

## BOUNDNESS OF INTERSECTION NUMBERS FOR ACTIONS BY TWO-DIMENSIONAL BIHOLOMORPHISMS

JAVIER RIBÓN 

*Instituto de Matemática e Estatística, Universidade Federal Fluminense,  
Campus do Gragoatá, Rua Marcos Valdemar de Freitas Reis s/n,  
24210-201 Niterói, Rio de Janeiro, Brazil  
(jribon@id.uff.br)*

(Received 11 November 2019; revised 17 November 2020; accepted 20 November 2020;  
first published online 22 January 2021)

*Abstract* We say that a group  $G$  of local (maybe formal) biholomorphisms satisfies the uniform intersection property if the intersection multiplicity  $(\phi(V), W)$  takes only finitely many values as a function of  $G$  for any choice of analytic sets  $V$  and  $W$  of complementary dimension. In dimension 2 we show that  $G$  satisfies the uniform intersection property if and only if it is finitely determined – that is, if there exists a natural number  $k$  such that different elements of  $G$  have different Taylor expansions of degree  $k$  at the origin. We also prove that  $G$  is finitely determined if and only if the action of  $G$  on the space of germs of analytic curves has discrete orbits.

*Keywords and phrases:* local diffeomorphism, intersection number, solvable group, nilpotent group

*2020 Mathematics Subject Classification:* Primary: 32H50; 37C85; Secondary: 37F75; 20G20; 20F16

### 1. Introduction

Consider a subgroup  $G$  of the group  $\text{Diff}(\mathbb{C}^n, 0)$  of local biholomorphisms defined in a neighbourhood of the origin of  $\mathbb{C}^n$ . We are interested in understanding the properties of the action of  $G$  on the local intersection multiplicities  $(V, W)$  (at the origin), where  $V$  and  $W$  are germs of analytic set of complementary dimension (see [13, Chapter 8] for the definition of intersection multiplicity and [9, Chapter 2.6] for the case of plane curves that is the most used in this paper). Given local complex analytic sets defined in a neighbourhood of 0 in  $\mathbb{C}^n$ , we study the behaviour of the function  $\phi \mapsto (\phi(V), W)$  defined in  $G$ . This can be considered a proxy for the study of the dynamics of the group and its mixing properties. More precisely, we want to identify the groups such that for any choice of fixed  $V$  and  $W$ , the function  $\phi \mapsto (\phi(V), W)$  of  $G$  takes only finitely many values. We say that such groups satisfy the uniform intersection (UI) property.

A motivation to study the UI property is provided by the following result from Shub and Sullivan:

**Theorem 1.1** ([23]). *Let  $f : U \rightarrow \mathbb{R}^m$  be a  $C^1$  map, where  $U$  is an open subset of  $\mathbb{R}^m$ . Suppose that 0 is an isolated fixed point of  $f^n$  for every  $n \in \mathbb{N}$ . Then the sequence*

$(I(f^n, 0))_{n \geq 1}$  of fixed-point indexes is bounded from above by a constant that does not depend on  $n$ .

The fixed-point index of  $f$  at the isolated fixed point  $0$  is the topological intersection multiplicity of the diagonal  $\Delta = \{(x, x) : x \in \mathbb{R}^m\}$  of  $\mathbb{R}^m \times \mathbb{R}^m$  with the graph  $\Gamma_f = \{(x, f(x)) : x \in U\}$  at  $(0, 0)$  in  $\mathbb{R}^m \times \mathbb{R}^m$  (see [22, section 7.8]). This definition coincides with the usual one as the degree of the map  $\frac{f(x)-x}{|f(x)-x|} : \mathbb{S}^{m-1}(r) \rightarrow \mathbb{S}^{m-1}$  for  $r > 0$  small enough. Consider the map  $F : \mathbb{R}^m \times U \rightarrow \mathbb{R}^m \times \mathbb{R}^m$  defined by the formula  $F(x, y) = (x, f(y))$ . Since  $\Gamma_{f^n} = F^n(\Delta)$ , the fixed-point index of  $f^n$  at the origin is the topological intersection multiplicity  $(\Delta, F^n(\Delta))$  at  $(0, 0) \in \mathbb{R}^m \times \mathbb{R}^m$ . Hence, Shub and Sullivan’s theorem can be interpreted as a result of uniform intersection. As an application of Theorem 1.1, they show that a  $C^1$  automorphism  $f : M \rightarrow M$  of a compact manifold  $M$ , whose sequence of Lefschetz numbers  $(L(f^n))_{n \geq 1}$  is unbounded, has infinitely many periodic points.

Theorem 1.1 inspired Arnold [4] to study the uniform intersection property for  $C^\infty$  and holomorphic maps, especially for diffeomorphisms. Arnold’s results were generalised by Seigal and Yakovenko [21] and Binyamini [7]. The degree of regularity is much higher than  $C^1$ , of course, but on the other hand the results can be applied to more general classes of maps and groups.

This paper completes in dimension 2 the program, partially carried out in [4, 7, 19, 21], of characterising the groups that satisfy the uniform intersection property (see Main Theorem).

The paper is inserted into a classical subject, namely the local study of asymptotics of topological complexity of intersections. The global case has been treated extensively in the literature [1, 2, 3, 5] and the references therein. Moreover, there are more than 15 problems Arnold’s book devoted to asymptotics of topological complexity of intersections in both the local and global settings [3]. For instance, several problems are devoted to estimating the topological invariants of intersections of the form  $(\phi^n(V), W)$  in terms of  $n$  when  $\phi$  is not necessarily a diffeomorphism (1988-8, 1990-20, 1992-12, 1994-49,...). Also considered is the problem of estimating the growth of  $\sharp \text{Fix}(f^n)$  for smooth and analytic diffeomorphisms of compact manifolds (1989-2, 1994-45).

Let us recap some results from the literature. In the context of subgroups of  $\text{Diff}(C^n, 0)$  (or its formal completion  $\widehat{\text{Diff}}(C^n, 0)$ ), with  $n \geq 2$ , we have the following inclusions:

$$\text{cyclic} \subset \text{finitely generated abelian} \subset \text{Lie} = F_{\text{dim}} \subset \text{UI} \subset \text{FD}. \tag{1}$$

Cyclic groups satisfy UI by a theorem of Arnold [4]. The result was generalised to finitely generated abelian groups by Seigal and Yakovenko [21]. Binyamini showed UI for subgroups of  $\widehat{\text{Diff}}(C^n, 0)$  that can be embedded in a subgroup of  $\widehat{\text{Diff}}(C^n, 0)$  with a natural structure of finitely dimensional Lie group with finitely many connected components [7]. Moreover, he showed that every finitely generated abelian group admits such an embedding. The term  $F_{\text{dim}}$  stands for the set of subgroups of  $\widehat{\text{Diff}}(C^n, 0)$  whose Zariski closure has finite dimension; they satisfy UI by a theorem of the author [19]. These groups are exactly the groups that can be embedded in Lie groups (with finitely many connected components), but the use of intrinsic techniques allowed us to show, for instance, that virtually polycyclic subgroups of  $\widehat{\text{Diff}}(C^n, 0)$  (and in particular finitely generated nilpotent

subgroups) hold the property UI. Very roughly speaking, in all previous cases the problem is transferred to a finite-dimensional space where some finiteness result (the Skolem theorem on zeroes of quasipolynomials [24], or Noetherianity arguments) is used.

The term FD in sequence (1) stands for finite determination; we say that  $G$  is finitely determined if there exists  $k \in \mathbb{N}$  such that every element of  $G$  is determined by the  $k$  jet of its Taylor power-series expansion at the origin. This property has been studied by Baouendi et al. [6]; in particular, they showed that a Lie group (with finitely many connected components) of local analytic diffeomorphisms has the property of finite determination. Moreover, it is easy to verify that UI groups are always finitely determined for  $n \geq 2$  (Lemma 3.1).

Let us focus on the subsequence of inclusions  $F_{\dim} \subset UI \subset FD$ . There are examples of groups that satisfy UI but are not finite-dimensional. Consider the abelian subgroup

$$\mathcal{G} := \{(x, y + f(x)) : f \in \mathbb{C}\{x\}, f(0) = 0\}$$

of  $\text{Diff}(\mathbb{C}^2, 0)$ . Consider a finitely determined subgroup  $G$  of  $\mathcal{G}$  that is nonfinite-dimensional. For instance, we can consider the group generated by the diffeomorphisms  $\phi_j(x, y) = (x, y + d_j x + x^{j+1})$  for  $j \in \mathbb{N}$ . If the set  $\{d_1, d_2, \dots\}$  is linearly independent over  $\mathbb{Q}$ , then the first jet of an element  $\phi$  of  $G$  determines  $\phi$ . Moreover,  $G$  is infinite-dimensional, since the polynomials of the form  $d_j x + x^{j+1}$  for  $j \in \mathbb{N}$  are linearly independent over  $\mathbb{C}$ .  $G$  satisfies UI. A complete proof is provided in Proposition 7.1, but in order to illustrate how finite determination implies UI in this context, let us consider the intersection multiplicity

$$(\phi\{y = g(x)\}, \{y = g(x)\}) \tag{2}$$

for a smooth curve  $y = g(x)$ . Given  $\phi(x, y) = (x, y + f(x))$ , we have

$$\phi\{y = g(x)\} = \{y = f(x) + g(x)\}.$$

Thus expression (2) is equal to the vanishing order of  $f(x)$  at  $x = 0$  and hence it is bounded by hypothesis whenever it is different than  $\infty$ . Analogously we have  $F_{\dim} \subsetneq UI$  (see sequence (1)) in every dimension  $n \geq 2$ .

What happens with the inclusion  $UI \subset FD$ ? It is not difficult to find examples of subgroups of  $\text{Diff}(\mathbb{C}^n, 0)$  that satisfy FD but not UI in dimension  $n \geq 3$  (Lemma 3.2). The main result of this paper is the following theorem:

**Main Theorem.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then  $G$  satisfies the uniform intersection property if and only if  $G$  satisfies the finite determination property.*

This theorem characterises the groups of diffeomorphisms satisfying the uniform intersection property in the two-dimensional case. It unifies and extends the previous results in the literature, which showed that the following types of groups hold the uniform intersection property: cyclic groups [4], finitely generated abelian groups [21], Lie groups (with finitely many connected components) [7] and finitely generated virtually nilpotent groups and virtually polycyclic groups [19]. The finite determination condition is the culmination of the journey. It is much simpler than the previous best sufficient conditions to guarantee the uniform intersection property, namely Lie and  $F_{\dim}$  (equation (1)).

In order to show the UI property, finite determination is the simplest necessary condition. Surprisingly, it is also a sufficient condition.

We can characterise finitely determined subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  in terms of their actions on the space of curves.

**Theorem 1.2.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then  $G$  is finitely determined if and only if the action of  $G$  on the topological space of formal irreducible curves has discrete orbits.*

We say that a formal irreducible curve  $\gamma$  (see Definition 2.3) satisfies the property  $(\text{UI})_\gamma$  if its orbit is discrete (see Definition 4.4 for a description of the topology on the space of curves). We remark that the  $G$ -orbit of  $\gamma$  is discrete if and only if the function  $\phi \mapsto (\phi(\gamma), \gamma)$  (of  $G$ ) takes finitely many values (Lemma 4.1).

The Main Theorem holds true in the  $C^\infty$  setting, where we consider subgroups of the group  $\text{Diff}^\infty(\mathbb{R}^2, 0)$  of  $C^\infty$  local diffeomorphisms (defined in a neighbourhood of the origin of  $\mathbb{R}^2$ ) and germs of  $C^\infty$  subvarieties in the definition of the uniform intersection property. Indeed, given an ideal  $I$  of the ring  $C_0^\infty(\mathbb{R}^n)$  of germs of  $C^\infty$  functions, the set  $\hat{I}$  consisting of the Taylor power-series expansions of its elements at 0 is an ideal of the ring  $\mathbb{R}[[x_1, \dots, x_n]]$  of formal power series. Given an ideal  $J$  of  $C_0^\infty(\mathbb{R}^n)$ , we define the intersection of  $I$  and  $J$  as the intersection of  $\hat{I}$  and  $\hat{J}$  (see Definition 3.2). So we fall into the setting of the Main Theorem by considering the Taylor series expansion at the origin of diffeomorphisms and subvarieties. For instance, the Main Theorem implies that if  $G$  is a finitely determined subgroup of  $\text{Diff}^\infty(\mathbb{R}^2, 0)$  and  $\alpha$  and  $\beta$  are germs of smooth curves, then there exists  $M \in \mathbb{N}$  such that if the order of contact between  $\phi(\alpha)$  and  $\beta$  is greater than  $M$  and  $\phi \in G$ , then  $\phi(\alpha)$  and  $\beta$  are infinitely tangent at 0.

The implication  $\text{FD} \implies \text{UI}$  is the difficult part of the proof of the Main Theorem. As a first reduction, we show that the group  $G$  satisfies UI if and only if  $(\text{UI})_\gamma$  holds for any irreducible curve  $\gamma$  (Proposition 4.2).

Since  $\text{F}_{\dim}$  implies UI [19], we just need to consider groups that are nonfinite-dimensional. In order to understand such groups we introduce two results of independent interest about the properties of infinite-dimensional Lie subalgebras of the Lie algebra  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  of formal vector fields. First, we provide a classification of infinite-dimensional nilpotent Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ .

**Theorem 1.3.** *Let  $\mathfrak{g}$  be a nilpotent subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ . Then either  $\mathfrak{g}$  is finite-dimensional as a complex vector space or there exists  $X \in \mathfrak{g}$  that has a first integral in  $\hat{\mathcal{O}}_2 \setminus \mathbb{C}$  such that*

$$\mathfrak{g} \subset \{fX : f \in \hat{\mathcal{O}}_2 \text{ and } X(f) = 0\}. \tag{3}$$

*In particular, if  $\mathfrak{g}$  is infinite-dimensional, then  $\mathfrak{g}$  is abelian.*

It is useful to study infinite-dimensional nilpotent subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  (and thus infinite-dimensional nilpotent Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ ), since, for example, finitely determined subgroups of diffeomorphisms tangent to the identity are nilpotent (see Lemma 7.1).

It is possible to reduce the proof of the Main Theorem to the case of solvable groups  $G$ . Hence it is natural to work with the derived series of  $G$  and its Lie algebra. The next theorem implies that the property of being infinite-dimensional is preserved along the derived series:

**Theorem 1.4.** *Let  $\mathfrak{g}$  be a Lie subalgebra of  $\widehat{\mathfrak{X}}(\mathbb{C}^2, 0)$  such that  $\dim_{\mathbb{C}} \mathfrak{g} = \infty$ . Then either  $\mathfrak{g}$  is abelian or  $\dim_{\mathbb{C}} \mathfrak{g}' = \infty$ .*

Given a pair  $(G, \gamma)$  consisting of a finitely determined subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and a formal irreducible curve  $\gamma$ , we say that  $(J, \gamma)$  is a reduction pair for  $G$  if  $J$  is a subgroup of  $G$  and  $G$  satisfies  $(\text{UI})_{\gamma}$  if and only if  $J$  does. Fix an irreducible curve  $\gamma$ . It is possible to show that there exists a reduction pair  $(J, \gamma)$  for  $G$  such that one of the following situations is true:

- $\gamma$  is  $J$ -invariant;
- $J$  is finite-dimensional;
- $J$  is abelian.

The first case is trivial, since the function of  $J$  defined by  $\phi \mapsto (\phi(\gamma), \gamma)$  is constant and equal to  $\infty$ . In the second case,  $J$  satisfies UI as a consequence of the inclusion  $F_{\dim} \subset \text{UI}$  and more precisely of [19]. It remains to consider the case where  $J$  is abelian and infinite-dimensional. Then  $J$  is a subgroup of the exponential  $\exp(\mathfrak{g})$  of an infinite-dimensional abelian complex Lie algebra  $\mathfrak{g}$  of formal vector fields. Moreover,  $\mathfrak{g}$  has the special form determined in Theorem 1.3, since  $\dim_{\mathbb{C}} \mathfrak{g} = \infty$ . This situation was described already for  $X = \frac{\partial}{\partial y}$ . The proof of property UI can be reduced to the proof of such a case.

We consider several kinds of reductions  $(J, \gamma)$  of a pair  $(G, \gamma)$ . One reduction consists of replacing  $G$  with a finite index or suitable normal subgroup  $J$ . We can also reduce the pair by replacing  $G$  with the subgroup  $J$  of elements of  $G$  whose linear parts preserve the tangent direction  $\ell$  of  $\gamma$  at the origin. Furthermore, this reduction can be generalised by considering iterated tangents (or infinitely near points) of the curve  $\gamma$ . It is then natural to use desingularisation techniques (of curves and foliations) in the proof of the Main Theorem to obtain simpler expressions for the pair  $(G, \gamma)$ .

**2. Notation**

Let us introduce some notation that will be useful in the paper.

**2.1. Formal power series and curves**

**Definition 2.1.** We denote by  $\mathcal{O}_n$  (resp.,  $\hat{\mathcal{O}}_n$ ) the local ring  $\mathbb{C}\{z_1, \dots, z_n\}$  (resp.,  $\mathbb{C}[[z_1, \dots, z_n]]$ ) of convergent (resp., formal) power series with complex coefficients in  $n$  variables. We denote by  $\mathfrak{m}$  the maximal ideal of  $\hat{\mathcal{O}}_n$ . We define  $\hat{K}_n$  as the field of fractions of  $\hat{\mathcal{O}}_n$ .

**Definition 2.2.** Let  $f \in \hat{\mathcal{O}}_n$ . We define the  $k$ -jet  $j^k f$  as the polynomial  $P_k(z_1, \dots, z_n)$  of degree less or equal than  $k$  such that  $f - P_k \in \mathfrak{m}^{k+1}$ .

Let us define formal curves for an ambient space of dimension 2.

**Definition 2.3.** A formal curve is an ideal  $(f)$  of  $\hat{\mathcal{O}}_2$ , where  $f \neq 0$  belongs to the maximal ideal of  $\hat{\mathcal{O}}_2$ . We say that the curve is irreducible if  $f$  is an irreducible element of  $\hat{\mathcal{O}}_2$ .

**2.2. Formal diffeomorphisms and vector fields**

**Definition 2.4.** We define the group of formal diffeomorphisms in  $n$  variables as

$$\widehat{\text{Diff}}(\mathbb{C}^n, 0) = \{(\phi_1, \dots, \phi_n) \in \mathfrak{m} \times \dots \times \mathfrak{m} : (j^1\phi_1, \dots, j^1\phi_n) \in \text{GL}(n, \mathbb{C})\}.$$

The group operation is defined in such a way that the composition

$$(\rho_1, \dots, \rho_n) = (\phi_1, \dots, \phi_n) \circ (\eta_1, \dots, \eta_n)$$

satisfies  $j^k\rho_j = j^k(j^k\phi_j \circ (j^k\eta_1, \dots, j^k\eta_n))$  for all  $1 \leq j \leq n$  and  $k \in \mathbb{N}$ . The subgroup  $\text{Diff}(\mathbb{C}^n, 0) = \widehat{\text{Diff}}(\mathbb{C}^n, 0) \cap \mathcal{O}_n^n$  is called the *group of local biholomorphisms* in  $n$  variables.

**Remark 2.1.** The group  $\text{Diff}(\mathbb{C}^n, 0)$  consists of germs of biholomorphisms by the inverse function theorem.

**Definition 2.5.** Given  $\phi = (\phi_1, \dots, \phi_n) \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , we denote by  $j^1\phi$  the linear part at the origin  $(j^1\phi_1, \dots, j^1\phi_n)$  of  $\phi$ . Given a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , we define  $j^1G$  as the group  $\{j^1\phi : \phi \in G\}$  of linear parts of elements of  $G$ .

**Definition 2.6.** We denote by  $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$  the set of formal diffeomorphisms consisting of elements  $\phi = (\phi_1, \dots, \phi_n)$  such that  $j^1\phi$  is a unipotent linear isomorphism – that is, 1 is its unique eigenvalue. We define the group of formal diffeomorphisms *tangent to the identity* as

$$\widehat{\text{Diff}}_1(\mathbb{C}^n, 0) = \{\phi = (\phi_1, \dots, \phi_n) \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : j^1\phi = \text{Id}\}.$$

That is, it is the subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  of elements with identity linear part.

**Definition 2.7.** We define the Lie algebra of formal vector fields in  $n$  variables as

$$\hat{\mathfrak{X}}(\mathbb{C}^n, 0) = \left\{ \sum_{j=1}^n f_j(z_1, \dots, z_n) \frac{\partial}{\partial z_j} : f_1, \dots, f_n \in \mathfrak{m} \right\}.$$

It can be interpreted as a derivation of the  $\mathbb{C}$ -algebra  $\mathfrak{m}$ . If an element of  $\hat{\mathfrak{X}}(\mathbb{C}^n, 0)$  satisfies  $f_1, \dots, f_n \in \mathcal{O}_n$ , we say that it is a (singular) holomorphic local vector field. We denote by  $\mathfrak{X}(\mathbb{C}^n, 0)$  the set of holomorphic singular local vector fields.

**Definition 2.8.** Given  $X = \sum_{j=1}^n f_j(z_1, \dots, z_n) \frac{\partial}{\partial z_j} \in \hat{\mathfrak{X}}(\mathbb{C}^n, 0)$ , we say that  $X$  is *nilpotent* if  $j^1X := \sum_{j=1}^n j^1f_j(z_1, \dots, z_n) \frac{\partial}{\partial z_j}$  is a linear nilpotent vector field. Denote by  $\hat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$  the set of formal nilpotent vector fields.

**2.3. The central and the derived series**

**Definition 2.9.** Let  $H, L$  be subgroups of  $G$ . We define  $[H, L]$  as the group generated by the commutators  $[h, l] := hlh^{-1}l^{-1}$  for  $h \in H, l \in L$ . We define the derived series

$$G^{(0)} = G, G^{(1)} = [G^{(0)}, G^{(0)}], \dots, G^{(k+1)} = [G^{(k)}, G^{(k)}], \dots$$

of the group  $G$ . The group  $G^{(k)}$  is called the  $k$ th derived group of  $G$ . The derived group  $G^{(1)}$  is also denoted by  $G'$ .

We define by  $\ell(G) = \min(\{k \geq 0 : G^{(k)} = \{\text{Id}\} \cup \{\infty\}\})$  the *derived length* of  $G$ . We say that  $G$  is *solvable* if  $\ell(G) < \infty$ .

**Definition 2.10.** We define the lower central series

$$C^0 G = G, C^1 G = [C^0 G, G], \dots, C^{k+1} G = [C^k G, G], \dots$$

of the group  $G$ . We say that  $G$  is *nilpotent* if there exists  $k \geq 0$  such that  $C^k G = \{\text{Id}\}$ . Moreover,  $\min(\{k \geq 0 : C^k G = \{\text{Id}\} \cup \{\infty\}\})$  is called the *nilpotency class* of  $G$ .

**Definition 2.11.** Given Lie subalgebras  $\mathfrak{h}, \mathfrak{l}$  of a Lie algebra  $\mathfrak{g}$ , we define  $[\mathfrak{h}, \mathfrak{l}]$  as the Lie algebra generated by the Lie brackets  $[X, Y]$ , where  $X \in \mathfrak{h}$  and  $Y \in \mathfrak{l}$ . We can define the derived Lie algebra  $\mathfrak{g}'$ , the derived series  $(\mathfrak{g}^{(k)})_{k \geq 0}$ , the central lower series  $(C^k \mathfrak{g})_{k \geq 0}$  and nilpotent and solvable Lie algebras analogously as in Definitions 2.9 and 2.10.

### 3. Finite determination and uniform intersection

In this section we introduce the first connections between the finite determination property and the uniform intersection property. In particular, we show that the latter property implies the former in dimension  $n \geq 2$  but the reciprocal is not true for any  $n \geq 3$ . In particular, the analogue of the Main Theorem for  $n \geq 3$  does not hold true.

**Definition 3.1.** Given  $k \in \mathbb{N}$ , we say that a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is *k-finitely determined* if  $\{\phi \in G : j^k \phi = \text{Id}\} = \{\text{Id}\}$ . We say that  $G$  has the *finite determination property* (FD) if it is  $k$ -finitely determined for some  $k \in \mathbb{N}$ .

**Remark 3.1.** The elements of a  $k$ -finitely determined subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  are determined by their  $k$ -jets.

**Definition 3.2.** The intersection multiplicity of ideals  $I$  and  $J$  of  $\hat{\mathcal{O}}_n$  is defined as the dimension of the complex vector space  $\hat{\mathcal{O}}_n/(I, J)$ .

**Remark 3.2.** Formal schemes are given by ideals of the ring  $\hat{\mathcal{O}}_n$ . Definition 3.2 provides an upper bound for the usual definition of intersection multiplicity of formal schemes (or ideals) with their associated cycle structure (cf. [13, Proposition 8.2]). Moreover, both definitions coincide if one of them is equal to  $\infty$ . Thus the uniform intersection property with respect to Definition 3.2 implies UI for the usual definition of the intersection multiplicity.

**Definition 3.3.** We say that a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  has the *uniform intersection property* (UI) if for any pair  $\alpha, \beta$  of formal schemes, the subset  $\{(\phi(\alpha), \beta) : \phi \in G\}$  of  $\mathbb{N} \cup \{\infty\}$  is finite. Equivalently, given ideals  $I, J$  of  $\hat{\mathcal{O}}_n$ , the set  $\{(\phi^*(I), J) : \phi \in G\}$  is finite. In other words, the intersection multiplicities between the shifts of  $\alpha$  by the action of the group and  $\beta$  (different from infinity) are uniformly bounded.

**Lemma 3.1.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  ( $n \geq 2$ ) that satisfies UI. Then it also satisfies FD.*

**Proof.** Suppose that  $G$  does not satisfy the finite determination property. Then there exists a sequence  $(\phi_k)_{k \geq 1}$  of elements in  $G \setminus \{\text{Id}\}$  such that  $j^k \phi_k = \text{Id}$  for any  $k \in \mathbb{N}$ . Let  $N$  be a germ of holomorphic manifold of dimension  $n - 1$ . Up to considering a holomorphic change of coordinates, we can suppose that  $N$  is a linear subspace of  $\mathbb{C}^n$ . Given  $k \in \mathbb{N}$ , we define

$$B_k = \{\ell \in P(N) : \phi_k(\ell) \subset N\},$$

where  $P(N)$  is the set of lines in  $N$  through the origin – that is, the projective space of  $N$ . The set  $B_k$  either is equal to  $P(N)$  (and hence  $\phi_k(N) = N$ ) or is a closed nowhere dense subset of  $P(N)$ . We claim that  $\phi_k(N) = N$  for some  $k \geq k_0(N)$ . Otherwise there exists a line  $\ell$  contained in  $N$  such that  $\phi_k(\ell) \not\subset N$  for every element  $k$  of some infinite subset  $I$  of  $\mathbb{N}$ , by the Baire category theorem. In particular, we have  $(\phi_k(\ell), N) \neq \infty$  for any  $k \in I$ . The construction of the sequence  $(\phi_k)_{k \geq 1}$  implies that  $\lim_{k \rightarrow \infty} (\phi_k(\ell), N) = \infty$  and hence  $\{(\phi_k(\ell), N) : k \in I\}$  is an infinite set, contradicting UI.

Given  $c \in \mathbb{C}$ , the submanifold  $N_c = \{x_1 - cx_2 = 0\}$  of  $\mathbb{C}^n$  is invariant by  $\phi_k$  for  $k \geq k_0(c)$ , by the previous discussion. We define

$$A_m = \{c \in \mathbb{C} : \phi_k(N_c) = N_c \ \forall k \geq m\}.$$

Since  $\cup_{m \in \mathbb{N}} A_m = \mathbb{C}$ , it follows that  $A_{m_0}$  is uncountable for some  $m_0 \in \mathbb{N}$ . The equality of ideals  $(x_1 - cx_2) \circ \phi_k = (x_1 - cx_2)$  implies

$$(x_1 \circ \phi_k)x_2 - x_1(x_2 \circ \phi_k) = (u_{c,k}x_2 - x_2 \circ \phi_k)(x_1 - cx_2)$$

for all  $c \in A_{m_0}$  and  $k \geq m_0$ , where  $u_{c,k} = \frac{(x_1 - cx_2) \circ \phi_k}{x_1 - cx_2} \in \hat{\mathcal{O}}_n$  is a unit. Since  $(x_1 \circ \phi_k)x_2 - x_1(x_2 \circ \phi_k)$  vanishes in uncountable many hypersurfaces, we get  $(x_1 \circ \phi_k)x_2 - x_1(x_2 \circ \phi_k) = 0$  and then  $\frac{x_1}{x_2} \circ \phi_k \equiv \frac{x_1}{x_2}$  for any  $k \geq m_0$ . By proceeding analogously with the family  $\{x_1 - cx_2^2 = 0\}_{c \in \mathbb{C}}$ , we deduce that  $\frac{x_1}{x_2} \circ \phi_k \equiv \frac{x_1}{x_2}$  for every  $k \geq m_1$ . This implies  $x_1 \circ \phi_k = x_1$  and  $x_2 \circ \phi_k = x_2$  for every  $k \geq \max(m_0, m_1)$ . By considering  $x_j/x_l$  and  $x_j/x_l^2$  for  $1 \leq j < l \leq n$ , we obtain  $k_0 \in \mathbb{N}$  such that  $x_j \circ \phi_k = x_j$  for all  $1 \leq j \leq n$  and  $k \geq k_0$ . Therefore,  $\phi_k = \text{Id}$  for any  $k \geq k_0$ , contradicting the choice of  $(\phi_k)_{k \geq 1}$ . □

**Lemma 3.2.** *Let  $n \geq 3$ . There exists a subgroup  $G$  of  $\text{Diff}(\mathbb{C}^n, 0)$  that satisfies FD but does not hold UI.*

**Proof.** We denote

$$\phi_j(x_1, x_2, x_3, \dots, x_n) = (x_1, x_2 + d_j x_1^2 + x_3^{j+2}, x_3, \dots, x_n) \in \text{Diff}(\mathbb{C}^n, 0)$$

for any  $j \in \mathbb{N}$ , where  $\{d_1, d_2, \dots\}$  is linearly independent over  $\mathbb{Q}$ . We define the group  $G = \langle \phi_1, \phi_2, \dots \rangle$ . It is clearly abelian. The condition of linear independence of  $\{d_1, d_2, \dots\}$  implies that any diffeomorphism  $\phi \in G$  that is not equal to the identity map has a nonvanishing second jet. Therefore  $G$  satisfies FD.



Denote  $\alpha = \{x_1 = x_2 = 0\}$  and  $\beta = \{x_2 = x_4 = \dots = x_n = 0\}$ . We have

$$\begin{aligned}
 (\phi_j^{-1}(\alpha), \beta) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(x_1, x_2 + d_j x_1^2 + x_3^{j+2}, x_2, x_4, \dots, x_n)} \\
 &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{(x_1, x_2, x_3^{j+2}, x_4, \dots, x_n)} = j + 2
 \end{aligned}$$

for any  $j \in \mathbb{N}$ . Thus  $G$  does not satisfy UI. □

**Remark 3.3.** The group  $G$  defined in Lemma 3.2 is not finitely generated. This is fundamental in the example, since finitely generated abelian subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  satisfy UI [21].

**Remark 3.4.** The example in Lemma 3.2 illustrates a phenomenon that prevents the Main Theorem from being true for higher dimensions. The Zariski closure of the  $G$ -orbit of  $\alpha$  can have intermediate dimension (greater than  $\dim(\alpha)$  and less than  $n$ ). Indeed, in the example,  $\alpha$  is contained in the  $G$ -invariant analytic set  $\{x_1 = 0\}$ , but the subgroup  $\{\phi|_{\{x_1=0\}} : \phi \in G\}$  of  $\text{Diff}(\{x_1 = 0\}, 0)$  is not finitely determined.

#### 4. Intersection properties of curves

In this section we will see that the UI condition in dimension 2 is a property of orbits of curves. We will introduce some notation and techniques, mainly related to blowups, that allow us to reduce the proof of the Main Theorem to more manageable cases.

Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Given ideals  $I$  and  $J$ , the set  $\{(\phi^*(I), J) : \phi \in G\}$  is bounded if  $\sqrt{I}$  (or  $\sqrt{J}$ ) contains the maximal ideal  $(x, y)$ , since then  $(x, y)^k \subset I$  for some  $k \geq 1$ , and it is clear that

$$\dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(\phi^*(I), J)} \leq \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{I} \leq \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(x, y)^k} = k(k - 1)/2 < \infty$$

for any  $\phi \in G$ . The case  $I = 0$  or  $J = 0$  is also simple, since  $\dim_{\mathbb{C}} \hat{\mathcal{O}}_2/(\phi^*(I), J)$  is constant as a function of  $\phi \in G$ . Hence we can suppose  $\sqrt{I} = (f)$ ,  $\sqrt{J} = (g)$ , where  $f = f_1 \cdots f_j$ ,  $g = g_1 \cdots g_l$  and  $f_1, \dots, f_j$  (resp.,  $g_1, \dots, g_l$ ) are pairwise relatively prime irreducible elements of  $\hat{\mathcal{O}}_2$ . There exists  $k \in \mathbb{N}$  such that  $(f^k) \subset I \subset (f)$  and  $(g^k) \subset J \subset (g)$ . Given  $\phi \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ , we obtain

$$\begin{aligned}
 \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(\phi^*(I), J)} &\leq \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(f^k \circ \phi, g^k)} \leq k^2 \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(f \circ \phi, g)} = \\
 k^2 \sum_{1 \leq a \leq j, 1 \leq b \leq l} \dim_{\mathbb{C}} \frac{\hat{\mathcal{O}}_2}{(f_a \circ \phi, g_b)} &= k^2 \sum_{1 \leq a \leq j, 1 \leq b \leq l} (\phi^{-1}(\alpha_a), \beta_b),
 \end{aligned}$$

where  $\alpha_a$  (resp.,  $\beta_b$ ) is the formal irreducible curve of ideal  $(f_a)$  (resp.,  $(g_b)$ ). The previous discussion leads to the first reduction of the setting of the problem.

**Remark 4.1.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then  $G$  satisfies UI if and only if, given any pair of formal irreducible curves  $\alpha$  and  $\beta$ , the set  $\{(\phi(\alpha), \beta) : \phi \in G\}$  is finite.

**4.1. The action of a group on the space of curves**

So far we have seen that it suffices to consider curves to check the uniform intersection property for a group  $G$ . The main result in this subsection is Proposition 4.2, which provides a stronger property:  $G$  satisfies UI if and only if all the orbits of formal irreducible curves are discrete. We will define the natural topology in the space of curves. The orbit of a formal irreducible curve  $\gamma$  is discrete if and only if  $\{(\phi(\gamma), \gamma) : \phi \in G\}$  is finite.

Let  $\gamma$  be a formal irreducible curve given by an ideal  $(f)$ , where  $f$  is an irreducible element of  $\hat{\mathcal{O}}_2$ . Denote  $\gamma_0 = \gamma$  and  $p_0 = (0, 0) \in \mathbb{C}^2$ .

**Definition 4.1.** Let  $f \in \hat{\mathcal{O}}_2$ . We denote by  $m_0(f)$  the multiplicity of  $f$  at the origin. It is the integer number  $m \geq 0$  such that  $f \in (x, y)^m \setminus (x, y)^{m+1}$ . Given a formal irreducible curve  $\gamma$  given by an ideal  $(f)$ , we define the multiplicity of  $\gamma$  at the origin by  $m_0(\gamma) = m_0(f)$ . We say that  $\gamma$  is smooth if  $m_0(\gamma) = 1$ .

**Definition 4.2.** Let  $f$  be an irreducible element of  $\hat{\mathcal{O}}_2$  defining a formal irreducible curve  $\gamma$ . There exists  $(a, b) \in \mathbb{C}^2 \setminus \{(0, 0)\}$  such that

$$f - (ax + by)^{m_0(\gamma)} \in (x, y)^{m_0(\gamma)+1}$$

(cf. [9, Corollary 2.2.6, p. 46]). The line  $\ell = \{ax + by = 0\}$  is the *tangent direction* of  $\gamma$  at the origin.

Consider the blowup  $\pi_1 : \tilde{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$  of the origin of  $\mathbb{C}^2$ . Let  $p_1(\gamma)$  be the first infinitely near point of  $\gamma$  – that is, the point in  $\pi_1^{-1}(p_0)$  corresponding to the tangent direction  $\ell$  of  $\gamma$  at the origin. Denote by  $\gamma_1$  the strict transform of  $\gamma$ . More precisely, suppose  $\ell = \{y = 0\}$  and consider coordinates  $(x, t)$  in the first chart of the blowup of the origin. Then  $\pi_1(x, t) = (x, xt)$  is the expression of  $\pi_1$  in local coordinates. The formal irreducible curve  $\gamma_1$  is equal to the prime ideal  $(f(x, xt)/x^{m_0(\gamma)})$  of  $\mathbb{C}[[x, t]]$ . The point  $p_1(\gamma)$  has coordinates  $(x, t) = (0, 0)$ . We denote  $\mathcal{X}_0 = \mathbb{C}^2$  and  $\mathcal{X}_1 = \tilde{\mathbb{C}}^2$ . By repeating this process we obtain a sequence of blowups

$$\dots \xrightarrow{\pi_{m+2}} \mathcal{X}_{m+1} \xrightarrow{\pi_{m+1}} \mathcal{X}_m \xrightarrow{\pi_m} \dots \xrightarrow{\pi_3} \mathcal{X}_2 \xrightarrow{\pi_2} \mathcal{X}_1 \xrightarrow{\pi_1} \mathcal{X}_0, \tag{4}$$

where  $p_0 = (0, 0)$ ,  $\gamma_0 = \gamma$ ,  $\pi_{k+1}$  is the blowup of  $\mathcal{X}_k$  at  $p_k(\gamma)$  and  $p_{k+1}(\gamma)$  is the point in  $\pi_{k+1}^{-1}(p_k(\gamma))$  corresponding to the tangent direction of  $\gamma_k$  at  $p_k(\gamma)$  for  $k \geq 0$ . The strict transform  $\gamma_{k+1}$  of  $\gamma_k$  by the blowup  $\pi_{k+1}$  passes through the point  $p_{k+1}(\gamma)$  for any  $k \geq 0$ . Then  $(p_k(\gamma))_{k \geq 0}$  and  $(\gamma_k)_{k \geq 0}$  are the sequences of infinitely near points and strict transforms of  $\gamma$ , respectively. We denote  $p_k = p_k(\gamma)$  if  $\gamma$  is implicit.

**Definition 4.3.** Given  $k \geq 1$ , we say that  $p_k(\gamma)$  is a *trace point* if it belongs to exactly one irreducible component of  $(\pi_1 \circ \dots \circ \pi_k)^{-1}(0, 0)$ . This property depends just on  $p_1(\gamma), \dots, p_k(\gamma)$ .

The previous construction can be used to provide a natural topology in the space of curves.

**Definition 4.4.** Given a finite sequence  $\mathfrak{s} = (p_k)_{0 \leq k \leq n}$  ( $n \geq 0$ ) of infinitely near points, we define the set  $\mathcal{U}_{\mathfrak{s}}$  consisting of the formal irreducible curves  $\gamma$  such that  $p_k(\gamma) = p_k$  for

any  $0 \leq k \leq n$ . The sets of the form  $U_\varepsilon$  form the base of a topology on the set of formal irreducible curves.

**Remark 4.2.** The space of formal irreducible curves can be interpreted as a subset of the valuations of  $\mathbb{C}[[x, y]]$  that take values in  $\mathbb{R}^+ \cup \{\infty\}$ . This space is a tree, and we just considered the topology induced on the space of curves by any of the natural topologies of the valuative tree (weak, strong, thin, Hausdorff-Zariski) [12].

**Remark 4.3.** The space of formal irreducible curves is not second-countable with the topology in Definition 4.4. On the other hand, every formal irreducible curve has a countable neighbourhood base, so it is first-countable and in particular a sequential space.

**Definition 4.5.** Given  $k \geq 1$ , we denote by  $m_k(\gamma)$  the multiplicity of  $\gamma_k$  at  $p_k$ .

**Lemma 4.1.** Let  $(\alpha^n)_{n \geq 1}, (\beta^n)_{n \geq 1}$  be sequences of formal irreducible curves. Suppose that  $\alpha^n$  (resp.,  $\beta^n$ ) is in the  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ -orbit of  $\alpha^1$  (resp.,  $\beta^1$ ) for any  $n \in \mathbb{N}$ . Let  $\gamma$  be a formal irreducible curve. Then

- $\lim_{n \rightarrow \infty} \alpha^n = \gamma$  if and only if  $\lim_{n \rightarrow \infty} (\alpha^n, \gamma) = \infty$ ;
- $\lim_{n \rightarrow \infty} (\alpha^n, \gamma) = \lim_{n \rightarrow \infty} (\beta^n, \gamma) = \infty$  implies  $\lim_{n \rightarrow \infty} (\alpha^n, \beta^n) = \infty$ .

**Proof.** Given formal irreducible curves  $\alpha, \beta$ , we have

$$(\alpha, \beta) = m_0(\alpha)m_0(\beta) + \sum_{k=1}^{\infty} m_k(\alpha)m_k(\beta)\delta_k, \tag{5}$$

where  $\delta_k = 1$  if the first  $k$  infinitely near points of  $\alpha$  and  $\beta$  coincide and  $\delta_k = 0$  otherwise (cf. [14, Corollary 8.30]). The property  $\lim_{n \rightarrow \infty} \alpha^n = \gamma$  implies  $\lim_{n \rightarrow \infty} (\alpha^n, \gamma) = \infty$  by equation (5).

The sequences  $(m_k(\alpha^n))_{k \geq 0}$  are decreasing and do not depend on  $n$ . Hence the property  $\lim_{n \rightarrow \infty} (\alpha^n, \gamma) = \infty$  implies  $\lim_{n \rightarrow \infty} \alpha^n = \gamma$ .

Suppose  $\lim_{n \rightarrow \infty} (\alpha^n, \gamma) = \lim_{n \rightarrow \infty} (\beta^n, \gamma) = \infty$ . By the first part of the proof,  $\alpha^n$  and  $\beta^n$  share the first  $a_n$  infinitely near points, where  $\lim_{n \rightarrow \infty} a_n = \infty$ . As a consequence, we obtain  $\lim_{n \rightarrow \infty} (\alpha^n, \beta^n) = \infty$  by equation (5). □

**Remark 4.4.** A subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  satisfies the uniform intersection property if and only if every  $G$ -orbit of formal irreducible curves is discrete and closed by Lemma 4.1.

We can understand the uniform intersection property as a property of orbits of curves.

**Proposition 4.1.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Let  $\mathcal{O} = (\phi(\gamma))_{\phi \in G}$  be a  $G$ -orbit of formal irreducible curves. Then

- $\mathcal{O}$  is discrete and hence closed or
- $\overline{\mathcal{O}}$  is a Cantor set, meaning it is a closed set with no isolated points and has the cardinality of the continuum.

Moreover,  $\overline{\mathcal{O}}$  is a completely metrisable space.

**Proof.** Let  $F_m$  be the closed subset of the space of formal irreducible curves whose elements  $\alpha$  satisfy the condition that  $p_k(\alpha)$  is a trace point for any  $k \geq m$ . We can define

an ultrametric  $d$  in  $F_m$  that induces the topology of  $F_m$ , namely  $d(\alpha, \beta) = 0$  if  $\alpha = \beta$  and  $d(\alpha, \beta) = j^{-1}$  if  $\alpha \neq \beta$ , where  $j$  is the first natural number such that  $p_j(\alpha) \neq p_j(\beta)$ . Consider a Cauchy sequence  $(\alpha^n)_{n \geq 1}$  in  $F_m$ . There exists a sequence  $(p_k)_{k \geq 0}$  of infinitely near points such that given any  $k \geq 0$ , there exists  $r_k \in \mathbb{N}$  such that  $p_k = p_k(\alpha^n)$  for any  $n \geq r_k$ . Moreover,  $p_k$  is a trace point for any  $k \geq m$ . Since there are finitely many nontrace points in  $(p_k)_{k \geq 0}$ , we deduce that there exists a formal irreducible curve  $\beta$  such that  $p_k(\beta) = p_k$  for any  $k \geq 0$ . The construction of  $\beta$  implies  $\lim_{n \rightarrow \infty} \alpha^n = \beta$ . Since every Cauchy sequence converges,  $F_m$  is a complete metric space. Moreover,  $\gamma$  is contained in some  $F_m$  for some  $m \in \mathbb{N}$ , by the properties of the resolution of singularities of plane curves (see [25, Theorem 3.4.4]). Since  $\mathcal{O}$  is the orbit of  $\gamma$  by a group of formal diffeomorphisms, we get  $\mathcal{O} \subset F_m$  and hence  $\overline{\mathcal{O}} \subset F_m$ . Finally,  $\overline{\mathcal{O}}$  is a closed subset of a complete metric space and hence also a complete metric space.

Suppose that  $\mathcal{O}$  is discrete. Let  $\alpha \in \overline{\mathcal{O}}$ . There exists a sequence  $(\phi_j)_{j \geq 1}$  in  $G$  such that  $\lim_{j \rightarrow \infty} \phi_j(\gamma) = \alpha$ , by Remark 4.3. We get  $\lim_{j \rightarrow \infty} (\phi_j(\gamma), \alpha) = \infty$  by the first property in Lemma 4.1. Moreover, the second property of Lemma 4.1 implies  $\lim_{j \rightarrow \infty} (\phi_j(\gamma), \phi_{j+1}(\gamma)) = \infty$ . We deduce  $\lim_{j \rightarrow \infty} ((\phi_{j+1}^{-1} \circ \phi_j)(\gamma), \gamma) = \infty$  and then  $\lim_{j \rightarrow \infty} (\phi_{j+1}^{-1} \circ \phi_j)(\gamma) = \gamma$ . Since  $\mathcal{O}$  is discrete, we obtain  $\phi_j(\gamma) = \phi_{j+1}(\gamma)$  for  $j \in \mathbb{N}$  sufficiently big. In particular,  $\alpha$  belongs to  $\mathcal{O}$  for any  $\alpha \in \overline{\mathcal{O}}$ , and hence  $\mathcal{O}$  is closed.

Suppose that  $\mathcal{O}$  is nondiscrete from now on. We claim that  $\overline{\mathcal{O}}$  is a perfect set. Otherwise, there exists an isolated point  $\alpha$  in  $\overline{\mathcal{O}}$ . It is clear that  $\alpha$  is also an isolated point of  $\mathcal{O}$ . Since any  $\phi \in G$  induces a homeomorphism of  $\mathcal{O}$ , and the action of  $G$  on  $\mathcal{O}$  is transitive, we deduce that all points of  $\mathcal{O}$  are isolated, contradicting the supposition that  $\mathcal{O}$  is nondiscrete.

Sequences of infinitely near points can be identified with sequences of points in the complex projective space  $\mathbb{C}P^1$ . Thus such a set has the cardinality of the continuum, as does the set of formal irreducible curves. Since  $\overline{\mathcal{O}}$  is a perfect complete metric space, it has at least the cardinality of the continuum [15, Theorem VII.2.14Ac]. Therefore  $\overline{\mathcal{O}}$  has the cardinality of the continuum. □

**Corollary 4.1.** *Let  $G$  be a countable subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then a  $G$ -orbit of formal irreducible curves is closed if and only if it is discrete.*

**Definition 4.6.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and  $\gamma$  a formal curve. We say that  $G$  satisfies the property  $(UI)_\gamma$  if the  $G$ -orbit of  $\gamma$  is discrete – that is, if  $\{(\phi(\gamma), \gamma) : \phi \in G\}$  is finite.

The following result is a corollary of Remark 4.4 and Proposition 4.1:

**Proposition 4.2.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then  $G$  satisfies the uniform intersection property if and only if every  $G$ -orbit of formal irreducible curves is discrete. Equivalently,  $G$  satisfies  $UI$  if and only if it satisfies  $(UI)_\gamma$  for any formal irreducible curve  $\gamma$ .*

**Remark 4.5.** As a consequence of Proposition 4.2, the Main Theorem and Theorem 1.2 are equivalent. In practice, we are going to focus on the proof of the latter theorem from

now on. This is particular to dimension 2, since it is not clear that UI can be interpreted as a property of the orbits of subvarieties in higher dimension.

Next, we see that the uniform intersection property is invariant under finite extensions. This result has a technical interest in the sequel.

**Proposition 4.3.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and  $\gamma$  an irreducible formal curve. Consider a finite-index subgroup  $H$  of  $G$ . Then  $G$  satisfies  $(\text{UI})_\gamma$  if and only if  $H$  does.*

**Proof.** The sufficient condition is obvious. Suppose that  $H$  satisfies  $(\text{UI})_\gamma$ . Then the  $H$ -orbit of  $\gamma$  is discrete and closed, by Proposition 4.1. Since the  $G$ -orbit of  $\gamma$  is a finite union of discrete closed sets, it is discrete and closed. □

The next result is a corollary of Propositions 4.2 and 4.3:

**Corollary 4.2.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Let  $H$  be a finite-index subgroup of  $G$ . Then  $G$  satisfies UI if and only if  $H$  satisfies UI.*

**Remark 4.6.** It is straightforward to show that  $G$  satisfies UI if and only if a finite-index subgroup  $H$  does for any dimension.

**4.2. Reduction via blowup**

In this section we will obtain further reductions, since we will get simpler expressions for the elements of a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and a formal irreducible curve  $\gamma$  by considering blowups of the infinitely near points of  $\gamma$  (Propositions 4.4, 4.5 and 4.6).

First we consider the problem of lifting vector fields and diffeomorphisms by the blowup maps. We will use the notation of section 4.1.

**Remark 4.7.** Let  $\gamma$  be a formal irreducible curve. Let  $X \in \widehat{\mathfrak{X}}(\mathbb{C}^2, 0)$ . We can lift this vector field to  $\pi_1^{-1}(0, 0)$  as a transversally formal vector field. Moreover, if the tangent line  $\ell$  of  $\gamma$  at  $(0, 0)$  is  $j^1 X$ -invariant, then  $X$  can be lifted to a formal vector field  $\overline{X}_1 \in \widehat{\mathfrak{X}}(\mathcal{X}_1, p_1)$ . Analogously, if  $p_j$  is a singular point of  $\overline{X}_j$  for  $1 \leq j < k$ , we can lift  $X$  to a formal vector field  $\overline{X}_k$  at  $p_k$ . Moreover,  $\overline{X}_k$  belongs to  $\widehat{\mathfrak{X}}(\mathcal{X}_k, p_k)$  if  $p_k$  is a singular point of  $\overline{X}_k$ . In a similar way, if  $\ell$  is  $j^1 \phi$ -invariant for  $\phi \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ , we can lift  $\phi$  to an element  $\tau_1(\phi)$  of  $\widehat{\text{Diff}}(\mathcal{X}_1, p_1)$  and so on.

**Definition 4.7.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . We define  $G_{\gamma, k}$  as the subgroup of  $G$  whose elements  $\phi$  satisfy the condition that  $\phi(\gamma)$  and  $\gamma$  share the first  $k$  infinitely near points. We define  $G_\gamma = \{\phi \in G : \phi(\gamma) = \gamma\}$ .

Every element  $\phi$  of  $G$  induces an action in  $\pi_1^{-1}(p_0)$  by a Möbius transformation. Indeed,  $\phi$  belongs to  $G_{\gamma, 1}$  if and only if  $p_1$  is a fixed point of such an action. Thus we can lift  $G_{\gamma, 1}$  to obtain a subgroup  $\tilde{G}_{\gamma, 1}$  of  $\widehat{\text{Diff}}(\tilde{\mathbb{C}}^2, p_1)$ . Let  $\tau_1 : G_{\gamma, 1} \rightarrow \tilde{G}_{\gamma, 1}$  be the group isomorphism sending every element of  $G_{\gamma, 1}$  to its lift in  $\widehat{\text{Diff}}(\tilde{\mathbb{C}}^2, p_1)$ . All elements of  $\tau_1(G_{\gamma, k})$  fix the first  $k - 1$  infinitely near points of  $\gamma_1$ . By replacing  $G_{\gamma, 1}$  and  $\gamma$  with  $\tau_1(G_{\gamma, 2})$  and  $\gamma_1$ , respectively, we obtain a group  $\tilde{G}_{\gamma, 2}$  of formal local diffeomorphisms in a neighbourhood of the second infinitely near point  $p_2$  of  $\gamma$  and a group isomorphism  $\tau_2 : G_{\gamma, 2} \rightarrow \tilde{G}_{\gamma, 2}$ .

By repeating this construction with  $\tau_k(G_{\gamma,k+1})$  we obtain a group  $\tilde{G}_{\gamma,k+1}$  of formal local diffeomorphisms and a group isomorphism  $\tau_{k+1} : G_{\gamma,k+1} \rightarrow \tilde{G}_{\gamma,k+1}$  for  $k \geq 1$ .

**Remark 4.8.** Since  $(m_k(\phi(\gamma)))_{k \geq 0}$  does not depend on  $\phi \in \widehat{\text{Diff}}(\mathbb{C}^2, 0)$ , the set  $\{(\phi(\gamma), \gamma) : \phi \in G \setminus G_{\gamma,k}\}$  is bounded for any  $k \in \mathbb{N}$  by equation (5).

Next let us show that the properties FD and UI for subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  are invariant by blowup.

**Proposition 4.4.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Suppose that all elements of  $j^1 G$  fix a direction  $\ell$ . Then  $G$  has the finite determination property if and only if  $\tilde{G}_{\ell,1}$  has the finite determination property.*

**Proof.** Suppose  $\ell = \{y = 0\}$  without lack of generality. Suppose that  $G$  does not satisfy FD. Then there exists a sequence  $(\phi_k)_{k \geq 1}$  in  $G \setminus \{\text{Id}\}$  such that  $j^k \phi_k = \text{Id}$  for any  $k \in \mathbb{N}$ . Let  $\tilde{\phi}_k(x, t) = (\tilde{a}_k(x, t), \tilde{b}_k(x, t))$  be the lift of  $\phi_k(x, y) = (a_k(x, y), b_k(x, y))$  to  $\tilde{G}_{\ell,1}$ . We have

$$\tilde{\phi}_k(x, t) = \left( \tilde{a}_k(x, t), \tilde{b}_k(x, t) \right) = \left( a_k(x, xt), \frac{b_k(x, xt)}{a_k(x, xt)} \right)$$

for  $k \in \mathbb{N}$ . We obtain  $j^k \tilde{a}_k(x, xt) = x$ ,  $j^{k-1} \tilde{b}_k(x, t) = t$  and  $\tilde{\phi}_k \neq \text{Id}$  for any  $k \in \mathbb{N}$ . Therefore,  $\tilde{G}_{\ell,1}$  does not satisfy FD.

Suppose that  $\tilde{G}_{\ell,1}$  does not satisfy FD. There exists a sequence  $(\phi_k)_{k \geq 1}$  in  $G \setminus \{\text{Id}\}$  such that  $\phi_k \neq \text{Id}$  and  $j^k \tilde{\phi}_k \equiv \text{Id}$  for any  $k \in \mathbb{N}$ . Since we have

$$\phi_k(x, y) = \left( \tilde{a}_k \left( x, \frac{y}{x} \right), \tilde{a}_k \tilde{b}_k \left( x, \frac{y}{x} \right) \right),$$

we obtain  $j^{\lfloor \frac{k+1}{2} \rfloor - 1} a_k = x$  and  $j^{\lfloor \frac{k+2}{2} \rfloor - 1} b_k = y$ . Thus  $G$  does not hold FD. □

**Proposition 4.5.** *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Consider a formal irreducible curve  $\gamma$  and  $1 \leq j \leq k$ . Then  $G$  satisfies  $(\text{UI})_\gamma$  if and only if  $\tau_j(G_{\gamma,k})$  satisfies  $(\text{UI})_{\gamma_j}$ .*

**Proof.** Fix  $k \geq 1$ . The group  $G$  satisfies  $(\text{UI})_\gamma$  if and only if  $G_{\gamma,k}$  does, by Remark 4.8. Equation (5) implies

$$(\phi(\gamma), \gamma) = m_0(\gamma)^2 + (\tau_1(\phi)(\gamma_1), \gamma_1)$$

for any  $\phi \in G_{\gamma,1}$ . Thus  $G$  satisfies  $(\text{UI})_\gamma$  if and only if  $\tau_1(G_{\gamma,k})$  satisfies  $(\text{UI})_{\gamma_1}$ . By iterating the previous argument, we obtain that, given  $1 \leq j \leq k$ ,  $G$  satisfies  $(\text{UI})_\gamma$  if and only if  $\tau_j(G_{\gamma,k})$  satisfies  $(\text{UI})_{\gamma_j}$ . □

Next we see that blowups can be used to reduce the proof of the property  $(\text{UI})_\gamma$  to simpler settings.

**Definition 4.8.** Let  $G$  be a finitely determined subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and  $\gamma$  be a formal smooth curve. We say that the pair  $(G, \gamma)$  is *tame* if there exists a formal smooth curve  $\alpha$  transverse to  $\gamma$  that is  $G$ -invariant and the tangent direction of  $\gamma$  at  $(0, 0)$  is  $j^1 G$ -invariant.

Tame pairs are easier to handle. Indeed,  $j^1 G$  is diagonalisable and thus  $G'$  is contained in  $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ . Since an FD subgroup of  $\widehat{\text{Diff}}_1(\mathbb{C}^n, 0)$  is nilpotent (Lemma 7.1), it follows that  $G$  is solvable. This allows us to use the properties of solvable subgroups of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and solvable Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  in the next section to prove the Main Theorem.

**Proposition 4.6.** *Suppose that for any tame pair  $(G, \gamma)$ ,  $G$  satisfies the property  $(\text{UI})_\gamma$ . Then any finitely determined subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  satisfies UI.*

**Proof.** Consider a finitely determined subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  and a formal irreducible curve  $\gamma$ . It suffices to prove  $(\text{UI})_\gamma$ , by Proposition 4.2.

By the resolution of singularities of plane curves (see [25, Theorem 3.4.4]), we can suppose, up to making  $k$  blowups following infinitely near points of  $\gamma$  for  $k \in \mathbb{N}$  sufficiently big, that  $\gamma_k$  is smooth, that there exists a unique irreducible component of  $D := (\pi_1 \circ \dots \circ \pi_k)^{-1}(0, 0)$  (see equation (4)) containing the point  $p_k(\gamma)$  and that  $\gamma_k$  and  $D$  are transverse at  $p_k(\gamma)$ . Moreover, the germ of  $D$  at  $p_k(\gamma)$  is  $\tau_k(G_{\gamma, k+1})$ -invariant. By replacing  $G$  with  $G_{\gamma, k+1}$  and applying Proposition 4.4  $k$  times, we obtain that  $\tau_k(G_{\gamma, k+1})$  is finitely determined. Moreover,  $(\tau_k(G_{\gamma, k+1}), \gamma_k)$  is tame by construction. By hypothesis,  $\tau_k(G_{\gamma, k+1})$  satisfies  $(\text{UI})_{\gamma_k}$ . We deduce that  $G$  satisfies  $(\text{UI})_\gamma$  by Proposition 4.5.  $\square$

### 5. Lie algebras of formal vector fields

One of the ingredients in the proof of the Main Theorem is the classification of nilpotent and solvable Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  [16]. The main idea of this section is that an infinite-dimensional nilpotent (or solvable) Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  has a high degree of symmetry and very particular properties. Our next goal is to prove Theorem 1.3; let us introduce the setting of the proof.

**Remark 5.1.** The properties of the subfields of  $\hat{K}_2$  of first integrals of solvable Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  are fundamental to classifying them [16]. Let  $X \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0) \setminus \{0\}$ . We denote  $\mathcal{M} = \{f \in \hat{K}_2 : X(f) = 0\}$ . We have the following:

- If  $\mathcal{M} \cap \hat{\mathcal{O}}_2 \neq \mathbb{C}$ , then any element  $g$  of  $\mathcal{M}$  satisfies either  $g \in \hat{\mathcal{O}}_2$  or  $1/g \in \hat{\mathcal{O}}_2$ . Moreover,  $\mathcal{M} \cap \hat{\mathcal{O}}_2$  is equal to  $\mathbb{C}[[f_0]]$  for some  $f_0 \in \mathcal{M} \cap \mathfrak{m}$ .
- If  $\mathcal{M} \cap \hat{\mathcal{O}}_2 = \mathbb{C}$  but  $\mathcal{M} \cap \hat{K}_2 \neq \mathbb{C}$ , then we get  $\mathcal{M} = \mathbb{C}(f_0)$  for some  $f_0 \in \mathcal{M} \cap (\hat{K}_2 \setminus \mathbb{C})$ .

The previous results were proved by Mattei and Moussu [18] and Cerveau and Mattei [10], respectively, for the case  $X \in \mathfrak{X}(\mathbb{C}^2, 0)$ . They can be easily adapted for formal vector fields (cf. [20, Proposition 5.1]).

The next lemma provides a hint of how the existence of first integrals has an influence on whether or not a Lie subalgebra of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$  is finite-dimensional.

**Lemma 5.1.** *Let  $X \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0) \setminus \{0\}$  have no first integral in  $\hat{\mathcal{O}}_2 \setminus \mathbb{C}$ . Then the complex vector space*

$$\mathfrak{g} := \{fX : f \in \hat{K}_2, X(f) = 0 \text{ and } fX \in \hat{\mathfrak{X}}(\mathbb{C}^2, 0)\}$$

*is finite-dimensional.*

**Proof.** Denote  $\mathcal{M} = \{f \in \hat{K}_2 : X(f) = 0\}$ . We can suppose that  $X$  has nonconstant first integrals, since otherwise  $\dim_{\mathbb{C}} \mathfrak{g} = 1$ . Hence we have  $\mathcal{M} = \mathbb{C}(f_0)$ , where  $f_0 \notin \hat{\mathcal{O}}_2$  and  $1/f_0 \notin \hat{\mathcal{O}}_2$  by Remark 5.1.

The formal vector field  $X$  is of the form  $gX'$ , where the coefficients of  $X'$  do not have a common factor in  $\hat{\mathcal{O}}_2$ . It suffices to show  $\dim_{\mathbb{C}} V < \infty$ , where

$$V = \left\{ f \in \hat{\mathcal{O}}_2 : X \left( \frac{f}{g} \right) = 0 \right\}.$$

Consider the map

$$\begin{aligned} V &\xrightarrow{\Lambda} j^k \hat{\mathcal{O}}_2 \\ f &\mapsto j^k f. \end{aligned}$$

Let us show  $\text{Ker}(\Lambda) = \{0\}$  if  $k \gg 1$ . In this way we obtain  $\dim_{\mathbb{C}} V \leq \dim_{\mathbb{C}} j^k \hat{\mathcal{O}}_2 < \infty$ .

Since every irreducible component of  $f_0 = c$  for any  $c \in \mathbb{C}$  is  $X$ -invariant, there are infinitely many  $X$ -invariant formal irreducible curves. Therefore there exists a dicritic divisor  $D$  in the desingularisation of the dual form  $\omega$  of  $X$  ( $\omega(X) = 0$ ) – that is,  $D$  is not an invariant curve for the “formal foliation” defined by the strict transform of  $\omega$ . The function  $g$  vanishes along  $D$  with order  $m$  for some  $m \in \mathbb{N}$ . Consider  $k > m$ . Any element  $f$  of  $\text{Ker}(\Lambda)$  vanishes along  $D$  with order at least  $m + 1$ , and then  $f/g$  vanishes along  $D$ . Since  $D$  is dicritic and  $X(f/g) = 0$ , it follows that  $f/g = 0$  and then  $f = 0$ . □

**Proof.**[Proof of Theorem 1.3] Suppose  $\dim_{\hat{K}_2}(\mathfrak{g} \otimes_{\mathbb{C}} \hat{K}_2) = 2$ . Consider an element  $X$  of  $\mathcal{C}^k \mathfrak{g} \setminus \{0\}$ , where  $k + 1 \geq 1$  is the nilpotency class of  $\mathfrak{g}$ ; it belongs to the centre of  $\mathfrak{g}$ . Let  $Y$  be an element of  $\mathfrak{g}$  that is not of the form  $hX$  for some  $h \in \hat{K}_2$ . The elements of  $\mathfrak{g}$  are of the form

$$gX + hY,$$

where  $g, h \in \hat{K}_2$ . Since  $X$  is in the centre of  $\mathfrak{g}$ , we get  $[X, Y] = 0$  and then  $[X, gX + hY] = X(g)X + X(h)Y = 0$ . We deduce  $X(g) = X(h) = 0$  for any  $gX + hY \in \mathfrak{g}$ . The property

$$\underbrace{[Y, \dots, [Y, gX + hY]]}_{m \text{ times}} = Y^m(g)X + Y^m(h)Y \tag{6}$$

and  $\mathcal{C}^{k+1} \mathfrak{g} = \{0\}$  imply  $Y^{k+1}(g) = Y^{k+1}(h) = 0$  for any  $gX + hY \in \mathfrak{g}$ . Consider the complex vector space

$$V_j := \{g \in \hat{K}_2 : X(g) = 0 \text{ and } Y^j(g) = 0\}$$

for  $j \in \mathbb{N}$ . We have shown that an element  $gX + hY$  of  $\mathfrak{g}$  satisfies  $g, h \in V_{k+1}$ . We have  $V_1 = \mathbb{C}$ . The linear map

$$\begin{aligned} V_{j+1} &\xrightarrow{\Lambda_j} V_j \\ g &\mapsto Y(g) \end{aligned}$$

is well defined, since  $X(Y(g)) = Y(X(g)) = Y(0) = 0$  and  $Y^j(Y(g)) = 0$  for any  $g \in V_{j+1}$ . The kernel of  $\Lambda_j$  coincides with  $V_1$  and hence  $\dim_{\mathbb{C}} V_{j+1} \leq \dim_{\mathbb{C}} V_1 + \dim_{\mathbb{C}} V_j$ . We obtain



$\dim_{\mathbb{C}} V_j \leq j$  for any  $j \in \mathbb{N}$  by induction on  $j$ . Since  $\dim_{\mathbb{C}} \mathfrak{g} \leq 2 \dim V_{k+1} \leq 2k + 2$ , the Lie algebra  $\mathfrak{g}$  is finite-dimensional.

Suppose now  $\dim_{\hat{K}_2}(\mathfrak{g} \otimes_{\mathbb{C}} \hat{K}_2) = 1$  and  $\dim_{\mathbb{C}} \mathfrak{g} = \infty$ . Let  $X$  be a nonvanishing element of the centre of  $\mathfrak{g}$ . Every element  $Z$  of  $\mathfrak{g}$  is of the form  $fX$ , where  $f \in \hat{K}_2$  and satisfies  $[X, fX] = X(f)X = 0$ . We deduce

$$\mathfrak{g} \subset \{fX : f \in \hat{K}_2 \text{ and } X(f) = 0\}.$$

In particular,  $\mathfrak{g}$  is an abelian Lie algebra. Then  $X$  has a first integral in  $\hat{\mathcal{O}}_2 \setminus \mathbb{C}$  by Lemma 5.1. In particular, every first integral  $h$  of  $X$  in  $\hat{K}_2$  satisfies either  $h \in \hat{\mathcal{O}}_2$  or  $1/h \in \hat{\mathcal{O}}_2$ , by Remark 5.1. Replacing  $X = a(x, y)\partial/\partial x + b(x, y)\partial/\partial y$  if necessary with an element in  $\mathfrak{g}$  whose multiplicity  $\min(m_0(a), m_0(b))$  at the origin is minimal, every element of  $\mathfrak{g}$  is of the form  $(g/h)X$ , where  $g, h \in \hat{\mathcal{O}}_2$  and  $m_0(g) \geq m_0(h)$ . Since  $g/h$  is a first integral of  $X$  and either  $g/h$  or  $h/g$  belongs to  $\hat{\mathcal{O}}_2$ , it follows that  $g/h \in \hat{\mathcal{O}}_2$ . Hence we obtain property (3). □

Next we consider general Lie subalgebras of  $\hat{\mathfrak{X}}(\mathbb{C}^2, 0)$ .

**Proof.**[Proof of Theorem 1.4] Suppose  $\dim_{\mathbb{C}} \mathfrak{g}' < \infty$ . Our goal is to show that  $\mathfrak{g}$  is abelian. Let  $\{Y_1, \dots, Y_p\}$  be a basis of  $\mathfrak{g}'$ . Denote by  $\mathfrak{h}$  the kernel of the linear map

$$\begin{aligned} \mathfrak{g} &\rightarrow (\mathfrak{g}')^p \\ Z &\mapsto ([Z, Y_1], \dots, [Z, Y_p]). \end{aligned}$$

It satisfies  $\mathfrak{h} = \{Z \in \mathfrak{g} : [Z, Y] = 0 \ \forall Y \in \mathfrak{g}'\}$ . Moreover,  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$  by Jacobi's formula. Since  $(\mathfrak{g}')^p$  is finite-dimensional, we obtain  $\dim_{\mathbb{C}} \mathfrak{g}/\mathfrak{h} < \infty$ . It is clear that  $\mathfrak{h}$  is nilpotent, and since  $\dim_{\mathbb{C}} \mathfrak{h} = \infty$ , it follows that  $\mathfrak{h}$  is abelian by Theorem 1.3. Consider  $\{Z_1, \dots, Z_q\} \subset \mathfrak{g}$  such that  $\{Z_1 + \mathfrak{h}, \dots, Z_q + \mathfrak{h}\}$  is a basis of  $\mathfrak{g}/\mathfrak{h}$ . Denote by  $\mathfrak{j}$  the kernel of the linear map

$$\begin{aligned} \mathfrak{h} &\rightarrow (\mathfrak{g}')^q \\ Z &\mapsto ([Z, Z_1], \dots, [Z, Z_q]). \end{aligned}$$

The vector space  $\mathfrak{j}$  satisfies  $\dim_{\mathbb{C}} \mathfrak{h}/\mathfrak{j} < \infty$ . It is clearly contained in the centre  $Z(\mathfrak{g})$  of  $\mathfrak{g}$ , and hence  $\dim_{\mathbb{C}} \mathfrak{g}/Z(\mathfrak{g}) < \infty$  and  $\dim_{\mathbb{C}} Z(\mathfrak{g}) = \infty$ . Since  $Z(\mathfrak{g})$  is abelian and infinite-dimensional, Theorem 1.3 implies the existence of nontrivial elements  $X$  and  $fX$  of  $Z(\mathfrak{g})$  such that  $X(f) = 0$  and  $f \in \hat{\mathcal{O}}_2 \setminus \mathbb{C}$ . Given  $Z \in \mathfrak{g}$ , we have  $[Z, X] = 0$  and  $[Z, fX] = Z(f)X = 0$ . Thus  $f$  is a first integral of  $Z$ , and hence  $Z$  is of the form  $gX$ , where  $g \in \hat{K}_2$ . Since  $0 = [X, gX] = X(g)X$  for any  $gX \in \mathfrak{g}$ , we deduce that  $\mathfrak{g}$  is abelian. □

### 6. Algebraic properties of groups of local biholomorphisms

We need to relate the properties of a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$  with the properties of its Lie algebra of formal vector fields. In this section we explain how to define such Lie algebras and the properties of the Lie correspondence. The definitions and results were introduced in [16, 20] and are included here for the sake of simplicity.

**6.1. Finite-dimensional groups of formal diffeomorphisms**

Given a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , we can define its Zariski closure  $\overline{G}$  (cf. Definition 6.2), which is a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  that can be interpreted as a projective limit of linear algebraic groups. As a consequence, there is a natural definition of  $\dim(\overline{G})$  (cf. Definition 6.1), and we just define  $\dim(G) = \dim(\overline{G})$ .

One of the major points in the proof of the Main Theorem is that the next result allows us to reduce our study to infinite-dimensional groups:

**Theorem 6.1** ([19, Theorem 1.5]). *Let  $G$  be a finite-dimensional subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . Then  $G$  satisfies UI.*

Let us provide a rough idea of the proof of Theorem 6.1. Fix two ideals  $I$  and  $J$  of  $\mathcal{O}_n$ . The condition  $(\phi^*I, J) > m$  is equivalent to a system of polynomial equations on the coefficients of  $j^m\phi$ . A finite-dimensional subgroup  $G$  satisfies the condition that there exists  $k$  such that  $G$  is  $k$ -finitely determined and the map  $j^k\phi \mapsto j^l\phi$  defined in  $j^kG$  is a regular map for every  $l \geq k$  [19, Remark 3.4]. Thus the condition  $(\phi^*I, J) > m$  defines an increasing sequence  $(I_m)_{m \geq k}$  of ideals in a polynomial ring  $\mathbb{C}[w_1, \dots, w_p]$ , where the variables represent coefficients of degree less than or equal to  $k$  of elements of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . Noetherianity implies that these ideals stabilise, and hence there exists  $m_0 \in \mathbb{N}$  such that  $\phi \in G$  and  $(\phi^*I, J) > m_0$  implies  $(\phi^*I, J) = \infty$ .

**6.2. Zariski closure of a group of formal diffeomorphisms**

Given  $\phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , we can interpret  $\phi$  as a family  $(\phi_k)_{k \geq 1}$  of linear maps. More precisely,  $\phi_k$  is the element of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$  defined by

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{k+1} & \xrightarrow{\phi_k} & \mathfrak{m}/\mathfrak{m}^{k+1} \\ f + \mathfrak{m}^{k+1} & \mapsto & f \circ \phi + \mathfrak{m}/\mathfrak{m}^{k+1}. \end{array}$$

We define  $\pi_k : \widehat{\text{Diff}}(\mathbb{C}^n, 0) \rightarrow D_k$  by the formula  $\pi_k(\phi) = \phi_k$ , where

$$D_k := \{A \in \text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1}) : A(fg) = A(f)A(g) \ \forall f, g \in \mathfrak{m}/\mathfrak{m}^{k+1}\}.$$

Notice that  $D_k$  is the group of isomorphisms of the  $\mathbb{C}$ -algebra  $\mathfrak{m}/\mathfrak{m}^{k+1}$ . It is an algebraic subgroup of  $\text{GL}(\mathfrak{m}/\mathfrak{m}^{k+1})$ , since the equation  $A(fg) = A(f)A(g)$  is algebraic in the coefficients of the matrix  $A$ . We have  $D_k = \{\phi_k : \phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0)\}$  [20, Lemma 2.1].

We have a natural map  $\pi_{k,l} : D_k \rightarrow D_l$  for  $k \geq l$  defined by  $\pi_{k,l}(\phi_k) = \phi_l$ . The projective limit  $\varprojlim_{k \in \mathbb{N}} D_k$  is called a group of formal diffeomorphisms. Given  $f \in \mathfrak{m}$ , then  $(\phi_k(f + \mathfrak{m}^{k+1}))_{k \geq 1}$  belongs to  $\varprojlim_{k \in \mathbb{N}} \mathfrak{m}/\mathfrak{m}^{k+1}$ . We can interpret  $(\phi_k(f + \mathfrak{m}^{k+1}))_{k \geq 1}$  as an element of  $\mathfrak{m}$ , since  $\mathfrak{m}$  and  $\varprojlim_{k \in \mathbb{N}} \mathfrak{m}/\mathfrak{m}^{k+1}$  are naturally identified. Moreover, the map

$$\begin{array}{ccc} \varprojlim_{k \in \mathbb{N}} D_k & \rightarrow & \widehat{\text{Diff}}(\mathbb{C}^n, 0) \\ (\phi_k)_{k \geq 1} & \mapsto & ((\phi_k(z_1 + \mathfrak{m}^{k+1}))_{k \geq 1}, \dots, (\phi_k(z_n + \mathfrak{m}^{k+1}))_{k \geq 1}) \end{array}$$

is an anti-isomorphism of groups [20, Lemma 2.2].

We define  $G_k$  as the Zariski closure of  $\pi_k(G)$  in the linear algebraic group  $D_k$ . From the surjective nature of  $\pi_{k,l} : \pi_k(G) \rightarrow \pi_l(G)$ , it is possible to deduce that  $\pi_{k,l} : G_k \rightarrow G_l$

is a well-defined and surjective morphism of algebraic groups for all  $k \geq l \geq 1$  [20, Lemma 2.5]. The sequence  $(\dim G_k)_{k \geq 1}$  is increasing [19, Lemma 3.1]. Thus we can establish the following definitions:

**Definition 6.1.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . We define  $\dim G = \lim_{k \rightarrow \infty} \dim G_k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ . We say that  $G$  is *finite-dimensional* if  $\dim G < \infty$ .

**Definition 6.2** ([16]). Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . We define the *Zariski closure* (or the *pro-algebraic closure*)  $\overline{G}$  of  $G$  as the group  $\varprojlim_{k \in \mathbb{N}} G_k$ . Indeed, we have

$$\overline{G} = \{\phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \phi_k \in G_k \ \forall k \in \mathbb{N}\}.$$

We say that  $G$  is *pro-algebraic* if  $G = \overline{G}$ .

**Definition 6.3.** Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . The closure of  $G$  in the *Krull topology* (*m-adic topology*) is the subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  consisting of the elements  $\phi$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  such that there exists  $\eta(k) \in G$  satisfying  $j^k \phi = j^k \eta(k)$  for any  $k \in \mathbb{N}$ .

**Remark 6.1.**  $\overline{G}$  is closed in the Krull topology.

### 6.3. The Lie correspondence

We denote by  $L_k$  the Lie algebra of derivations of the  $\mathbb{C}$ -algebra  $\mathfrak{m}/\mathfrak{m}^{k+1}$ . Let  $X \in \widehat{\mathfrak{X}}(\mathbb{C}^n, 0)$ . The map  $X_k$  defined by

$$\begin{array}{ccc} \mathfrak{m}/\mathfrak{m}^{k+1} & \xrightarrow{X_k} & \mathfrak{m}/\mathfrak{m}^{k+1} \\ f + \mathfrak{m}^{k+1} & \mapsto & X(f) + \mathfrak{m}/\mathfrak{m}^{k+1} \end{array}$$

belongs to  $L_k$ . The Lie algebra  $\widehat{\mathfrak{X}}(\mathbb{C}^n, 0)$  can be identified with  $\varprojlim_{k \in \mathbb{N}} L_k$ , analogously to how  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is identified with  $\varprojlim D_k$ . Moreover,  $L_k$  is the Lie algebra of  $D_k$  for any  $k \in \mathbb{N}$ . Given  $X = (X_k)_{k \geq 1} \in \varprojlim L_k$ , we can define  $\exp(X) = (\exp(X_k))_{k \geq 1} \in \varprojlim D_k$ . The previous definition implies

$$f \circ \exp(X) = \sum_{k=0}^{\infty} \frac{X^k(f)}{k!},$$

where  $X^0(f) = f$  and  $X^{k+1}(f) = X(X^k(f))$  for  $k \geq 0$ . In particular, we obtain

$$\exp(X)(z_1, \dots, z_n) = \left( \sum_{k=0}^{\infty} \frac{X^k(z_1)}{k!}, \dots, \sum_{k=0}^{\infty} \frac{X^k(z_n)}{k!} \right).$$

Given a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , we define the Lie algebra of  $G$  (or  $\overline{G}$ ) as the Lie algebra  $\varprojlim_{k \in \mathbb{N}} \mathfrak{g}_k$ , where  $\mathfrak{g}_k$  is the Lie algebra of  $G_k$ . It is clearly closed in the Krull topology (the definition is analogous to Definition 6.3). The Lie algebra  $\mathfrak{g}$  of  $G$  satisfies

$$\mathfrak{g} = \{X \in \widehat{\mathfrak{X}}(\mathbb{C}^n, 0) : \exp(tX) \in \overline{G} \ \forall t \in \mathbb{C}\} \tag{7}$$

(cf. [16, Proposition 2]). We define the connected component of identity  $\overline{G}_0 = \varprojlim_{k \in \mathbb{N}} G_{k,0}$ , where  $G_{k,0}$  is the connected component of  $\text{Id}$  of  $G_k$ .

**Proposition 6.1** ([16, Proposition 2], [20, Proposition 2.3, Remark 2.9]). *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ . Then  $\overline{G}_0$  is a finite-index normal pro-algebraic subgroup of  $\overline{G}$  generated by  $\exp(\mathfrak{g})$ . Moreover, if  $G$  consists of unipotent formal diffeomorphisms, then  $\exp : \mathfrak{g} \rightarrow \overline{G}$  is a bijection and  $\mathfrak{g}$  consists of formal nilpotent vector fields.*

**Remark 6.2** (cf. [11, 17]). The exponential provides a bijection between  $\widehat{\mathfrak{X}}_N(\mathbb{C}^n, 0)$  and  $\widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ .

**Remark 6.3** ([19]). A subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is finite-dimensional if and only if its Lie algebra  $\mathfrak{g}$  is finite-dimensional.

**Definition 6.4.** Given  $\phi \in \widehat{\text{Diff}}_u(\mathbb{C}^n, 0)$ , we denote by  $\log \phi$  the unique formal nilpotent vector field such that  $\phi = \exp(\log \phi)$ . It is called the *infinitesimal generator* of  $\phi$ .

**Remark 6.4.** We have  $\overline{\langle \phi \rangle} = \{\exp(t \log \phi) : t \in \mathbb{C}\}$  for any  $\phi \in \widehat{\text{Diff}}_u(\mathbb{C}^2, 0)$  [20, Remark 2.11]. In particular, if  $\phi$  is a unipotent element of a subgroup  $G$  of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$ , then the formal vector field  $\log \phi$  belongs to the Lie algebra  $\mathfrak{g}$  of  $G$  as a consequence of  $\overline{\langle \phi \rangle} \subset \overline{G}$ .

**Remark 6.5.** Let  $\phi \in \widehat{\text{Diff}}_u(\mathbb{C}^2, 0)$ . Consider a formal irreducible curve  $\gamma$ . Then  $\gamma$  is  $\phi$ -invariant if and only if is  $\log \phi$ -invariant. The necessary condition is obvious. Let us show the sufficient condition. The group  $H_\gamma = \{\eta \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \eta(\gamma) = \gamma\}$  is pro-algebraic [20, Remark 2.8]. Thus  $\overline{\langle \phi \rangle} = \{\exp(t \log \phi) : t \in \mathbb{C}\}$  is contained in  $H_\gamma$ . Since the one parameter flow of  $\log \phi$  preserves  $\gamma$ , it follows that  $\gamma$  is  $\log \phi$ -invariant.

Instead of the derived series of groups of formal diffeomorphisms and Lie algebras of formal vector fields, it is more natural to consider its projective versions. More precisely, we define

$$\overline{G}^{(k)} = \{\phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \phi_j \in G_j^{(k)} \ \forall j \in \mathbb{N}\},$$

$$\overline{\mathfrak{g}}^{(k)} = \{X \in \widehat{\mathfrak{X}}(\mathbb{C}^n, 0) : X_j \in \mathfrak{g}_j^{(k)} \ \forall j \in \mathbb{N}\}$$

and

$$\overline{C}^k G = \{\phi \in \widehat{\text{Diff}}(\mathbb{C}^n, 0) : \phi_j \in C^k G_j \ \forall j \in \mathbb{N}\},$$

$$\overline{C}^k \mathfrak{g} = \{X \in \widehat{\mathfrak{X}}(\mathbb{C}^n, 0) : X_j \in C^k \mathfrak{g}_j \ \forall j \in \mathbb{N}\}$$

for  $k \geq 0$ . We have  $\overline{G}^{(0)} = \overline{C}^0 G = \overline{G}$  and  $\overline{\mathfrak{g}}^{(0)} = \overline{C}^0 \mathfrak{g} = \mathfrak{g}$ .

**Remark 6.6** ([16, Lemmas 5 and 6], [20, Proposition 2.8]). The group  $\overline{G}^{(k)}$  is pro-algebraic and is equal to the closure of  $(\overline{G})^{(k)}$  in the Krull topology for any  $k \in \mathbb{N}$ . Analogously,  $\overline{\mathfrak{g}}^{(k)}$  is the closure in the Krull topology of  $\mathfrak{g}^{(k)}$  for any  $k \in \mathbb{N}$ .

The Lie correspondence is well behaved for the derivation.

**Proposition 6.2** ([16, Proposition 3]). *Let  $G$  be a subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  that is connected – that is,  $\overline{G} = \overline{G}_0$ . Then  $\overline{\mathfrak{g}}^{(k)}$  is the Lie algebra of  $\overline{G}^{(k)}$  and  $\overline{C}^k \mathfrak{g}$  is the Lie algebra of  $\overline{C}^k G$  for any  $k \geq 0$ .*

**Remark 6.7.** Since

$$\overline{\pi_k(G^{(j)})} = (\overline{\pi_k(G)})^{(j)} = G_k^{(j)} \text{ and } \overline{\pi_k(\mathcal{C}^j G)} = \mathcal{C}^j \overline{\pi_k(G)} = \mathcal{C}^j G_k$$

for  $j \geq 0$  and  $k \geq 1$  (cf. [8, Corollary 1, p. 60]), the Zariski closure  $\overline{G^{(j)}}$  of  $G^{(j)}$  is equal to  $\overline{G}^{(j)}$  and  $\overline{\mathcal{C}^j G} = \overline{\mathcal{C}^j G}$  for  $j \geq 0$ . In particular,  $G$  is abelian (resp., nilpotent, solvable) if and only if  $\overline{G}$  is abelian (resp., nilpotent, solvable). Proposition 6.2 implies that if  $G$  is abelian (resp., nilpotent, solvable), then  $\mathfrak{g}$  is abelian (resp., nilpotent, solvable). The reciprocal also holds if  $\overline{G} = \overline{G}_0$ .

**7. Proof of the Main Theorem**

In this section we show that FD implies UI in dimension 2. We start by showing the result for nilpotent groups (Proposition 7.1), and then we proceed with the general case.

**Proposition 7.1.** *Let  $G$  be an FD nilpotent subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^2, 0)$ . Then  $G$  satisfies UI.*

**Proof.** Consider a formal irreducible curve  $\gamma$ . Let us show that  $G$  satisfies (UI) $_{\gamma}$ . By Propositions 4.4 and 4.5 we can suppose that  $\gamma$  is smooth up to replacing  $\gamma$  and  $G$  with  $\gamma_j$  and  $\tau_j(G_{\gamma,j})$ , respectively, for some  $j \geq 1$  (see Definition 4.7 and the discussion afterward).

We can suppose that  $G$  is infinite-dimensional by Theorem 6.1. Hence the Lie algebra  $\mathfrak{g}$  of  $G$  is infinite-dimensional and nilpotent, by Remarks 6.3 and 6.7. We deduce that  $\mathfrak{g}$  is abelian and such that

$$\mathfrak{g} \subset \{gX : g \in \hat{\mathcal{O}}_2 \text{ and } X(g) = 0\}$$

for some  $X \in \mathfrak{g} \setminus \{0\}$  that has a first integral in  $\hat{\mathcal{O}}_2 \setminus \mathbb{C}$  by Theorem 1.3. The group  $\overline{G}_0$  is generated by  $\exp(\mathfrak{g})$  (Proposition 6.1), and since  $\mathfrak{g}$  is abelian,  $\overline{G}_0$  is abelian and  $\overline{G}_0 = \exp(\mathfrak{g})$ .

Suppose that  $\gamma$  is  $X$ -invariant. Then it is  $\mathfrak{g}$ -invariant and we obtain  $\phi(\gamma) = \gamma$  for any  $\phi \in \overline{G}_0$ . Since  $\overline{G}_0$  is a finite-index subgroup of  $\overline{G}$  by Proposition 6.1, the  $\overline{G}$ -orbit of  $\gamma$  is finite and both  $\overline{G}$  and  $G$  satisfy (UI) $_{\gamma}$ . Thus we suppose that  $\gamma$  is not  $X$ -invariant from now on.

Let  $(p_k)_{k \geq 1}$  be the sequence of infinitely near points of  $\gamma$  (see section 4.2). Consider the lift  $\overline{X}_k$  of  $X$  at  $p_k$  for  $k \geq 0$  (see Remark 4.7). Since  $\gamma$  is not  $X$ -invariant, there exists  $a \geq 1$  such that  $p_a$  is the first infinitely near point of  $\gamma$  satisfying the condition that  $\overline{X}_a$  is nonsingular at  $p_a$ . Moreover,  $\overline{X}_a$  preserves the irreducible components of the divisor  $(\pi_1 \circ \dots \circ \pi_a)^{-1}(0, 0)$  of the blowup process passing through  $p_a$  (cf. section 4.2). Since  $\overline{X}_a$  is nonsingular at  $p_a$ , there is a unique such component  $D$ . Moreover, since  $\gamma$  is smooth,  $\gamma_a$  is transverse to  $D$  at  $p_a$ . Hence there exists a formal coordinate system  $(x, y)$  centred at  $p_a$  such that  $\overline{X}_a = \frac{\partial}{\partial y}$ ,  $D = \{x = 0\}$  and  $\gamma_a = \{y = 0\}$ .

We denote  $J = G_{\gamma,a}$  (see Definition 4.7) and  $H = J \cap \overline{J}_0$ . Let  $\mathfrak{h}$  be the Lie algebra of  $J$ . The inclusion  $J \subset G$  implies  $\mathfrak{h} \subset \mathfrak{g}$ , and in particular  $\mathfrak{h}$  is abelian. As a consequence of the characterisation of  $\mathfrak{h}$  given by equation (7),  $p_j$  is a singular point of  $(gX)_j$  for all  $gX \in \mathfrak{h}$  and  $0 \leq j \leq a$ . Since  $p_j$  is a singular point of  $\overline{X}_j$ , it is also singular for the lift of

any element of  $\mathfrak{g}$  at  $p_j$  for every  $0 \leq j < a$ . The lift  $(\overline{gX})_a$  of  $gX \in \mathfrak{g}$  is singular at  $a$  if and only if  $g$  belongs to the maximal ideal  $\mathfrak{m}$  of  $\hat{\mathcal{O}}_2$ . The previous discussion implies

$$\mathfrak{h} \subset \{gX \in \mathfrak{g} : g \in \hat{\mathcal{O}}_2 \cap \mathfrak{m}\}.$$

Since  $\overline{J}_0$  is a finite-index normal subgroup of  $\overline{J}$ ,  $H$  is a finite-index normal subgroup of  $J$ . The Zariski closure  $\overline{H}$  of  $H$  is a finite-index normal subgroup of  $\overline{J}$  [19, Lemma 2.4]. A finite-index subgroup of  $\overline{J}$  always contains  $\overline{J}_0$  [19, Lemma 2.1]. Since  $\overline{J}_0$  is pro-algebraic by Proposition 6.1, we obtain  $\overline{J}_0 \subset \overline{H} \subset \overline{J}_0$  and then  $\overline{H} = \overline{J}_0$ . In particular, we deduce  $\overline{H} = \overline{H}_0$ . The property  $\overline{H} = \overline{J}_0$  implies that the group  $\overline{H}$  is generated by  $\exp(\mathfrak{h})$  (Proposition 6.1). Since  $\mathfrak{h}$  is abelian, the groups  $H$  and  $\overline{H}$  are abelian,  $\overline{H} = \exp(\mathfrak{h})$  and in particular  $H \subset \exp(\mathfrak{h})$ . Thus any  $\phi \in \tau_a(H)$  is of the form

$$\phi(x, y) = \exp\left(f(x) \frac{\partial}{\partial y}\right) = (x, y + f(x)), \tag{8}$$

where  $f(x) \in \mathbb{C}[[x]] \cap (x)$ , and we obtain  $(\phi(\gamma_a), \gamma_a) = m_0(f)$ . Since  $\tau_a(H)$  satisfies FD by Proposition 4.4, we deduce that  $\tau_a(H)$  satisfies  $(UI)_{\gamma_a}$ . Therefore  $H$  satisfies  $(UI)_{\gamma}$  by Proposition 4.5. Since  $H$  is a finite-index subgroup of  $J$ , the group  $J$  satisfies  $(UI)_{\gamma}$  by Proposition 4.3. Finally,  $G$  satisfies  $(UI)_{\gamma}$  by Remark 4.8.  $\square$

**Remark 7.1.** The orbit of a formal curve by a finitely determined subgroup of  $\widehat{\text{Diff}}(\mathbb{C}^n, 0)$  is not necessarily discrete for  $n \geq 3$ . Notice that even if we suppose that  $G$  is abelian and the Lie algebra is of the type in Theorem 1.3, the analogue of equation (8) does not guarantee that the orbit of a curve is discrete. A major problem is that the orbit of the curve can be nontranscendental (see also Remark 3.4). An example is given by the group  $G$  in Lemma 3.2 for  $n = 3$ . It is generated by elements of the form

$$\phi(x_1, x_2, x_3) = (x_1, x_2 + d_j x_1^2 + x_3^{j+2}, x_3) = \exp\left((d_j x_1^2 + x_3^{j+2}) \frac{\partial}{\partial x_2}\right),$$

and the orbit of the  $x_3$ -axis is nondiscrete and nontranscendental (it is contained in  $\{x_1 = 0\}$ ). This makes possible the existence of first integrals  $d_j x_1^2 + x_3^{j+2}$  of the Lie algebra whose vanishing multiplicity is equal to 2, but the restriction to the  $x_3$ -axis has vanishing multiplicity equal to  $j + 2$ .

**Lemma 7.1.** *Let  $G$  be an FD subgroup of  $\widehat{\text{Diff}}_1(\mathbb{C}^n, 0)$ . Then  $G$  is nilpotent.*

**Proof.** Let  $\phi, \eta \in G$  such that  $j^k \phi = \text{Id}$ . We have

$$j^{k+1}(\phi^{-1} \circ \eta^{-1}) = j^{k+1} \eta^{-1} - (j^{k+1} \phi - \text{Id})$$

and hence

$$j^{k+1}(\eta \circ \phi^{-1} \circ \eta^{-1}) - \text{Id} = -(j^{k+1} \phi - \text{Id}).$$

Since  $[\phi, \eta] = \phi \circ (\eta \circ \phi^{-1} \circ \eta^{-1})$ , we deduce

$$j^{k+1}[\phi, \eta] - \text{Id} = (j^{k+1} \phi - \text{Id}) - (j^{k+1} \phi - \text{Id}) = 0 \implies j^{k+1}[\phi, \eta] = \text{Id}.$$

As a consequence,  $\mathcal{C}^k G$  is contained in  $\{\phi \in G : j^{k+1}\phi = \text{Id}\}$ . Since  $G$  is FD, the latter group is trivial for  $k \gg 1$  and hence  $G$  is nilpotent.  $\square$

The following result is an immediate corollary of Proposition 7.1 and Lemma 7.1:

**Corollary 7.1.** *Let  $G$  be an FD subgroup of  $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ . Then  $G$  satisfies UI.*

*Proof of the Main Theorem.* The property UI implies the finite determination property by Lemma 3.1.

Let us show the necessary condition. It suffices to show the property  $(\text{UI})_\gamma$  for a tame pair  $(G, \gamma)$ , by Proposition 4.6. Since  $G \cap \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$  is finitely determined, it satisfies  $(\text{UI})_\gamma$  by Corollary 7.1. In particular, there exists  $M \in \mathbb{N}$  such that  $(\phi(\gamma), \gamma) < M$  for any  $\phi \in (G \cap \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)) \setminus G_\gamma$ . We deduce that  $G_{\gamma, M} \cap \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$  is contained in  $G_\gamma$ . As a consequence and up to replacing  $G$  with  $G_{\gamma, M}$ , we can suppose that  $(G, \gamma)$  is tame and  $G \cap \widehat{\text{Diff}}_1(\mathbb{C}^2, 0) \subset G_\gamma$  by Remark 4.8. Moreover, up to replacing  $G$  with its finite-index subgroup  $G \cap \overline{G}_0$ , we can suppose  $\overline{G} = \overline{G}_0$  by Proposition 4.3.

Since  $(G, \gamma)$  is tame, the group  $j^1 G$  is diagonalisable, and in particular the derived group  $G'$  is contained in  $\widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ . The group  $G'$  is nilpotent by Lemma 7.1. We can suppose that  $G$  is nonabelian and infinite-dimensional by Proposition 7.1 and Theorem 6.1, respectively. Since  $\overline{G} = \overline{G}_0$ , Remark 6.7 implies that  $\mathfrak{g}$  is nonabelian. Such a property, together with  $\dim_{\mathbb{C}} \mathfrak{g} = \infty$  (Remark 6.3), implies  $\dim_{\mathbb{C}} \mathfrak{g}^{(1)} = \infty$  by Theorem 1.4. The Lie algebra  $\mathfrak{h}$  of  $G'$  is equal to  $\overline{\mathfrak{g}}^{(1)}$  by Remark 6.7 and Proposition 6.2. The Lie algebra  $\mathfrak{h}$  is nilpotent (Remark 6.7) and satisfies  $\dim_{\mathbb{C}} \mathfrak{h} \geq \dim_{\mathbb{C}} \mathfrak{g}^{(1)} = \infty$ . Hence we obtain

$$\mathfrak{h} \subset \{fX : f \in \hat{\mathcal{O}}_2 \text{ and } X(f) = 0\}$$

for some  $X \in \mathfrak{h} \setminus \{0\}$  such that  $X$  has a first integral  $g$  in  $\mathfrak{m} \setminus \{0\}$ , by Theorem 1.3. The vector field  $X$  is of the form  $hX'$ , where  $h \in \hat{\mathcal{O}}_2$  and the coefficients of  $X'$  have no common factors in  $\hat{\mathcal{O}}_2$ . Given a nontrivial element  $fX$  of  $\mathfrak{h}$ , there are finitely many  $fX$ -invariant curves. Indeed, the formal irreducible  $fX$ -invariant curves are the curves  $h_j = 0$  ( $1 \leq j \leq k$ ) where  $fgh = h_1^{n_1} \cdots h_k^{n_k}$  is the irreducible decomposition of  $fgh$  in  $\hat{\mathcal{O}}_2$ . Choose  $\phi \in G' \setminus \{\text{Id}\}$ . The set of formal irreducible  $\phi$ -invariant curves coincides with the set of formal irreducible  $\log \phi$ -invariant curves by Remark 6.5. Since  $\log \phi$  belongs to  $\mathfrak{h} \setminus \{0\}$  by Remark 6.4, there are finitely many formal irreducible  $\phi$ -invariant curves and hence finitely many formal irreducible  $G'$ -invariant curves.

Since  $G' \subset \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$  and  $\gamma$  is  $G \cap \widehat{\text{Diff}}_1(\mathbb{C}^2, 0)$ -invariant, the curve  $\gamma$  is  $G'$ -invariant. Since  $G'$  is a normal subgroup of  $G$ , every curve in the  $G$ -orbit of  $\gamma$  is also  $G'$ -invariant. Therefore the  $G$ -orbit of  $\gamma$  is finite, and it is obvious that  $G$  satisfies  $(\text{UI})_\gamma$ .  $\square$

**References**

[1] V. I. ARNOLD, Dynamics of complexity of intersections, *Bol. Soc. Bras. Mat., Nova Sér.*, **21**(1) (1990), 1–10.  
 [2] V. I. ARNOLD, Dynamics of intersections, in *Analysis, et Cetera*, pp. 77–84 (Academic Press, Boston, MA, 1990).

- [3] V. I. ARNOLD, ed., *Arnold's Problems* (Springer, Berlin, 2004). Translated and revised edition of the 2000 Russian original.
- [4] V. I. ARNOLD, Bounds for Milnor numbers of intersections in holomorphic dynamical systems, in *Topological Methods in Modern Mathematics: Proceedings of a Symposium in Honor of John Milnor's Sixtieth Birthday, Held at the State University of New York at Stony Brook, USA, June 14-June 21, 1991*, pp. 379–390 (Publish or Perish, Inc., Houston, TX, 1993).
- [5] M. ARTIN AND B. MAZUR, On periodic points, *Ann. of Math. (2)* **81** (1965), 82–99.
- [6] M. S. BAOUENDI, L. PREISS ROTHSCHILD, J. WINKELMANN AND D. ZAITSEV, Lie group structures on groups of diffeomorphisms and applications to CR manifolds, *Ann. Inst. Fourier (Grenoble)*, **54**(5) (2004), 1279–1303, 2004.
- [7] G. BINYAMINI, Finiteness properties of formal Lie group actions, *Transform. Groups*, **20**(4) (2015), 939–952.
- [8] A. BOREL, *Linear Algebraic Groups*, 2nd ed., *Graduate Texts in Mathematics* no. 126, (Springer-Verlag, New York, 1991).
- [9] E. CASAS-ALVERO, *Singularities of Plane Curves* (Cambridge University Press, 2000).
- [10] D. CERVEAU AND J.-F. MATTEI, Formes intégrables holomorphes singulières, *Astérisque*, **97**:(1982).
- [11] J. ÉCALLE, Théorie itérative: introduction à la théorie des invariants holomorphes, *J. Math. Pures Appl. (9)* **54** (1975), 183–258.
- [12] C. FAVRE AND M. JONSSON, *The Valuative Tree Lecture Notes in Mathematics* no. 1853 (Springer-Verlag, Berlin, 2004).
- [13] W. FULTON, *Intersection Theory*, 2nd ed. (Springer, Berlin, 1998).
- [14] Y. ILYASHENKO AND S. YAKOVENKO, *Lectures on Analytic Differential Equations Graduate Studies in Mathematics* no. 86 (American Mathematical Society, Providence, RI, 2008).
- [15] A. LEVY, *Basic Set Theory* (Dover Publications, Mineola, NY, 2002). Reprint of the 1979 original.
- [16] M. MARTELO AND J. RIBÓN, Derived length of solvable groups of local diffeomorphisms, *Math. Ann.* **358**(3) (2014), 701–728.
- [17] J. MARTINET AND J.-P. RAMIS, Classification analytique des équations différentielles non linéaires résonnantes du premier ordre, *Ann. Sci. Éc. Norm. Supér. (4)*, **4**(16) (1983), 571–621.
- [18] J. -F. MATTEI AND R. MOUSSU, Holonomie et intégrales premières, *Ann. Sci. Éc. Norm. Supér. (4)*, **13**(4) (1980), 469–523.
- [19] J. RIBÓN, Finite dimensional groups of local diffeomorphisms, *Israel J. Math.* **227**(1) (2018), 289–329.
- [20] J. RIBÓN, The solvable length of groups of local diffeomorphisms, *J. Reine Angew. Math.* **752** (2019), 105–139.
- [21] A. L. SEIGAL AND S. YAKOVENKO, Local dynamics of intersections: V. I. Arnold's theorem revisited, *Israel J. Math.* **201**(2) (2014), 813–833.
- [22] A. R. SHASTRI, *Elements of Differential Topology* (CRC Press, Boca Raton, FL, 2011). With a foreword by F. T. Farrell.
- [23] M. SHUB AND D. SULLIVAN, A remark on the Lefschetz fixed point formula for differentiable maps, *Topology* **13** (1974), 189–191.
- [24] T. SKOLEM, Ein Verfahren zur Behandlung gewisser exponentialer Gleichungen und diophantischer Gleichungen, in *8th Scand. Math. Congr., Stockholm, 1934*, pp. 163–188 (Håkan Ohlssons Boktryckeri, Lund, 1935).
- [25] C. T. C. WALL, *Singular Points of Plane Curves*, London Mathematical Society Student Texts no. **63** (Cambridge University Press, 2004).