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A MULTIFUNCTION GENERALIZATION OF GALE'S ASCOLI THEOREM

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1. Introduction

Our purpose is to improve the Gale-type multifunction Ascoli theorem of Mancuso (1971; page 470). This latter supposes the range space to be normal and Hausdorff, and therefore does not contain Gale's theorem (1950; page 304). To obtain a multifunction theorem containing Gale's theorem (also Mancuso's theorem), we return to Gale's essential hypotheses. Thus, we assume the regularity of the range space in the sufficiency direction, and, in the necessity direction, we assume the domain to be a k-space and the range to be a regular Hausdorff space. We dispense with the "point-like" condition imposed by Mancuso. Unexplained terminology and notation is that of Mancuso's paper.

2. Multifunctions with values in a regular space

Let X, Y be topological spaces. A multifunction $f: X \to Y$ is *lower* (*upper*) *semi-continuous* if $f^{-}(U)$ ($f^{+}(U)$) is open in X whenever U is open in Y. Consequently, f is continuous if and only if f is both lower and upper semi-continuous.

LEMMA 1. *Let f: $X \to Y$ be a multifunction on a topological space X to a regular space Y. Then f is lower semi-continuous if, for each $x \in X$ and each open set U of Y such that $fx \cap U \neq \varnothing$, there exists a neighbourhood N of x such that $fz \cap \bar{U} \neq \varnothing$ for all $z \in N$.*

PROOF. Let G be an open set of Y and let $x \in f^{-}(G)$. Then $fx \cap G \neq \varnothing$. Let $y \in fx \cap G$. By the regularity, there is an open set U in Y such that $y \in U \subset \bar{U} \subset G$. Since $fx \cap U \neq \varnothing$, there exists, by hypothesis, a neighbourhood N of x such that, if $z \in N, fz \cap \bar{U} \neq \varnothing$. So $fz \cap G \neq \varnothing$, that is, $z \in f^{-}(G)$.

LEMMA 2. *Let f: $X \to Y$ be a point-compact multifunction on a topo* *logical space X to a regular space Y. Then f is upper semi-continuous if, for each $x \in X$ and each open set U of Y such that $fx \subset U$, there exists a neighbourhood N of x such that $fz \subset \bar{U}$ for all $z \in N$.*

PROOF. Let G be an open set of Y and let $x \in f^+(G)$. Then $fx \subset G$. Since fx is compact and Y is regular, there is an open set U in Y such that $fx \subset U \subset \bar{U} \subset G$. By hypothesis, there exists a neighbourhood N of x such that if $z \in N$, then $fz \subset \bar{U}$. So $fz \subset G$, that is, $z \in f^+(G)$.

The following lemma generalizes Lemma 2.4 of Mancuso (1971; page 467), and will be crucial in the proof of our Ascoli theorem:

LEMMA 3. *Let $f: X \to Y$ be a continuous multifunction on a topological space X to a regular space Y. If $fx \cap U \neq \varnothing$ for some open set U in Y, there exists a neighbourhood V of x such that $fz \cap U \neq \varnothing$ for all $z \in \bar{V}$.*

PROOF. Let $y \in fx \cap U$. By the regularity there is an open set W in Y such that $y \in W \subset \bar{W} \subset U$. Since f is lower semi-continuous, there exists a neighbourhood V of x such that $V \subset f^-(W)$. Let $z \in \bar{V}$ and suppose that $fz \cap U = \varnothing$. Then $fz \cap \bar{W} = \varnothing$ so that $fz \subset Y - \bar{W}$. Since f is upper semi-continuous, there is a neighbourhood H of z such that $H \subset f^+(Y - \bar{W})$. Since $z \in \bar{V}$, there is $v \in H \cap V$. Then $fv \subset Y - \bar{W}$ and therefore $fv \cap W = \varnothing$. This is a contradiction, because $v \in V \subset f^-(W)$.

3. Condition (G)

Let X, Y be topological spaces and let $F \subset Y^{mX}$. The set of subsets of F of the form $\{f : f(K) \subset U \text{ and } fx \cap V \neq \varnothing \text{ for all } x \in K\}$, where K is a compact subset of X and U, V are open in Y, is an open subbase for the *compact open topology* τ_c on F [7, page 47]. It is clear that τ_c is larger than the pointwise topology τ_p on F. The family F is called *collectively continuous* if $\cup_{f \in F} f^+(B)$ and $\cup_{f \in F} f^-(B)$ are closed in X whenever B is closed in Y, or, equivalently, $\cap_{f \in F} f^+(B)$ and $\cap_{f \in F} f^-(B)$ are open in X whenever B is open in Y (Mancuso (1971; page 469)). We say that F satisfies the *condition* (*G*) if each closed subset of (F, τ_c) is collectively continuous.

LEMMA 4. *Let F be a family of point-compact multifunctions on a topological space X to a regular space Y. If F satisfies (G), then the members of the τ_p-closure $\bar{F}$ of F, in the set of point-compact multifunctions of Y^{mX}, are continuous.*

PROOF. Let $f \in \bar{F}$. To show that f is lower semi-continuous, suppose $fx \cap V \neq \varnothing$, where V is open in Y. By Lemma 1, it suffices to show that there exists a neighbourhood N of x such that $fz \cap \bar{V} \neq \varnothing$ for all $z \in N$. Let $y \in fx \cap V$ and let G be an open set in Y such that $y \in G \subset \bar{G} \subset V$. Then $F_0 = \{h : h \in F \text{ and } hx \cap \bar{G} \neq \varnothing\}$ is closed in (F, τ_c). Since F satisfies (G), F_0 is collectively continuous. Then $N = \bigcap_{h \in F_0} h^-(V)$ is an open set in X containing x. We claim that $z \in N$ implies $fz \cap \bar{V} \neq \varnothing$. In fact, suppose that $fz \cap \bar{V} = \varnothing$. Then $fz \subset Y - \bar{V}$, and, since fz is compact, there exists an open set W such that $fz \subset W \subset \bar{W} \subset Y - \bar{V}$. Since $M = \{g: g \in Y^{mX}, g$ is point-compact, $gz \subset W$ and $gx \cap G \neq \varnothing\}$ is a τ_p neighbourhood of f, there exists $f' \in F \cap M$. Then $f' \in F_0$, and, since $z \in N$, $f'z \cap V \neq \varnothing$. This is a contradiction, because $f'z \subset W$ and $W \cap V = \varnothing$.

To show that f is upper semi-continuous, suppose $fx \subset V$, where V is open in Y. By Lemma 2, it suffices to show that there exists a neighbourhood N of x such that $fz \subset \bar{V}$ for all $z \in N$. Since fx is compact there exists an open set G in Y such that $fx \subset G \subset \bar{G} \subset V$. Then $F_0 = \{h: h \in F$ and $hx \subset \bar{G}\}$ is closed in (F,τ_c). Since F satisfies (G), F_0 is collectively continuous. Then $N = \bigcap_{h \in F_0} h^+(V)$ is an open set in X containing x. We claim that $z \in N$ implies $fz \subset \bar{V}$. In fact, let $z \in N$, $y \in fz$ and let W be any neighbourhood of y. Since $M = \{g: g \in Y^{mX}$, g is point-compact, $gx \subset G$ and $gz \cap W \neq \varnothing\}$ is a τ_p-neighbourhood of f, there exists $f' \in M \cap F$. Then $f'z \cap W \neq \varnothing$ and $f' \in F_0$, so, because $z \in N$, we have $f'z \subset V$. Thus $W \cap V \neq \varnothing$, so $y \in \bar{V}$. Since y is an arbitrary point of fz, $fz \subset \bar{V}$.

COROLLARY 1. *Let F be a family of point-compact multifunctions on a topological space X to a regular space Y. If F satisfies* (G), *then the members of F are continuous.*

COROLLARY 2. *Let F be a family of functions on a topological space X to a regular space Y. If F satisfies* (G), *then the members of the τ_p-closure of F in Y^X are continuous.*

LEMMA 5. *Let F be a family of point-compact multifunctions on a topological space X to a regular space Y. If F satisfies* (G), *then the τ_p-closure and the τ_c-closure of F, in the set of point-compact multifunctions of Y^{mX}, are identical.*

PROOF. Let $\bar{F}$, $\tilde{F}$ denote the τ_p-closure, the τ_c-closure of F, respectively, in the set of point compact multifunctions of Y^{mX}. It must be shown that $f_0 \in \bar{F}$ implies $f_0 \in \tilde{F}$. It will suffice to show that every τ_c-neighbourhood of f_0 of the form $\bigcap_{i=1}^n \{f : f \in Y^{mX}, f$ is point-compact, $f(K_i) \subset U_i$ and $fx \cap U_i' \neq \varnothing$ for all $x \in K_i\}$, where $K_i \subset X$ is compact and $U_i, U_i' \subset Y$ are open $(i = 1, \cdots, n)$ intersects F. Since f_0 is continuous (Lemma 4) and point-compact, $f_0(K_i)$ is a compact subset of U_i $(i = 1, \cdots, n)$ (Berge (1965; page 110)). Then there exists, for each $i = 1, \cdots, n$, an open set V_i in Y such that $f_0(K_i) \subset V_i \subset \bar{V}_i \subset U_i$. Since $f_0 x \cap U_i' \neq \varnothing$ for all $x \in K_i$ $(i = 1, \cdots, n)$, by the Lemma 1 of Smithson (1971; page 47), there exists an open set V_i' in Y such that $\bar{V}_i' \subset U_i'$ and $f_0 x \cap V_i' \neq \varnothing$ for all $x \in K_i$. Choose a fixed index i. For $x \in K_i$ let $H_x = \{f : f \in Y^{mX}, f$ is point-compact, $fx \subset \bar{V}_i$ and $fx \cap \bar{V}_i' \neq \varnothing\}$. Then H_x is a closed τ_p-neighbourhood of f_0, and therefore $F_x = H_x \cap F$ is a non-empty closed subset of (F, τ_c). Since F satisfies (G), F_x is collectively continuous. Then $N_x = \bigcap_{f \in F_x} f^+(U_i) \cap \bigcap_{f \in F_x} f^-(U_i')$ is open in X, and contains x, because $f \in F_x$ implies $fx \subset \bar{V}_i \subset U_i$ and $fx \cap U_i' \supset fx \cap \bar{V}_i \neq \emptyset$. Since K_i is compact, there is a finite sequence $\{x_j^i\}_{1 \leqq j \leqq k_i}$ in K_i and corresponding neighbourhoods $H_{x_j^i}, N_{x_j^i}$ of f_0, x_j^i, respectively, such that $K_i \subset \bigcup_{j=1}^{k_i} N_{x_j^i}$. Then $M = \cap_{i=1}^{n} \cap_{j=1}^{k_i} H_{xj}$ is a τ_p-neighbourhood of f_0. Hence there exists $f' \in M \cap F$. The proof will be complete if we show that, for any $i = 1, \cdots, n$, $f'(K_i) \subset U_i$ and $f'x \cap U_i' \neq \emptyset$ for all, $x \in K_i$. Let $x \in K_i$. There exists $j(1 \leqq j \leqq k_i)$ such that $x \in N_{x_j^i} \subset \bigcap_{f \in F_{x_j^i}} f^-(U_i')$ and therefore $fx \cap U_i' \neq \emptyset$ for all $f \in F_{x_j^i}$. Since $f' \in M \cap F \subset H_{x_j^i} \cap F = F_{x_j^i}$ we have $f'x \cap U_i' \neq \emptyset$. On the other hand, $x \in N_{x_j^i} \subset \bigcap_{f \in F_{x_j^i}} f^+(U_i)$. Then $fx \subset U_i$ for all $f \in F_{x_j^i}$, and, in particular, $f'x \subset U_i$. Since x is an arbitrary point of K_i, $f'(K_i) \subset U_i$.

Corollary 3. *Let F be a family of point-compact multifunctions on a topological space X to a regular space Y. If F satisfies (G), then $\tau_p = \tau_c$ on F.*

Proof. It suffices to show that, on F, τ_p is larger than τ_c. Let H be closed in (F, τ_c). Obviously H satisfies (G), so $\bar{H} = \tilde{H}$. Hence $H = \tilde{H} \cap F = \bar{H} \cap F$ is closed in (F, τ_p).

4. Ascoli theorem

A subset E of a topological space X is *k-closed* if $E \cap K$ is closed in K for every compact subset K of X. A topological space is a *k-space* if every k-closed subset is closed Cohen (1954; page 79). Locally compact spaces and spaces satisfying the first countability axiom are familiar examples of k-spaces.

Lemma 6. *Let F_0 be the family of all point-compact multifunctions on a set X to a Hausdorff space Y. Then (F_0, τ_p) is Hausdorff.*

Proof. Let f, g be distinct members of F_0; we may suppose a $y \in fx - gx$ for some $x \in X$ and some $y \in Y$. We can then construct disjoint neighbourhoods of y, gx, from which we define disjoint neighbourhoods of f, g.

Let F_0 be the family of all point-compact multifunctions on a set X to a topological space Y, and let $F \subset F_0$. We will say that a subset B of F_0 *covers* F if $\bar{F} \subset B$, where $\bar{F}$ denotes the τ_p-closure of F in F_0. If X, Y are topological spaces, the symbol $\mathscr{C}(X, Y)(C(X, Y))$ will denote the subfamily consisting of all continuous multifunctions (continuous functions) of F_0. If Y is regular and $F \subset F_0$ satisfies (G), then $\mathscr{C}(X, Y)$ covers F (Lemma 4). If, further, F consists of functions, then $C(X, Y)$ covers F (Corollary 2).

The following lemma appears as Corollary 3 in Fox and Morales (to appear):

Lemma 7. *Let $\{X_a\}_{a \in A}$ be a non-empty family of compact spaces. The set of all point-compact multifunctions of $P\{X_a : a \in A\}$ is τ_p-compact.*

Let $\{X_a\}_{a\in A}$ be a family of non-empty sets and let $F \subset P\{X_a: a \in A\}$. For $a \in A$, we write $F[a] = \bigcup_{f\in F} fa$.

LEMMA 8. *Let $\{X_a\}_{a\in A}$ be a non-empty family of topological spaces. If $F \subset P\{X_a: a\in A\}$ then $\bar{F} \subset P\{\overline{F[a]}: a \in A\}$, where $\bar{F}$ denotes the τ_p-closure of F in $P\{X_a: a\in A\}$.*

PROOF. Let $f \in \bar{F}$. Let $a \in A$ be arbitrary, let $y \in fa$, and let V be an arbitrary neighbourhood of y. Since $M = \{h: h \in P\{X_a: a \in A\}$ and $ha \cap V \neq \varnothing\}$ is a τ_p-neighbourhood of f, there exists $f' \in M \cap F$. Then $f'a \cap V \neq \varnothing$ and $f'a \subset F[a]$, and therefore $F[a] \cap V \neq \varnothing$. So $y \in \overline{F[a]}$. Since y is an arbitrary point of fa, we have $fa \subset \overline{F[a]}$.

THEOREM. *Let F be a family of point-compact continuous multifunctions on a topological space X to a regular space Y, and let B be a family of point-compact multifunctions of Y^{mX} which covers F. The conditions*

(a) *F is closed in (B, τ_c),*

(b) *$F[x]$ is compact for all $x \in X$, and*

(c) *F satisfies (G),*

are sufficient for the τ_c-compactness of F. If, further, X is a k-space and Y is Hausdorff, then the condition (a), (b) *and* (c) *are necessary for the τ_c-compactness of F.*

PROOF. *Sufficiency.* Let F_0, F_0' be the sets of all point-compact multifunctions of Y^{mX}, $P\{\overline{F[x]}: x \in X\}$, respectively. If $\bar{F}$ denotes the τ-$_p$closure of F in F_0, Lemma 8 implies $\bar{F} \subset F_0' \subset F_0$. Since Y is regular, (b) implies the compactness of $\overline{F[x]}$ for all $x \in X$. Then, by Lemma 7, F_0' is τ_p-compact, and so $\bar{F}$ is τ_p-compact. Let $\tilde{F}$ denote the τ_c-closure of F in F_0. By Lemma 5, (c) implies $\tilde{F} = \bar{F}$. But $\bar{F} \subset B$ and (a) implies $F = \tilde{F} \cap B$, thus $F = \bar{F}$ is τ_p-compact. Since (c) implies $(F, \tau_p) = (F, \tau_c)$ (Corollary 3), F is τ_c-compact.

Necessity. Since (B, τ_c) is Hausdorff (Lemma 6), F is closed in (B, τ_c). Since the projections pr_x are point-compact and continuous, the $F[x]$ are compact. To prove that F satisfies (G) it will suffice to show that F is collectively continuous. Since X is a k-space, it will suffice to show that, if W is closed in Y, then $\bigcup_{f\in F} f^+(W)$ and $\bigcup_{f\in F} f^-(W)$ are k-closed. Let K be an arbitrary compact subset of X and write $S = \bigcup_{f\in F} f^+(W) \cap K$, $S' = \bigcup_{f\in F} f^-(W) \cap K$.

We will show that if $x \in K - S$, then $x \notin \bar{S} \cap K$. We have $fx \cap (Y - W) \neq \varnothing$ for all $f \in F$. Let $f \in F$. By Lemma 3, there exists a neighbourhood N_f of x such that $fz \cap (Y - W) \neq \varnothing$ for all $z \in \bar{N}_f$. Then $K_f = \bar{N}_f \cap K$ is compact, and therefore $M_f = \{h: h \in F$ and $hz \cap (Y - W) \neq \varnothing$ for all $z \in K_f\}$ is open in (F, τ_c) and contains f. Since F is compact, there is a finite sequence $\{f_i\}_{1 \leq i \leq n}$ in F such that $F \subset \bigcup_{i=1}^n M_{f_i}$. Let $K^* = \bigcap_{i=1}^n K_{f_i}$. Then, if $f \in F$, $fz \cap (Y - W) \neq \varnothing$ for all $z \in K^*$, and therefore $K^* \cap (\bigcup_{f\in F} f^+(W)) = \varnothing$. On the other hand,

$N^* = \bigcap_{i=1}^{n} N_{f_i}$ is a neighbourhood of x such that $N^* \cap K \subset K^*$. Then $N^* \cap S = \emptyset$, proving the assertion.

We will show that if $x \in K - S'$, then $x \notin \bar{S}' \cap K$. We have $fx \subset Y - W$ for all $f \in F$. Let $f \in F$. There exists an open set V_f in Y such that $fx \subset V_f \subset \bar{V}_f \subset Y - W$, and there exists a neighbourhood N_f of x such that $f(N_f) \subset V_f$. Then $\overline{f(N_f)} \subset Y - W$. Let $K_f = \bar{N}_f \cap K$. Then K_f is compact, and therefore $M_f = \{h\colon h \in F \text{ and } h(K_f) \subset Y - W\}$ is open in (F, τ_c). Since $f(\bar{N}_f) \subset \overline{f(N_f)}$ Ponomarev (1960; page 120), $f \in M_f$. There is a finite sequence $\{f_i\}_{1 \leqq i \leqq n}$ in F such that $F \subset \bigcup_{i=1}^{n} M_{fi}$. Let $K^* = \bigcap_{i=1}^{n} K_{fi}$. Then, if $f \in F$, $f(K^*) \subset Y - W$, and therefore $K^* \cap (\bigcup_{f \in F} f^{-}(W)) = \emptyset$. Let $N^* = \bigcap_{i=1}^{n} N_{fi}$. Then N^* is a neighbourhood of x such that $N^* \cap K \subset K^*$; therefore $N^* \cap S' = \emptyset$, and the proof is complete.

COROLLARY 4. *Let F be a family of point-compact continuous multifunctions on a topological space X to a regular space Y. The conditions*

(a) *F is closed in $(\mathscr{C}(X, Y), \tau_c)$,*
(b) *$F[x]$ is compact for all $x \in X$, and*
(c) *F satisfies (G),*

are sufficient for the τ_c-compactness of F. If, further, X is a k-space and Y is Hausdorff, then the conditions (a), (b) *and* (c) *are necessary for the τ_c-compactness of F.*

COROLLARY 5. *Let F be a family of continuous functions on a topological space X to a regular space Y. The conditions*

(a) *F is closed in $(C(X, Y), \tau_c)$,*
(b) *$F[x]$ is compact for all $x \in X$, and*
(c) *F satisfies (G),*

are sufficient for the τ_c-compactness of F. If, further, X is a k-space and Y is Hausdorff, then the conditions (a), (b) *and* (c) *are necessary for the τ_c-compactness of F.*

REMARKS 1. Corollary 5 is the essential Theorem 1 of Gale (1950; page 304).

2. The point-like condition of Mancuso played two roles, one of which we have by-passed by working entirely in the space of point-compact multifunctions. The second role was to assure the non-emptiness of the sets F_0 of our Lemma 4 (logically superfluous). However, without evoking the point-like condition, we may if we wish, prove these sets F_0 non-empty. For example, in the first case, we consider $M = \{g\colon g \in Y^{mX}, g \text{ is point-compact and } gx \cap G \neq \emptyset\}$, which is a τ_p-neighbourhood of f, so there exists $f' \in F \cap M \subset F_0$.

References

C. Berge (1965), *Topological Spaces*, (The MacMillan Company, New York (1965).)

D. E. Cohen (1954), 'Spaces with weak topology', *Quart. J. Math. Oxford* Ser. (2) **5**, 77-80.

G. Fox and P. Morales (to appear), 'A general Tychonoff theorem for multifunctions', *Canad. Math. Bull.*

D. Gale (1950), 'Compact sets of functions and functions rings', *Proc. Amer. Math. Soc.* **1**, 303–308.

V. J. Mancuso (1971), 'An Ascoli theorem for multi-valued functions', *J. Austral. Math. Soc.* **12**, 466–472.

V. I. Ponomarev (1960), 'Properties of topological spaces preserved under multi-valued continuous mappings', *Math. Sb.* **5** (N.S.) 51 (93), 515–536 = *Amer. Math. Soc. Transl.* Ser (2) **38** (1968), 119–140.

R. E. Smithson (1971), 'Topologies on sets of relations', *J. Math. Natur. Sci. and Math.* (Lahore) **11**, 43–50.

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