

POINTS IN A FOLD

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Abstract

When a page, represented by the interval $[0, 1]$, is folded right over left n times, the right-hand fold contains a sequence of points. We specify these points and the order in which they appear in each fold. We also determine exactly where in the folded structure any point in $[0, 1]$ appears and, given any point on the bottom line of the structure, which point lies at each level above it.

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1. Introduction

When the line joining 0 to 1 (or a rectangular piece of paper with $[0, 1]$ at its edge) is folded in half, right over left, the point $1/2$ is at the fold. When the line is then folded in half again, right over left, the points $1/4$ and $3/4$ are at this second fold, in that order, as indicated by the horizontal arrow, from the outside of the fold to its interior in Figure 1. We have ignored the fact that, strictly speaking, a line with no thickness, when halved and completely folded back upon itself, would have all points superimposed, not juxtaposed at the fold.

Proceeding in this way, the sequence of points from the outside of the third fold to the inside becomes $\langle 1/8, 7/8, 5/8, 3/8 \rangle$. Their placing, prior to the third fold, is indicated by the vertical arrow in Figure 1.

DEFINITION 1.1 (*n th fold sequence*). After we have folded the line n times, the sequence of points from the outside of the n th fold to the inside is designated as the *n th fold sequence*, S_n , and the j th term in S_n is designated as $S_{n,j}$.

EXAMPLE 1.2. The sequences after 2, 3 and 4 folds are:

$$\begin{aligned} S_2 &= \langle S_{2,1}, S_{2,2} \rangle = \left\langle \frac{1}{4}, \frac{3}{4} \right\rangle, \\ S_3 &= \langle S_{3,1}, S_{3,2}, S_{3,3}, S_{3,4} \rangle = \left\langle \frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8} \right\rangle, \\ S_4 &= \langle S_{4,1}, S_{4,2}, S_{4,3}, S_{4,4}, S_{4,5}, S_{4,6}, S_{4,7}, S_{4,8} \rangle = \left\langle \frac{1}{16}, \frac{15}{16}, \frac{9}{16}, \frac{7}{16}, \frac{5}{16}, \frac{11}{16}, \frac{13}{16}, \frac{3}{16} \right\rangle. \end{aligned}$$

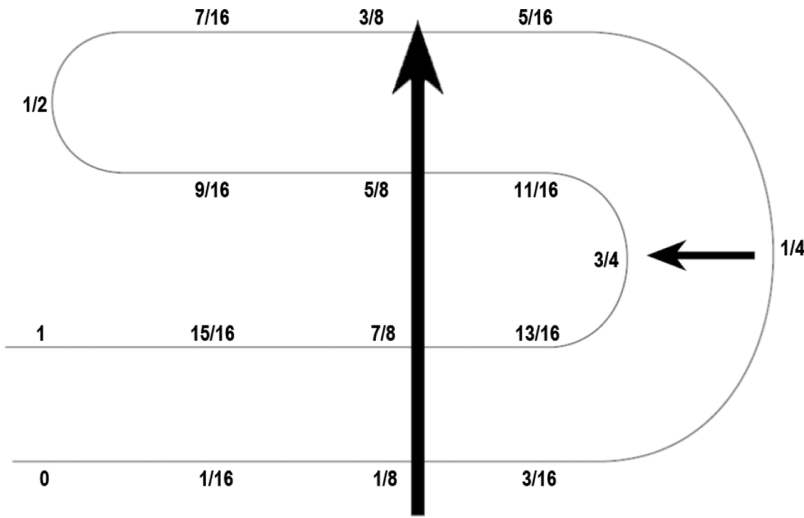


FIGURE 1. Points in a fold after two folds.

The line after n folds has the following properties:

- the n th fold, S_n , is on the right;
- the ends of the line appear at the bottom on the left;
- the first fold, S_1 , then the second fold, S_2 , all the way through to the $(n - 1)$ th fold, S_{n-1} , appear immediately above the ends of the line, in ascending order.

For points found in the folds in $[0, 1]$, we will determine:

- the j th term, $S_{n,j}$, in S_n in terms of n and j (Theorems 3.2 and 3.4);
- the relationship between points in the same fold (Theorems 3.5 and 3.6);
- the exact position in the folded structure of any point of $[0, 1]$ (Theorems 3.7 and 3.9);
- given a point on the bottom line of the structure, which point lies at any level above it (Theorem 3.1).

The line when unfolded shows a sequence of \vee and \wedge creases, which form the paperfolding sequence which has been studied in [1, 2–4, 7].

2. Folds and prefolds

We now show one of the properties of S_n .

THEOREM 2.1 (Points of S_n). *The set of points in S_n is*

$$\left\{ \frac{2k + 1}{2^n} \mid 0 \leq k < 2^{n-1} \right\}.$$

PROOF. Each new fold places a crease midway between creases from earlier folds as well as a new crease midway between 0 and the first crease and another crease midway between 1 and the last crease. Thus the n th fold places a new crease at $1/2^n$ and further new creases thereafter at intervals of $1/2^{n-1}$ units. Thus for $k = 0, 1, 2, \dots, 2^{n-1} - 1$, points in the n th fold occur at the locations

$$\frac{1}{2^n} + \frac{k}{2^{n-1}} = \frac{2k+1}{2^n}. \quad \square$$

We note that S_n is the sequence of middle points of the set of overlaid intervals, each of length $1/2^{n-1}$, formed after $n - 1$ folds and listed from bottom to top. To help determine this order we introduce the concept of *prefolds*. We begin with the definition of an *n th prefold sequence*.

DEFINITION 2.2 (n th prefold). Fold the line from 0 to 1 n times, where $n \geq 0$. Let j be an integer such that $0 < j \leq 2^n$ and a be a real number such that $0 < a < 1/2^n$. Then we have the following properties.

- (1) $F_n(a, j)$ is the j th term from the bottom, lying at, or directly above a ; we call $F_n(a, j)$ the j th term of the n th prefold sequence at a .
- (2) $F_n(a)$ is the sequence of all terms in the n th prefold sequence at a , that is,

$$F_n(a) = \langle F_n(a, 1), F_n(a, 2), F_n(a, 3), \dots, F_n(a, 2^n) \rangle.$$

- (3) $\widetilde{F}_n(a)$ is the reverse n th prefold sequence at a , that is,

$$\widetilde{F}_n(a) = \langle F_n(a, 2^n), \dots, F_n(a, 3), F_n(a, 2), F_n(a, 1) \rangle.$$

By Definition 2.2, $F_n(a, 1) = a$ and we can express S_n in terms of F_{n-1} .

THEOREM 2.3. For $n > 0$,

$$S_n = F_{n-1}\left(\frac{1}{2^n}\right) = \left\langle F_{n-1}\left(\frac{1}{2^n}, 1\right), F_{n-1}\left(\frac{1}{2^n}, 2\right), F_{n-1}\left(\frac{1}{2^n}, 3\right), \dots, F_{n-1}\left(\frac{1}{2^n}, 2^{n-1}\right) \right\rangle.$$

EXAMPLE 2.4. For $n = 3$,

$$S_3 = \left\langle F_2\left(\frac{1}{8}, 1\right), F_2\left(\frac{1}{8}, 2\right), F_2\left(\frac{1}{8}, 3\right), F_2\left(\frac{1}{8}, 4\right) \right\rangle = \left\langle \frac{1}{8}, \frac{7}{8}, \frac{5}{8}, \frac{3}{8} \right\rangle.$$

We now derive connections between prefolds.

LEMMA 2.5 (Form of $F_n(a)$). For $0 < a < 1/2^n$,

$$F_n(a) = \left\langle F_{n-1}(a), \widetilde{F}_{n-1}\left(\frac{1}{2^{n-1}} - a\right) \right\rangle.$$

PROOF. The n th fold occurs at $1/2^n$. It involves $F_{n-1}((1/2^{n-1}) - a)$ for $0 < a < 1/2^n$ being rotated anticlockwise around $1/2^n$ and then placed atop $F_{n-1}(a)$ to form $F_n(a)$, giving the result. \square

Lemma 2.5 gives a recursion for F_n .

THEOREM 2.6 (Recursive expression of prefolds).

$$F_n(a, j) = \begin{cases} F_{n-1}(a, j), & \text{if } 1 \leq j \leq 2^{n-1}, \\ F_{n-1}((1/2^{n-1}) - a, 2^n - j + 1), & \text{if } 2^{n-1} < j \leq 2^n. \end{cases}$$

THEOREM 2.7 (Prefold expression of folds).

$$\begin{aligned} S_n &= \left\langle F_{n-2}\left(\frac{1}{2^n}\right), \widetilde{F_{n-2}\left(\frac{3}{2^n}\right)} \right\rangle \\ &= \left\langle F_{n-2}\left(\frac{1}{2^n}, 1\right), F_{n-2}\left(\frac{1}{2^n}, 2\right), \dots, F_{n-2}\left(\frac{1}{2^n}, 2^{n-2}\right), \right. \\ &\quad \left. F_{n-2}\left(\frac{3}{2^n}, 2^{n-2}\right), F_{n-2}\left(\frac{3}{2^n}, 2^{n-2} - 1\right), \dots, F_{n-2}\left(\frac{3}{2^n}, 1\right) \right\rangle. \end{aligned}$$

3. The main result

THEOREM 3.1. *Let $s \leq 2^n$.*

- (i) *If $s = j + 1$ where $j = \sum_{p=1}^r 2^{t_p} \leq 2^n$ with $r \geq 0$ and $t_1 > t_2 > \dots > t_r > 0$ if $r > 0$, then*

$$F_n(a, s) = \sum_{p=1}^r \frac{1}{2^{t_p}} + a.$$

- (ii) *If $s = j$ with j as in (i), then*

$$F_n(a, s) = \frac{1}{2^{r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - a.$$

PROOF. The proof is by induction on n . If $n = 1$ and $j = 0$ then $r = 0$ and $F_n(a, j + 1) = a$, so we have (i). If $n = 1$ and $j = 2$ then $r = t_r = 1$ and $F_n(a, j) = 1 - a$, so we have (ii).

Assume the result for n .

If $1 \leq s \leq 2^n$, by Theorem 2.6(i), $F_{n+1}(a, s) = F_n(a, s)$. The induction hypothesis for $s = j$ gives (ii) with $n + 1$ for n and, for $s = j + 1 \leq 2^n$, it gives (i) with $n + 1$ for n .

If $2^n < s \leq 2^{n+1}$, we have $t_1 = n$ and by Theorem 2.6(ii),

$$F_{n+1}(a, s) = F_n\left(\frac{1}{2^n} - a, 2^{n+1} - s + 1\right).$$

If $s = j + 1$, we can write

$$2^{n+1} - (j + 1) + 1 = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} 2^p \right) + 2^{t_r}$$

because the double sum is the sum of the powers of 2 from 2^{t_1-1} to 2^{t_r+1} omitting $2^{t_2}, 2^{t_3}, \dots, 2^{t_{r-1}}$. By the induction hypothesis,

$$F_{n+1}(a, j + 1) = \frac{1}{2^{t_r-1}} - \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} \frac{1}{2^p} \right) + a - \frac{1}{2^n},$$

where the double sum is the sum of the powers of $\frac{1}{2}$ from $1/2^{t_1-1}$ to $1/2^{t_r+1}$, omitting $1/2^{t_2}, 1/2^{t_3}, \dots, 1/2^{t_{r-1}}$. This shows $F_{n+1}(a, s)$ is as in (i) because

$$\frac{1}{2^{t_r-1}} - \frac{1}{2^n} = \frac{1}{2^r} + \frac{1}{2^{r+1}} + \dots + \frac{1}{2^{t_1-1}} + \frac{1}{2^{t_1}}.$$

If $s = j$, we can write

$$2^{n+1} - j + 1 = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} 2^p \right) + 2^{t_r} + 1.$$

By the induction hypothesis,

$$F_{n+1}(a, j) = \sum_{i=1}^{r-1} \left(\sum_{p=t_i-1}^{t_{i+1}+1} \frac{1}{2^p} \right) + \frac{1}{2^{t_r}} + \frac{1}{2^n} - a,$$

where the double sums are as in the $s = j + 1$ case. Since

$$\frac{1}{2^r} + \frac{1}{2^n} = \frac{1}{2^{t_r-1}} - \left(\frac{1}{2^{t_r+1}} + \dots + \frac{1}{2^{t_1-1}} + \frac{1}{2^{t_1}} \right),$$

$F_{n+1}(a, s)$ is as in (ii). □

THEOREM 3.2. *Let $s \leq 2^n$.*

- (i) *If $s = j + 1$ where $j = \sum_{p=1}^r 2^{t_p} \leq 2^n$ with $r \geq 0$ and $t_1 > t_2 > \dots > t_r > 0$ if $r > 0$, then*

$$S_{n,s} = \sum_{p=1}^r \frac{1}{2^{t_p}} + \frac{1}{2^n}.$$

- (ii) *If $s = j$ with j as in (i), then*

$$S_{n,s} = \frac{1}{2^{t_r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}.$$

PROOF. The result follows from Theorem 3.1 with $a = 1/2^n$. □

EXAMPLE 3.3. Since $11 = 2^3 + 2 + 1$, we can take $r = 2, t_1 = 3$ and $t_2 = 1$ in Theorem 3.2(i) to give

$$S_{6,11} = \frac{1}{2^3} + \frac{1}{2} + \frac{1}{2^6} = \frac{41}{64}.$$

Similarly, since $26 = 2^4 + 2^3 + 2$, we can take $r = 3, t_1 = 4$ and $t_2 = 3$ in Theorem 3.2(ii) to give

$$S_{6,26} = \frac{3}{2} - \left(\frac{1}{2^4} + \frac{1}{2^3} + \frac{1}{2} + \frac{1}{2^6} \right) = \frac{51}{64}.$$

The next result gives some special cases of Theorem 3.2.

THEOREM 3.4. *Let $p \in \mathbb{N}$.*

- (i) $S_{n,2^{p-1}} = 1 - (1/2^{p-1}) + (1/2^n)$ if $n \geq 1, 0 < p \leq n$.
- (ii) $S_{n,2^p} = (1/2^{p-1}) - (1/2^n)$ if $n > 1, 0 < p < n$.
- (iii) $S_{n,2^{p+1}} = (1/2^p) + (1/2^n)$ if $n > 2, 0 < p < n$.
- (iv) $S_{n,2^{p+2}} = 1 - (1/2^p) - (1/2^n)$ if $n > 2, 0 < p < n$.

We can relate consecutive elements of S_n .

THEOREM 3.5. *If $j = \sum_{p=1}^r 2^{t_p} \leq 2^{n-1}, r > 0$ and $t_1 > t_2 > \dots > t_r > 0$, then*

- (i) $S_{n,j+1} + S_{n,j} = 3/2^{t_r}$,
- (ii) $S_{n,j-1} + S_{n,j} = 1$.

PROOF. (i) This follows from Theorem 3.2.

(ii) As $j - 1 = \sum_{p=1}^{r-1} 2^{t_p} + 2^{t_r-1} + 2^{t_r-2} + \dots + 2 + 1$, by Theorem 3.2(i),

$$S_{n,j-1} = \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} + \sum_{i=1}^{t_r-1} \frac{1}{2^i} + \frac{1}{2^n} = \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} + 1 - \frac{1}{2^{t_r}} + \frac{1}{2^n}.$$

The result follows because

$$S_{n,j} = \frac{1}{2^{t_r}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}. \quad \square$$

THEOREM 3.6. *For $n \geq 2$,*

$$S_{n,j} = S_{n,2^{n-1}-j+1} + \frac{(-1)^j}{2^{n-1}}.$$

PROOF. For j even, if $j = \sum_{p=1}^r 2^{t_p} \leq 2^{n-1}$ where $r > 0$ and $t_1 > t_2 > \dots > t_r > 0$, then

$$2^{n-1} - j + 1 = \sum_{i=t_1+1}^{n-2} 2^i + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} 2^p \right) + 2^{t_r} + 1.$$

By Theorem 3.2,

$$S_{n,2^{n-1}-j+1} = \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} \frac{1}{2^p} \right) + \frac{1}{2^{t_r}} + \frac{1}{2^n}$$

and

$$S_{n,j} = \frac{1}{2^{t_r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n}.$$

So

$$S_{n,2^{n-1}-j+1} - S_{n,j} = \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=t_1}^{t_r+1} \frac{1}{2^i} - \frac{1}{2^{t_r}} + \frac{1}{2^{n-1}} = -\frac{1}{2^{n-1}}.$$

For j odd, let $j = \sum_{p=1}^r 2^{t_p} + 1$ with $r > 0$ and $t_1 > t_2 > \dots > t_r > 0$. Then

$$2^{n-1} - j + 1 = \sum_{i=t_1+1}^{n-2} 2^i + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} 2^p \right) + 2^{t_r}.$$

So, by Theorem 3.2,

$$S_{n,2^{n-1}-j+1} = \frac{1}{2^{t_r-1}} - \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} + \sum_{i=1}^{r-1} \left(\sum_{p=t_{i+1}+1}^{t_i-1} \frac{1}{2^p} \right) - \frac{1}{2^n}$$

and

$$S_{n,j} = \sum_{p=1}^r \frac{1}{2^{t_p}} + \frac{1}{2^n}.$$

Thus

$$S_{n,2^{n-1}-j+1} - S_{n,j} = \frac{1}{2^{t_r-1}} - \sum_{i=t_1+1}^{n-2} \frac{1}{2^i} - \sum_{i=t_1}^{t_r+1} \frac{1}{2^i} - \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}. \quad \square$$

The next two theorems allow us to determine exactly where in the n times folded structure any point b of $[0, 1]$ appears.

THEOREM 3.7. *If $0 \leq b \leq 1$, $b \neq (2k + 1)/2^{n+1}$, then b lies on level j of the n times folded structure at, or directly above, a , where $0 \leq a < 1/2^n$.*

(i) *If $\lfloor 2^n b \rfloor$ is even, that is, $\lfloor 2^n b \rfloor = \sum_{p=1}^s 2^{u_p}$ where $s = 0$, or $s = 1$ and $u_1 = n$, or $n > u_1 > u_2 > \dots > u_s > 0$, then*

$$a = b - \frac{\lfloor 2^n b \rfloor}{2^n} \quad \text{and} \quad j = 1 + \sum_{p=1}^s 2^{n-u_s-p+1}.$$

(ii) *If $\lfloor 2^n b \rfloor$ is odd, that is, for some m , $0 \leq m < n$, $\lfloor 2^n b \rfloor = 2^{n-m} - 1 - \sum_{p=1}^v 2^{w_p}$ where $v = 0$ or $v > 0$ and $n - m - 1 > w_1 > w_2 > \dots > w_s > 0$, then*

$$a = \frac{\lfloor 2^n b \rfloor}{2^n} + \frac{1}{2^n} - b \quad \text{and} \quad j = \sum_{p=1}^v 2^{n-w_v-p+1} + 2^{m+1}.$$

PROOF. Note that a and j are such that $0 \leq a < 1/2^n$, $1 \leq j \leq 2^n$ and $b = F_n(a, j)$.

(i) Let $j = \sum_{p=1}^r 2^{t_p} + 1$, where $n > t_1 > t_2 > \dots > t_r > 0$ if $r > 0$. By Theorem 3.1(i),

$$b = \sum_{p=1}^r \frac{1}{2^{t_p}} + a.$$

Consequently $\lfloor 2^n b \rfloor = \sum_{p=1}^r 2^{n-t_p}$. If $\lfloor 2^n b \rfloor = \sum_{p=1}^s 2^{u_p}$, then $r = s$ and $t_p = n - u_{s-p+1}$ for $1 \leq p \leq s$. So

$$j = 1 + \sum_{p=1}^s 2^{n-u_{s-p+1}}.$$

As $t_1 < n$, j can only be odd if $u_s > 0$, that is, $\lfloor 2^n b \rfloor$ is even.

(ii) Let $j = \sum_{p=1}^r 2^{t_p}$, where $r > 0$ and $t_1 > t_2 > \dots > t_r > 0$. By Theorem 3.1(ii),

$$b = \frac{1}{2^{t_{r-1}}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - a.$$

Consequently $\lfloor 2^n b \rfloor = 2^{n-t_{r-1}} - 1 - \sum_{p=1}^{r-1} 2^{n-t_p}$. If j is even then $\lfloor 2^n b \rfloor$ must be odd. If, for some m with $0 \leq m < n$, we have $\lfloor 2^n b \rfloor = 2^{n-m} - 1 - \sum_{p=1}^v 2^{w_p}$ with $v = 0$ or $v > 0$ and $n - m - 1 > w_1 > w_2 > \dots > w_p > 0$, then

$$2^{n-m} - \sum_{p=1}^v 2^{w_p} = 2^{n-t_{r-1}} - 1 - \sum_{p=1}^{r-1} 2^{n-t_p}.$$

As $n - m > w_1 + 1$ and $n - t_r + 1 > n - t_{r-1} + 1$, we must have $t_r = m + 1$, $r = v + 1$ and $t_p = n - w_{v-p+1}$ for $0 \leq p \leq r - 1$. So

$$j = \sum_{p=1}^v 2^{n-w_{v-p+1}} + 2^{m+1}. \quad \square$$

EXAMPLE 3.8. If $n = 5$ and

$$b = \frac{9}{32} + \frac{\pi}{180} = \frac{2^5 - 2^4 - 2^2 - 2 - 1}{2^5} + \frac{\pi}{180}.$$

then $\lfloor 2^5 b \rfloor = 2^5 - 2^4 - 2^2 - 2 - 1 = 9$. So by Theorem 3.7(ii) with $n = 5$, $m = 0$, $v = 3$, $w_1 = 4$, $w_2 = 2$ and $w_3 = 1$,

$$a = \frac{5}{16} - \frac{9}{32} - \frac{\pi}{180} = \frac{1}{32} - \frac{\pi}{180} \quad \text{and} \quad j = 2^4 + 2^3 + 2 = 26.$$

If $n = 5$ and

$$b = \frac{5}{16} + \frac{\pi}{180},$$

then $\lfloor 2^5 b \rfloor = 2^3 + 2 = 10$. So by Theorem 3.7(i) with $s = 2, u_1 = 3$ and $u_2 = 1$,

$$a = \frac{5}{16} + \frac{\pi}{180} - \frac{5}{16} = \frac{\pi}{180} \quad \text{and} \quad j = 1 + 2^2 + 2^4 = 21.$$

THEOREM 3.9. Let $S_{n,j} = (2k + 1)/2^n$.

(i) If $k = \sum_{p=1}^s 2^{u_p}$ where $n - 1 > u_1 > u_2 > \dots > u_s > 0$ or $s = 1$ and $u_1 = n - 1$, then

$$j = 1 + \sum_{p=1}^s 2^{n-1-u_{s-p+1}}.$$

(ii) If $k = 2^{n-1-m} - 1 - \sum_{p=1}^v 2^{w_p}$ where $v = 0$ or $v > 0$ and $n - m - 2 > w_1 > w_2 > \dots > w_v > 0$, then

$$j = 2^{m+1} + \sum_{p=1}^v 2^{n-1-w_{v-p+1}}.$$

PROOF. If $S_{n,j} = (2k + 1)/2^n$, then $F_{n-1}(1/2^n, j) = (2k + 1)/2^n$ and $\lfloor 2^{n-1}(2k + 1)/2^n \rfloor = k$, so we use Theorem 3.7 with $(2k + 1)/2^n$ for b and $n - 1$ for n .

(i) If $k = \sum_{p=1}^s 2^{u_p}$ where $s = 1$ and $u_1 = n - 1$ or $n - 1 > u_1 > u_2 > \dots > u_s > 0$, then

$$j = 1 + \sum_{p=1}^s 2^{n-1-u_{s-p+1}}.$$

(ii) If $k = 2^{n-1-m} - 1 - \sum_{p=1}^v 2^{w_p}$ where $v = 0$ or $v > 0$ and $n - m - 2 > w_1 > w_2 > \dots > w_v > 0$, then

$$j = 2^{m+1} + \sum_{p=1}^v 2^{n-1-w_{v-p+1}}. \quad \square$$

EXAMPLE 3.10. For $n = 5, k = 9, m = 0, v = 2, w_1 = 2$ and $w_2 = 1$, we have $S_{5,14} = 19/32$.

For $n = 5, k = 14, u_1 = 3, u_2 = 2, u_3 = 1$ and $s = 3$, we have $S_{5,15} = 29/32$.

4. Folds and the triangular array $T(n, k)$

We thank the referee for drawing our attention to Sigrist’s triangular array $T(n, i)$ [6] which also arises from paperfolding. Our sequences $S_{n,i}$ are related to $T(n, i)$ by

$$T(n, i) = 2^{n-1} S_{n,i} + \frac{1}{2}.$$

As a result, our Theorem 3.2 gives the following result for $T(n, i)$.

THEOREM 4.1. *Let $s \leq 2^n$.*

(i) *If $s = j + 1$ where $j = \sum_{p=1}^r 2^{t_p} \leq 2^n, r \geq 0$ and $t_1 > t_2 > \dots > t_r > 0$ if $r > 0$, then*

$$T(n, s) = \sum_{p=1}^r 2^{n-t_p-1} + 1.$$

(ii) *If $s = j$, with j as in (i), then*

$$T(n, s) = 2^{n-t_r} - \sum_{p=1}^{r-1} 2^{n-t_p-1}.$$

PROOF. The statements follow from Theorem 3.2. If s is as in (i), then

$$T(n, s) = 2^{n-1} \left(\sum_{p=1}^r \frac{1}{2^{t_p}} + \frac{1}{2^n} \right) + \frac{1}{2} = \sum_{p=1}^r 2^{n-t_p-1} + 1.$$

If s is as in (ii), then

$$T(n, s) = 2^{n-1} \left(\frac{1}{2^{t_r-1}} - \sum_{p=1}^{r-1} \frac{1}{2^{t_p}} - \frac{1}{2^n} \right) + \frac{1}{2} = 2^{n-t_r} - \sum_{p=1}^{r-1} 2^{n-t_p-1}. \quad \square$$

Similarly, Theorems 3.4 and 3.5 give the following results.

THEOREM 4.2. *We have:*

- (i) $T(n, 2^p - 1) = 2^{n-1} - 2^{n-p} + 1$ if $n \geq 1, 0 < p \leq n$;
- (ii) $T(n, 2^p) = 2^{n-p}$ if $n > 1, 0 < p < n$;
- (iii) $T(n, 2^p + 1) = 2^{n-p-1} + 1$ if $n > 2, 0 < p < n$;
- (iv) $T(n, 2^p + 2) = 2^{n-1} - 2^{n-p-1}$ if $n > 2, 0 < p < n$.

THEOREM 4.3. *If $j = \sum_{p=1}^r 2^{t_p} \leq 2^{n-1}, r > 0$ and $t_1 > t_2 > \dots > t_r > 0$, then:*

- (i) $T(n, j + 1) + T(n, j) = 3 \cdot 2^{n+t_r-1} + 1$;
- (ii) $T(n, j - 1) + T(n, j) = 2^{n-1} + 1$.

Write out the elements of $T(n, i)$ as

$$\begin{aligned} T(1, 1) &= a(1), \\ T(2, 1) &= a(2), T(2, 2) = a(3), \\ T(3, 1) &= a(4), T(3, 2) = a(5), T(3, 3) = a(6), \\ &\vdots \end{aligned}$$

Shallit [5] observed that $a(n)$ is a 2-regular sequence defined by the recursion

$$\begin{aligned} a(4n) &= a(2n), \\ a(4n + 2) &= a(2n) - a(2n + 1) + a(4n + 1), \\ a(4n + 3) &= a(2n + 1), \end{aligned}$$

$$a(8n + 1) = -2a(2n + 1) + 3a(4n + 1),$$

$$a(8n + 5) = -a(2n + 1) + 2a(4n + 1).$$

As, clearly, $T(n, i) = a(2^{n-1} + i - 1)$, we have $a(2^{n-1} + i - 1) = 2^{n-1}S_{n,i} + 1/2$. This leads to the following result.

THEOREM 4.4. *We have:*

- (i) $2S_{n,4j+1} = S_{n-1,2j+1}$ if $n > 2$;
- (ii) $2S_{n,4j+3} = S_{n-1,2j+1} - S_{n-1,2j+2} + 2S_{n,4j+2}$ if $n > 2$;
- (iii) $2S_{n,4j+4} = S_{n-1,2j+2}$ if $n > 2$;
- (iv) $2S_{n,8j+2} = 3S_{n-1,4j+2} - S_{n-2,2j+2}$ if $n > 3$;
- (v) $4S_{n,8j+6} = 4S_{n-1,4j+2} - S_{n-2,2j+2}$ if $n > 3$.

PROOF. If $n > 2$, then $a(2^{n-1} + 4j) = a(2^{n-2} + 2j)$, so $2^{n-1}S_{n,4j+1} = 2^{n-2}S_{n-1,2j+1}$, giving (i). A similar approach gives the remaining results. \square

References

- [1] B. P. Bates, M. W. Bunder and K. P. Tognetti, ‘Mirroring and interleaving in the paperfolding sequence’, *Appl. Anal. Discrete Math.* **4** (2010), 96–118.
- [2] F. M. Dekking, M. Mendès France and A. J. van der Poorten, ‘Folds!’, *Math. Intelligencer* **4** (1982), I: 130–138; II: 173–181; III: 190–195.
- [3] M. Mendès France, J. Shallit and A. J. van der Poorten, ‘On lacunary formal power series and their continued fraction expansion’, in: *Number Theory in Progress: Proceedings of the International Conference in Honour of A. Schinzel, Zakopane 1997*, Vol. 1 (eds. K. Györy, H. Iwaniec and J. Urbanowicz) (de Gruyter, Berlin, 1999), 321–326.
- [4] M. Mendès France and A. J. van der Poorten, ‘Arithmetic and analytical properties of paper folding sequences’, *Bull. Aust. Math. Soc.* **24** (1981), 123–131.
- [5] J. Shallit, *The Online Encyclopedia of Integer Sequences*, A281589 #17, <https://oeis.org/A281589>.
- [6] R. Sigrist, *The Online Encyclopedia of Integer Sequences*, A281589 #5, <https://oeis.org/A281589>.
- [7] A. J. van der Poorten and J. O. Shallit, ‘Folded continued fractions’, *J. Number Theory* **40** (1992), 237–250.

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