = \beta, \\ \bar{\partial} x = \bar{\gamma}, \end{cases} \tag{24}$$

where β, γ are $(p+1, p)$ -forms on X . The system of equations (24) has a solution if and only if the following three statements hold:

- (1) $\bar{\partial}\beta + \partial\bar{\gamma} = 0$.
- (2) The ∂ -equation $\partial x = \beta$ has a solution.
- (3) The $\bar{\partial}$ -equation $\bar{\partial} x = \bar{\gamma}$ has a solution.

Proof. If the system (24) has a solution η , then both the ∂ - and $\bar{\partial}$ -equations have solutions. And it is easy to see that

$$\bar{\partial}\beta + \partial\bar{\gamma} = \bar{\partial}\partial\eta + \partial\bar{\partial}\eta = 0.$$

Conversely, let η_1 be a solution of the ∂ -equation, η_2 a solution of the $\bar{\partial}$ -equation with β, γ satisfying $\bar{\partial}\beta + \partial\bar{\gamma} = 0$, which yields that $\partial\eta_1 = \beta$.

We claim that there exists some $\tau \in A^{p,p-1}(X)$ such that $\eta_2 + \bar{\partial}\tau$ satisfies the system (24). In fact, it is obvious that $\eta_2 + \bar{\partial}\tau$ satisfies the second equation of (24). As to the first one, we only need to show that $\beta - \partial\eta_2$ is $\partial\bar{\partial}$ -exact. It is easy to check that

$$\bar{\partial}(\beta - \partial\eta_2) = \bar{\partial}\beta + \partial\bar{\gamma} = 0.$$

And note that $\beta - \partial\eta_2$ is ∂ -exact by the equality

$$\beta - \partial\eta_2 = \partial(\eta_1 - \eta_2).$$

From the $\partial\bar{\partial}$ -Lemma, there exists some $\tau \in A^{p,p-1}(X)$ such that $\beta - \partial\eta_2 = \partial\bar{\partial}\tau$, equivalently saying that $\eta_2 + \bar{\partial}\tau$ satisfies the first equation of (24). Therefore, the claim is proved and $\eta_2 + \bar{\partial}\tau$ is the solution of the system (24). □

COROLLARY 2.10. *Let X be a complex manifold satisfying the $\partial\bar{\partial}$ -lemma. The system of equations:*

$$\begin{cases} \partial x = \beta, \\ \bar{\partial} x = \bar{\beta}, \end{cases} \tag{25}$$

where β is a $(p+1, p)$ -form on X , has a real solution if and only if the following two statements hold:

- (1) $\bar{\partial}\beta + \partial\bar{\beta} = 0$.
- (2) The $\bar{\partial}$ -equation $\bar{\partial} x = \bar{\beta}$ has a solution.

Proof. If η_2 is a solution of the $\bar{\partial}$ -equation and $\bar{\partial}\beta + \partial\bar{\beta} = 0$, it is clear that $\bar{\eta}_2$ satisfies the ∂ -equation $\partial\bar{\eta}_2 = \beta$. Then Lemma 2.9 assures that the system (25) of equations admits a solution, denoted by η . And $\frac{\eta+\bar{\eta}}{2}$ will be a real solution of the system (25). \square

Since β and $\bar{\gamma}$ involved in this paper are mostly $\bar{\partial}$ -exact and ∂ -exact, respectively, such as (23), we have:

OBSERVATION 2.11. *Let $\beta = \bar{\partial}\zeta$ and $\bar{\gamma} = \partial\xi$ for some suitable-type complex differential forms ζ and ξ , respectively, which automatically fulfill the condition (1) in Lemma 2.9. The conditions (2) and (3) rely on the equalities*

$$\begin{cases} \partial\beta = \partial(\bar{\partial}\zeta) = 0, \\ \bar{\partial}\bar{\gamma} = \bar{\partial}(\partial\xi) = 0. \end{cases} \tag{26}$$

Then the $\partial\bar{\partial}$ -lemma will produce μ and ν , satisfying the equations

$$\begin{cases} \partial\bar{\partial}\mu = \bar{\partial}\zeta = \beta, \\ \bar{\partial}\partial\nu = \partial\xi = \bar{\gamma}. \end{cases} \tag{27}$$

The combined expression

$$\bar{\partial}\mu + \partial\nu \tag{28}$$

is our choice for the solution of the system (24), which will be slightly modified to

$$\bar{\partial}\mu + \partial\bar{\mu}$$

as the real solution of the system (25), when β happens to equal to γ .

Recall a useful fact that $\bar{\partial}^*\mathbb{G}_{\bar{\partial}}y$ is the unique solution, minimizing the L^2 -norms of all the solutions, of the equation

$$\bar{\partial}x = y$$

on a compact complex manifold if the equation admits one, where x, y are complex differential forms of pure types and the operator $\mathbb{G}_{\bar{\partial}}$ denotes the corresponding Green's operator of the $\bar{\partial}$ -Laplacian \square . In the Kähler case, we choose

$$\bar{\partial}\mu = \partial^*\mathbb{G}_{\partial}(\bar{\partial}\zeta) = \partial^*\mathbb{G}_{\partial}\beta \quad \text{and} \quad \partial\nu = \bar{\partial}^*\mathbb{G}_{\bar{\partial}}(\partial\xi) = \bar{\partial}^*\mathbb{G}_{\bar{\partial}}\bar{\gamma},$$

where \mathbb{G}_{∂} and $\mathbb{G}_{\bar{\partial}}$ coincide, with a uniform symbol \mathbb{G} used afterwards. Then an explicit solution of the system (24) can be taken as

$$x = \partial^*\mathbb{G}\bar{\partial}\zeta + \bar{\partial}^*\mathbb{G}\partial\xi = \partial^*\mathbb{G}\beta + \bar{\partial}^*\mathbb{G}\bar{\gamma}.$$

When β happens to equal to γ , one takes the real solution of the system (25) as

$$x = \partial^*\mathbb{G}\bar{\partial}\zeta + \bar{\partial}^*\mathbb{G}\partial\bar{\zeta} = \partial^*\mathbb{G}\beta + \bar{\partial}^*\mathbb{G}\bar{\beta} \tag{29}$$

and accordingly, notices that the operator \mathbb{G} is real in this case.

By these, one is able to obtain the main result of this section:

THEOREM 2.12. *The system of equations*

$$\begin{cases} d(e^{t\varphi|t\bar{\varphi}}(\omega(t))) = 0, \\ \omega(t) = \overline{\omega(t)}, \\ \omega(0) = \omega_0, \end{cases} \quad (30)$$

admits a smooth solution $\omega(t) \in A^{1,1}(X_0)$, where ω_0 is a Kähler metric on the complex manifold X_0 . Therefore, we can construct a smooth Kähler metric $e^{t\varphi|t\bar{\varphi}}(\omega(t))$ on X_t .

Proof. We are going to present such an explicit expression for the solution of the obstruction equation (23), whose existence is assured by Proposition 2.7 and the remarks after it, with the initial metric $\omega(0) = \omega_0$. The first-order system of equations

$$\begin{cases} \bar{\partial}\omega_1 = -\partial(\varphi_1 \lrcorner \omega_0), \\ \partial\omega_1 = -\bar{\partial}(\bar{\varphi}_1 \lrcorner \omega_0), \end{cases}$$

admits an explicit real solution, as given by (29),

$$\omega_1 = \partial\bar{\partial}^* \mathbb{G}(\varphi_1 \lrcorner \omega_0) + \bar{\partial}\partial^* \mathbb{G}(\bar{\varphi}_1 \lrcorner \omega_0).$$

By induction, we may assume that (23) has an explicit real solution ω_k for $k \leq N-1$. Based the construction (29) above, one gets a real solution of the N th order equation (23) $_N$

$$\omega_N = (\bar{\varphi} \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega)_N + \left(\partial\bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega) + \bar{\partial}\partial^* \mathbb{G}(\bar{\varphi} \lrcorner \omega) \right)_N,$$

using Proposition 2.6.(1).

Hence, we complete the induction and get a formal solution $\omega(t)$ of (30) with explicit expressions. By the Hölder $C^{k,\alpha}$ -convergence and regularity argument in Sections 2.3 and 2.4, the formal power series $\omega(t)$ constructed above is smooth and solves the system (30) of equations. \square

2.3 Hölder convergence

Consider an important power series in deformation theory of complex structures

$$A(t) = \frac{\beta}{16\gamma} \sum_{m=1}^{\infty} \frac{(\gamma t)^m}{m^2} := \sum_{m=1}^{\infty} A_m t^m, \quad (31)$$

where β, γ are positive constants to be determined. The power series (31) converges for $|t| < \frac{1}{\gamma}$ and has a nice property:

$$A^i(t) \ll \left(\frac{\beta}{\gamma} \right)^{i-1} A(t). \quad (32)$$

See [39, Lemma 3.6 and its Corollary in Chapter 2] for these basic facts. Here, we use the following notation: For the series with real positive coefficients

$$a(t) = \sum_{m=1}^{\infty} a_m t^m, \quad b(t) = \sum_{m=1}^{\infty} b_m t^m,$$

say that $a(t)$ dominates $b(t)$, written as $b(t) \ll a(t)$, if $b_m \leq a_m$. But for a power series of (bundle-valued) complex differential forms

$$\eta(t) = \sum_{i+j \geq 0}^{\infty} \eta_{i,j} t^i \bar{t}^j,$$

the notation

$$\|\eta(t)\|_{k,\alpha} \ll A(t)$$

means

$$\sum_{i+j=m} \|\eta_{i,j}\|_{k,\alpha} \leq A_m$$

with the $C^{k,\alpha}$ -norm $\|\cdot\|_{k,\alpha}$ as defined on [39, p. 159]. In this manner, for a power series of Beltrami differential $\psi(t) = \sum_{i+j=1}^{\infty} \psi_{i,j} t^i \bar{t}^j$, the notation

$$\|\psi(t)\|_{k,\alpha} \ll A(t)$$

indicates $\sum_{i+j=m} \|\psi_{i,j}\|_{k,\alpha} \leq A_m$, and similarly for $\bar{\psi}(t)$. This notation is also used to compare two or more power series of (bundle-valued) complex differential forms degree by degree, such as $\|\psi(t)\|_{k,\alpha} \ll \|\eta(t)\|_{k,\alpha} \cdot \|\rho(t)\|_{k,\alpha}$ for three such power series.

For any complex differential form ϕ , we have two a priori elliptic estimates

$$\|\bar{\partial}^* \phi\|_{k-1,\alpha} \leq C_1 \|\phi\|_{k,\alpha} \tag{33}$$

and

$$\|\mathbb{G}\phi\|_{k,\alpha} \leq C_{k,\alpha} \|\phi\|_{k-2,\alpha}, \tag{34}$$

where \mathbb{G} is the associated Green’s operator to the operator $\bar{\partial}$, $k > 1$, C_1 and $C_{k,\alpha}$ depend on only on k and α , not on ϕ . (See [39, Proposition 2.3 in Chapter 4].)

According to the proof of Theorem 2.12, for $r \geq 1$, one real solution of the obstruction equation (23) is

$$\omega_r = (\bar{\varphi} \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega)_r + \left(\partial \bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega) + \bar{\partial} \partial^* \mathbb{G}(\bar{\varphi} \lrcorner \omega) \right)_r, \tag{35}$$

and obviously

$$\omega^{(r)} = (\bar{\varphi} \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega)^{(r)} + \left(\partial \bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega) + \bar{\partial} \partial^* \mathbb{G}(\bar{\varphi} \lrcorner \omega) \right)^{(r)} \tag{36}$$

are real complex differential forms. Here for a power series of (bundle-valued) complex differential forms

$$\eta(t) = \sum_{i+j=0}^{\infty} \eta_{i,j} t^i \bar{t}^j,$$

one denotes by $\eta^{(r)}$ the summation

$$\sum_{i+j=1}^r \eta_{i,j} t^i \bar{t}^j := \eta_1 + \dots + \eta_r.$$

Recall the holomorphic family

$$\varphi(t) = \sum_{i=1}^{\infty} \varphi_i t^i := \sum_{j=1}^{\infty} \varphi_j(t), \tag{37}$$

for $|t| < \rho$ a small positive constant, of Beltrami differentials representing Kuranishi family develop as in (9) and (10):

$$\varphi_1(t) = t\eta,$$

where η is a base of for $\mathbb{H}^{0,1}(X_0, T_{X_0}^{1,0})$, and for $i \geq 2$,

$$\varphi_i = \frac{1}{2} \bar{\partial}^* \mathbb{G} \sum_{j+l=i} [\varphi_j, \varphi_l].$$

Then it satisfies a nice convergence property:

$$\|\varphi(t)\|_{k,\alpha} \ll A(t)$$

as given in the proof of [39, Proposition 2.4 in Chapter 4]. In this proof, β and $\frac{\beta}{\gamma}$ should satisfy that

$$\beta \geq b, \quad \frac{\beta}{\gamma} \leq b_k, \tag{38}$$

where the constants $b, b_k > 0$ and b_k depends on k . Here, we follow the idea of proving the convergence of $\varphi(t)$ there to obtain that

$$\|\omega^{(r)}\|_{k,\alpha} \ll A(t), \text{ for any } r \geq 1,$$

which implies the desired convergence $\|\omega^{(+\infty)}\|_{k,\alpha} \ll A(t)$ immediately. Assume that they are chosen so that

$$\|\omega^{(r-2)}\|_{k,\alpha}, \|\omega^{(r-1)}\|_{k,\alpha} \ll A(t).$$

By the expression (36) and the two a priori elliptic estimates (33) and (34), one has

$$\begin{aligned} \|\omega^{(r)}\|_{k,\alpha} &= \|(\bar{\varphi}\varphi \lrcorner \omega^{(r-2)} - \varphi \lrcorner \bar{\varphi} \lrcorner \omega^{(r-2)})^{(r)} + \left(\partial\bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega^{(r-1)}) + \bar{\partial}\partial^* \mathbb{G}(\bar{\varphi} \lrcorner \omega^{(r-1)})\right)^{(r)} \\ &\quad + (\bar{\varphi}\varphi \lrcorner \omega_0 - \varphi \lrcorner \bar{\varphi} \lrcorner \omega_0)^{(r)} + \left(\partial\bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega_0) + \bar{\partial}\partial^* \mathbb{G}(\bar{\varphi} \lrcorner \omega_0)\right)^{(r)}\|_{k,\alpha} \\ &\ll 2\|\bar{\varphi}^{(r-2)}\|_{k,\alpha} \|\varphi^{(r-2)}\|_{k,\alpha} \|\omega^{(r-2)}\|_{k,\alpha} + 2C_1 C_{k,\alpha} \|\varphi^{(r-1)}\|_{k,\alpha} \|\omega^{(r-1)}\|_{k,\alpha} \\ &\quad + 2\|\bar{\varphi}^{(r-1)}\|_{k,\alpha} \|\varphi^{(r-1)}\|_{k,\alpha} \|\omega_0\|_{k,\alpha} + 2C_1 C_{k,\alpha} \|\varphi^{(r)}\|_{k,\alpha} \|\omega_0\|_{k,\alpha}. \end{aligned}$$

Then, we use induction and (32) to get:

$$\begin{aligned} \|\omega^{(r)}\|_{k,\alpha} &\ll 2(A(t) + C_1 C_{k,\alpha}) A^2(t) + 2\|\omega_0\|_{k,\alpha} (A(t) + C_1 C_{k,\alpha}) A(t) \\ &\ll 2(A(t) + C_1 C_{k,\alpha}) \left(\frac{\beta}{\gamma} + \|\omega_0\|_{k,\alpha}\right) A(t). \end{aligned}$$

Hence, we may choose $\beta, \gamma, \|\omega_0\|_{k,\alpha}$ so that the following inequalities hold:

$$2(A(t) + C_1 C_{k,\alpha}) \left(\frac{\beta}{\gamma} + \|\omega_0\|_{k,\alpha}\right) < 1,$$

Equation (38) and $\omega^{(1)}, \omega^{(2)} \ll A(t)$ according to the above formulation, which are obviously possible as long as t is small and we notice that the Hölder norm depends on the choice of the local coordinate charts when defined a differential manifold as pointed out on [28, p. 275]. Therefore, for small t , $\omega(t)$ converges in the $C^{k,\alpha}$ -norm and thus its positivity follows.

2.4 Regularity argument

In this subsection, we proceed to the regularity argument for the power series constructed as above since there is possibly no uniform lower bound for the convergence radius obtained in the last subsection in the $C^{k,\alpha}$ -norm as k converges to $+\infty$. We resort to the elliptic operator, the $\bar{\partial}$ -Laplacian

$$\square = \bar{\partial}^* \bar{\partial} + \bar{\partial} \bar{\partial}^*. \quad (39)$$

Here, the dual operators are defined with respect to the fixed original Kähler metric ω_0 . By the classical Kähler identity, Hodge decomposition and the induced commutativity of the associated operators, one has

$$\begin{aligned} \square \omega &= \square(\bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega) + \square \bar{\partial} \bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega) + \overline{\square \bar{\partial} \bar{\partial}^* \mathbb{G}(\varphi \lrcorner \omega)} \\ &= \square(\bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega) + \bar{\partial} \bar{\partial}^* (\varphi \lrcorner \omega) + \overline{\bar{\partial} \bar{\partial}^* (\varphi \lrcorner \omega)} \end{aligned}$$

according to the solution (35).

Consequently, ω is a solution of the two-order partial differential equation

$$\square \omega - \square(\bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \bar{\partial} \bar{\partial}^* (\varphi \lrcorner \omega) - \bar{\partial} \bar{\partial}^* (\bar{\varphi} \lrcorner \omega) = 0. \quad (40)$$

Similarly to the argument on [28, p. 281], writing out the last three terms in the left-hand side of (40) locally, we can find that the expressions of their principal parts (i.e., the highest-order terms) contain factors from $\bar{\varphi} \varphi$, φ , or $\bar{\varphi}$. Since $\varphi(t) \rightarrow 0$ as $t \rightarrow 0$, taking sufficiently small ϵ -disk $\Delta_\epsilon \subseteq \mathbb{C}$, we can assume that the equation (40) is a linear elliptic partial differential equation of ω on X_0 when noticing the ellipticity of an operator only concerns about its principal part. Thus, the interior estimates [20] for elliptic systems of partial differential equations give rise to the desired regularity of $\omega(t)$, that is, $\omega(t)$ is smooth on X_0 for each t smaller than a uniform upper bound to be obtained similarly to [28, Appendix 8]. Then $\omega(t)$ can be regarded as a real analytic family of $(1,1)$ -forms in t and it is smooth on t by [30, Proposition 2.2.3].

§3. Stability of balanced structures with $(n-1, n)$ th mild $\bar{\partial} \bar{\partial}^*$ -lemma

In this section, we will discuss the local stability problem of balanced structures, satisfying various $\bar{\partial} \bar{\partial}^*$ -lemmata. Recall that a *balanced metric* ω on an n -dimensional complex manifold is a real positive $(1,1)$ -form, satisfying that

$$d(\omega^{n-1}) = 0,$$

and a complex manifold is called *balanced* if there exists a balanced metric ω on it. Note that the existence of a balanced metric ω is equivalent to that of a d -closed real positive $(n-1, n-1)$ -form Ω with the relation $\Omega = \omega^{n-1}$ (see [38, (4.8)]).

3.1 The $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma and examples

We are going to study a new kind of “ $\partial\bar{\partial}$ -lemma,” its relations with various analogous conditions in the literature and its examples involved.

Let X be a compact complex manifold of (complex) dimension n with the following commutative diagram

$$\begin{array}{ccccc}
 & & H_{\partial}^{n-1,n}(X) & & \\
 & \nearrow \iota_{BC,\partial}^{n-1,n} & & \nwarrow \iota_{\partial,A}^{n-1,n} & \\
 H_{BC}^{n-1,n}(X) & \xrightarrow{\iota_{BC,A}^{n-1,n}} & & & H_A^{n-1,n}(X) \\
 & \searrow \iota_{BC,\bar{\partial}}^{n-1,n} & & \nearrow \iota_{\bar{\partial},A}^{n-1,n} & \\
 & & H_{\bar{\partial}}^{n-1,n}(X) & &
 \end{array}$$

Bott–Chern and Aeppli cohomology groups of X are defined as

$$H_{BC}^{\bullet,\bullet}(X) =: \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im} \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) =: \frac{\ker \partial \bar{\partial}}{\text{im} \partial + \text{im} \bar{\partial}},$$

respectively, while $H_{\partial}^{\bullet,\bullet}(X)$ is defined similarly. Note that the inequalities

$$\dim_{\mathbb{C}} H_A^{n-1,n}(X) \leq \dim_{\mathbb{C}} H_{\partial}^{n-1,n}(X) \leq \dim_{\mathbb{C}} H_{BC}^{n-1,n}(X)$$

hold on any compact complex manifold [10, Corollary 3.3].

DEFINITION 3.1. The compact complex manifold X satisfies the $(n - 1, n)$ th mild $\partial\bar{\partial}$ - lemma, if the mapping $\iota_{BC,\partial}^{n-1,n} : H_{BC}^{n-1,n}(X) \rightarrow H_{\partial}^{n-1,n}(X)$, induced by the identity map, is injective. Equivalently, for any $(n - 2, n)$ -complex differential form ξ , there exists an $(n - 2, n - 1)$ -form θ such that

$$\partial\bar{\partial}\theta = \partial\xi.$$

Since $\iota_{BC,\partial}^{n-1,n} : H_{BC}^{n-1,n}(X) \rightarrow H_{\partial}^{n-1,n}(X)$ is always surjective, the following conditions are equivalent:

$$\begin{aligned}
 \text{the } (n - 1, n)\text{th mild } \partial\bar{\partial}\text{-lemma} &\iff \iota_{BC,\partial}^{n-1,n} : H_{BC}^{n-1,n}(X) \rightarrow H_{\partial}^{n-1,n}(X) \text{ an isomorphism} \\
 &\iff \dim_{\mathbb{C}} H_{BC}^{n-1,n}(X) = \dim_{\mathbb{C}} H_{\partial}^{n-1,n}(X).
 \end{aligned}$$

Let us give an example of a non- $\partial\bar{\partial}$ -manifold which satisfies the (1,2)-mild $\partial\bar{\partial}$ -lemma.

EXAMPLE 3.2 ([40, p. 90] and [27, Example 1]). Let X be the manifold in the case (ii) of the completely-solvable Nakamura manifold as given in [7, Example 3.1]. Then the manifold X satisfies the (1,2)-mild $\partial\bar{\partial}$ -lemma, but not the $\partial\bar{\partial}$ -lemma.

Proof. It is shown in [7, Table 5 in Appendix A] and [27, the case B in Example 1] that the bases of $H_{BC}^{2,1}(X)$ and $H_{\bar{\partial}}^{2,1}(X)$ can be illustrated as follows:

$$\begin{aligned}
 H_{BC}^{2,1}(X) &= \langle [dz_{12\bar{3}}]_{BC}, [e^{-2z_1} dz_{12\bar{2}}]_{BC}, [e^{2z_1} dz_{13\bar{3}}]_{BC}, [dz_{12\bar{3}}]_{BC}, [dz_{13\bar{2}}]_{BC} \rangle, \\
 H_{\bar{\partial}}^{2,1}(X) &= \langle [dz_{12\bar{3}}]_{\bar{\partial}}, [e^{-2z_1} dz_{12\bar{2}}]_{\bar{\partial}}, [e^{2z_1} dz_{13\bar{3}}]_{\bar{\partial}}, [dz_{12\bar{3}}]_{\bar{\partial}}, [dz_{13\bar{2}}]_{\bar{\partial}} \rangle,
 \end{aligned}$$

which indicates that $\iota_{BC, \bar{\partial}}^{2,1}$ is actually an isomorphism. It is obvious from [7, Table 6 in Appendix A] that $\dim_{\mathbb{C}} H_{BC}^{1,1}(X) = 3$ and $\dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X) = 5$, which implies that the $\partial\bar{\partial}$ -lemma doesn't hold on X . \square

There are three more similar conditions in relevance with the local stability of balanced structures. *The $(n - 1, n)$ th weak $\partial\bar{\partial}$ -lemma* on the compact complex manifold X , introduced by Fu–Yau [23], says that if for any real $(n - 1, n - 1)$ -form ψ such that $\bar{\partial}\psi$ is ∂ -exact, there exists an $(n - 2, n - 1)$ -form θ , satisfying

$$\partial\bar{\partial}\theta = \bar{\partial}\psi.$$

And *the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma*, proposed by Angella–Ugarte [10], states that the mapping $\iota_{BC,A}^{n-1,n} : H_{BC}^{n-1,n}(X) \rightarrow H_A^{n-1,n}(X)$, induced by the identity map, is injective, which is equivalent to that for any ∂ -closed $(n - 1, n)$ -form Γ of the type $\Gamma = \partial\xi + \bar{\partial}\psi$, there exists an $(n - 2, n - 1)$ -form θ such that

$$\partial\bar{\partial}\theta = \Gamma.$$

Angella–Ugarte [10, Theorem 3.1] show that the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma amounts to the sGG condition, carefully studied in [42], and the vanishing of the first $\partial\bar{\partial}$ -degree $\Delta^1(X)$, introduced in [8], with the deformation openness of the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma proved in [10, Proposition 4.8]. They also show in [10, Corollary 3.3] the equivalence:

$$\begin{aligned} \text{the } (n - 1, n)\text{th strong } \partial\bar{\partial}\text{-lemma} &\iff \dim_{\mathbb{C}} H_{BC}^{n-1,n}(X) = \dim_{\mathbb{C}} H_A^{n-1,n}(X) \\ &\iff \dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_A^{0,1}(X) \\ &\iff b_1 = 2 \dim_{\mathbb{C}} H_A^{0,1}(X). \end{aligned}$$

Besides, the condition that the induced mapping $\iota_{BC, \bar{\partial}}^{n-1,n} : H_{BC}^{n-1,n}(X) \rightarrow H_{\bar{\partial}}^{n-1,n}(X)$ by the identity map is injective, is presented by Angella–Ugarte [9] to study local conformal balanced structures and global ones, which we may call *the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma*.

After a simple check, we have the following observation:

OBSERVATION 3.3. *The compact complex manifold X satisfies the $(n - 1, n)$ th strong $\partial\bar{\partial}$ -lemma if and only if both of the mild one and the dual mild one hold on X .*

And the mild one and the dual mild one both imply the weak one. All the four “ $\partial\bar{\partial}$ -lemmata” hold if the compact complex manifold X satisfies the standard $\partial\bar{\partial}$ -lemma.

We refer the readers to [7, 32, 45, 53] as the background materials on the theory of nilmanifolds and solvmanifolds to be focused on in the rest of this subsection.

Recall that a *nilmanifold M with left-invariant complex structure* is a compact quotient of a simply-connected nilpotent Lie group G of real even dimension by a lattice Γ of maximal rank, whose Lie algebra \mathfrak{g} admits an integrable complex structure J . It is clear that the *invariant complex structure J on G* descends to M in a natural way and it is given by an endomorphism $J : \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra \mathfrak{g} such that $J^2 = -\mathbf{1}$, satisfying the “Nijenhuis condition”

$$[JX, JY] = J[JX, Y] + J[X, JY] + [X, Y],$$

for any $X, Y \in \mathfrak{g}$.

Let $\mathfrak{g}_{\mathbb{C}}$ be the complexification of \mathfrak{g} and $\mathfrak{g}_{\mathbb{C}}^*$ its dual. We denote by $\mathfrak{g}^{1,0}$ and $\mathfrak{g}^{0,1}$ the eigenspaces corresponding to the eigenvalues $\pm\sqrt{-1}$ of J as an endomorphism of $\mathfrak{g}_{\mathbb{C}}^*$, respectively. The decomposition

$$\mathfrak{g}_{\mathbb{C}}^* = \mathfrak{g}^{1,0} \oplus \mathfrak{g}^{0,1}$$

gives rise to a natural bigraduation on the complexified exterior algebra

$$\bigwedge^* \mathfrak{g}_{\mathbb{C}}^* = \bigoplus_{p,q} \bigwedge^{p,q} \mathfrak{g}^* = \bigoplus_{p,q} \bigwedge^p \mathfrak{g}^{1,0} \oplus \bigoplus_{p,q} \bigwedge^q \mathfrak{g}^{0,1}.$$

We will still use the Chevalley–Eilenberg differential d of the Lie algebra to denote its extension to the complexified exterior algebra, that is, $d: \bigwedge^* \mathfrak{g}_{\mathbb{C}}^* \rightarrow \bigwedge^{*+1} \mathfrak{g}_{\mathbb{C}}^*$. It is well known that the endomorphism J is a complex structure if and only if

$$d\mathfrak{g}^{1,0} \subset \bigwedge^{2,0} \mathfrak{g}^* \oplus \bigwedge^{1,1} \mathfrak{g}^*.$$

As for nilpotent Lie algebras \mathfrak{g} , Salamon [46] proves the equivalence: J is a *complex structure* on \mathfrak{g} if and only if $\mathfrak{g}^{1,0}$ has a basis $\{\varpi^i\}_{i=1}^n$ such that $d\varpi^1 = 0$ and

$$d\varpi^i \in \mathcal{I}(\varpi^1, \dots, \varpi^{i-1}), \text{ for } i = 2, \dots, n,$$

where $\mathcal{I}(\varpi^1, \dots, \varpi^{i-1})$ is the ideal in $\bigwedge^* \mathfrak{g}_{\mathbb{C}}^*$ generated by $\{\varpi^1, \dots, \varpi^{i-1}\}$. The work [16] shows that a complex structure J is *nilpotent* if and only if $\mathfrak{g}^{1,0}$ admits a basis $\{\varpi^i\}_{i=1}^n$ with $d\varpi^1 = 0$ and

$$d\varpi^i \in \wedge^2 \langle \varpi^1, \dots, \varpi^{i-1}, \bar{\varpi}^1, \dots, \bar{\varpi}^{i-1} \rangle, \text{ for } i = 2, \dots, n;$$

otherwise J is *non-nilpotent*. *Abelian complex structures* satisfy additionally that $d\mathfrak{g}^{1,0} \subset \bigwedge^{1,1} \mathfrak{g}^*$. A nilpotent complex structure is *complex parallelizable* if and only if $d\mathfrak{g}^{1,0} \subset \bigwedge^{2,0} \mathfrak{g}^*$.

Inspired by Ugarte–Villacampa [53, Proposition 3.4 and Corollary 3.5], one has:

PROPOSITION 3.4. *Let $M = \Gamma \backslash G$ be a $2n$ -dimensional nilmanifold endowed with an invariant complex structure J (i.e., complex n -dimensional M) and \mathfrak{g} the Lie algebra of G . If (\mathfrak{g}, J) satisfies the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma and $H_{\bar{\partial}}^{n-2, n}(M, J) \cong H_{\bar{\partial}}^{n-2, n}(\mathfrak{g}, J)$, then (M, J) satisfies the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma.*

Proof. Let ξ be an $(n-2, n)$ -complex differential form on M . From the isomorphism $H_{\bar{\partial}}^{n-2, n}(M, J) \cong H_{\bar{\partial}}^{n-2, n}(\mathfrak{g}, J)$ and [53, Remark 3.3], there exists an $(n-2, n-1)$ -form θ on M such that

$$\xi = \xi_{\nu} + \bar{\partial}\theta,$$

where ξ_{ν} denotes the image of ξ under the symmetrization process $A^{p,q}(M) \rightarrow \bigwedge^{p,q}(\mathfrak{g}^*)$. The $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma on (\mathfrak{g}, J) implies that

$$\bar{\partial}\bar{\partial}\tilde{\theta} = \partial\xi_{\nu}$$

for some $\tilde{\theta} \in \bigwedge^{n-2, n-1}(\mathfrak{g}^*)$. It follows that

$$\partial\xi = \partial\bar{\partial}(\tilde{\theta} + \theta). \quad \square$$

COROLLARY 3.5. *Let $M = \Gamma \backslash G$ be a $2n$ -dimensional nilmanifold endowed with an invariant abelian complex structure J . Then M satisfies the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma.*

Proof. It is known that the isomorphism $H_{\bar{\partial}}^{p,q}(M, J) \cong H_{\bar{\partial}}^{p,q}(\mathfrak{g}, J)$ holds for any (p, q) with the abelian complex structure J (cf. [14, Remark 4], [45, Theorem 1.10] and the discussion ahead of [53, Remark 3.3]). And the abelian complex structure J implies that $\partial(\bigwedge^{n-2,n}(\mathfrak{g}^*)) = 0$. \square

Similarly with [53, Proposition 3.2], we have

PROPOSITION 3.6. *Let $M = \Gamma \backslash G$ be a $2n$ -dimensional nilmanifold endowed with an invariant complex structure J with \mathfrak{g} the Lie algebra of G . If (\mathfrak{g}, J) does not satisfy the $(n - 1, n)$ -th mild $\partial\bar{\partial}$ -lemma, then (M, J) does not satisfy the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma either.*

Proof. Suppose that the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma holds on (M, J) , that is, for any $(n - 2, n)$ -form ξ on M , there exists an $(n - 2, n - 1)$ -form θ such that

$$\partial\bar{\partial}\theta = \partial\xi.$$

Using the symmetrization process (cf. the proof of [53, Proposition 3.2]) on the both sides, we have

$$\partial\bar{\partial}\theta_{\nu} = \partial\xi_{\nu},$$

which contradicts the assumption on the Lie algebra level (\mathfrak{g}, J) . Here, the equalities

$$(\partial\alpha)_{\nu} = \partial\alpha_{\nu} \quad \text{and} \quad (\bar{\partial}\alpha)_{\nu} = \bar{\partial}\alpha_{\nu}$$

for any $\alpha \in A^{p,q}(M)$ are used as in the proof of [53, Proposition 3.2]. \square

The following result of the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma is almost the same as the mild one in Propositions 3.4 and 3.6, for which we omit the proofs.

PROPOSITION 3.7. *Let $M = \Gamma \backslash G$ be a $2n$ -dimensional nilmanifold endowed with an invariant complex structure J and \mathfrak{g} the Lie algebra of G . If (\mathfrak{g}, J) satisfies the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma and $H_{BC}^{n-1,n}(M, J) \cong H_{BC}^{n-1,n}(\mathfrak{g}, J)$, then (M, J) satisfies the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma. Similarly if (\mathfrak{g}, J) does not satisfy the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma, then (M, J) does not satisfy the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma.*

EXAMPLE 3.8. The complex structure in the category (i) of [53, Proposition 2.3], that is, the complex parallelizable case of complex dimension 3, satisfies the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma and the dual mild one, but does not satisfy the mild one. The Iwasawa manifold belongs to the category (i).

Proof. Let J be the complex structure in the category (i), which satisfies the $(2, 3)$ th weak $\partial\bar{\partial}$ -lemma by the proof of [53, Proposition 3.6]. It is easy to check that, on the Lie algebra level,

$$\partial(\bigwedge^{1,3}(\mathfrak{g}^*)) = \langle \omega^{12\bar{1}\bar{2}\bar{3}} \rangle, \quad \bar{\partial}(\bigwedge^{1,2}(\mathfrak{g}^*)) = 0 \quad \text{and} \quad \bar{\partial}(\bigwedge^{2,2}(\mathfrak{g}^*)) = 0.$$

However,

$$\partial\omega^{3\bar{1}\bar{2}\bar{3}} = \omega^{12\bar{1}\bar{2}\bar{3}} \notin \partial\bar{\partial}(\bigwedge^{1,2}(\mathfrak{g}^*)) = 0,$$

and thus (2, 3)th mild $\partial\bar{\partial}$ -lemma does not hold on the nilmanifold, from Proposition 3.6. Here, $\omega^{3\bar{1}\bar{2}\bar{3}}$ denotes $\omega^3 \wedge \bar{\omega}^1 \wedge \bar{\omega}^2 \wedge \bar{\omega}^3$ as the notation used in [53]. Also, it is clear that (\mathfrak{g}, J) satisfies the (2, 3)th dual mild $\partial\bar{\partial}$ -lemma, with $H_{BC}^{p,q}(M, J) \cong H_{BC}^{p,q}(\mathfrak{g}, J)$ for any p, q assured by [6, Theorem 3.8]. Therefore, the (2, 3)-th dual mild $\partial\bar{\partial}$ -lemma holds on the nilmanifold by Proposition 3.7. \square

An invariant balanced Hermitian structure F on a nilmanifold M is the one, coming from a balanced Hermitian structure on the Lie algebra \mathfrak{g}^* .

PROPOSITION 3.9. *Let M be a six-dimensional nilmanifold endowed with an invariant balanced Hermitian structure (J, F) . Then (M, J) satisfies the (2, 3)th mild $\partial\bar{\partial}$ -lemma if and only if J is abelian or non-nilpotent, and (M, J) satisfies the (2, 3)th dual mild $\partial\bar{\partial}$ -lemma if and only if J is complex parallelizable.*

Proof. It is known, from [53, Proposition 3.6], (M, J) satisfies the (2, 3)th weak $\partial\bar{\partial}$ -lemma if and only if J is abelian, complex parallelizable or non-nilpotent. Example 3.8 shows that the complex parallelizable case satisfies the (2, 3)th dual mild $\partial\bar{\partial}$ -lemma but does not satisfy the mild one. Corollary 3.5 and [9, Proposition 2.9] say that a $2n$ -dimensional nilmanifold endowed with an invariant abelian complex structure satisfies the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma but never satisfies the $(n - 1, n)$ th dual mild $\partial\bar{\partial}$ -lemma, especially for the situation of the six-dimension (i.e., complex dimension 3) here. Hence, the only left to check is the non-nilpotent case.

As shown in the proof of [53, Proposition 3.6], the equality holds

$$\partial\bar{\partial}(\bigwedge^{1,2}(\mathfrak{g}^*)) = \langle \omega^{13\bar{1}\bar{2}\bar{3}} \rangle = \partial(\bigwedge^{1,3}(\mathfrak{g}^*)),$$

according to the category (iii) of [53, Proposition 2.3], and the natural inclusion

$$(\bigwedge^{*,*}(\mathfrak{g}^*), \bar{\partial}) \hookrightarrow (A^{*,*}(M), \bar{\partial})$$

induces an isomorphism in the $\bar{\partial}$ -cohomology in this case. Therefore, (M, J) with the non-nilpotent complex structure J satisfies the (2, 3)-th mild $\partial\bar{\partial}$ -lemma by Proposition 3.4. Meanwhile, it should be noted that

$$\bar{\partial}\omega^{13\bar{2}\bar{3}} = \pm\sqrt{-1}\omega^{12\bar{1}\bar{2}\bar{3}} \quad \text{and} \quad \partial\omega^{12\bar{1}\bar{2}\bar{3}} = 0,$$

according to the category (iii) of [53, Proposition 2.3]. Hence, we have

$$\text{the } \partial\text{-closed (2, 3)-form } \bar{\partial}\omega^{13\bar{2}\bar{3}} = \pm\sqrt{-1}\omega^{12\bar{1}\bar{2}\bar{3}} \notin \partial\bar{\partial}(\bigwedge^{1,2}(\mathfrak{g}^*)) = \langle \omega^{13\bar{1}\bar{2}\bar{3}} \rangle.$$

Then Proposition 3.7 tells us that (M, J) with J non-nilpotent does not satisfy the (2, 3)th dual mild $\partial\bar{\partial}$ -lemma. \square

Explicit examples of different complex structures in Proposition 3.9 have been provided in [53]. Directly from Observation 3.3 and Proposition 3.9, one has:

COROLLARY 3.10. *Let M be a six-dimensional nilmanifold endowed with an invariant balanced Hermitian structure (J, F) . Then the (2, 3)th strong $\partial\bar{\partial}$ -lemma does not hold on M except for a torus, that is, the mild one and dual mild one never hold simultaneously on M except the torus case. Especially, the $(n - 1, n)$ th mild $\partial\bar{\partial}$ -lemma and the dual mild one are unrelated.*

REMARK 3.11. The (2,3)th strong $\partial\bar{\partial}$ -lemma can hold on the completely-solvable balanced Nakamura manifolds, with one concrete example given in [10, Example 4.10], which is of complex dimension 3 and satisfies $\dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_A^{0,1}(X) = 1$.

Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact n -dimensional complex manifolds with the reference fiber $\pi^{-1}(t_0) = X_0$ and the general fibers $X_t := \pi^{-1}(t)$. Here, B denotes a sufficiently small domain in \mathbb{R}^k . Fu–Yau [23, Theorem 6] show that the balanced structure is deformation open, assuming that the $(n-1, n)$ th weak $\partial\bar{\partial}$ -lemma holds on the general fibers X_t for $t \neq 0$. Angella–Ugarte [10, Theorem 4.9] prove that if X_0 admits a locally conformal balanced metric and satisfies the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma, then X_t is balanced for t small. Our main result in this section, whose proof is postponed to the next subsection, is

THEOREM 3.12. *Let X_0 be a compact balanced manifold of complex dimension n , satisfying the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma. Then X_t also admits a balanced metric for t small.*

It is well known from [3] that small deformation of the Iwasawa manifold, which satisfies the (2,3)th weak $\partial\bar{\partial}$ -lemma but does not satisfy the mild one from Example 3.8, may not be balanced. Thus, the condition “ $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma” in Theorem 3.12 cannot be replaced by the weak one. It is an obvious generalization of Wu’s result [55, Theorem 5.13] that the balanced condition is preserved under small deformation if the reference fiber satisfies the $\partial\bar{\partial}$ -lemma. Based on Corollary 3.5 and Proposition 3.9, one obtains:

COROLLARY 3.13. *Let M be a $2n$ -dimensional nilmanifold endowed with an invariant abelian balanced Hermitian structure. Then small deformation of M is also balanced. Moreover, in the case of six-dimension, this result still holds when M is endowed with the non-nilpotent balanced Hermitian structure.*

Examples and results such as [10, Proposition 4.4, Remark 4.6, Remark 4.7, and Example 4.10] and [23, Corollary 8 and Corollary 9] become consequences of Theorem 3.12 and Corollary 3.13. And it is interesting to note that six-dimensional nilmanifolds endowed with an invariant abelian or non-nilpotent balanced Hermitian structure provide solutions of the Strominger system with respect to the Bismut connection or the Chern connection in the anomaly cancellation condition, respectively. See [53, Section 5] and [52, Section 4] for more details.

EXAMPLE 3.14 ([53, Example 3.7]). Ugarte–Villacampa constructed an explicit family of nilmanifolds with invariant complex structures I_λ for $\lambda \in [0, 1)$ (of complex dimension 3), with the fixed underlying manifold the Iwasawa manifold. The complex structure of the reference fiber I_0 is abelian and admits an invariant balanced metric, satisfying the (2,3)th mild $\partial\bar{\partial}$ -lemma by Proposition 3.9. The complex structures of I_λ for $\lambda \neq 0$ are nilpotent from [15, Corollary 2], but neither complex-parallelizable nor abelian. And thus they do not satisfy the (2,3)th weak $\partial\bar{\partial}$ -lemma by [53, Proposition 3.6]. However, the nilmanifolds I_λ for $\lambda \neq 0$ admit balanced metrics.

The example above proves that neither the $(n-1, n)$ th weak $\partial\bar{\partial}$ -lemma nor the mild one is deformation open. And it shows that the condition in [23, Theorem 6] is not a necessary one for the deformation openness of balanced structures as mentioned in [53, the discussion ahead of Example 3.7]. Fortunately, Corollary 3.13 can be applied to this example. See also [10, Remark 4.7], where Corollary 3.13 can also be applied.

Meanwhile, from Corollary 3.5 and [9, Proposition 2.9], a $2n$ -dimensional nilmanifold endowed with an invariant abelian complex structure satisfies the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma but never satisfies the $(n-1, n)$ th dual mild $\partial\bar{\partial}$ -lemma. It implies that the deformation openness of balanced structures with the reference fiber a $2n$ -dimensional nilmanifold endowed with an invariant abelian balanced Hermitian structure can be obtained by Corollary 3.13, but it cannot be derived from [10, Theorem 4.9].

Besides, it is known that the deformation invariance of the dimensions of the $(n-1, n-1)$ th Bott–Chern group $H_{BC}^{n-1, n-1}(X_t)$ can assure the deformation openness of balanced structures as shown in [10, Proposition 4.1]. See also Proposition 4.1, which is a kind of generalization of this result. However, [10, Example 4.10] shows that small deformation of a completely-solvable Nakamura threefold, which is balanced and satisfies the $(2, 3)$ th strong $\partial\bar{\partial}$ -lemma, is also balanced. The $(2, 2)$ th Bott–Chern number varies along this deformation. Fortunately, Theorem 3.12 is applicable to this case and also possibly to some cases with deformation variance of $(n-1, n-1)$ th Bott–Chern number.

Finally, from the perspective of Theorem 3.12, we may have a clear picture of Angella–Ugarte’s result [10, Theorem 4.9], which states that if X_0 admits a locally conformal balanced metric and satisfies the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma, then X_t is balanced for t small. Actually, the $(n-1, n)$ th strong $\partial\bar{\partial}$ -lemma decomposes into the mild one and the dual mild one, according to Observation 3.3. A locally conformal balanced metric can be transformed into a balanced one by the $(n-1, n)$ -th dual mild $\partial\bar{\partial}$ -lemma, from [9, Theorem 2.5]. Then the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma assures that the deformation openness of balanced structures starts from the transformed balanced metric on the reference fiber, thanks to Theorem 3.12.

3.2 Proof for stability of balanced structures with mild $\partial\bar{\partial}$ -lemma

In this subsection, we will prove the local stability Theorem 3.12 of balanced structures with the balanced reference fiber X_0 with the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma. Similarly to the Kähler case, we will reduce the proof to Kuranishi family as described in the beginning of Section 2.

Our goal is to construct a power series $\Omega(t) \in A^{n-1, n-1}(X_0)$ such that

$$\begin{cases} d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega(t))) = 0, \\ \Omega(t) = \overline{\Omega(t)}, \\ \Omega(0) = \omega^{n-1}, \end{cases} \quad (41)$$

and show the Hölder $C^{k, \alpha}$ -convergence and regularity of the power series $\Omega(t)$. Then it is clear that $e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega(t))$ will be a positive $(n-1, n-1)$ -form on X_t for t small, due to the positivity of its initial complex differential form ω^{n-1} and the convergence argument. By the real property of $e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}$ as in [43, Lemma 2.8], it suffices to solve the following system of equations

$$\begin{cases} d(e^{\iota_\varphi|_{\iota_{\bar{\varphi}}}}(\Omega(t))) = 0, \\ \Omega(0) = \omega^{n-1}, \end{cases} \quad (42)$$

since $\frac{\Omega(t) + \overline{\Omega(t)}}{2}$ from one solution $\Omega(t)$ of (42) becomes one of the system (41). The resolution of the system (42) below is a bit different from the one for the Kähler case, which relies more on the form type $(n-1, n-1)$.

As both $e^{t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}$ and $e^{t\varphi}$ are invertible operators when t is sufficiently small, it follows that for any $\Omega \in A^{n-1,n-1}(X_0)$,

$$e^{t\varphi}|_{\iota\bar{\varphi}}(\Omega) = e^{t\varphi} \circ e^{t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-t\varphi} \circ e^{t\varphi}|_{\iota\bar{\varphi}}(\Omega). \tag{43}$$

Set

$$\tilde{\Omega} = e^{-t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-t\varphi} \circ e^{t\varphi}|_{\iota\bar{\varphi}}(\Omega), \tag{44}$$

where Ω and $\tilde{\Omega}$ are apparently one-to-one correspondence. And it is easy to check that the operator

$$e^{-t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-t\varphi} \circ e^{t\varphi}|_{\iota\bar{\varphi}}$$

preserves the form types and thus $\tilde{\Omega}$ is still an $(n-1, n-1)$ -form. In fact, for any (p, q) -form α on X_0 , we will find

$$\begin{aligned} & e^{-t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-t\varphi} \circ e^{t\varphi}|_{\iota\bar{\varphi}}(\alpha) \\ &= \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner d\bar{z}^{j_1} \wedge \dots \wedge (\mathbb{1} - \bar{\varphi}\varphi) \lrcorner d\bar{z}^{j_q} \in A^{p,q}(X_0), \end{aligned}$$

where $\alpha = \alpha_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q} dz^{i_1} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \wedge \dots \wedge d\bar{z}^{j_q}$. Together with (43) and (44), we obtain that

$$\begin{aligned} d(e^{t\varphi}|_{\iota\bar{\varphi}}(\Omega)) &= d \circ e^{t\varphi} \circ e^{t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= e^{t\varphi} \circ (\bar{\partial} + [\partial, \iota_\varphi] + \partial) \circ e^{t(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= e^{t\varphi} \circ (\bar{\partial} + [\partial, \iota_\varphi] + \partial)(\tilde{\Omega} + \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega})), \end{aligned} \tag{45}$$

where Proposition 1.3 is used in the second equality of (45) and the third equality results from the form type of $\tilde{\Omega}$.

By the invertibility of the operator $e^{t\varphi}$ and the form-type comparison, the equation $d(e^{t\varphi}|_{\iota\bar{\varphi}}(\Omega)) = 0$ amounts to

$$\begin{cases} (\bar{\partial} + [\partial, \iota_\varphi])\tilde{\Omega} = 0, \\ \partial\tilde{\Omega} + (\bar{\partial} + \partial \circ \iota_\varphi) \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) = 0. \end{cases} \tag{46}$$

Then the second equation in (46) and Lemma 2.5 imply

$$\begin{aligned} \iota_\varphi \circ \partial\tilde{\Omega} &= -\iota_\varphi \circ (\bar{\partial} + \partial \circ \iota_\varphi) \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= -(\bar{\partial} \circ \iota_\varphi - \iota_{\frac{1}{2}[\varphi, \varphi]} + \iota_\varphi \circ \partial \circ \iota_\varphi) \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= -(\bar{\partial} \circ \iota_\varphi + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi + \frac{1}{2}\iota_\varphi \circ \iota_\varphi \circ \partial) \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}) \\ &= -(\bar{\partial} \circ \iota_\varphi + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi) \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}), \end{aligned} \tag{47}$$

where the form type of $\tilde{\Omega}$ is also used in the fourth equality of (47). Substituting (47) into (46), one obtains that (46) is equivalent to

$$\begin{cases} (\bar{\partial} + \partial \circ \iota_\varphi + \bar{\partial} \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega} = 0, \\ (\bar{\partial} + \bar{\partial} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \partial \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega} = 0. \end{cases} \tag{48}$$

For the resolution of $\partial\bar{\partial}$ -equations, we need a lemma due to [41, Theorem 4.1]:

LEMMA 3.15. *Let (X, ω) be a compact Hermitian complex manifold with the pure-type complex differential forms x and y . Assume that the $\partial\bar{\partial}$ -equation*

$$\partial\bar{\partial}x = y \tag{49}$$

admits a solution. Then an explicit solution of the $\partial\bar{\partial}$ -equation (49) can be chosen as

$$(\partial\bar{\partial})^*\mathbb{G}_{BC}y,$$

which uniquely minimizes the L^2 -norms of all the solutions with respect to ω . Besides, the equalities hold

$$\mathbb{G}_{BC}(\partial\bar{\partial}) = (\partial\bar{\partial})\mathbb{G}_A \quad \text{and} \quad (\partial\bar{\partial})^*\mathbb{G}_{BC} = \mathbb{G}_A(\partial\bar{\partial})^*,$$

where \mathbb{G}_{BC} and \mathbb{G}_A are the associated Green's operators of \square_{BC} and \square_A , respectively. Here, \square_{BC} is defined in (7) and \square_A is the second Kodaira–Spencer operator (often also called Aeppli Laplacian)

$$\square_A = \partial^*\bar{\partial}^*\bar{\partial}\partial + \bar{\partial}\partial\partial^*\bar{\partial}^* + \bar{\partial}\partial^*\partial\bar{\partial}^* + \partial\bar{\partial}^*\bar{\partial}\partial^* + \bar{\partial}\bar{\partial}^* + \partial\partial^*.$$

Proof. We shall use the Hodge decomposition of \square_{BC} on X :

$$A^{p,q}(X) = \ker \square_{BC} \oplus \text{Im}(\partial\bar{\partial}) \oplus (\text{Im} \partial^* + \text{Im} \bar{\partial}^*), \tag{50}$$

whose three parts are orthogonal to each other with respect to the L^2 -scalar product defined by ω , combined with the equality

$$\mathbf{1} = \mathbb{H}_{BC} + \square_{BC}\mathbb{G}_{BC},$$

where \mathbb{H}_{BC} is the harmonic projection operator. And it should be noted that

$$\ker \square_{BC} = \ker \partial \cap \ker \bar{\partial} \cap \ker (\partial\bar{\partial})^*. \tag{51}$$

Then two observations follow:

- (1) $\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}$.
- (2) $\mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC}$.

It is clear that (1) implies (2). Actually, (1) yields

$$\mathbb{G}_{BC}\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC} = \mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}\mathbb{G}_{BC}.$$

A routine check shows that

$$\mathbb{G}_{BC}\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC} = (\mathbf{1} - \mathbb{H}_{BC})\partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC} = \partial\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC}$$

and

$$\mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}\mathbb{G}_{BC} = \mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*(\mathbf{1} - \mathbb{H}_{BC}) = \mathbb{G}_{BC}\partial\bar{\partial}(\partial\bar{\partial})^*;$$

while, the statement (1) is proved by a direct calculation:

$$\square_{BC}\partial\bar{\partial}(\partial\bar{\partial})^* = (\partial\bar{\partial})(\partial\bar{\partial})^*(\partial\bar{\partial})(\partial\bar{\partial})^* = \partial\bar{\partial}(\partial\bar{\partial})^*\square_{BC}.$$

Hence, one has

$$(\partial\bar{\partial})(\partial\bar{\partial})^*\mathbb{G}_{BC}y = \mathbb{G}_{BC}(\partial\bar{\partial})(\partial\bar{\partial})^*y = \mathbb{G}_{BC}\square_{BC}y = (\mathbf{1} - \mathbb{H}_{BC})y = y,$$

where $y \in \text{Im} \partial\bar{\partial}$ due to the solution-existence of the $\partial\bar{\partial}$ -equation.

To see that the solution $(\partial\bar{\partial})^*\mathbb{G}_{BC}y$ is the unique L^2 -norm minimum, we resort to the Hodge decomposition of the operator \square_A :

$$A^{p,q}(X) = \ker \square_A \oplus (\text{Im } \partial + \text{Im } \bar{\partial}) \oplus \text{Im } (\partial\bar{\partial})^*, \tag{52}$$

where $\ker \square_A = \ker(\partial\bar{\partial}) \cap \ker \partial^* \cap \ker \bar{\partial}^*$. Let z be an arbitrary solution of the $\partial\bar{\partial}$ -equation (49), which decomposes into three components $z_1 + z_2 + z_3$ with respect to the Hodge decomposition (52) of \square_A . By the Hodge theory of \square_A , the equality holds

$$\ker(\partial\bar{\partial}) = \ker \square_A \oplus (\text{Im } \partial + \text{Im } \bar{\partial}),$$

which implies that $\partial\bar{\partial}(z_1 + z_2) = 0$. Hence, it follows that

$$\partial\bar{\partial}z = \partial\bar{\partial}z_3 = y.$$

After noticing that $\bar{\partial}^* z_3 = \partial^* z_3 = 0$, we get

$$(\partial\bar{\partial})^*y = (\partial\bar{\partial})^*\partial\bar{\partial}z_3 = \square_A z_3,$$

which implies that

$$\mathbb{G}_A(\partial\bar{\partial})^*y = \mathbb{G}_A \square_A z_3 = (\mathbf{1} - \mathbb{H}_A)z_3 = z_3.$$

Then it is obvious that

$$\square_{BC}(\partial\bar{\partial}) = \partial\bar{\partial}(\partial\bar{\partial})^*\partial\bar{\partial} = (\partial\bar{\partial})\square_A \tag{53}$$

with $\mathbb{G}_{BC}(\partial\bar{\partial}) = (\partial\bar{\partial})\mathbb{G}_A$ established as well. Taking adjoint operators of both sides in (53), we will find that

$$(\partial\bar{\partial})^*\square_{BC} = \square_A(\partial\bar{\partial})^*,$$

implying the equality

$$(\partial\bar{\partial})^*\mathbb{G}_{BC} = \mathbb{G}_A(\partial\bar{\partial})^*.$$

And thus, we obtain that

$$z_3 = \mathbb{G}_A(\partial\bar{\partial})^*y = (\partial\bar{\partial})^*\mathbb{G}_{BC}y.$$

Therefore,

$$\|z\|^2 = \|z_1\|^2 + \|z_2\|^2 + \|z_3\|^2 \geq \|z_3\|^2 = \|(\partial\bar{\partial})^*\mathbb{G}_{BC}y\|^2,$$

and the equality holds if and only if $z_1 = z_2 = 0$, that is, $z = z_3 = (\partial\bar{\partial})^*\mathbb{G}_{BC}y$. □

Now we arrive at:

PROPOSITION 3.16. *Let ω be a balanced metric on X_0 , which satisfies the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma. Then the system (41) of equations is formally solved.*

Proof. It suffices to resolve (48) with initial value $\tilde{\Omega}(0) = \omega^{n-1}$. Actually, the solution $\tilde{\Omega}(t)$ of the system (48), satisfying that $\tilde{\Omega}(0) = \omega^{n-1}$, corresponds to the one of (42), by the relation below as in (44)

$$\tilde{\Omega}(t) = e^{-\iota_{(\mathbf{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} \circ e^{-\iota_{\varphi}} \circ e^{\iota_{\varphi}}|_{\iota_{\bar{\varphi}}}(\Omega(t)).$$

Therefore, we focus on

$$\begin{cases} (\bar{\partial} + \partial \circ \iota_\varphi + \bar{\partial} \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) = 0, \\ (\partial + \bar{\partial} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \partial \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) = 0, \\ \tilde{\Omega}(0) = \omega^{n-1}, \end{cases}$$

and solve this system of equations also by the iteration method used in the Kähler case.

Assume that $\tilde{\Omega}(t)$ can develop into a power series as follows:

$$\begin{cases} \tilde{\Omega}(t) = \sum_{k=0}^\infty \tilde{\Omega}_k, \\ \tilde{\Omega}_k = \sum_{i+j=k} \tilde{\Omega}_{i,j} t^i \bar{t}^j, \end{cases}$$

where $\tilde{\Omega}_k$ is the k -degree homogeneous part in the expansion of $\tilde{\Omega}(t)$ and $\tilde{\Omega}_{i,j}$ are all smooth $(n-1, n-1)$ -forms on X_0 . Here, we follow the notation described before Proposition 2.7. And in (37) the holomorphic family of integrable Beltrami differentials on X_0 develops a power series of Beltrami differentials in t as

$$\varphi(t) = \sum_{i=1}^\infty \varphi_i t^i.$$

Hence, we need to solve

$$\begin{cases} \left((\bar{\partial} + \partial \circ \iota_\varphi + \bar{\partial} \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \frac{1}{2}\partial \circ \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) \right)_k = 0, \\ \left((\partial + \bar{\partial} \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}} + \partial \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}})\tilde{\Omega}(t) \right)_k = 0, \\ \tilde{\Omega}(0) = \omega^{n-1}, \end{cases} \tag{54}$$

for any $k \geq 0$.

The case $k = 0$ of the equations (54) holds since ω is a balanced metric. By induction, we assume that the system (54) of equations is solved for each $k \leq l$ and the solutions are denoted by $\{\tilde{\Omega}_k\}_{k \leq l} \in A^{n-1, n-1}(X_0)$. By the form-type consideration, one observes that the $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma produces μ and ν , such that the following equalities

$$\begin{cases} \partial\bar{\partial}\mu = -\bar{\partial}\left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))\right)_{l+1}, \\ \bar{\partial}\partial\nu = -\partial\left(\left(\iota_\varphi + \frac{1}{2}\iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}\right)\tilde{\Omega}(t)\right)_{l+1} \end{cases}$$

hold as the ones (27) in Observation 2.11. Then Lemma 3.15 enables us to determine the two explicit solutions

$$\begin{cases} \mu = -(\partial\bar{\partial})^* \mathbb{G}_{BC} \bar{\partial}\left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))\right)_{l+1}, \\ \nu = (\partial\bar{\partial})^* \mathbb{G}_{BC} \partial\left(\left(\iota_\varphi + \frac{1}{2}\iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}\right)\tilde{\Omega}(t)\right)_{l+1}. \end{cases} \tag{55}$$

Hence, the $(l+1)$ th order of the system (54) is solved, yielding that

$$\tilde{\Omega}_{l+1} + \left(\iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))\right)_{l+1} = \bar{\partial}\mu + \partial\nu,$$

in the same manner as (28), where μ and ν are given by (55). Therefore, we complete the proof. \square

Then we come to the Hölder convergence argument for $\tilde{\Omega}(t)$. For $l = 1, 2, \dots$, one l -degree canonical formal solution of (54) is given by induction:

$$\begin{aligned} \tilde{\Omega}_l = & - \left(\iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)_l - \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)_l \\ & + \partial(\partial\bar{\partial})^* \mathbb{G}_{BC} \partial \left(\left(\iota_\varphi + \frac{1}{2} \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}} \right) \tilde{\Omega}(t) \right)_l \end{aligned} \tag{56}$$

and obviously

$$\begin{aligned} \tilde{\Omega}^{(l)} = & - \left(\iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)^{(l)} - \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)^{(l)} \\ & + \partial(\partial\bar{\partial})^* \mathbb{G}_{BC} \partial \left(\left(\iota_\varphi + \frac{1}{2} \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}} \right) \tilde{\Omega}(t) \right)^{(l)}. \end{aligned} \tag{57}$$

In this proof, we follow the notations in Section 2.3.

We use an important a priori estimate for the three terms in the left-hand side of the above equality: for any complex differential form ϕ ,

$$\|\mathbb{G}_{BC}\phi\|_{k,\alpha} \leq C_{k,\alpha} \|\phi\|_{k-4,\alpha}, \tag{58}$$

where $k > 3$ and $C_{k,\alpha}$ depends on only on k and α , not on ϕ . See [28, Appendix Theorem 7.4] for example. Assume that

$$\|\tilde{\Omega}^{(r-1)}\|_{k,\alpha}, \|\tilde{\Omega}^{(r-2)}\|_{k,\alpha}, \|\tilde{\Omega}^{(r-3)}\|_{k,\alpha} \ll A(t).$$

So, by the expression (57) and the a priori estimate (58),

$$\begin{aligned} & \left\| - \left(\iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)^{(r)} \right\|_{k,\alpha} \\ = & \left\| - (\varphi \lrcorner (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi} \lrcorner \tilde{\Omega}^{(r-2)})^{(r)} - (\varphi \lrcorner (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi} \lrcorner \tilde{\Omega}_0)^{(r)} \right\|_{k,\alpha} \\ \ll & 2 \|\varphi^{(r-2)}\|_{k,\alpha}^2 \cdot \|\tilde{\Omega}^{(r-2)}\|_{k,\alpha} + 2 \|\varphi^{(r-1)}\|_{k,\alpha}^2 \cdot \|\tilde{\Omega}_0\|_{k,\alpha} \\ \ll & 2 \left(\frac{\beta}{\gamma} \right) \left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha} \right) A(t), \\ & \left\| - \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)^{(r)} \right\|_{k,\alpha} \\ = & \left\| \bar{\partial}(\partial\bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left((\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi} \lrcorner \tilde{\Omega}^{(r-1)} + (\mathbb{1} - \bar{\varphi}\varphi)^{-1} \bar{\varphi} \lrcorner \tilde{\Omega}_0 \right)^{(r)} \right\|_{k,\alpha} \\ \ll & C_1^2 C_{k,\alpha} \left(2 \|\varphi^{(r-1)}\|_{k,\alpha} \|\tilde{\Omega}^{(r-1)}\|_{k,\alpha} + 2 \|\varphi^{(r)}\|_{k,\alpha} \|\tilde{\Omega}_0\|_{k,\alpha} \right) \\ \ll & 2 C_1^2 C_{k,\alpha} \left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha} \right) A(t) \end{aligned}$$

and similarly,

$$\begin{aligned} & \left\| \partial(\partial\bar{\partial})^* \mathbb{G}_{BC} \partial \left(\left(\iota_\varphi + \frac{1}{2} \iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1} - \bar{\varphi}\varphi)^{-1}\bar{\varphi}} \right) \tilde{\Omega}(t) \right)^{(r)} \right\|_{k,\alpha} \\ \ll & C_1^2 C_{k,\alpha} \left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha} \right) \left(1 + \left(\frac{\beta}{\gamma} \right)^2 \right) A(t). \end{aligned}$$

Hence, we choose $\beta, \gamma, \|\tilde{\Omega}_0\|_{k,\alpha}$ so that the following inequalities hold:

$$2\left(\frac{\beta}{\gamma}\right)\left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha}\right), 2C_1^2 C_{k,\alpha}\left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha}\right), C_1^2 C_{k,\alpha}\left(\frac{\beta}{\gamma} + \|\tilde{\Omega}_0\|_{k,\alpha}\right)\left(1 + \left(\frac{\beta}{\gamma}\right)^2\right) < \frac{1}{3},$$

(38) for the integrable Beltrami differential $\varphi(t)$ and

$$\|\tilde{\Omega}^{(1)}\|_{k,\alpha}, \|\tilde{\Omega}^{(2)}\|_{k,\alpha}, \|\tilde{\Omega}^{(3)}\|_{k,\alpha} \ll A(t)$$

according the above formulation, which are obviously possible as t is small. Thus, we obtain that

$$\|\tilde{\Omega}^{(r)}\|_{k,\alpha} \ll A(t), \text{ for any } r \geq 1,$$

which implies the desired convergence $\|\tilde{\Omega}^{(+\infty)}\|_{k,\alpha} \ll A(t)$ immediately.

Finally, inspired by the elliptic argument [28, Appendix 8], we will apply the interior estimate to obtain the regularity of $\tilde{\Omega}(t)$, which is a local problem. By the canonical formal solution expression (56) of (54), one just needs to use the following strongly elliptic second-order pseudo-differential equation

$$\begin{aligned} \square \tilde{\Omega}(t) = & -\square\left(\iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))\right) - \bar{\partial}\bar{\partial}^*\bar{\partial}(\partial\bar{\partial})^*\mathbb{G}_{BC}\bar{\partial}\left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t))\right) \\ & + \square\partial(\partial\bar{\partial})^*\mathbb{G}_{BC}\partial\left(\left(\iota_\varphi + \frac{1}{2}\iota_\varphi \circ \iota_\varphi \circ \iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}\right)\tilde{\Omega}(t)\right), \end{aligned} \tag{59}$$

where \square is the $\bar{\partial}$ -Laplacian defined by (39). Recall that an elliptic partial differential operator of order $2m$ is pseudo-differential and so is its inverse, whose order becomes $-2m$ as a pseudo-differential operator.

We cover $X := X_0$ by coordinates neighborhoods $X_j, j = 1, \dots, J$. Let $z_j = (z_j^1, \dots, z_j^n)$ be the local holomorphic coordinates on X_j with $z_j^k = x_j^k + \sqrt{-1}x_j^{k+n}$ and put $t = x^{2n+1} + \sqrt{-1}x^{2n+2} \in \Delta_\epsilon$ a ϵ -disk in \mathbb{C} . By these $2n + 2$ real coordinates, $X_j \times \Delta_\epsilon$ is identified with an open set U_j of a $(2n + 2)$ -dimensional torus \mathbb{T}^{2n+2} .

Choose a partition of unity subordinate to X_j , that is, a set $\{\rho_j\}$ of C^∞ functions $\rho_j(x)$ on X so that $\text{supp } \rho_j \subset X_j$ and for any $x \in X, \sum_{j=1}^J \rho_j(x) \equiv 1$. For each $l = 1, 2, \dots$, choose a smooth function $\eta^l(t)$ with values in $[0, 1]$:

$$\eta^l(t) \equiv \begin{cases} 1, & \text{for } |t| \leq \left(\frac{1}{2} + \frac{1}{2^{l+1}}\right)r; \\ 0, & \text{for } |t| \geq \left(\frac{1}{2} + \frac{1}{2^l}\right)r, \end{cases} \tag{60}$$

where r is a positive constant to be determined. Notice that r is crucially used to give the uniform bound for the convergence radius of $\tilde{\Omega}(t)$.

Set

$$\rho_j^l(x, t) = \rho_j(x)\eta^l(t). \tag{61}$$

Recall in the proof for $\varphi(t)$ in [28, Appendix 8], one should also set

$$\chi_j^l(x, t) = \chi_j(x)\eta^l(t),$$

where the smooth function $\chi_j(x)$ with $\text{supp } \chi_j \subset X_j$ is identically equal to 1 on some neighborhood of the support of ρ_j . But here we will replace its role directly by $\eta^l(t)$ to avoid the trouble caused by the presence of Green's operator \mathbb{G}_{BC} .

First we will prove that $\eta^3\tilde{\Omega}$ is $C^{k+1,\alpha}$. Consider the equation:

$$\square(\Delta_i^h \rho_j^3 \tilde{\Omega}) = F_1^1 + F_2^1 + F_3^1, \tag{62}$$

where F_1^1, F_2^1, F_3^1 denote the three terms with respect to the ones in the right-hand side of (59) after the corresponding operations, respectively, and $i = 1, \dots, 2n$. Here Δ_i^h is the difference quotient as [28, Appendix (8.14)]. In particular,

$$F_2^1 = -\Delta_i^h \left(\rho_j^3 \bar{\partial} \bar{\partial}^* \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right) \right) + \text{lower-order terms of } \tilde{\Omega}(t).$$

In this proof, the ‘‘order’’ refers to the one of a pseudo-differential operator. Since \square is an elliptic linear differential operator whose principal part is of diagonal type, by [28, Appendix Theorem 2.3] and (62), one obtains the a priori estimate

$$\begin{aligned} \|\Delta_i^h \rho_j^3 \tilde{\Omega}\|_{k,\alpha} &\leq C_k (\|F_1^1 + F_2^1 + F_3^1\|_{k-2,\alpha} + \|\Delta_i^h \rho_j^3 \tilde{\Omega}\|_0) \\ &\leq C_k (\|F_1^1\|_{k-2,\alpha} + \|F_2^1\|_{k-2,\alpha} + \|F_3^1\|_{k-2,\alpha} + \|\Delta_i^h \rho_j^3 \tilde{\Omega}\|_0), \end{aligned} \tag{63}$$

where C_k is a positive constant, possibly depending on k . Now let us estimate the first three terms in the right-hand side of the above inequality. Here, we just estimate the second one, the most troublesome one, since the other two terms are quite analogous.

We need an equality for the Green’s operator: Let $E := E(x, D)$ be an elliptic linear partial differential operator on a smooth manifold and G_E, \mathbb{H}_E its associated Green’s operator and orthogonal projection to $\ker E$ defined such as in [28, Appendix Definition 7.2]. Then for any smooth function f and differential form ϖ on this manifold,

$$fG_E\varpi = G_E(f\varpi) - G_E(f\mathbb{H}_E\varpi) + \mathbb{H}_E(fG_E\varpi) + \text{lower-order terms of } \varpi. \tag{64}$$

Applying this equality to $\rho_j^3 \bar{\partial} \bar{\partial}^* \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)$ with $E = \square_{BC}$, $f = \rho_j^3$ and $\varpi = \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)$, one obtains

$$\begin{aligned} &\rho_j^3 \bar{\partial} \bar{\partial}^* \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right) \\ &= \bar{\partial} \bar{\partial}^* \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\rho_j^3 \tilde{\Omega}(t)) \right) + \text{lower-order terms of } \tilde{\Omega}(t), \end{aligned} \tag{65}$$

where we use the fact that $\bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right)$ is $\bar{\partial} \bar{\partial}$ -exact and the equalities (50) and (51). Thus, by (65) and the useful formula

$$\eta^{2l-1} \cdot \eta^{2l+1} = \eta^{2l+1}, \quad l = 1, 2, \dots, \tag{66}$$

one gets the estimate on the first term of F_2^1 :

$$\begin{aligned} &\| -\Delta_i^h \left(\rho_j^3 \bar{\partial} \bar{\partial}^* \bar{\partial}(\partial \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(\mathbb{1}-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right) \right) \|_{k-2,\alpha} \\ &\leq L \|\eta^1 \varphi\|_0 \|\Delta_i^h(\rho_j^3 \tilde{\Omega}(t))\|_{k,\alpha} + L' \|\eta^3 \tilde{\Omega}(t)\|_{k,\alpha} \|\eta^1 \varphi\|_{k,\alpha}, \end{aligned}$$

where L, L' are positive numbers. Hence, one obtains:

$$\|F_2^1\|_{k-2,\alpha} \leq M_2^1 A(r) \|\Delta_i^h(\rho_j^3 \tilde{\Omega}(t))\|_{k,\alpha} + M_k \|\eta^3 \tilde{\Omega}(t)\|_{k,\alpha} \|\eta^1 \varphi\|_{k,\alpha}, \tag{67}$$

where M_2^1, M_k are positive numbers. Similarly, we are able to get the analogous estimates for F_1^1 and F_3^1 with the positive constants M_1^1 and M_3^1 . Thus,

$$\begin{aligned} & \|\Delta_i^h(\rho_j^3 \tilde{\Omega}(t))\|_{k,\alpha} \\ & \leq C_k(M_1^1 + M_2^1 + M_3^1)A(r)\|\Delta_i^h(\rho_j^3 \tilde{\Omega}(t))\|_{k,\alpha} + C_k\|\rho_j^3 \tilde{\Omega}(t)\|_1 + C_k M_k \|\eta^3 \tilde{\Omega}(t)\|_{k,\alpha} \|\eta^1 \varphi\|_{k,\alpha}. \end{aligned} \tag{68}$$

Choose a sufficiently small r such that

$$C_k(M_1^1 + M_2^1 + M_3^1)A(r) \leq \frac{1}{2} \tag{69}$$

and if $|t| < r$

$$\tilde{\Omega}(t), \varphi(t) \in C^{k,\alpha}. \tag{70}$$

Then

$$\eta^3 \tilde{\Omega}(t), \eta^1 \varphi(t) \in C^{k,\alpha}.$$

Therefore, by (68), one knows that

$$\|\Delta_i^h(\rho_j^3 \tilde{\Omega}(t))\|_{k,\alpha} \leq 2C_k\|\rho_j^3 \tilde{\Omega}(t)\|_1 + 2C_k M_k \|\rho_j^3 \tilde{\Omega}(t)\|_{k,\alpha} \|\eta^1 \varphi\|_{k,\alpha},$$

where the right-hand side is bounded and independent of h . Hence, we have proved

$$\rho_j^3 \tilde{\Omega}(t) \in C^{k+1,\alpha}$$

by [28, Appendix Lemma 8.2.(iii)]. Summing with respect j , one knows that $\eta^3 \tilde{\Omega}(t) \in C^{k+1,\alpha}$ by (61).

Next, we shall prove that $\eta^5 \tilde{\Omega}$ is $C^{k+2,\alpha}$. Consider the equation:

$$\square(\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega})) = F_1^2 + F_2^2 + F_3^2,$$

where $D_\beta = \frac{\partial}{\partial x^\beta}$ and F_1^2, F_2^2, F_3^2 denote the three terms with respect to the ones in the right-hand side of (59) after the corresponding operations, respectively. In particular,

$$F_2^2 = D_\beta F_2^1 + \text{lower-order terms of } \tilde{\Omega}(t).$$

By [28, Appendix Theorem 2.3], one obtains the a priori estimate

$$\begin{aligned} \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega})\|_{k,\alpha} & \leq C_k(\|F_1^2 + F_2^2 + F_3^2\|_{k-2,\alpha} + \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega})\|_0) \\ & \leq C_k(\|F_1^2\|_{k-2,\alpha} + \|F_2^2\|_{k-2,\alpha} + \|F_3^2\|_{k-2,\alpha} + \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega})\|_0), \end{aligned} \tag{71}$$

where C_k is the same as in (63). Now let us estimate the first three terms in the right-hand side of the above inequality. Here, we just estimate the second one since the other two terms are quite analogous. We use the equality (64) for the Green's operator again. Then one gets the estimate on the first term of F_2^2 :

$$\begin{aligned} & \left\| -\Delta_i^h D_\beta \left(\rho_j^5 \bar{\partial} \bar{\partial}^* \bar{\partial}(\bar{\partial} \bar{\partial})^* \mathbb{G}_{BC} \bar{\partial} \left(\iota_{(1-\bar{\varphi}\varphi)^{-1}\bar{\varphi}}(\tilde{\Omega}(t)) \right) \right) \right\|_{k-2,\alpha} \\ & \leq L\|\eta^3 \varphi\|_0 \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega}(t))\|_{k,\alpha} + L''\|\eta^5 \tilde{\Omega}(t)\|_{k+1,\alpha} \|\eta^3 \varphi\|_{k+1,\alpha}, \end{aligned}$$

where L, L'' are positive numbers. Hence, one obtains:

$$\|F_2^2\|_{k-2,\alpha} \leq M_2^1 A(r) \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega}(t))\|_{k,\alpha} + M_{k+1} \|\eta^5 \tilde{\Omega}(t)\|_{k+1,\alpha} \|\eta^3 \varphi\|_{k+1,\alpha}, \tag{72}$$

where M_2^1, M_{k+1} are positive numbers. Crucially, M_2^1 is exactly the same as in (67), since the range of the function $\eta^l(t)$ is $[0, 1]$ by (60), although M_{k+1} is possibly different from M_k in (67). Similarly, we are able to get the analogous estimates for F_1^2 and F_3^2 with the positive constants M_1^1 and M_3^1 , which are the same as in the above argument for $\eta^3\tilde{\Omega}(t) \in C^{k+1,\alpha}$. Thus, by (71) and (72),

$$\begin{aligned} & \|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega}(t))\|_{k,\alpha} \\ & \leq C_k(M_1^1 + M_2^1 + M_3^1)A(r)\|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega}(t))\|_{k,\alpha} + C_k\|\rho_j^5 \tilde{\Omega}(t)\|_2 \\ & \quad + C_k M_{k+1}\|\eta^5 \tilde{\Omega}(t)\|_{k+1,\alpha}\|\eta^3 \varphi\|_{k+1,\alpha}. \end{aligned} \tag{73}$$

Choose the same sufficiently small r as in the above argument for $\eta^3\tilde{\Omega}(t) \in C^{k+1,\alpha}$, given by (69) and (70). Therefore, by (73),

$$\|\Delta_i^h D_\beta(\rho_j^5 \tilde{\Omega}(t))\|_{k,\alpha} \leq 2C_k\|\rho_j^5 \tilde{\Omega}(t)\|_2 + 2C_k M_{k+1}\|\eta^5 \tilde{\Omega}(t)\|_{k+1,\alpha}\|\eta^3 \varphi\|_{k+1,\alpha},$$

where the right-hand side is bounded when we use the formula (66), $\eta^3\tilde{\Omega}(t) \in C^{k+1,\alpha}$ that we just proved in the above argument and also the fact that $\eta^3\varphi(t) \in C^{k+1,\alpha}$ proved in [28, Appendix 8]. Hence, we have proved

$$D_\beta(\rho_j^5 \tilde{\Omega}(t)) \in C^{k+1,\alpha}$$

by [28, Appendix Lemma 8.2.(iii)] and thus

$$\rho_j^5 \tilde{\Omega}(t) \in C^{k+2,\alpha}.$$

Summing with respect j , one obtains that $\eta^5\tilde{\Omega}(t) \in C^{k+2,\alpha}$ by (61). Notice that in this procedure r has not been replaced.

We can also prove that, for any $l = 1, 2, \dots$, $\eta^{2l+1}\tilde{\Omega}$ is $C^{k+l,\alpha}$, where r can be chosen independent of l . Since $\eta^{2l+1}(t)$ is identically equal to 1 on $|t| < \frac{r}{2}$ which is independent of l , $\tilde{\Omega}(t)$ is C^∞ on X_0 with $|t| < \frac{r}{2}$. Then $\tilde{\Omega}(t)$ can be considered as a real analytic family of $(n-1, n-1)$ -forms in t and it is smooth on t by [30, Proposition 2.2.3] again.

§4. Stability of p -Kähler structures

This section is to prove a local stability theorem of p -Kähler structures with deformation invariance of Bott–Chern numbers. We will first study obstruction of extension for d -closed forms and then the un-obstruction of real extension for transverse form via its two equivalent definitions.

Consider the differentiable family $\pi : \mathcal{X} \rightarrow B$ of compact complex n -dimensional manifolds over a sufficiently small domain in \mathbb{R}^k with the reference fiber $X_0 := \pi^{-1}(0)$ and the general fibers $X_t := \pi^{-1}(t)$. Here, we fix a family of hermitian metrics on X_t .

4.1 Obstruction of extension for d -closed and $\partial\bar{\partial}$ -closed forms

Inspired by Wu’s result [55, Theorem 5.13], one has:

PROPOSITION 4.1. *Let r and s be non-negative integers. Assume that the reference fiber X_0 satisfies the $\partial\bar{\partial}$ -lemma. Then any d -closed (r, s) -form Ω_0 and $\partial_0\bar{\partial}_0$ -closed (r, s) -form Ψ_0 on X_0 can be extended unobstructed to a d -closed (r, s) -form Ω_t and a $\partial_t\bar{\partial}_t$ -closed (r, s) -form Ψ_t on its small differentiable deformation X_t , respectively.*

Proof. By use of the extension map $e^{\iota_\varphi|\iota_{\bar{\varphi}}}$, we can construct two (r, s) -forms $e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Omega_0)$ and $e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Psi_0)$ on X_t , starting with Ω_0 and Ψ_0 , respectively.

Let F_t be the orthogonal projection to \mathbb{F}_t , the kernel of

$$\square_{BC,t} = \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t + \bar{\partial}_t^* \partial_t \partial_t^* \bar{\partial}_t + \partial_t^* \bar{\partial}_t \bar{\partial}_t^* \partial_t + \bar{\partial}_t^* \bar{\partial}_t + \partial_t^* \partial_t$$

and \mathbb{G}_t denote the associated Green’s operator with respect to a smooth family of Hermitian metrics on X_t . Then F_t and \mathbb{G}_t are C^∞ differentiable in t since the $\dim \mathbb{F}_t$ is deformation invariant, thanks to [55, Theorem 5.12]. Therefore, the desired d -closed (r, s) -form is

$$\Omega_t = (\partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* \mathbb{G}_t + F_t)(e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Omega_0)) \tag{74}$$

as long as one notices that

$$\Omega_t|_{t=0} = (\square_{BC,0} \mathbb{G}_0 + F_0)\Omega_0 = \Omega_0$$

by the Hodge decomposition of the operator $\square_{BC,0}$ and the d -closedness of Ω_0 .

The construction of Ψ_t is quite similar to the one of Ω_t in (74). Denote by \tilde{F}_t the orthogonal projection to $\tilde{\mathbb{F}}_t$, the kernel of

$$\square_{A,t} = \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t^* \partial_t^* \partial_t \bar{\partial}_t + \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t \partial_t^* \partial_t \bar{\partial}_t^* + \bar{\partial}_t \bar{\partial}_t^* + \partial_t \partial_t^*$$

and by $\tilde{\mathbb{G}}_t$ the associated Green’s operator with respect to a smooth family of Hermitian metrics on X_t . By the same token, \tilde{F}_t and $\tilde{\mathbb{G}}_t$ are also C^∞ differentiable in t since the $\dim \tilde{\mathbb{F}}_t$ is deformation invariant. Then the construction of Ψ_t goes as follows:

$$\Psi_t = \left((\partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \partial_t \bar{\partial}_t \bar{\partial}_t^* \partial_t^* + \bar{\partial}_t \partial_t^* \partial_t \bar{\partial}_t^* + \bar{\partial}_t \bar{\partial}_t^* + \partial_t \partial_t^*) \tilde{\mathbb{G}}_t + \tilde{F}_t \right) \left(e^{\iota_\varphi|\iota_{\bar{\varphi}}}(\Psi_0) \right),$$

where it is easy to see that

$$\Psi_t|_{t=0} = (\square_{A,0} \tilde{\mathbb{G}}_0 + \tilde{F}_0)\Psi_0 = \Psi_0. \tag{□}$$

REMARK 4.2. It follows easily from the proposition above that any small deformation of a pluriclosed manifold, satisfying the $\partial\bar{\partial}$ -lemma, is still pluriclosed. Recall that a compact complex manifold is called *pluriclosed* if it admits a $\partial\bar{\partial}$ -closed positive $(1, 1)$ -form. Moreover, it follows from the proof that the theorem still holds when the $\partial\bar{\partial}$ -lemma assumption is replaced by the infinitesimal deformation invariance of (r, s) -Bott–Chern and Aeppli numbers, respectively. These results are possibly known to experts.

We can also prove this proposition by another way inspired by the results of [10, 23, 55] in the $(n - 1, n - 1)$ -case, that is, any d -closed $(n - 1, n - 1)$ -form Ω on a complex manifold X satisfying the $\partial\bar{\partial}$ -lemma can be extended unobstructed to a d -closed $(n - 1, n - 1)$ -form on the small differentiable deformation X_t of X .

Let $f_t : X_t \rightarrow X_0$ be a diffeomorphism depending on t with $f_0 = \text{identity}$. And then one obtains a d -closed $(2n - 2)$ -form $\Omega_t = f_t^* \Omega$ on X_t , which is decomposed as

$$\Omega_t = \Omega_t^{n-2,n} + \Omega_t^{n-1,n-1} + \Omega_t^{n,n-2}$$

with respect to the complex structure on X_t . It is easy to check the following properties:

- (1) $\Omega_t^{n-1,n-1}$ approaches to Ω as $t \rightarrow 0$.
- (2) $\Omega_t^{n-2,n}$ and $\Omega_t^{n,n-2}$ approach to 0 as $t \rightarrow 0$.

Recall that [55, Theorem 5.12] or [8, Corollary 3.7] says that if X_0 satisfies the $\partial\bar{\partial}$ -lemma, so does the general fiber X_t . So we can choose an $(n-2, n-1)$ -form Ψ_1 and an $(n-1, n-2)$ -form Ψ_2 on X_t such that

$$\begin{aligned} \partial_t \bar{\partial}_t \Psi_1 &= \bar{\partial}_t \Omega_t^{n-1, n-1} = -\partial_t \Omega_t^{n-2, n}, \\ -\partial_t \bar{\partial}_t \Psi_2 &= \partial_t \Omega_t^{n-1, n-1} = -\bar{\partial}_t \Omega_t^{n, n-2}, \end{aligned}$$

where Ψ_1 and Ψ_2 can be set as

$$\Psi_1, \Psi_2 \perp_{\omega_t} \ker(\partial_t \bar{\partial}_t).$$

Put

$$\tilde{\Omega}_t = \Omega_t^{n-1, n-1} + \partial_t \Psi_1 + \bar{\partial}_t \Psi_2.$$

Obviously, $\tilde{\Omega}_t$ is a d -closed $(n-1, n-1)$ -form on X_t . Then Fu–Li–Yau proved the following highly nontrivial estimates in [22, Sections 4 and 5]: for some $0 < \alpha < 1$

$$\|\partial_t \Psi_1\|_{C^0(\omega_t)} \leq C \|\partial_t \Omega_t^{n-2, n}\|_{C^{0, \alpha}(\omega_t)}$$

and similarly

$$\|\bar{\partial}_t \Psi_2\|_{C^0(\omega_t)} \leq C \|\bar{\partial}_t \Omega_t^{n, n-2}\|_{C^{0, \alpha}(\omega_t)},$$

where C is a uniform constant. By use of these two estimates, one knows that $\tilde{\Omega}_t$ is indeed d -closed extension of Ω since $\partial_t \Omega_t^{n-2, n}$ and $\bar{\partial}_t \Omega_t^{n, n-2}$ approaches to zero uniformly as $t \rightarrow 0$.

It is easy to see that Theorem 1.5 is impossible to obtain by Fu–Yau’s result since the proof would rely on the deformation openness of $(n-1, n)$ th mild $\partial\bar{\partial}$ -lemma, which contradicts with Ugarte–Villacampa’s Example 3.14.

4.2 Un-obstruction of extension for transverse forms

In this subsection, we study some basic properties and local stabilities of p -Kähler structures, which seem more pertinent to the nature of the stability problem of complex structures.

Let V be a complex vector space of complex dimension n with its dual space V^* , namely the space of complex linear functionals over V . Denote the complexified space of the exterior m -vectors of V^* by $\bigwedge_{\mathbb{C}}^m V^*$, which admits a natural direct sum decomposition

$$\bigwedge_{\mathbb{C}}^m V^* = \sum_{r+s=m} \bigwedge^{r, s} V^*,$$

where $\bigwedge^{r, s} V^*$ is the complex vector space of (r, s) -forms on V^* . The case $m = 1$ exactly reads

$$\bigwedge_{\mathbb{C}}^1 V^* = V^* \oplus \overline{V^*},$$

where the natural isomorphism $V^* \cong \bigwedge^{1, 0} V^*$ is used. Let $q \in \{1, \dots, n\}$ and $p = n - q$. Obviously, the complex dimension N of $\bigwedge^{q, 0} V^*$ equals to the combination number C_n^q . After a basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\bigwedge^{q, 0} V^*$ is fixed, the canonical Plücker

embedding as in [24, p. 209] is given by

$$\begin{aligned} \rho: G(q, n) &\hookrightarrow \mathbb{P}(\wedge^{q,0} V^*) \\ \Lambda &\mapsto [\dots, \Lambda_i, \dots]. \end{aligned}$$

Here, $G(q, n)$ denotes the Grassmannian of q -planes in the vector space V^* and $\mathbb{P}(\wedge^{q,0} V^*)$ is the projectivization of $\wedge^{q,0} V^*$. A q -plane in V^* can be represented by a decomposable $(q, 0)$ -form $\Lambda \in \wedge^{q,0} V^*$ up to a nonzero complex number, and $\{\Lambda_i\}_{i=1}^N$ are exactly the coordinates of Λ under the fixed basis $\{\beta_i\}_{i=1}^N$. *Decomposable $(q, 0)$ -forms* are those forms in $\wedge^{q,0} V^*$ that can be expressed as $\gamma_1 \wedge \dots \wedge \gamma_q$ with $\gamma_i \in V^* \cong \wedge^{1,0} V^*$ for $1 \leq i \leq q$. Set

$$k = (N - 1) - pq \tag{75}$$

to be the codimension of $\rho(G(q, n))$ in $\mathbb{P}(\wedge^{q,0} V^*)$, whose locus characterizes the decomposable $(q, 0)$ -forms in $\mathbb{P}(\wedge^{q,0} V^*)$.

Now, we list several positivity notations and refer the readers to [18, 25, 26] for more details. A (q, q) -form Θ in $\wedge^{q,q} V^*$ is defined to be *strictly positive (resp. positive)* if

$$\Theta = \sigma_q \sum_{i,j=1}^N \Theta_{i\bar{j}} \beta_i \wedge \bar{\beta}_j,$$

where Θ_{ij} is a positive (resp. semi-positive) hermitian matrix of the size $N \times N$ with $N = C_n^q$ under the basis $\{\beta_i\}_{i=1}^N$ of the complex vector space $\wedge^{q,0} V^*$ and σ_q is defined to be the constant $2^{-q}(\sqrt{-1})^{q^2}$. According to this definition, the fundamental form of a hermitian metric on a complex manifold is actually a strictly positive $(1, 1)$ -form everywhere. A (p, p) -form $\Gamma \in \wedge^{p,p} V^*$ is called *weakly positive* if the volume form

$$\Gamma \wedge \sigma_q \tau \wedge \bar{\tau}$$

is positive for every nonzero decomposable $(q, 0)$ -form τ of V^* ; a (q, q) -form $\Upsilon \in \wedge^{q,q} V^*$ is said to be *strongly positive* if Υ is a convex combination

$$\Upsilon = \sum \gamma_s \sqrt{-1} \alpha_{s,1} \wedge \bar{\alpha}_{s,1} \wedge \dots \wedge \sqrt{-1} \alpha_{s,q} \wedge \bar{\alpha}_{s,q},$$

where $\alpha_{s,i} \in V^*$ and $\gamma_s \geq 0$. As shown in [18, Chapter III, Section 1.A], the sets of weakly positive and strongly positive forms are closed convex cones, and by definition, the weakly positive cone is dual to the strongly positive cone via the pairing

$$\bigwedge^{p,p} V^* \times \bigwedge^{q,q} V^* \longrightarrow \mathbb{C};$$

all weakly positive forms are real. An element $\Xi \in \wedge^{p,p} V^*$ is called *transverse*, if the volume form

$$\Xi \wedge \sigma_q \tau \wedge \bar{\tau}$$

is strictly positive for every nonzero decomposable $(q, 0)$ -form τ of V^* . There exist many various names for this terminology and we refer to [4, Appendix] for a list.

These positivity notations on complex vector spaces can be extended pointwise to complex differential forms on a complex manifold. Let M be a complex manifold of dimension n . Then:

DEFINITION 4.3 ([2, Definition 1.11], for example). Let p be an integer, $1 \leq p \leq n$. Then M is called a p -Kähler manifold if there exists a d -closed transverse (p, p) -form on M .

The duality between the weakly positive and strongly positive cones of forms is used to define corresponding positivities for currents.

DEFINITION 4.4. A current T of bidegree (q, q) on M is strongly positive (resp. positive) if the pairing $\langle T, u \rangle \geq 0$ for all weakly positive (resp. strongly positive) test forms $u \in A^{p,p}(M)$ at each point. Clearly, each positive current is real.

We are going to discuss several basics of transverse forms, such as the equivalent characterizations of one-Kählerness and $(n - 1)$ -Kählerness by a slightly different approach from Alessandrini-Andreatta [2, Proposition 1.15]. Let V be furnished with a Hermitian inner product and V^* with the dual inner product, which will extend to $\bigwedge_{\mathbb{C}}^m V^*$. Denote by $\bigwedge_{\mathbb{R}}^{p,p} V$ the (real) vector space of real (p, p) -forms of V^* , which consists of invariant complex (p, p) -forms of V^* under conjugation. Then it is well known that, for every $\Omega \in \bigwedge_{\mathbb{R}}^{p,p} V^*$, there exist real numbers $\{\lambda_1, \dots, \lambda_N\}$ and an orthogonal basis $\{\eta_1, \dots, \eta_N\}$ of $\bigwedge^{p,0} V^*$, satisfying $|\eta_j|^2 = 2^p$ for $1 \leq j \leq N$, such that

$$\Omega = \sigma_p \sum_{j=1}^N \lambda_j \eta_j \wedge \bar{\eta}_j,$$

which is called the *canonical form* for real (p, p) -forms (See [26] for more details). The *positive (resp. negative) index* of Ω is the number of positive (resp. negative) ones in $\{\lambda_1, \dots, \lambda_N\}$.

PROPOSITION 4.5. Let Ω be a transverse (p, p) -form of V^* . Then the positive index of Ω is no less than $N - k$, where k is given by (75).

Proof. It is clear that

$$\Omega = \sigma_p(\lambda_1 \eta_1 \wedge \bar{\eta}_1 + \dots + \lambda_N \eta_N \wedge \bar{\eta}_N)$$

for some real numbers $\{\lambda_1, \dots, \lambda_N\}$ and some orthogonal basis $\{\eta_1, \dots, \eta_N\}$ of $\bigwedge^{p,0} V^*$, satisfying that $|\eta_j|^2 = 2^p$ for $1 \leq j \leq N$.

Suppose that the positive index $\Omega \leq N - k - 1$. Without loss of generality, we may assume that $\{\lambda_{N-k}, \dots, \lambda_N\}$ are all nonpositive real numbers.

Apparently, $\bigwedge^{p,0} V^*$ and $\bigwedge^{q,0} V^*$ are dual vector spaces. And thus, the zero loci of $\eta_1, \dots, \eta_{N-(k+1)}$ define $N - (k + 1)$ hyperplanes in $\mathbb{P}(\bigwedge^{q,0} V^*)$, denoted by $\eta_1^\perp, \dots, \eta_{N-(k+1)}^\perp$, respectively. Since the codimension of $\rho(G(q, n))$ is k and the complex plane $\eta_1^\perp \cap \dots \cap \eta_{N-(k+1)}^\perp$ is of dimension k , there exists some nonzero decomposable $(q, 0)$ -form τ , which lies in $\eta_1^\perp \cap \dots \cap \eta_{N-(k+1)}^\perp$. Then it follows that

$$\Omega \wedge \sigma_q \tau \wedge \bar{\tau} \leq 0,$$

which contradicts with the definition “transverse.” □

COROLLARY 4.6 ([2, Proposition 1.15]). A complex manifold M is one-Kähler if and only if M is Kähler; M is $(n - 1)$ -Kähler if and only if M is balanced.

Proof. Both cases follow from the expression formula (75),

$$N - k = N - (N - 1) + (n - 1) \cdot 1 = n. \quad \square$$

EXAMPLE 4.7. There exists a transverse (p, p) -form Ω of V^* , whose positive index is exactly $N - k$.

Actually, let $\{\eta_1, \dots, \eta_N\}$ still denote the orthogonal basis of $\bigwedge^{p,0} V^*$ as above. We shall construct an element Ω in $\bigwedge_{\mathbb{R}}^{p,p} V^*$ below

$$\Omega = \sigma_p \sum_{i=1}^{N-k} \lambda_i \eta_i \wedge \bar{\eta}_i$$

with $\lambda_1, \dots, \lambda_{N-k} > 0$. It is clear that the dimension of the complex plane $\eta_1^\perp \cap \dots \cap \eta_{N-k}^\perp$ is $k - 1$. Hence, we can slightly change the orthogonal basis $\{\eta_1, \dots, \eta_N\}$ to another one with the same feature, such that $\eta_1^\perp \cap \dots \cap \eta_{N-k}^\perp$ has no intersection with $\rho(G(q, n))$. Let us check that Ω is transverse. It is easy to see that

$$\Omega \wedge \sigma_q \tau \wedge \bar{\tau} \geq 0$$

for each nonzero decomposable $(q, 0)$ -form τ . Now suppose that $\Omega \wedge \sigma_q \tau \wedge \bar{\tau} = 0$ for some decomposable $(q, 0)$ -form $\tau \neq 0$. Then it follows that

$$\begin{cases} \eta_1 \wedge \tau = 0, \\ \vdots \\ \eta_{N-k} \wedge \tau = 0, \end{cases}$$

which implies that $\tau \in \eta_1^\perp \cap \dots \cap \eta_{N-k}^\perp$. This contradicts with the choice of the orthogonal basis $\{\eta_1, \dots, \eta_N\}$. Therefore, Ω is a transverse (p, p) -form.

EXAMPLE 4.8 (See also the example on [26, p. 50]). There exists a transverse (p, p) -form Ω of V^* , whose negative index is larger than 0.

In fact, let us slightly modify the example above. Consider the element in $\bigwedge_{\mathbb{R}}^{p,p} V^*$,

$$\Omega = \sigma_p(\lambda_1 \eta_1 \wedge \bar{\eta}_1 + \dots + \lambda_{N-k} \eta_{N-k} \wedge \bar{\eta}_{N-k} + \lambda_{N-k+1} \eta_{N-k+1} \wedge \bar{\eta}_{N-k+1}), \tag{76}$$

where $\lambda_1, \dots, \lambda_{N-k} > 0$, λ_{N-k+1} is some negative number to be fixed later, and the complex plane $\eta_1^\perp \cap \dots \cap \eta_{N-k}^\perp$ has no intersection with $\rho(G(q, n))$ in $\mathbb{P}(\bigwedge^{q,0} V^*)$. Construct a function over $\rho(G(q, n))$, defined by

$$f([\tau]) = \frac{\sigma_p(\lambda_1 \eta_1 \wedge \bar{\eta}_1 + \dots + \lambda_{N-k} \eta_{N-k} \wedge \bar{\eta}_{N-k}) \wedge \sigma_q \tau \wedge \bar{\tau}}{\sigma_p \eta_{N-k+1} \wedge \bar{\eta}_{N-k+1} \wedge \sigma_q \tau \wedge \bar{\tau}}.$$

It is easy to check that f is well-defined on $\rho(G(q, n))$. The function f has positive values when $[\tau] \in \rho(G(q, n)) \setminus \eta_{N-k+1}^\perp$ and attains to $+\infty$ when $[\tau] \in \rho(G(q, n)) \cap \eta_{N-k+1}^\perp$. Then f can obtain its minimum value over $\rho(G(q, n))$ by an elementary analysis, which is denoted by $a > 0$. Let $-a < \lambda_{N-k+1} < 0$. Then Ω constructed in (76) is transverse.

In [3], Alessandrini–Bassanelli proved that $(n - 1)$ -Kählerian property is not preserved under the small deformations for balanced manifolds nor, more generally, for p -Kähler manifolds ($p > 1$), while, based on the above argument on the p -Kähler structures and Proposition 4.1, we have the following local stability theorem of p -Kählerian structures:

THEOREM 4.9. For any positive integer $p \leq n - 1$, any small differentiable deformation X_t of a p -Kähler manifold X_0 satisfying the $\partial\bar{\partial}$ -lemma is still p -Kählerian.

Alessandrini–Bassanelli [5, Section 4] constructed a smooth proper modification \tilde{X} of $\mathbb{C}\mathbb{P}^5$, which will be p -Kähler for $2 \leq p \leq 5$, but non-Kähler. It is clear that the non-Kähler Moishezon fivefold \tilde{X} is p -Kähler for $p = 2, 3$ (the most interesting parts in this theorem), satisfying the $\partial\bar{\partial}$ -lemma due to [17], which indicates that Theorem 4.9 does not just concern Kähler or balanced $\partial\bar{\partial}$ -structures. It is worth noticing that Alessandrini–Bassanelli conjectured on [5, p. 299] that a p -Kählerian complex manifold is also q -Kähler for $p \leq q \leq n$. Besides, Theorem 4.9 might help to produce examples of p -Kähler $\partial\bar{\partial}$ -manifolds, which are not in the Fujiki class, since being in the Fujiki class is not an open property under deformations thanks to [12] and [33].

We will present two proofs for this theorem and use the following trivial lemma of Calculus in both proofs.

LEMMA 4.10. *Let $f(z, t)$ be a real continuous function on $K \times \Delta_\epsilon$, where K is a compact set and $\Delta_\epsilon = \{t \in \mathbb{R}^k \mid |t| < \epsilon\}$. Assume that*

$$f(z, 0) > 0, \quad \text{for } z \in K.$$

Then there exists some positive number $\delta > 0$, such that

$$f(z, t) > 0,$$

for $z \in K$ and $t \in \Delta_\delta$.

Proof. It is clear that for each $z \in K$, there exists an open neighborhood U_z in K and some $\delta_z > 0$, such that

$$f(z, t) > 0,$$

for $z \in U_z$ and $t \in \Delta_{\delta_z}$. Compactness of K enables us to find a finite open covering of K , say U_{z_1}, \dots, U_{z_m} . Then we may set δ to be

$$\min\{\delta_{z_1}, \dots, \delta_{z_m}\}.$$

Therefore, it follows that

$$f(z, t) > 0,$$

for $z \in K$ and $t \in \Delta_\delta$. □

Next, we proceed to the first proof of Theorem 4.9, which is based on an equivalent definition of transverse (p, p) -forms via strongly positive currents and their extension property.

Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact complex manifolds with the reference fiber $X_0 := \pi^{-1}(0)$ and the general fibers $X_t := \pi^{-1}(t)$. It is known from [43, Lemma 2.8] that the extension map in (1.2)

$$e^{\iota_{\varphi(t)}|_{\iota_{\overline{\varphi(t)}}}} : A^{n-p, n-q}(X_0) \rightarrow A^{n-p, n-q}(X_t)$$

is a linear isomorphism, depending smoothly on t , and its inverse map $e^{-\iota_{\varphi(t)}|_{-\iota_{\overline{\varphi(t)}}}}$ is defined by (13). Since the dual spaces of $A^{n-p, n-q}(X_0)$ and $A^{n-p, n-q}(X_t)$ are exactly the spaces of (p, q) -currents on X_0 and X_t , set as $\mathfrak{D}^{p, q}(X_0)$ and $\mathfrak{D}^{p, q}(X_t)$, with the weak topologies respectively, the adjoint map $(e^{-\iota_{\varphi(t)}|_{-\iota_{\overline{\varphi(t)}}})^*$, given by

$$(e^{-\iota_{\varphi(t)}|_{-\iota_{\overline{\varphi(t)}}})^* : \mathfrak{D}^{p, q}(X_0) \rightarrow \mathfrak{D}^{p, q}(X_t), \tag{77}$$

is defined by the following formula:

$$\left\langle (e^{-\iota_{\varphi(t)}|-\iota_{\overline{\varphi(t)}}})^* T, \Omega_t \right\rangle = \left\langle T, e^{-\iota_{\varphi(t)}|-\iota_{\overline{\varphi(t)}}}(\Omega_t) \right\rangle, \tag{78}$$

where T is a (p, q) -current on X_0 , Ω_t is an $(n - p, n - q)$ -form on X_t and the pairing $\langle \bullet, \bullet \rangle$ is the natural pairing between currents and smooth complex differential forms of pure type on X_0 or X_t . It is clear that every $(n - p, n - q)$ -form Ω_t on X_t can be expressed as $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega)$, for some $(n - p, n - q)$ -form Ω on X_0 , due to the linear isomorphism $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$. Then the formula (78) now reads:

$$\left\langle (e^{-\iota_{\varphi(t)}|-\iota_{\overline{\varphi(t)}}})^* T, e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega) \right\rangle = \langle T, \Omega \rangle, \tag{79}$$

which can be regarded as the defining formula of the adjoint map (77). It is easy to see that this adjoint map is a linear homeomorphism, depending smoothly on t . Using this map, one obtains the following extension proposition, to be also of independent interest.

PROPOSITION 4.11. *The natural map $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$ sends each kind of positive (p, p) -forms (defined in this subsection) on X_0 bijectively onto the corresponding one on X_t ; the map $(e^{-\iota_{\varphi(t)}|-\iota_{\overline{\varphi(t)}}})^*$ sends strongly positive and positive (p, p) -currents homeomorphically onto the corresponding ones on X_t , respectively.*

Proof. What we need to show is that the map $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$ actually sends smooth forms or currents, satisfying some positive condition, to the ones with the same positivity, since it can be similarly proved that the inverse map of $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$ shares the same property.

It is clear that the map $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$ sends strongly positive (p, p) -forms X_0 to the strongly positive ones on X_t , by the very definition of $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$ and strong positivity. As to the positive case, we may choose a positive (p, p) -form Ω on X_0 , which can be locally written as

$$\Omega = \sigma_p \sum_{|I|=|J|=p} \Omega_{I\bar{J}} dz^I \wedge d\bar{z}^J,$$

where $\Omega_{I\bar{J}}$ is a semi-positive hermitian matrix of the size $N \times N$ with $N = C_n^p$ everywhere and varies with respect to the local coordinates $\{z^i\}_{i=1}^n$. The local expression of $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega)$ amounts to

$$\sigma_p \sum_{|I|=|J|=p} \Omega_{I\bar{J}} e^{\iota_{\varphi(t)}}(dz^I) \wedge \overline{e^{\iota_{\varphi(t)}}(dz^J)},$$

where

$$\left\{ e^{\iota_{\varphi(t)}}(dz^I) \wedge \overline{e^{\iota_{\varphi(t)}}(dz^J)} \right\}_{|I|=|J|=p}$$

is a local basis of (p, p) -forms on X_t as shown in [43, Lemma 2.4] for example. And thus, $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega)$ is clearly a positive (p, p) -form on X_t . To see the case of weak positivity, let Ω be a weakly positive (p, p) -form on X_0 . We need to show that $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega)$ is weakly positive on X_t . Fix a point w on X_t and a strongly positive $(n - p, n - p)$ -form

$$\eta_t \in \bigwedge^{n-p, n-p} T_{w, X_t}^{*(1,0)}.$$

Let $\mathcal{X} \stackrel{(\rho, \pi)}{\simeq} X_0 \times B$ be the diffeomorphism for the differentiable family $\pi : \mathcal{X} \rightarrow B$, which induces the integrable Beltrami differential form $\varphi(t)$ (cf.[13]). Then η_t can be expressed as

$e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\eta)$ for some

$$\eta \in \bigwedge^{n-p, n-p} T_{\rho(w), X_0}^{*(1,0)}.$$

Besides, the equality holds

$$e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega) \Big|_w \wedge \eta_t = e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega) \Big|_w \wedge e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\eta) = e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega|_{\rho(w)} \wedge \eta).$$

From the very definition of weakly positive (p, p) -form Ω on X_0 , it follows that

$$\Omega|_{\rho(w)} \wedge \eta$$

is a positive (n, n) -form at $\rho(w)$ on X_0 and thus

$$e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega) \Big|_w \wedge \eta_t$$

is also a positive (n, n) -form at w on X_t , by the definition of $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}$. Therefore, the (p, p) -form $e^{\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}}(\Omega)$ is weakly positive on X_t , since w and η_t can be arbitrarily chosen.

The statements on the currents can be proved directly from the Definition 4.4 for the positivities of currents, the formula (79) and the results of smooth forms shown above. \square

To study local stabilities of transverse (p, p) -forms, we need an equivalent characterization of them, as in [1, Claim on Page 5] or implicit in [2, The proofs of Lemma 1.22 and Theorem 1.17]: a (p, p) -form Ω is transverse on an n -dimensional compact complex manifold M if and only if

$$\int_M \Omega \wedge T > 0,$$

for every nonzero strongly positive $(n - p, n - p)$ -current T on it.

PROPOSITION 4.12. *Let $\pi : \mathcal{X} \rightarrow B$ be a differentiable family of compact complex n -dimensional manifolds and Ω_t a family of real (p, p) -forms on X_t , depending smoothly on t . Assume that Ω_0 is a transverse (p, p) -form on X_0 . Then Ω_t is also transverse on X_t for t small.*

Proof. Fix a hermitian metric ω_0 on X_0 and define a real function $f(T, t)$ as follows:

$$f(T, t) = \int_{X_t} \Omega_t \wedge (e^{-\iota_{\varphi(t)}|\iota_{\overline{\varphi(t)}}})^* T,$$

where T varies in the space of strongly positive $(n - p, n - p)$ -currents on X_0 , satisfying

$$\int_{X_0} \omega_0^p \wedge T = 1,$$

and $t \in \Delta_\epsilon$. As pointed out in [2, Proposition 1.7] or [47, Proposition I.5], the space of strongly positive $(n - p, n - p)$ -currents T on X_0 , satisfying $\int_{X_0} \omega_0^p \wedge T = 1$, is a compact set, denoted by K here. Then it is obvious that the function $f(T, t)$ is continuous on $K \times \Delta_\epsilon$. The assumption that the real (p, p) -form Ω_0 is transverse on X_0 implies that

$$f(T, 0) > 0,$$

for each $T \in K$. Hence, Lemma 4.10 provides a positive number $\delta > 0$, such that

$$f(T, t) > 0,$$

for $T \in K$ and $t \in \Delta_\delta$. This indeed shows that Ω_t is transverse for $t \in \Delta_\delta$, by Proposition 4.11 and the following application of the formula (79):

$$\begin{aligned} & \int_{X_t} \left(e^{\iota_{\varphi(t)} | \iota_{\overline{\varphi(t)}}} (\omega_0) \right)^p \wedge (e^{-\iota_{\varphi(t)} | -\iota_{\overline{\varphi(t)}}})^* T \\ &= \int_{X_t} e^{\iota_{\varphi(t)} | \iota_{\overline{\varphi(t)}}} (\omega_0^p) \wedge (e^{-\iota_{\varphi(t)} | -\iota_{\overline{\varphi(t)}}})^* T = \int_{X_0} \omega_0^p \wedge T = 1, \end{aligned}$$

where $e^{\iota_{\varphi(t)} | \iota_{\overline{\varphi(t)}}} (\omega_0)$ can be regarded as a fixed hermitian metric on X_t . □

The first proof of Theorem 4.9. Let Ω_0 be a d -closed transverse (p, p) -form on the p -Kähler manifold X_0 . Proposition 4.1 assures that there exists a d -closed real extension Ω_t of Ω_0 on X_t , depending smoothly on t . Therefore, by Proposition 4.12, Ω_t is actually a d -closed transverse (p, p) -form on X_t for small t , which implies that X_t is a p -Kähler manifold. □

Then one also has another proof:

The second proof of Theorem 4.9. Let Ω_0 be a d -closed transverse (p, p) -form on X_0 and Ω_t its d -closed real extension on X_t constructed in (74) by Proposition 4.1. To prove the theorem, we just need: there exists a uniform small constant $\varepsilon > 0$, such that for any $t \in \Delta_\varepsilon$ and any nonzero decomposable $(q, 0)$ -form τ at any given point $x \in X_0$ with $p + q = n$,

$$\Omega_t(x) \wedge \sigma_q e^{\iota_\varphi}(\tau) \wedge e^{\iota_{\overline{\varphi}}}(\overline{\tau}) > 0, \tag{80}$$

where $\varphi := \varphi(x, t)$ is induced by the small differentiable deformation. Since e^{ι_φ} induces an isomorphism between decomposable $(q, 0)$ -forms of X_0 and those of X_t , Ω_t will be the desired d -closed transverse (p, p) -form on X_t .

In fact, let ω be a Hermitian metric on X_0 . For any fixed point $x \in X_0$, we define a continuous function $f_x(t, [\tau])$ on $\Delta_\varepsilon \times Y_x$ by

$$f_x(t, [\tau]) := \frac{\Omega_t(x) \wedge \sigma_q e^{\iota_\varphi}(\tau) \wedge e^{\iota_{\overline{\varphi}}}(\overline{\tau})}{|\tau|_{\omega(x)}^2 \cdot \omega(x)^n}, \tag{81}$$

where $Y_x = \rho(G(q, n))|_x \subset \mathbb{P}(\wedge^{q,0} T_{X_0}^*|_x)$ is compact. Notice that $\Omega_t \wedge \sigma_q e^{\iota_\varphi}(\tau) \wedge e^{\iota_{\overline{\varphi}}}(\overline{\tau})$ can be considered as an (n, n) -form on X_0 . Then by the transversality of Ω_0 ,

$$f_x(0, [\tau]) = \frac{\Omega_0(x) \wedge \sigma_q \tau \wedge \overline{\tau}}{|\tau|_{\omega(x)}^2 \cdot \omega(x)^n} > 0.$$

Thus, by the continuity of $f_x(t, [\tau])$ on t and $[\tau]$, Lemma 4.10 gives rise to a constant $\epsilon_x > 0$ depending only on x , such that

$$f(x, \overline{\Delta}_{\epsilon_x/2}, Y_x) := f_x(\overline{\Delta}_{\epsilon_x/2}, Y_x) > 0.$$

Let $\{U_j\}_{j=1}^J$ be trivializing covering of $\wedge^{q,0} T_{X_0}^*$, and choose any $x \in U_j$ for some j . Then one can identify Y_x and Y_y for any point $y \in U_j$, and f_y is defined on Y_x . So by Lemma 4.10,

there exists an open neighborhood $V_x \subset U_j$ of x such that

$$f(V_x, \overline{\Delta}_{\epsilon_x/2}, Y_x) > 0$$

since f is continuous on $\overline{\Delta}_{\epsilon_x/2} \times Y_x$.

Noticing that X_0 is compact, one obtains a finite open covering V_{x_i} , $i = 1, \dots, m$, $X_0 = \cup_{i=1}^m V_{x_i}$. Set

$$\epsilon := \min_{1 \leq i \leq m} \epsilon_{x_i}/2 > 0.$$

Then

$$f(x, \overline{\Delta}_\epsilon, Y_x) = f_x(\overline{\Delta}_\epsilon, Y_x) > 0$$

for any $x \in X_0$.

Therefore, by the definition (81), for any $|t| \leq \epsilon$,

$$\Omega_t(x) \wedge \sigma_q e^{t\varphi(x)}(\tau) \wedge e^{t\bar{\varphi}(x)}(\bar{\tau}) = f_x(t, [\tau]) |\tau|_{\omega(x)}^2 \cdot \omega(x)^n > 0$$

for any nonzero decomposable $(q, 0)$ -form τ at x . This is exactly the desired inequality (80). □

REMARK 4.13. Proposition 4.12 and the second proof of Theorem 4.9 actually show that any smooth real extension of a transverse (p, p) -form is still transverse, which also plays an important role in [44]. So the obstruction to extend a d -closed transverse (p, p) -form on a compact complex manifold lies in the d -closedness. Hence, by Remark 4.2, the condition of the $\partial\bar{\partial}$ -lemma in Theorem 4.9 can be replaced by the deformation invariance of (p, p) -Bott–Chern numbers. Moreover, this condition may be weakened as some kind of the $\partial\bar{\partial}$ -lemma if the power series method works, just similarly to what we have done in the balanced case in Section 3.

Finally, we have to mention an interesting conjecture proposed by Demailly–Păun for the “global” stability of Kähler structures, that is, the Kähler property should be open for the countable Zariski topology on the base.

CONJECTURE 4.14 ([19, Conjecture 5.1]). *Let $X \rightarrow S$ be a deformation of compact complex manifolds over an irreducible base S . Assume that one of the fibres X_{t_0} is Kähler. Then there exists a countable union $S' \subsetneq S$ of analytic subsets in the base such that X_t is Kähler for $t \in S \setminus S'$. Moreover, S can be chosen so that the Kähler cone is invariant over $S \setminus S'$, under parallel transport by the Gauss–Manin connection.*

Appendix A: Proof of Proposition 2.6

One chooses a holomorphic coordinate chart (z^1, \dots, z^n) on the complex manifold X throughout this proof.

(1) Set $\phi = \phi_t^s dz^t \frac{\partial}{\partial z^s}$ and $\alpha = \frac{1}{p!q!} \alpha_{I\bar{J}} dz^I \wedge d\bar{z}^J$ locally. One calculates:
 $\phi \lrcorner \bar{\phi} \lrcorner \alpha - (\phi \lrcorner \bar{\phi}) \lrcorner \alpha$

$$\begin{aligned} &= \phi \lrcorner \left(\sum_{1 \leq u \leq q} \frac{1}{p!q!} \bar{\phi}_t^{\bar{j}_u} \alpha_{I\bar{J}} dz^I \wedge \dots \wedge (dz^t)_{j_u} \wedge \dots \right) - \left(\phi_t^t \bar{\phi}_t^{\bar{s}} dz^l \frac{\partial}{\partial \bar{z}^s} \right) \lrcorner \alpha \\ &= \phi \lrcorner \left(\sum_{1 \leq u \leq q} \frac{1}{p!q!} \bar{\phi}_t^{\bar{j}_u} \alpha_{I\bar{J}} dz^I \wedge \dots \wedge (dz^t)_{j_u} \wedge \dots \right) \\ &\quad - \sum_{1 \leq u \leq q} \frac{1}{p!q!} \phi_t^t \bar{\phi}_t^{\bar{j}_u} \alpha_{I\bar{J}} dz^I \wedge \dots \wedge (d\bar{z}^l)_{j_u} \wedge \dots \\ &= \sum_{\substack{1 \leq v \leq p \\ 1 \leq u \leq q}} \frac{1}{p!q!} \phi_t^{i_v} \bar{\phi}_t^{\bar{j}_u} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^l)_{i_v} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \dots \wedge (dz^t)_{j_u} \wedge \dots, \end{aligned}$$

while

$$\begin{aligned} &\bar{\phi} \lrcorner \phi \lrcorner \alpha - (\bar{\phi} \lrcorner \phi) \lrcorner \alpha \\ &= \bar{\phi} \lrcorner \left(\sum_{1 \leq v \leq p} \frac{1}{p!q!} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^l)_{i_v} \wedge \dots \wedge d\bar{z}^J \right) - \left(\bar{\phi}_t^{\bar{i}} \phi_t^s dz^t \frac{\partial}{\partial z^s} \right) \lrcorner \alpha \\ &= \bar{\phi} \lrcorner \left(\sum_{1 \leq v \leq p} \frac{1}{p!q!} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^l)_{i_v} \wedge \dots \wedge d\bar{z}^J \right) \\ &\quad - \sum_{1 \leq v \leq p} \frac{1}{p!q!} \bar{\phi}_t^{\bar{i}} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (dz^t)_{i_v} \wedge \dots \wedge d\bar{z}^J \\ &= \sum_{\substack{1 \leq v \leq p \\ 1 \leq u \leq q}} \frac{1}{p!q!} \phi_t^{i_v} \bar{\phi}_t^{\bar{j}_u} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^l)_{i_v} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \dots \wedge (dz^t)_{j_u} \wedge \dots. \end{aligned}$$

(2) This follows directly from the generalized commutator formula (22) and

$$\partial(\phi \lrcorner \phi \lrcorner \bar{\phi}) = 0.$$

(3) This is proved in [13, Appendix B].

(4) Set $\phi = \phi_t^s dz^t \frac{\partial}{\partial z^s}$ and $\alpha = \frac{1}{p!q!} \alpha_{I\bar{J}} dz^I \wedge d\bar{z}^J$ locally. Thus,

$$\begin{aligned} &\bar{\phi} \lrcorner \bar{\phi} \lrcorner \phi \lrcorner \alpha - \bar{\phi} \lrcorner \bar{\phi} \lrcorner \phi \lrcorner \alpha \\ &= \bar{\phi} \lrcorner \bar{\phi} \lrcorner \left(\sum_{1 \leq v \leq p} \frac{1}{p!q!} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^t)_{i_v} \wedge \dots \wedge d\bar{z}^J \right) \\ &\quad - 2\bar{\phi} \lrcorner \left(\sum_{1 \leq u < l \leq q} \frac{1}{p!q!} \bar{\phi}_i^{\bar{j}_u} \bar{\phi}_j^{\bar{j}_l} \alpha_{I\bar{J}} dz^I \wedge \dots \wedge (dz^i)_{j_u} \wedge \dots \wedge (dz^j)_{j_l} \wedge \dots \right) \\ &= 2 \sum_{\substack{1 \leq v \leq p \\ 1 \leq u \leq q}} \frac{1}{p!q!} \bar{\phi}_i^{\bar{j}_u} \bar{\phi}_j^{\bar{j}_l} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (dz^j)_{i_v} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \dots \wedge (dz^i)_{j_u} \wedge \dots \\ &\quad + 2 \sum_{\substack{1 \leq v \leq p \\ 1 \leq u < l \leq q}} \frac{1}{p!q!} \bar{\phi}_i^{\bar{j}_u} \bar{\phi}_j^{\bar{j}_l} \phi_t^{i_v} \alpha_{I\bar{J}} \dots \wedge (d\bar{z}^t)_{i_v} \wedge \dots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \dots \wedge (dz^i)_{j_u} \wedge \dots \wedge (dz^j)_{j_l} \wedge \dots \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{\substack{1 \leq v \leq p \\ 1 \leq u < l \leq q}} \frac{1}{p!q!} \bar{\phi}_i^{\bar{j}_u} \bar{\phi}_j^{\bar{j}_l} \phi_v^{i_v} \alpha_{I\bar{J}} \cdots \wedge (d\bar{z}^t)_{i_v} \wedge \cdots \wedge dz^{i_p} \wedge d\bar{z}^{j_1} \cdots \wedge (dz^i)_{j_u} \wedge \cdots \wedge (dz^j)_{j_l} \wedge \cdots \\
 & -2 \left(\sum_{1 \leq u < l \leq q} \frac{1}{p!q!} \phi_i^{\bar{j}_u} \bar{\phi}_j^{\bar{j}_l} \alpha_{I\bar{J}} dz^I \wedge \cdots \wedge (d\bar{z}^t)_{j_u} \wedge \cdots \wedge (dz^j)_{j_l} \wedge \cdots \right. \\
 & \left. + \sum_{1 \leq u < l \leq q} \frac{1}{p!q!} \bar{\phi}_i^{\bar{j}_u} \phi_j^{\bar{j}_l} \alpha_{I\bar{J}} dz^I \wedge \cdots \wedge (dz^i)_{j_u} \wedge \cdots \wedge (d\bar{z}^t)_{j_l} \wedge \cdots \right) \\
 & = 2(\bar{\phi} \lrcorner \phi \bar{\phi} \lrcorner \alpha - \bar{\phi} \phi \lrcorner \bar{\phi} \lrcorner \alpha).
 \end{aligned}$$

Appendix B: Proof of Observation 2.8

We will omit the sub-index in many places without danger of confusion. By use of (3) and (2) in Proposition 2.6, the integrability condition (11) and the commutator formula (22) repeatedly, one has

$$\begin{aligned}
 & \bar{\partial}(\varphi \lrcorner \omega) \\
 & = \frac{1}{2}[\varphi, \varphi] \lrcorner \omega + \varphi \lrcorner \bar{\partial} \omega \\
 & = -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner \partial \omega + \varphi \lrcorner \partial(\varphi \lrcorner \omega) + \varphi \lrcorner (\bar{\partial}(\bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \partial(\varphi \lrcorner \omega)) \\
 & = -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner (\partial(\varphi \bar{\varphi} \lrcorner \omega - \bar{\varphi} \lrcorner \varphi \lrcorner \omega) - \bar{\partial}(\bar{\varphi} \lrcorner \omega)) + \varphi \lrcorner \bar{\partial}(\bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \bar{\varphi} \lrcorner \omega) \\
 & = -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner (\partial(\varphi \bar{\varphi} \lrcorner \omega - \bar{\varphi} \lrcorner \varphi \lrcorner \omega) - \bar{\partial}(\bar{\varphi} \lrcorner \omega)) \\
 & \quad + \bar{\partial}(\varphi \lrcorner \bar{\varphi} \varphi \lrcorner \omega) - \frac{1}{2}[\varphi, \varphi] \lrcorner \bar{\varphi} \varphi \lrcorner \omega - \varphi \lrcorner \frac{1}{2}[\varphi, \varphi] \lrcorner \bar{\varphi} \lrcorner \omega - \varphi \lrcorner \varphi \lrcorner \bar{\partial}(\bar{\varphi} \lrcorner \omega).
 \end{aligned}$$

Then (1) and (4) in Proposition 2.6 yield that

$$\begin{aligned}
 & \bar{\partial}(\varphi \lrcorner \omega) \\
 & = -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner (\partial(\varphi \bar{\varphi} \lrcorner \omega) + \bar{\partial}(\bar{\varphi} \lrcorner \omega)) + \frac{1}{2} \bar{\partial}(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) + \frac{1}{2} \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega) \\
 & = -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner (\partial(\varphi \bar{\varphi} \lrcorner \omega) + \bar{\partial}(\bar{\varphi} \lrcorner \omega)) + \frac{1}{2} \varphi \lrcorner [\varphi, \varphi] \lrcorner \bar{\varphi} \lrcorner \omega + \frac{1}{2} \varphi \lrcorner \varphi \lrcorner \bar{\partial}(\bar{\varphi} \lrcorner \omega) + \frac{1}{2} \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega) \\
 & = -\frac{1}{2} (\varphi \lrcorner \partial(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \varphi \lrcorner \varphi \lrcorner \partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega)).
 \end{aligned}$$

Note that the equality $\iota_\varphi \circ \iota_{[\varphi, \varphi]} = \iota_{[\varphi, \varphi]} \circ \iota_\varphi$ holds (cf. [13, p. 361]). Then the last equality but one above is also equal to

$$\begin{aligned}
 & -\frac{1}{2} \varphi \lrcorner \varphi \lrcorner (\partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega) + \bar{\partial}(\bar{\varphi} \lrcorner \omega)) + \frac{1}{2} [\varphi, \varphi] \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega + \frac{1}{2} \varphi \lrcorner \varphi \lrcorner \bar{\partial}(\bar{\varphi} \lrcorner \omega) + \frac{1}{2} \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega) \\
 & = \frac{1}{2} (\varphi \lrcorner \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega) + 2(\varphi \lrcorner \partial(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \varphi \lrcorner \varphi \lrcorner \partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega))).
 \end{aligned}$$

Hence, we get the equality

$$\begin{aligned} & -\frac{1}{2}(\varphi \lrcorner \partial(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \varphi \lrcorner \varphi \lrcorner \partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega)) \\ & = \frac{1}{2}(\varphi \lrcorner \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega) + 2(\varphi \lrcorner \partial(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \varphi \lrcorner \varphi \lrcorner \partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega))), \end{aligned}$$

which implies that

$$\begin{aligned} \bar{\partial}(\varphi \lrcorner \omega) & = -\frac{1}{2}(\varphi \lrcorner \partial(\varphi \lrcorner \varphi \lrcorner \bar{\varphi} \lrcorner \omega) - \varphi \lrcorner \varphi \lrcorner \partial(\varphi \lrcorner \bar{\varphi} \lrcorner \omega)) \\ & = \frac{1}{6}\varphi \lrcorner \varphi \lrcorner \varphi \lrcorner \partial(\bar{\varphi} \lrcorner \omega). \end{aligned}$$

Therefore, $\bar{\partial}(\varphi \lrcorner \omega)_N = 0$ reduces to the equality

$$\bar{\partial}(\varphi \lrcorner \omega)_k = 0, \quad \text{for } k = 1, 2, 3,$$

which are easily to be checked by the above formulation. This concludes the proof of Observation 2.8.

Acknowledgements. We would like to express our gratitude to Professors Kefeng Liu, Huitao Feng, and Fangyang Zheng for their constant encouragement and help, Lucia Alessandrini, Xiaojun Huang, Dan Popovici, and Luis Ugarte for their interest and useful comments. The first author would like to thank Dr. Xiaoshan Li, Jie Tu, and Yi Wang for several useful discussions and Wanke Yin for his lectures on positive currents at the seminar. This work also benefits from NCTS of Taiwan University, The Institute of Mathematical Sciences at the Chinese University of Hong Kong, the Mathematics Department of UCLA and the Institut de Mathematiques de Toulouse.

REFERENCES

- [1] L. Alessandrini, *Proper modifications of generalized p -Kähler manifolds*, J. Geom. Anal. **27** (2017), no. 2, 947–967.
- [2] L. Alessandrini and M. Andreatta, *Closed transverse (p, p) -forms on compact complex manifolds*, Compos. Math. **61** (1987), no. 2, 181–200. Erratum 63 (1987), no. 1, 143.
- [3] L. Alessandrini and G. Bassanelli, *Small deformations of a class of compact non-Kähler manifolds*, Proc. Amer. Math. Soc. **109** (1990), no. 4, 1059–1062.
- [4] L. Alessandrini and G. Bassanelli, *Compact p -Kähler manifolds*, Geom. Dedicata **38** (1991), 199–210.
- [5] L. Alessandrini and G. Bassanelli, *Positive $\partial\bar{\partial}$ -closed currents and non-Kähler geometry*, J. Geom. Anal. **2** (1992), 291–361.
- [6] D. Angella, *The cohomologies of the Iwasawa manifold and its small deformations*, J. Geom. Anal. **23** (2013), no. 3, 1355–1378.
- [7] D. Angella and H. Kasuya, *Bott–Chern cohomology of solvmanifolds*, Ann. Glob. Anal. Geom. **52** (2017), no. 4, 363–411.
- [8] D. Angella and A. Tomassini, *On the $\partial\bar{\partial}$ -lemma and Bott–Chern cohomology*, Invent. Math. **192** (2013), no. 1, 71–81.
- [9] D. Angella and L. Ugarte, *Locally conformal Hermitian metrics on complex non-Kähler manifolds*, Mediterr. J. Math. **13** (2016), no. 4, 2105–2145.
- [10] D. Angella and L. Ugarte, *On small deformations of balanced manifolds*, Differ. Geom. Appl. **54** (2017), no. part B, 464–474.
- [11] S. Barannikov and M. Kontsevich, *Frobenius manifolds and formality of Lie algebras of polyvector fields*, Int. Math. Res. Not. (1998), no. 4, 201–215.
- [12] F. Campana, *The class \mathcal{C} is not stable by small deformation*, Math. Ann. **290** (1991), 19–30.
- [13] H. Clemens, *Geometry of formal Kuranishi theory*, Adv. Math. **198** (2005), 311–365.
- [14] F. Console and A. Fino, *Dolbeault cohomology of compact nilmanifolds*, Transform. Groups **6** (2001), no. 2, 111–124.

- [15] F. Console, A. Fino, and Y.-S. Poon, *Stability of abelian complex structures*, Int. J. Math. **17** (2006), no. 4, 401–416.
- [16] L. Cordero, M. Fernández, A. Gray, and L. Ugarte, *Compact nilmanifolds with nilpotent complex structures: Dolbeault cohomology*, Trans. Amer. Math. Soc. **352** (2000), 5405–5433.
- [17] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), 245–274.
- [18] J.-P. Demailly, *Complex analytic and differential geometry* (2012), book online <http://www-fourier.ujf-grenoble.fr/demailly/books.html>.
- [19] J.-P. Demailly and M. Paun, *Numerical characterization of the Kähler cone of a compact Kähler manifold*, Ann. Math. **159** (2004), 1247–1274.
- [20] A. Douglis and L. Nirenberg, *Interior estimates for elliptic systems of partial differential equations*, Comm. Pure Appl. Math. **8** (1955), 503–538.
- [21] R. Friedman, “On threefolds with trivial canonical bundle”. in Complex geometry and Lie theory (Sundance, UT, 1989), *Proc. Sympos. Pure Math., Vol. 53*, Amer. Math. Soc., Providence, RI, 1991, 103–134.
- [22] J. Fu, J. Li, and S.-T. Yau, *Balanced metrics on non-Kähler Calabi–Yau threefolds*, J. Differ. Geom. **20** (2012), 81–129.
- [23] J. Fu and S.-T. Yau, *A note on small deformations of balanced manifolds*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 13–14, 793–796.
- [24] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, NY, 1978.
- [25] R. Harvey, “Holomorphic chains and their boundaries”. *Several complex variables* (Proc. Sympos. Pure Math., Vol. XXX, Part 1, Williams Coll., Williamstown, MA., 1975). Amer. Math. Soc., Providence, RI, 1977, 309–382.
- [26] R. Harvey and A. Knapp, *Positive (p,p) -forms, Wirtinger’s Inequality and Currents*, Procs Tulane Univ. Program on Value Distribution Theory in Complex Analysis and Related Topics in Differential Geometry 1972-73, Dekker, New York, NY, 1974, 43–62.
- [27] H. Kasuya, *Techniques of computations of Dolbeault cohomology of solvmanifolds*, Math. Z. **273** (2013), no. 1–2, 437–447.
- [28] K. Kodaira, *Complex Manifolds and Deformations of Complex Structures*, Grundlehren der Math. Wiss. 283, Springer, New York, NY, 1986.
- [29] K. Kodaira and D. Spencer, *On deformations of complex analytic structures. III. Stability theorems for complex structures*, Ann. Math. Sec. Ser. **71** (1960), 43–76.
- [30] S. Krantz and H. Parks, *A Primer of Real Analytic Functions*, 2nd ed., Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Boston, Inc., Boston, MA, 2002.
- [31] M. Kuranishi, *New proof for the existence of locally complete families of complex structures*, Proc. Conf. Complex Analysis (Minneapolis, 1964). Springer, Berlin, Germany, 1965, 142–154.
- [32] A. Latorre, L. Ugarte, and R. Villacampa, *On the Bott–Chern cohomology and balanced Hermitian nilmanifolds*, Int. J. Math. **25** (2014), no. 6, 1450057, 24 pp.
- [33] C. Lebrun and Y.-S. Poon, *Twisors, Kähler manifolds, and bimeromorphic geometry II*, J. Amer. Math. Soc. **5** (1992), no. 2, 317–325.
- [34] Y. Li, *On deformations of generalized complex structures: The generalized Calabi–Yau case*, October 2005. arXiv:0508030v2.
- [35] K. Liu and S. Rao, *Remarks on the Cartan formula and its applications*, Asian J. Math **16** (2012), no. 1, 157–170.
- [36] K. Liu, S. Rao, and X. Yang, *Quasi-isometry and deformations of Calabi–Yau manifolds*, Invent. Math. **199** (2015), no. 2, 423–453.
- [37] K. Liu, X. Sun, S.-T. Yau, “Recent development on the geometry of the Teichmüller and moduli spaces of Riemann surfaces”. *Surveys in differential geometry. Geometry of Riemann surfaces and their moduli spaces*, Surv. Differ. Geom., **14**, Int. Press, Somerville, MA, 2009, 221–259.
- [38] M. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **149** (1982), 261–295.
- [39] J. Morrow and K. Kodaira, *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London, 1971.
- [40] I. Nakamura, *Complex parallelisable manifolds and their small deformations*, J. Differ. Geom. **10** (1975), 85–112.

- [41] D. Popovici, *Aeppli cohomology classes associated with Gauduchon metrics on compact complex manifolds*, Bull. Soc. Math. France **143** (2015), no. 4, 763–800.
- [42] D. Popovici and L. Ugarte, *Compact complex manifolds with small Gauduchon cone*, Proc. London Math. Soc. (3) **116** (2018), no. 5, 1161–1186.
- [43] S. Rao and Q. Zhao, *Several special complex structures and their deformation properties*, J. Geom. Anal. **28** (2018), no. 4, 2984–3047.
- [44] S. Rao, X. Wan, and Q. Zhao, *On local stabilities of p -Kähler structures*, Compos. Math. **155** (2019), no. 3, 455–483.
- [45] S. Rollenske, *Geometry of nilmanifolds with left-invariant complex structure and deformations in the large*, Proc. London Math. Soc. **99** (2009), 425–460.
- [46] S. Salamon, *Complex structures on nilpotent Lie algebras*, J. Pure Appl. Algebra **157** (2001), 311–333.
- [47] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), 225–255.
- [48] X. Sun, *Deformation of canonical metrics I*, Asian J. Math. **16** (2012), no. 1, 141–155.
- [49] X. Sun and S.-T. Yau, *Deformation of Kähler–Einstein metrics*, Surveys in geometric analysis and relativity, Adv. Lect. Math. (ALM), **20**, Int. Press, Somerville, MA, 2011, 467–489.
- [50] G. Tian, *Smoothness of the universal deformation space of compact Calabi–Yau manifolds and its Petersson–Weil metric*, Mathematical aspects of string theory (San Diego, Calif., 1986), Adv. Ser. Math. Phys., **1**, World Sci. Publishing, Singapore, Asia, 1987, 629–646.
- [51] A. Todorov, *The Weil–Petersson geometry of the moduli space of $SU(n \geq 3)$ (Calabi–Yau) manifolds I*, Comm. Math. Phys. **126** (1989), no. 2, 325–346.
- [52] L. Ugarte and R. Villacampa, *Non-nilpotent complex geometry of nilmanifolds and heterotic supersymmetry*, Asian J. Math. **18** (2014), no. 2, 229–246.
- [53] L. Ugarte and R. Villacampa, *Balanced Hermitian geometry on 6-dimensional nilmanifolds*, Forum Math. **27** (2015), no. 2, 1025–1070.
- [54] C. Voisin, *Hodge Theory and Complex Algebraic Geometry. I*, Translated from the French original by Leila Schneps. Cambridge Studies in Advanced Mathematics, **76**. Cambridge University Press, Cambridge, MA, 2002.
- [55] C.-C. Wu, *On the geometry of superstrings with torsion*, Ph.D. thesis, Harvard University, 2006, 60 pp.
- [56] Q. Zhao and S. Rao, *Applications of deformation formula of holomorphic one-forms*, Pacific J. Math. **266** (2013), no. 1, 221–255.
- [57] Q. Zhao and S. Rao, *Extension formulas and deformation invariance of Hodge numbers*, C. R. Math. Acad. Sci. Paris **353** (2015), no. 11, 979–984.

Sheng Rao

School of Mathematics and statistics

Wuhan University

Wuhan 430072

China

Department of Mathematics

University of California at Los Angeles

Los Angeles, California 90095-1555

USA

likeanyone@whu.edu.cn

Xueyuan Wan

Mathematical Science Research Center

Chongqing University of Technology

Chongqing 400054

China

xwan@cqut.edu.cn

Quanting Zhao

School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences

Central China Normal University

Wuhan 430079

P.R. China

zhaoquanting@126.com; zhaoquanting@mail.ccnu.edu.cn