Adv. Appl. Prob. **47,** 506–529 (2015)
Printed in Northern Ireland

BURN-IN PROCEDURE BASED ON A DEPENDENT COVARIATE PROCESS

JI HWAN CHA,* *Ewha Womans University*
GIANPAOLO PULCINI,* ** *Istituto Motori*

Abstract

Burn-in is a method of 'elimination' of initial failures (infant mortality). In the conventional burn-in procedures, to burn-in a component or a system means to subject it to a fixed time period of simulated use prior to actual operation. Then those which fail during the burn-in procedure are scrapped and only those which survived the burn-in procedure are considered to be of satisfactory quality. Thus, in this case, the only information used for the elimination procedure is the lifetime of the corresponding item. In this paper we consider a new burn-in procedure which additionally employs a dependent covariate process in the elimination procedure. Through the comparison with the conventional burn-in procedure, we show that the new burn-in procedure is preferable under commonly satisfied conditions. The problem of determining the optimal burn-in parameters is also considered and the properties of the optimal parameters are derived. A numerical example is provided to illustrate the theoretical results obtained in this paper.

Keywords: Burn-in; dependent covariate process; positive quadrant dependent; negative quadrant dependent; degradation phenomena; heterogeneous population; catastrophic failure

2010 Mathematics Subject Classification: Primary 60K10
Secondary 62P30

1. Introduction

Burn-in is a method of 'elimination' of initial failures (infant mortality). Usually, to burn-in a component or a system means to subject it to a fixed time period of simulated use prior to actual operation. That is, before delivery to the customers, the components or systems are operated under operating conditions that approximate at best the working conditions in field operation. Then those which fail during the burn-in procedure will be scrapped or repaired and only those which survived the burn-in procedure will be considered to be of satisfactory quality. These will then be shipped to the customers or put into field operation. Under the assumption of decreasing or bathtub-shaped failure rate functions, various problems of determining optimal burn-in have been intensively studied in the literature (e.g. Mi (1994), Cha (2000), (2001)). Due to the high failure rate in the early stages of a component's life, burn-in has been widely accepted as an effective method of screening out these initial failures. An introduction to this important area of reliability engineering can be found in Jensen and Petersen (1982) and Kuo and Kuo (1983).

Received 16 January 2013; revision received 26 March 2014.
* Postal address: Department of Statistics, Ewha Womans University, Seoul, 120-750, Korea.
Email address: jhcha@ewha.ac.kr
** Postal address: Istituto Motori, Consiglio Nazionale delle Ricerche, Napoli, Italy.
Email address: g.pulcini@im.cnr.it

As burn-in is usually costly, one of the major problems is to determine the duration of this procedure. The best (usually in terms of costs involved) time to stop the burn-in process for a given criterion is called the optimal burn-in time. In the literature, in addition to different reliability performance criteria (see, e.g. Kim and Kuo (2009)), various cost structures have been proposed, and the corresponding problem of finding the optimal burn-in time has been considered (see, e.g. Clarotti and Spizzichino (1990), Mi (1994), Cha (2000), and Cha and Finkelstein (2011)). See Block and Savits (1997) for excellent survey of research in this area.

As described above, in the studies on conventional burn-in procedures, the only information used for the elimination procedure is the corresponding 'lifetime' of the item. That is, if the lifetime is not sufficiently large then the corresponding item is eliminated from the population that will be put into field use. However, if there is an observable evolving covariate process which is *dependent* on the lifetime of the component then this information can additionally be employed for the elimination procedure.

In this paper we consider a new burn-in procedure which employs a dependent covariate process in the elimination procedure with the aim of improving the reliability performance of the items in the population that passes the burn-in. In spite of the practical importance, to the authors' best knowledge, this type of burn-in procedure has not been considered in the literature. We show that the new burn-in procedure is preferable to the conventional burn-in procedure. We will also consider the problem of determining the optimal burn-in parameters and derive the properties of the optimal parameters. A numerical example which illustrates the application of the theoretical results is given.

The paper is organized as follows. In Section 2 the basic probabilistic setup for considering the new burn-in procedure is established. The conditions for justifying the new burn-in procedure are defined and the properties of the covariate process for satisfying such conditions are derived. For this purpose, a new conditional dependence concept between a random variable and a stochastic process is defined in this section. In Section 3 the problem of determining the optimal joint burn-in parameters is considered and the properties of the optimal burn-in parameters are derived. In Section 4, assuming the gamma process as the corresponding covariate process, the detailed theoretical results are described and an illustrative example is given. Finally, in Section 5, some meaningful remarks are given.

2. Dependent covariate process

In order to employ the covariate process to the burn-in decision problem, there should be a certain type of dependency between the lifetime and the covariate process. In this section, in order to further our discussions on the new burn-in procedure, a probabilistic model for stochastic dependence between the lifetime and the covariate process will be built.

2.1. Dependence concept

We start our discussion by introducing a dependence concept. Let X and Y be two dependent random variables. The most basic and weakest condition for the positive (negative) dependence between X and Y is that $\operatorname{cov}(X, Y) \geq (\leq)\, 0$. A stronger definition for the dependence between X and Y is the 'positive and negative quadrant dependencies' defined as follows.

Definition 1. (*Lehmann (1966). Positive and negative quadrant dependencies (PQD, NQD).*) Two random variables X and Y are PQD if the following inequality holds:

$$\mathbb{P}\{X > x, Y > y\} \geq \mathbb{P}\{X > x\}\mathbb{P}\{Y > y\} \quad \text{for all } x \text{ and } y. \tag{1}$$

If (1) holds with the inequality sign reversed, then X and Y are NQD.

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

Note that the condition for the negative quadrant dependency can equivalently be expressed as follows:

$$\begin{aligned}&\mathbb{P}\{X > x, Y > y\} \leq \mathbb{P}\{X > x\}\mathbb{P}\{Y > y\}\\ &\qquad \Longleftrightarrow \quad \mathbb{P}\{X > x, Y \leq y\} \geq \mathbb{P}\{X > x\}\mathbb{P}\{Y \leq y\}.\end{aligned} \tag{2}$$

It is notable that PQD (NQD) is shown to be a stronger notion of dependence than the positive (negative) covariance, but weaker than the 'association' which is a main concept of positive dependence discussed in Barlow and Proschan (1981). See also Nelsen (1992) and Shaked and Spizzichino (1998) for these concepts and other forms of positive (negative) dependence.

We will now discuss a new dependence concept which will be employed in our further discussions on the new burn-in procedure. Let T be the lifetime of the item randomly selected from the population and $\{W(t),\ t \geq 0\}$ be a 'dependent' covariate process. Here, the type of dependency will be precisely defined later in this subsection. In practice, the covariate processes are most often monotonically increasing. For example, the 'wear', 'erosion', 'corrosion', or 'degradation' processes of an item are monotonically increasing covariate processes. Therefore, throughout this paper, we will basically assume that the covariate process $\{W(t),\ t \geq 0\}$ is monotonically increasing. However, the discussions can be straightforwardly modified for the case when the covariate process is monotonically decreasing. When $\{W(t),\ t \geq 0\}$ is monotonically increasing, the following new burn-in procedure will be considered.

Joint burn-in procedure for an increasing covariate. An item randomly chosen from the population is operated for time $b > 0$, and if the item fails then it is discarded. Furthermore, depending on the observed value of the covariate at time b, $W(b)$, the item is eliminated or it is put into field operation as follows:

(i) if $W(b) \leq w$ then the item is put into field operation,

(ii) if $W(b) > w$ then the item is eliminated,

where w is a fixed constant.

Note that the above new burn-in procedure is composed of two stages: (stage 1) ordinary time burn-in; (stage 2) elimination based on the value of the covariate at time b. Then, in order to justify and employ the above joint burn-in procedure, the second elimination procedure (stage 2) should further improve the quality of the population that has passed stage 1. Now, to assess this, we have to define a conditional version of the NQD in (2).

Definition 2. (*Conditional negative quadrant dependency (CNQD) with respect to $T > b$.*) The lifetime T and the covariate process $\{W(t),\ t \geq 0\}$ are NQD on condition that $\{T > b\}$ if the inequality

$$\begin{aligned}&\mathbb{P}\{T > b + t, W(b) \leq w \mid T > b\}\\ &\qquad \geq \mathbb{P}\{T > b + t \mid T > b\}\mathbb{P}\{W(b) \leq w \mid T > b\} \quad \text{for all } t \text{ and } w,\end{aligned} \tag{3}$$

holds for all fixed b.

Intuitively, (3) can be interpreted as follows. Given that the item has survived the interval $[0, b]$, the 'remaining lifetime' and the covariate at time b are NQD. That is, the shorter the covariate at time b, the longer the remaining lifetime. This condition can also be practically justified, as the lower the degradation of an item at a time point generally implies the longer remaining lifetime in practice.

Let T_b be the residual lifetime of an item that has passed the usual time burn-in (stage 1) and $T_{b,w}$ be that of an item that has passed the joint burn-in procedure defined above. Rearranging (3), it obviously holds that

$$\mathbb{P}\{T > b + t \mid T > b, W(b) \leq w\} \geq \mathbb{P}\{T > b + t \mid T > b\} \quad \text{for all } t \text{ and } w,$$

which then implies that $T_{b,w} \geq_{\text{st}} T_b$ for all fixed b and w, where '$\geq_{\text{st}}$' represents the usual stochastic order between two random variables (Shaked and Shanthikumar (2007)). Therefore, (3) implies the following important property: for any fixed burn-in time b and elimination level w, the joint burn-in outperforms the usual time burn-in in the sense that the joint burn-in further improves the survival probability. Therefore, if the lifetime T and the increasing covariate process $\{W(t),\ t \geq 0\}$ satisfy (3), then it is reasonable to perform the joint burn-in procedure.

In the above discussion, we assumed that the covariate process is monotonically increasing. However, in practice, the covariate can be any monotonically decreasing quality measure that can represent the state of the item's performance. For instance, the accuracy of performing the requested jobs and certain quality measures (e.g. the efficiency of an electronic device) which are randomly decreasing with time are several practical examples of such covariate processes. We denote such a monotonically 'decreasing covariate process' by $\{V(t),\ t \geq 0\}$. In such a case, the larger $V(t)$ implies the better performance of the item. Thus, in this case, the joint burn-in procedure defined above should be modified by replacing the conditions '$W(b) \leq w$' and '$W(b) > w$' with '$V(b) \geq v$' and '$V(b) < w$', respectively. Now, letting $W(t) \equiv -V(t)$ and $w \equiv -v$, the covariate $W(t) \equiv -V(t)$ is increasing and the condition of CNQD (with respect to $T > b$) in (3) can be written as

$$\begin{aligned}\mathbb{P}\{T > b + t, V(b) \geq v \mid T > b\}& \\ \geq \mathbb{P}\{T > b + t \mid T > b\}\mathbb{P}\{V(b) \geq v \mid T > b\}& \quad \text{for all } t \text{ and } v.\end{aligned}$$

In this case, the lifetime T and the covariate process $\{V(t),\ t \geq 0\}$ can be defined as PQDt on condition that $\{T > b\}$ (CPQD with respect to $T > b$).

The discussions in the following will be focused only on the increasing covariate process $\{W(t),\ t \geq 0\}$. However, as shown above, they can be straightforwardly modified for the case with decreasing covariate process $\{V(t),\ t \geq 0\}$.

2.2. Dependence structure and basic property

In this subsection, for our further discussions on optimal burn-in, the composition of population and a more detailed dependence structure will be formulated. Burn-in has been widely accepted as an effective method of screening out the initial failures *due to the large failure rate in the early stages of a component's life*. Thus, the 'sufficient condition' for employing the burn-in is the initially decreasing failure rate. An important question arises: why does the failure rate initially decrease? It is observed that a population of the manufactured items is often composed of two subpopulations: the subpopulation with normal lifetimes (main distribution) and the subpopulation with relatively shorter lifetimes ('freak' distribution). In practice, items belonging to the 'freak distribution' can be produced along with the items of the main distribution, due to, for example, defective resources and components, human errors, unstable production environment caused by uncontrolled significant quality factors, etc. (see Jensen and Petersen (1982) and Kececioglu and Sun (1997)). In this case, the freak distribution generally exhibits a greater failure rate than the main distribution, which results in a mixture of stochastically ordered subpopulations (see Cha and Finkelstein (2011), (2012)). As stated in Badía *et al*. (2003) and Finkelstein (2008), the mixture of ordered failure rates is the main cause of the decreasing population failure rate. From this point of view, in order to consider the application of burn-in, it would be natural to assume that the whole population is the mixture of two 'stochastically ordered' subpopulations: the strong subpopulation (i.e. the subpopulation with normal lifetimes) and the weak subpopulation (i.e. the subpopulation with relatively shorter lifetimes).

Furthermore, the dependency between the lifetime and covariate process can be well formulated via the mixture setting described above. Often in practice, the covariate itself can also cause the failure of the item. For example, items can fail due to degradation or wear when the accumulated degradation exceeds the predetermined threshold level $\kappa > 0$. Thus, in the following discussions, we will assume that the item also fails when $W(t) > \kappa$. In this case, if the elimination level w satisfies $w \geq \kappa$, then the joint burn-in time obviously reduces to the ordinary burn-in time.

Therefore, in the following, we assume that there are two causes of failure: (i) catastrophic failure due to shock or normal ageing (cause I); (ii) failure by the 'increasing covariate process' $\{W(t),\ t \geq 0\}$ (such as wear, erosion, and corrosion) (cause II). However, the following discussions can easily be extended to the case when the covariate does not cause the failure of the item and there is only cause I (see Remark 3). In some instances in the following discussions, we will use the terms 'covariate process' and 'degradation process', interchangeably. The real field data for the items with these types of two causes of failure can be observed in, e.g. Huang and Askin (2003), where an electronic device has two kinds of failure mode: solder/Cu pad interface fracture (a catastrophic failure) and light intensity degradation (a degradation failure). See also Bocchetti *et al.* (2009) for a competing risks model with two failure modes: the catastrophic failure due to thermal cracking and the failure by wear.

Denote the time to the failure of an item from the 'strong subpopulation' due to catastrophic failure (cause I) by $T_{\mathrm{C}1}$ and its absolutely continuous cumulative distribution function (CDF), the probability density function (PDF), and the failure rate function by $F_{\mathrm{C}1}(t)$, $f_{\mathrm{C}1}(t)$ and $\lambda_{\mathrm{C}1}(t)$, respectively. Similarly, the time to the failure of an component from the 'weak subpopulation' due to cause I, the corresponding CDF, PDF, and the failure rate function are denoted by $T_{\mathrm{C}2}$, $F_{\mathrm{C}2}(t)$, $f_{\mathrm{C}2}(t)$, and $\lambda_{\mathrm{C}2}(t)$, accordingly.

Let $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, be the process of the covariate (accumulated degradation) of an item selected from strong and weak subpopulations, respectively. That is, $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, are the corresponding conditional covariate processes of $\{W(t),\ t \geq 0\}$. Denote the time to the failure of a component from the strong subpopulation due to cause II by $T_{\mathrm{D}1}$ and that of a component from the weak subpopulation by $T_{\mathrm{D}2}$. The corresponding survival function (SF), CDF, PDF, and the failure rate function are denoted by $\bar{F}_{\mathrm{D}i}(t)$, $F_{\mathrm{D}i}(t)$, $f_{\mathrm{D}i}(t)$, and $\lambda_{\mathrm{D}i}(t)$, $i = 1, 2$, respectively. Obviously,

$$\bar{F}_{\mathrm{D}i}(t) = \mathbb{P}\{T_{\mathrm{D}i} > t\} = \mathbb{P}\{W_i(t) \leq \kappa\}, \qquad i = 1, 2.$$

We assume that the covariate processes $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, possess *independent* and *possibly nonstationary increments*.

The initial ($t = 0$) composition of our mixed population is as follows: the proportion of the strong items is π, whereas the proportion of the weak items is $1 - \pi$, which means that the distribution of the discrete *frailty* random variable Z with realizations 1 and 2 in this case is

$$\pi(z) = \begin{cases} \pi, & z = 1, \\ 1 - \pi, & z = 2, \end{cases}$$

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

and 1 and 2 correspond to the strong and the weak subpopulations, respectively. From the above setting, the mixture (population) survival function is given by

$$\bar{F}_{\mathrm{m}}(t) = \pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t),$$

where $\bar{F}_1(t) = \bar{F}_{\mathrm{C}1}(t)\bar{F}_{\mathrm{D}1}(t)$ and $\bar{F}_2(t) = \bar{F}_{\mathrm{C}2}(t)\bar{F}_{\mathrm{D}2}(t)$. Therefore, the two causes of failures are 'conditionally' independent. However, clearly, the two causes of failures are unconditionally dependent. Under the above assumptions, the PDF of the mixed population is given by

$$f_{\mathrm{m}}(t) = \pi f_1(t) + (1-\pi) f_2(t),$$

where $f_i(t) = f_{\mathrm{C}i}(t)\bar{F}_{\mathrm{D}i}(t) + \bar{F}_{\mathrm{C}i}(t) f_{\mathrm{D}i}(t)$, $i = 1, 2$, and the failure rate is

$$\lambda_{\mathrm{m}}(t) = \rho(t)\lambda_1(t) + (1-\rho(t))\lambda_2(t),$$

where $\lambda_i(t) = \lambda_{\mathrm{C}i}(t) + \lambda_{\mathrm{D}i}(t)$, $i = 1, 2$, and $\rho(t) = [\pi \bar{F}_1(t)/\pi \bar{F}_1(t) + (1-\pi)\bar{F}_2(t)]$. See, e.g. Finkelstein (2008) and Cha and Finkelstein (2012) for more general discussions on the mixture populations.

In the previous subsection, it was mentioned that the reasonable condition for justifying the consideration of joint burn-in is given by (3) (CNQD) and, at the same time, the subpopulations should be stochastically ordered as mentioned in the first part of this subsection. Then Theorem 1 below provides a sufficient condition of the covariate process $\{W(t),\ t \ge 0\}$ for satisfying all these properties. For our discussions, we need some basic definitions and preliminary lemmas, which can be found in Shaked and Shanthikumar (2007).

Definition 3. Let X and Y be two nonnegative continuous random variables with the corresponding CDFs $F_X(t)$ and $F_Y(t)$, SFs $\bar{F}_X(t)$ and $\bar{F}_Y(t)$, PDFs $f_X(t)$ and $f_Y(t)$, and the failure rate functions $\lambda_X(t)$ and $\lambda_Y(t)$, respectively.

(i) If $f_X(t)/f_Y(t)$ decreases over the union of the supports of X and Y (here $a/0$ is taken to be equal to ∞ whenever $a > 0$) then X is smaller than Y in the likelihood ratio order, denoted by $X \le_{\mathrm{lr}} Y$.

(ii) If $\lambda_X(t) \ge \lambda_Y(t)$ for all $t \ge 0$ then X is smaller than Y in the failure rate order, denoted by $X \le_{\mathrm{fr}} Y$.

(iii) If $F_Y(t) \le F_X(t)$ for all $t \ge 0$ then X is smaller than Y in the usual stochastic order, denoted by $X \le_{\mathrm{st}} Y$.

Lemma 1. (i) *If X and Y are two nonnegative continuous random variables such that $X \le_{\mathrm{lr}} Y$ then $X \le_{\mathrm{fr}} Y$.*

(ii) *If X and Y are two nonnegative continuous random variables such that $X \le_{\mathrm{fr}} Y$ then $X \le_{\mathrm{st}} Y$.*

(iii) *If $X \le_{\mathrm{st}} Y$ and $g(\cdot)$ is any increasing [decreasing] function then $g(X) \le_{\mathrm{st}} [\ge_{\mathrm{st}}] g(Y)$.*

As mentioned before, natural conditions for considering the joint burn-in are that the subpopulations should be stochastically ordered and (3) (CNQD) should be satisfied. Clearly, the basis for these conditions could be verbally stated as 'the weak items fail (with respect to cause I) earlier than the strong ones' and 'the weak items deteriorate faster than the strong ones (with respect to cause II)'. This basis is precisely stipulated in the following theorem as assumptions.

Theorem 1. *Suppose that $\lambda_{C1}(t) \le \lambda_{C2}(t)$ for all $t \ge 0$, $W_1(t+s) - W_1(t) \le_{st} W_2(t+s) - W_2(t)$ for all $t, s \ge 0$, and $W_1(t) \le_{lr} W_2(t)$ for all $t \ge 0$. Then*

(i) *the subpopulations are stochastically ordered in the sense of failure rate ordering:*

$$\lambda_{C1}(t) + \lambda_{D1}(t) \le \lambda_{C2}(t) + \lambda_{D2}(t) \quad \textit{for all } t \ge 0;$$

(ii) $\{W(t),\ t \ge 0\}$ *satisfies the condition CNQD in (3):*

$$\begin{aligned}&\mathbb{P}\{T > b+t, W(b) \le w \mid T > b\}\\ &\qquad \ge \mathbb{P}\{T > b+t \mid T > b\}\mathbb{P}\{W(b) \le w \mid T > b\} \quad \textit{for all } t \textit{ and } w\end{aligned}$$

for all fixed b.

Proof. Observe that

$$\lambda_{Di}(t) = \lim_{\Delta t \to 0} \frac{\mathbb{P}\{t < T_{Di} \le t + \Delta t \mid T_{Di} > t\}}{\Delta t}, \qquad i = 1, 2.$$

For any fixed $\Delta t > 0$,

$$\begin{aligned}&\mathbb{P}\{t < T_{Di} \le t + \Delta t \mid T_{Di} > t\}\\ &\quad = \mathbb{P}\{W_i(t+\Delta t) > \kappa \mid W_i(t) \le \kappa\}\\ &\quad = \int_0^{\kappa} \mathbb{P}\{W_i(t+\Delta t) > \kappa \mid W_i(t) = y\} f_{W_i(t) \mid W_i(t) \le \kappa}(y)\,\mathrm{d}y\\ &\quad = \int_0^{\kappa} \mathbb{P}\{W_i(t+\Delta t) - W_i(t) > \kappa - y\} f_{W_i(t) \mid W_i(t) \le \kappa}(y)\,\mathrm{d}y\\ &\quad = \int_0^{\kappa} g_i(y) f_{W_i(t) \mid W_i(t) \le \kappa}(y)\,\mathrm{d}y,\end{aligned}$$

where $g_i(y) \equiv \mathbb{P}\{W_i(t+\Delta t) - W_i(t) > \kappa - y\}$, $i = 1, 2$, which is an increasing function of y. On the other hand,

$$f_{W_i(t) \mid W_i(t) \le \kappa}(y) = \frac{f_i(y;t)}{\mathbb{P}\{W_i(t) \le \kappa\}}, \qquad i = 1, 2,$$

and

$$\frac{f_{W_1(t) \mid W_1(t) \le \kappa}(y)}{f_{W_2(t) \mid W_2(t) \le \kappa}(y)} = \frac{\mathbb{P}\{W_2(t) \le \kappa\}}{\mathbb{P}\{W_1(t) \le \kappa\}} \frac{f_1(y;t)}{f_2(y;t)}$$

is decreasing in y due to the assumption $W_1(t) \le_{lr} W_2(t)$, where $f_i(y;t)$ is the PDF of $W_i(t)$, $i = 1, 2$. Thus, we can conclude that $\{W_1(t) \mid W_1(t) \le \kappa\} \le_{lr} \{W_2(t) \mid W_2(t) \le \kappa\}$.

Now, for any fixed $\Delta t > 0$, we have

$$\begin{aligned}\mathbb{P}\{t < T_{\mathrm{D1}} \le t + \Delta t \mid T_{\mathrm{D1}} > t\} &= \int_0^{\kappa} g_1(y) f_{W_1(t) \mid W_1(t) \le \kappa}(y)\,\mathrm{d}y \\ &\le \int_0^{\kappa} g_2(y) f_{W_1(t) \mid W_1(t) \le \kappa}(y)\,\mathrm{d}y \\ &\le \int_0^{\kappa} g_2(y) f_{W_2(t) \mid W_2(t) \le \kappa}(y)\,\mathrm{d}y \\ &= \mathbb{P}\{t < T_{\mathrm{D2}} \le t + \Delta t \mid T_{\mathrm{D2}} > t\},\end{aligned}$$

where the first inequality holds due to the fact that

$$W_1(t+s) - W_1(t) \le_{\mathrm{st}} W_2(t+s) - W_2(t) \quad \text{for all } t, s \ge 0,$$

and the second inequality holds due to the fact that

$$\{W_1(t) \mid W_1(t) \le \tau\} \le_{\mathrm{lr}} \{W_2(t) \mid W_2(t) \le \tau\} \quad \text{for all } t \ge 0,$$

and Lemma 1(iii). Therefore, we can conclude that $\lambda_{\mathrm{D1}}(t) \le \lambda_{\mathrm{D2}}(t)$ for all $t \ge 0$, and (i) is proved.

Theorem 1(ii) will now be proved. For $w \in (-\infty, 0] \cup [\kappa, \infty)$, it obviously holds that

$$\mathbb{P}\{T > b+t, W(b) \le w \mid T > b\} = \mathbb{P}\{T > b+t \mid T > b\}\mathbb{P}\{W(b) \le w \mid T > b\} \quad \text{for all } t.$$

For $w \in (0, \kappa)$, (3) is equivalent to

$$\mathbb{P}\{T > b+t \mid T > b, W(b) \le w\} \ge \mathbb{P}\{T > b+t \mid T > b\}, \tag{4}$$

where

$$\begin{aligned}&\mathbb{P}\{T > b+t \mid T > b, W(b) \le w\} \\ &\quad = \sum_{i=1}^{2} \mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\}\mathbb{P}\{Z = i \mid T > b, W(b) \le w\},\end{aligned} \tag{5}$$

and

$$\mathbb{P}\{T > b+t \mid T > b\} = \sum_{i=1}^{2} \mathbb{P}\{T > b+t \mid Z = i, T > b\}\mathbb{P}\{Z = i \mid T > b\}.$$

Thus, we can see that both $\mathbb{P}\{T > b+t \mid T > b, W(b) \le w\}$ and $\mathbb{P}\{T > b+t \mid T > b\}$ are given by the weighted averages of $\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\}$, $i = 1, 2$, and $\mathbb{P}\{T > b+t \mid Z = i, T > b\}$, $i = 1, 2$, respectively. To show the inequality in (4), we now analyze the corresponding elements and weights (proportions) of these weighted averages. Let us now assume that the following inequalities hold:

(a) $\mathbb{P}\{T > b+t \mid Z = 1, T > b\} \ge \mathbb{P}\{T > b+t \mid Z = 2, T > b\}$,

(b) $\mathbb{P}\{T > b+t \mid Z = 1, T > b, W(b) \le w\} \ge \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}$,

(c) $\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\} \ge \mathbb{P}\{T > b+t \mid Z = i, T > b\}$, $i = 1, 2$,

(d) $\mathbb{P}\{Z = 1 \mid T > b, W(b) \le w\} \ge \mathbb{P}\{Z = 1 \mid T > b\}$, (these inequalities will be proved successively). If (a)–(d) are true then the inequality in (4) can be shown by considering the two exclusive cases separately: case I, when $\mathbb{P}\{T > b + t \mid Z = 1, T > b\} \le \mathbb{P}\{T > b + t \mid Z = 2, T > b, W(b) \le w\}$, and case II, when $\mathbb{P}\{T > b+t \mid Z = 1, T > b\} > \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}$.

In case I, from (a) and (b) we have

$$\begin{aligned}\mathbb{P}\{T > b+t \mid Z = 2, T > b\} &\le \mathbb{P}\{T > b+t \mid Z = 1, T > b\}\\ &\le \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}\\ &\le \mathbb{P}\{T > b+t \mid Z = 1, T > b, W(b) \le w\}.\end{aligned}$$

From these inequalities, we can see that the elements in the weighted average of $\mathbb{P}\{T > b + t \mid T > b, W(b) \le w\}$ are greater than (or equal to) those in the weighted average of $\mathbb{P}\{T > b+t \mid T > b\}$ and, therefore, the inequality in (4) obviously holds.

In case II, from (c),

$$\begin{aligned}\mathbb{P}\{T > b+t \mid Z = 2, T > b\} &\le \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}\\ &< \mathbb{P}\{T > b+t \mid Z = 1, T > b\}\\ &\le \mathbb{P}\{T > b+t \mid Z = 1, T > b, W(b) \le w\}.\end{aligned}$$

Also, we have $\mathbb{P}\{Z = 1 \mid T > b, W(b) \le w\} \ge \mathbb{P}\{Z = 1 \mid T > b\}$ (see (d)). Therefore,

$$\begin{aligned}&\mathbb{P}\{T > b+t \mid T > b, W(b) \le w\}\\ &\quad= \mathbb{P}\{T > b+t \mid Z = 1, T > b, W(b) \le w\}\mathbb{P}\{Z = 1 \mid T > b, W(b) \le w\}\\ &\qquad+ \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}\mathbb{P}\{Z = 2 \mid T > b, W(b) \le w\}\\ &\quad\ge \mathbb{P}\{T > b+t \mid Z = 1, T > b, W(b) \le w\}\mathbb{P}\{Z = 1 \mid T > b\}\\ &\qquad+ \mathbb{P}\{T > b+t \mid Z = 2, T > b, W(b) \le w\}\mathbb{P}\{Z = 2 \mid T > b\}\\ &\quad\ge \mathbb{P}\{T > b+t \mid Z = 1, T > b\}\mathbb{P}\{Z = 1 \mid T > b\}\\ &\qquad+ \mathbb{P}\{T > b+t \mid Z = 2, T > b\}\mathbb{P}\{Z = 2 \mid T > b\}\\ &\quad= \mathbb{P}\{T > b+t \mid T > b\}.\end{aligned}$$

Therefore, the inequality in (4) also holds in this case.

For completeness of the proof, it is now sufficient to show (a)–(d).

(a) In the result (i) of this theorem, it already has been shown that

$$\lambda_{C1}(t) + \lambda_{D1}(t) \le \lambda_{C2}(t) + \lambda_{D2}(t) \quad \text{for all } t \ge 0.$$

Then, the inequality $\mathbb{P}\{T > b+t \mid Z = 1, T > b\} \ge \mathbb{P}\{T > b+t \mid Z = 2, T > b\}$ directly follows.

(b) If we set $\kappa \equiv w$ then

$$\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\} = \mathbb{P}\{T > b+t \mid Z = i, T > b\}, \qquad i = 1, 2.$$

However, as the result (i) of this theorem holds for any value of $\kappa > 0$, we have the desired result.

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

(c) Observe that

$$
\begin{aligned}
&\mathbb{P}\{T > b+t \mid Z = i, T > b\}\\
&\quad = \exp\biggl(-\int_0^t \lambda_{Ci}(b+u)\,\mathrm{d}u\biggr)\mathbb{P}\{W_i(b+t) < \kappa \mid W_i(b) \le \kappa\}\\
&\quad = \exp\biggl(-\int_0^t \lambda_{Ci}(b+u)\,\mathrm{d}u\biggr)\int_0^\kappa \mathbb{P}\{W_i(b+t) - W_i(b) < \kappa - y\} f_{\{W_i(b)\mid W_i(b)\le\kappa\}}(y)\,\mathrm{d}y,
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\}\\
&\quad = \exp\biggl(-\int_0^t \lambda_{Ci}(b+u)\,\mathrm{d}u\biggr)\mathbb{P}\{W_i(b+t) < \kappa \mid W_i(b) \le \kappa, W_i(b) \le w\}\\
&\quad = \exp\biggl(-\int_0^t \lambda_{Ci}(t+u)\,\mathrm{d}u\biggr)\int_0^w \mathbb{P}\{W_i(b+t) - W_i(b) < \kappa - y\} f_{W_i(b)\mid W_i(b)\le w}(y)\,\mathrm{d}y,
\end{aligned}
\tag{6}
$$

where $\mathbb{P}\{W_i(b+t) - W_i(b) < \kappa - y\}$ is a decreasing function of y. On the other hand,

$$
f_{W_i(b)\mid W_i(b)\le\kappa}(y) = \frac{f_i(y;b)}{\mathbb{P}\{W_i(b)\le\kappa\}}, \qquad i = 1, 2
$$

and

$$
\frac{f_{W_i(b)\mid W_i(b)\le w}(y)}{f_{W_i(b)\mid W_i(b)\le\kappa}(y)} = \frac{\mathbb{P}\{W_i(b)\le\kappa\}}{\mathbb{P}\{W_i(b)\le w\}}\,\frac{f_i(y;b)\,\mathbf{1}_{\{y\le w\}}}{f_i(y;b)\,\mathbf{1}_{\{y\le\kappa\}}}
$$

is given by $\mathbb{P}\{W_i(b)\le\kappa\}/\mathbb{P}\{W_i(b)\le w\}$ for $y \le w$ and is 0 for $w < y \le \kappa$. Thus, the ratio of the conditional PDFs of $(W_i(b) \mid W_i(b) \le w)$ and $(W_i(b) \mid W_i(b) \le \kappa)$ is decreasing, which implies that

$$
(W_i(b) \mid W_i(b) \le w) \le_{\mathrm{lr}} (W_i(b) \mid W_i(b) \le \kappa),
$$

and, by (ii) and (iii) of Lemma 1,

$$
\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \le w\} \ge \mathbb{P}\{T > b+t \mid Z = i, T > b\}.
$$

(d) Note that, as mentioned before, the joint burn-in is composed of two stages: (stage 1) ordinary time burn-in; (stage 2) elimination based on the wear amount. In stage 2, the probabilities of survival (i.e. successfully passing stage 2) are given by $\mathbb{P}\{W_1(b)\le w\}/\mathbb{P}\{W_1(b)\le\kappa\}$ and $\mathbb{P}\{W_2(b)\le w\}/\mathbb{P}\{W_2(b)\le\kappa\}$, for the strong and weak subpopulations, respectively. Thus, if

$$
\frac{\mathbb{P}\{W_1(b)\le w\}}{\mathbb{P}\{W_1(b)\le\kappa\}} \ge \frac{\mathbb{P}\{W_2(b)\le w\}}{\mathbb{P}\{W_2(b)\le\kappa\}}
$$

then the proportion of strong items in the population is higher after a joint burn-in with burn-in parameters (b, w) than that after the ordinary time burn-in of length b. However, from the proof of result (i), we see that if $W_1(t) \le_{\mathrm{lr}} W_2(t)$ for all $t \ge 0$ then $(W_1(t) \mid W_1(t) \le \kappa) \le_{\mathrm{lr}} (W_2(t) \mid W_2(t) \le \kappa)$. Therefore, we have

$$
\frac{\mathbb{P}\{W_1(b)\le w\}}{\mathbb{P}\{W_1(b)\le\kappa\}} \ge \frac{\mathbb{P}\{W_2(b)\le w\}}{\mathbb{P}\{W_2(b)\le\kappa\}},
$$

which implies that $\mathbb{P}\{Z = 1 \mid T > b, W(b) \le w\} \ge \mathbb{P}\{Z = 1 \mid T > b\}$.

Remark 1. Suppose that the increments of the covariate process are absolutely continuous random variables. Then $\mathbb{P}\{W_i(b+t) - W_i(b) < \kappa - y\}$ is a strictly decreasing function of y. In this case, for $w \in (0, \kappa)$, the strict inequality

$$\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \leq w\} > \mathbb{P}\{T > b+t \mid Z = i, T > b\}, \qquad i = 1, 2,$$

holds, and we have the following strict inequality:

$$\mathbb{P}\{T > b+t \mid T > b, W(b) \leq w\} > \mathbb{P}\{T > b+t \mid T > b\}.$$

Remark 2. If $W_1(t+s) - W_1(t) \leq_{\mathrm{lr}} W_2(t+s) - W_2(t)$, for all $t, s \geq 0$, then the conditions on $\{W(t),\ t \geq 0\}$ in Theorem 1 are satisfied. Therefore, the conditions in Theorem 1 are weaker than the condition that '$W_1(t+s) - W_1(t) \leq_{\mathrm{lr}} W_2(t+s) - W_2(t)$, for all $t, s \geq 0$'. Note that $X \leq_{\mathrm{lr}} Y$ implies the stochastic orders of residual random variables, i.e. $(X - t \mid X > t) \leq_{\mathrm{hr}} (Y - t \mid Y > t)$ and $(X - t \mid X > t) \leq_{\mathrm{st}} (Y - t \mid Y > t)$, whereas $X \leq_{\mathrm{st}} Y$ does not necessarily. Therefore, the class of degradation processes satisfying the conditions in Theorem 1 is larger than that of degradation processes satisfying $W_1(t+s) - W_1(t) \leq_{\mathrm{lr}} W_2(t+s) - W_2(t)$, for all $t, s \geq 0$.

Based on our discussions in this section, we can see that the conditions stated in Theorem 1 are reasonable assumptions for justifying the application of the joint burn-in.

Remark 3. When the covariate itself does not cause the failure of the item (i.e. $\kappa = \infty$ and there is no cause II), we have

$$\begin{aligned}\mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \leq w\} &= \exp\biggl(-\int_0^t \lambda_{\mathrm{C}i}(b+u)\,\mathrm{d}u\biggr)\\ &= \mathbb{P}\{T > b+t \mid Z = i, T > b\}\end{aligned}$$

and, thus, (a)–(d) in the proof of Theorem 1 still hold. Accordingly, Theorem 1 still holds in this case. Thus, even when there is no cause II, the elimination based on the covariate process further improves the survival probability of the item, and the joint burn-in procedure is still justified.

3. Optimal joint burn-in procedure

In this section we will now consider the problem of finding the optimal joint burn-in parameters. As burn-in is usually a costly procedure, most often, the optimal burn-in is determined to minimize the corresponding cost function. We will now adopt a cost structure to determine the optimal burn-in, which is similar to those in Mi (1994) and Cha (2000).

Before discussing the optimization problem, we need to describe the population distribution after the joint burn-in. From (5), it can be seen that the mixture survival function of the burned-in items, that is the (conditional) probability for an item that has survived the joint burn-in to survive a further time t, is given by

$$\begin{aligned}&\bar{F}_{\mathrm{m}}(t \mid b, w)\\ &\quad \equiv \mathbb{P}\{T > b+t \mid T > b, W(b) \leq w\}\\ &\quad = \sum_{i=1}^{2} \mathbb{P}\{T > b+t \mid Z = i, T > b, W(b) \leq w\}\mathbb{P}\{Z = i \mid T > b, W(b) \leq w\},\end{aligned}$$

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

where, from (6),

$$\begin{aligned}
&\mathbb{P}\{T > b+t \mid Z=i, T>b, W(b)\le w\} \\
&\quad = \exp\biggl(-\int_0^t \lambda_{\mathrm{C}i}(t+u)\,\mathrm{d}u\biggr)\int_0^w \mathbb{P}\{W_i(b+t)-W_i(b) < \kappa - y\} \\
&\qquad \times f_{W_i(b)\mid W_i(b)\le w}(y)\,\mathrm{d}y, \qquad i=1,2,
\end{aligned}$$

and the weights

$$\begin{aligned}
&\mathbb{P}\{Z=1 \mid T>b, W(b)\le w\} \\
&\quad = \frac{\pi\bar{F}_{\mathrm{C}1}(b)\mathbb{P}\{W_1(b)\le w\}}{\pi\bar{F}_{\mathrm{C}1}(b)\mathbb{P}\{W_1(b)\le w\}+(1-\pi)\bar{F}_{\mathrm{C}2}(b)\mathbb{P}\{W_2(b)\le w\}},
\end{aligned}$$

$$\begin{aligned}
&\mathbb{P}\{Z=2 \mid T>b, W(b)\le w\} \\
&\quad = \frac{(1-\pi)\bar{F}_{\mathrm{C}2}(b)\mathbb{P}\{W_2(b)\le w\}}{\pi\bar{F}_{\mathrm{C}1}(b)\mathbb{P}\{W_1(b)\le w\}+(1-\pi)\bar{F}_{\mathrm{C}2}(b)\mathbb{P}\{W_2(b)\le w\}}
\end{aligned}$$

are, respectively, the proportion of strong and weak items in the population of items which survived the joint burn-in with parameters (b, w).

For a notational convenience, we will use the following notation in the following discussions:

$$\bar{F}_{\mathrm{C}i}(t \mid b) \equiv \exp\biggl(-\int_0^t \lambda_{\mathrm{C}i}(t+u)\,\mathrm{d}u\biggr), \qquad i=1,2,$$

$$\bar{F}_{\mathrm{D}i}(t \mid b, w) \equiv \int_0^w \mathbb{P}\{W_i(b+t)-W_i(b) < \kappa - y\} f_{W_i(b)\mid W_i(b)\le w}(y)\,\mathrm{d}y, \qquad i=1,2,$$

$$\pi(b, w) \equiv \mathbb{P}\{Z=1 \mid T>b, W(b)\le w\} = \frac{\pi\bar{F}_{\mathrm{C}1}(b)\mathbb{P}\{W_1(b)\le w\}}{p(b, w)},$$

where

$$p(b, w) \equiv \pi\bar{F}_{\mathrm{C}1}(b)\mathbb{P}\{W_1(b)\le w)\}+(1-\pi)\bar{F}_{\mathrm{C}2}(b)\mathbb{P}\{W_2(b)\le w\},$$

which is the probability for a generic item of the population to survive the joint burn-in.

Let τ be the mission time during the field operation. A component is chosen at random from our initial population and the joint burn-in procedure with parameters (b, w) is applied. If it survives the joint burn-in procedure then it is put into the field operation, otherwise the corresponding component is discarded and another new one is chosen from the population, and so on. This procedure is repeated until the first surviving component is obtained. Let c_{sr} be the shop replacement cost (actually, it is the cost of a new item) and c_0 be the cost for conducting the joint burn-in procedure for one time. Let $c_1(b, w)$, as a function of (b, w), be the expected cost for eventually obtaining a component which has survived the joint burn-in. Then

$$c_1(b, w) = \frac{c_0 + c_{sr}[1-p(b, w)]}{p(b, w)},$$

where $1/p(b, w)$ is the total number of trials until the first 'success'. Assume that if a mission (of length τ) is successful (in field operation) then the gain $K > 0$ is 'earned'; otherwise a penalty $C > 0$ is imposed. Then the expected cost during the field operation is

$$c_2(b, w) = -K\bar{F}_{\mathrm{m}}(\tau \mid b, w) + CF_{\mathrm{m}}(\tau \mid b, w) = -(K+C)\bar{F}_{\mathrm{m}}(\tau \mid b, w) + C$$

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

and the total expected cost $c(b, w)$ is

$$c(b, w) = c_1(b, w) + c_2(b, w) = \frac{c_0 + c_{sr}[1 - p(b, w)]}{p(b, w)} - (K + C)\bar{F}_{\mathrm{m}}(\tau \mid b, w) + C. \quad (7)$$

Note that, for any $w \geq \kappa$, the joint burn-in with burn-in parameters (b, w) corresponds to the ordinary time burn-in with burn-in time b. Furthermore, $c(b, 0) = \infty$ for any $b > 0$. Therefore, the parameter space should be restricted to $\{(b, w) \mid b \geq 0,\ w \in (0, \kappa]\}$. The objective is now to find the joint optimal burn-in parameters (b^*, w^*) that satisfy

$$c(b^*, w^*) = \min_{b \geq 0, w \in (0, \kappa]} c(b, w).$$

Then, in order to find the joint optimal burn-in parameters (b^*, w^*) efficiently, the following procedures will be applied.

(i) At the first stage, we fix the elimination level $w \in (0, \kappa]$ and find the optimal $b^*(w)$ that satisfies

$$c(b^*(w), w) = \min_{b \geq 0} c(b, w). \quad (8)$$

(ii) At the second stage, we search for w^* that satisfies

$$c(b^*(w^*), w^*) = \min_{w \in (0, \kappa]} c(b^*(w), w).$$

Then the joint optimal solution is given by $(b^*(w^*), w^*)$, since the above procedure implies that

$$c(b^*(w^*), w^*) \leq \begin{cases} c(b^*(w), w) & \text{for all } w \in (0, \kappa], \\ c(b, w) & \text{for all } b \geq 0,\ w \in (0, \kappa]. \end{cases}$$

Following the procedure described above, first search for the optimal $b^*(w)$ satisfying (8) for each fixed w. In this case, as the burn-in time has no given finite upper bound, the existence of a finite upper bound for $b^*(w)$ for each fixed w would make the search for $b^*(w)$ substantially more efficient. For the existence of a finite upper bound for $b^*(w)$, compared with the conditions in Theorem 1, we need additional conditions on the degradation process $\{W_1(t),\ t \geq 0\}$ in Theorem 2: $W_1(t_1) \leq_{\mathrm{lr}} W_1(t_2)$, $W_1(t_1 + s) - W_1(t_1) \leq_{\mathrm{st}} W_1(t_2 + s) - W_1(t_2)$, for any $t_2 > t_1$ and $s \geq 0$. These conditions imply that the degradation rate of the component increases as the age of the component increases, which is, practically, a reasonable assumption.

Theorem 2. *Assume that $\lambda_{\mathrm{C}1}(t) \leq \lambda_{\mathrm{C}2}(t)$ for all $t \geq 0$, and that $\lambda_{\mathrm{C}1}(t)$ is strictly increasing with $\lambda_{\mathrm{C}1}(\infty) \equiv \lim_{t \to \infty} \lambda_{\mathrm{C}1}(t) = \infty$. Let $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, be any covariate processes which possess the independent increments property. Suppose that $W_1(t + s) - W_1(t) \leq_{\mathrm{st}} W_2(t + s) - W_2(t)$ for all $t, s \geq 0$, $W_1(t) \leq_{\mathrm{lr}} W_2(t)$ for all $t \geq 0$, and, further, that $W_1(t_1) \leq_{\mathrm{lr}} W_1(t_2)$, $W_1(t_1 + s) - W_1(t_1) \leq_{\mathrm{st}} W_1(t_2 + s) - W_1(t_2)$ for any $t_2 > t_1$ and $s \geq 0$. Then, for each fixed $w \in (0, \kappa]$, the upper bound for $b^*(w)$, which is denoted by $b_{\mathrm{U}}(w)$, is given by the unique solution of the equation:*

$$\psi(b \mid w) \equiv \bar{F}_{\mathrm{C}1}(\tau \mid b)\bar{F}_{\mathrm{D}1}(\tau \mid b, w) - \bar{F}_{\mathrm{C}2}(\tau \mid 0)\bar{F}_{\mathrm{D}2}(\tau \mid 0, w) = 0, \quad (9)$$

where $\psi(b \mid w)$ is strictly decreasing in b with

$$\psi(0 \mid w) > 0 \quad \textit{and} \quad \psi(\infty \mid w) \equiv \lim_{b \to \infty} \psi(b \mid w) < 0.$$

Proof. It will be shown that for a fixed elimination level $w \in (0, \kappa]$, $c(0, w) < c(b, w)$, for $b > b_{\mathrm{U}}(w)$, where $b_{\mathrm{U}}(w)$ is the unique solution of (9). Note that, as $p(b, w)$ is strictly decreasing in $b \geq 0$ for a fixed elimination level $w \in (0, \kappa]$, $c_1(b, w)$ is strictly increasing in $b \geq 0$. Accordingly, it is now sufficient to show that $c_2(0, w) < c_2(b, w)$, for $b > b_{\mathrm{U}}(w)$, or, equivalently, $\bar{F}_{\mathrm{m}}(\tau \mid 0, w) > \bar{F}_{\mathrm{m}}(\tau \mid b, w)$, for $b > b_{\mathrm{U}}(w)$.

Fix $w \in (0, \kappa]$. Observe that $\bar{F}_{\mathrm{m}}(\tau \mid b, w)$ is of the form of the weighted average of $\bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w)$ and $\bar{F}_{\mathrm{C2}}(\tau \mid b)\bar{F}_{\mathrm{D2}}(\tau \mid b, w)$, i.e.

$$\bar{F}_{\mathrm{m}}(\tau \mid b, w) = \pi(b, w)\bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w) + (1 - \pi(b, w))\bar{F}_{\mathrm{C2}}(\tau \mid b)\bar{F}_{\mathrm{D2}}(\tau \mid b, w). \tag{10}$$

Here, the weight $\pi(b, w)$ can be rewritten as

$$\begin{aligned}
&\pi(b, w) \\
&\quad= \frac{\pi \exp(-\int_0^b \lambda_{\mathrm{C1}}(u)\,\mathrm{d}u)\mathbb{P}\{W_1(b) \leq w\}}{p(b, w)} \\
&\quad= \frac{\pi \exp(-\int_0^b \lambda_{\mathrm{C1}}(u)\,\mathrm{d}u)\exp(-\int_0^b \lambda_{\mathrm{D1}}(u; w)\,\mathrm{d}u)}{\pi \exp(-\int_0^b \lambda_{\mathrm{C1}}(u)\,\mathrm{d}u)\exp(-\int_0^b \lambda_{\mathrm{D1}}(u; w)\,\mathrm{d}u) + (1-\pi)\exp(-\int_0^b \lambda_{\mathrm{C2}}(u)\,\mathrm{d}u)\exp(-\int_0^b \lambda_{\mathrm{D2}}(u; w)\,\mathrm{d}u)},
\end{aligned}$$

where $\lambda_{\mathrm{D}i}(u; w)$ is the failure rate of the lifetime $T_{\mathrm{D}i}$ when the threshold level κ is replaced by the elimination level w. As the result (i) of Theorem 1 holds for any positive $\kappa > 0$, we have $\lambda_{\mathrm{D1}}(t; w) \leq \lambda_{\mathrm{D2}}(t; w)$ for all $t \geq 0$. Then

$$\begin{aligned}
\pi(b, w) = \bigg[1 + \frac{1-\pi}{\pi} \exp\bigg(-\int_0^b [\lambda_{\mathrm{C2}}(u) - \lambda_{\mathrm{C1}}(u)]\,\mathrm{d}u\bigg) \\
\times \exp\bigg(-\int_0^b [\lambda_{\mathrm{D2}}(u; w) - \lambda_{\mathrm{D1}}(u; w)]\,\mathrm{d}u\bigg)\bigg]^{-1},
\end{aligned}$$

and, therefore, $\pi(b, w)$ is increasing in b.

Now we look at the corresponding components in the weighted average of (10), say $\bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w)$ and $\bar{F}_{\mathrm{C2}}(\tau \mid b)\bar{F}_{\mathrm{D2}}(\tau \mid b, w)$. Due to the assumption that $\lambda_{\mathrm{C1}}(t)$ is strictly increasing with $\lambda_{\mathrm{C1}}(\infty) \equiv \lim_{t\to\infty} \lambda_{\mathrm{C1}}(t) = \infty$, $\bar{F}_{\mathrm{C1}}(\tau \mid b)$ is strictly decreasing in b, with $\lim_{b\to\infty} \bar{F}_{\mathrm{C1}}(\tau \mid b) = 0$. Furthermore, due to the assumption that

$$W_1(t_1) \leq_{\mathrm{lr}} W_1(t_2), \qquad W_1(t_1 + s) - W_1(t_1) \leq_{\mathrm{st}} W_1(t_2 + s) - W_1(t_2),$$

for any $t_2 > t_1$ and $s \geq 0$, for any $b' < b''$,

$$\begin{aligned}
\bar{F}_{\mathrm{D1}}(\tau \mid b', w) &= \int_0^w \mathbb{P}\{[W_1(b' + \tau) - W_1(b')] \leq \kappa - y\} f_{W_1(b') \mid W_1(b') \leq w}(y)\,\mathrm{d}y \\
&\geq \int_0^w \mathbb{P}\{[W_1(b'' + \tau) - W_1(b'')] \leq \kappa - y\} f_{W_1(b') \mid W_1(b') \leq w}(y)\,\mathrm{d}y \\
&\geq \int_0^w \mathbb{P}\{[W_1(b'' + \tau) - W_1(b'')] \leq \kappa - y\} f_{W_1(b'') \mid W_1(b'') \leq w}(y)\,\mathrm{d}y \\
&= \bar{F}_{\mathrm{D1}}(\tau \mid b'', w).
\end{aligned}$$

Then the function $\bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w)$ is strictly decreasing in b, with $\lim_{b\to\infty} \bar{F}_{\mathrm{C1}}(\tau \mid b) \times \bar{F}_{\mathrm{D1}}(\tau \mid b, w) = 0$, and, therefore, there exists a value $b_{\mathrm{U}}(w)$ such that

$$\bar{F}_{\mathrm{C2}}(\tau \mid 0)\bar{F}_{\mathrm{D2}}(\tau \mid 0, w) > \bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w) \quad \text{for all } b > b_{\mathrm{U}}(w).$$

Clearly, $b_{\mathrm{U}}(w)$ is the unique solution of the equation

$$\psi(b \mid w) \equiv \bar{F}_{\mathrm{C}1}(\tau \mid b)\bar{F}_{\mathrm{D}1}(\tau \mid b, w) - \bar{F}_{\mathrm{C}2}(\tau \mid 0)\bar{F}_{\mathrm{D}2}(\tau \mid 0, w) = 0.$$

To complete the proof, we have to show that $b_{\mathrm{U}}(w)$ is actually the upper bound for $b^*(w)$, that is, that for fixed w the optimal burn-in time $b^*(w)$ cannot be greater than $b_{\mathrm{U}}(w)$. Similar to the procedure given in the poof of Theorem 1, we have

$$\begin{aligned}
\bar{F}_{\mathrm{D}1}(\tau \mid b, w) &= \int_0^w \mathbb{P}\{[W_1(b+t) - W_1(b)] \le \kappa - y\} f_{W_1(b) \mid W_1(b) \le w}(y)\,\mathrm{d}y \\
&\ge \int_0^w \mathbb{P}\{[W_2(b+t) - W_2(b)] \le \kappa - y\} f_{W_1(b) \mid W_1(b) \le w}(y)\,\mathrm{d}y \\
&\ge \int_0^w \mathbb{P}\{[W_2(b+t) - W_2(b)] \le \kappa - y\} f_{W_2(b) \mid W_2(b) \le w}(y)\,\mathrm{d}y \\
&= \bar{F}_{\mathrm{D}2}(\tau \mid b, w) \quad \text{for all } b \ge 0
\end{aligned}$$

and, thus,

$$\begin{aligned}
\bar{F}_{\mathrm{C}1}(\tau \mid 0)\bar{F}_{\mathrm{D}1}(\tau \mid 0, w) &\ge \bar{F}_{\mathrm{C}2}(\tau \mid 0)\bar{F}_{\mathrm{D}2}(\tau \mid 0, w) \\
&> \bar{F}_{\mathrm{C}1}(\tau \mid b)\bar{F}_{\mathrm{D}1}(\tau \mid b, w) \\
&\ge \bar{F}_{\mathrm{C}2}(\tau \mid b)\bar{F}_{\mathrm{D}2}(\tau \mid b, w)
\end{aligned} \tag{11}$$

for all $b > b_{\mathrm{U}}(w)$. From (11), we can see that the minimum of the elements of the weighted average in $\bar{F}_{\mathrm{m}}(\tau \mid 0, w)$ is greater than the maximum of the elements of the weighted average in $\bar{F}_{\mathrm{m}}(\tau \mid b, w)$ for all $b > b_{\mathrm{U}}(w)$. Therefore, we have

$$\bar{F}_{\mathrm{m}}(\tau \mid 0, w) > \bar{F}_{\mathrm{m}}(\tau \mid b, w) \quad \text{for all } b > b_{\mathrm{U}}(w).$$

As $c_1(b, w)$ is strictly increasing in $b \ge 0$, this inequality now means that $c(0, w) < c(b, w)$, for $b > b_{\mathrm{U}}(w)$, and, thus, any $b \in (b_{\mathrm{U}}(w), \infty)$ cannot be the optimal burn-in time for fixed w. Therefore, the solution of (9) is the upper bound for the optimal burn-in time $b^*(w)$ for fixed w.

In Theorem 2, only the increasing failure rate of $\lambda_{\mathrm{C}1}(t)$ is considered. However, actually, the results hold for a more general class of failure rates. For this discussion, we first need to define the notion of the eventually (ultimately) increasing function (Mi (2003)).

Definition 4. The failure rate $\lambda(x)$ is eventually increasing if there exists $0 \le x_0 < \infty$ such that $\lambda(x)$ strictly increases in $x > x_0$.

For the eventually increasing failure rate $\lambda(x)$, the first and the second wear-out points t^* and t^{**} are defined in Mi (2003) as

$$\begin{aligned}
t^* &= \inf\{t \ge 0\colon \lambda(x) \text{ is nondecreasing in } x \ge t\}, \\
t^{**} &= \inf\{t \ge 0\colon \lambda(x) \text{ strictly increases in } x \ge t\}.
\end{aligned}$$

Then, clearly, the strictly increasing failure rate is a special case of the eventually increasing failure rate when $t^* = t^{**} = 0$.

Corollary 1. *Assume that $\lambda_{\mathrm{C}1}(t) \le \lambda_{\mathrm{C}2}(t)$ for all $t \ge 0$, and that $\lambda_{\mathrm{C}1}(t)$ is eventually increasing with the second wear-out point t^{**} and $\lambda_{\mathrm{C}1}(\infty) \equiv \lim_{t\to\infty} \lambda_{\mathrm{C}1}(t) = \infty$. Under the same assumption for the covariate processes as those stated in Theorem 2, for each fixed*

https://doi.org/10.1239/aap/1435236985 Published online by Cambridge University Press

$w \in (0, \kappa]$, the upper bound $b_{\mathrm{U}}(w)$, $b_{\mathrm{U}}(w) \in [t^{**}, \infty)$, is given by the unique solution of the equation in the interval $[t^{**}, \infty)$:

$$\psi(b \mid w) \equiv \bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w) - \bar{F}_{\mathrm{C2}}(\tau \mid t^{**})\bar{F}_{\mathrm{D2}}(\tau \mid t^{**}, w) = 0, \qquad b \in [t^{**}, \infty),$$

where $\psi(b \mid w)$ is strictly decreasing in $b \in [t^{**}, \infty)$ with $\psi(t^{**} \mid w) > 0$ and $\psi(\infty \mid w) \equiv \lim_{b\to\infty} \psi(b \mid w) < 0$.

Proof. Due to the assumption that $\lambda_{\mathrm{C1}}(t)$ is eventually increasing with the second wear-out point t^{**} and $\lambda_{\mathrm{C1}}(\infty) \equiv \lim_{t\to\infty} \lambda_{\mathrm{C1}}(t) = \infty$, $\bar{F}_{\mathrm{C1}}(\tau \mid b)$ is strictly decreasing in b in the interval $b \in [t^{**}, \infty)$ with $\lim_{b\to\infty} \bar{F}_{\mathrm{C1}}(\tau \mid b) = 0$. The function $\bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w)$ is strictly decreasing for $b \in [t^{**}, \infty)$ with $\lim_{b\to\infty} \bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid B(b, w)) = 0$, and, therefore, there exists $b_{\mathrm{U}}(w)$ such that

$$\bar{F}_{\mathrm{C2}}(\tau \mid t^{**})\bar{F}_{\mathrm{D2}}(\tau \mid t^{**}, w) > \bar{F}_{\mathrm{C1}}(\tau \mid b)\bar{F}_{\mathrm{D1}}(\tau \mid b, w) \quad \text{for all } b > b_{\mathrm{U}}(w),$$

where $b_{\mathrm{U}}(w) \in [t^{**}, \infty)$. Then, by the similar arguments to those of the proof of Theorem 2, it can be shown that

$$\bar{F}_{\mathrm{m}}(\tau \mid t^{**}, w) > \bar{F}_{\mathrm{m}}(\tau \mid b, w) \quad \text{for all } b > b_{\mathrm{U}}(w).$$

Therefore, $b_{\mathrm{U}}(w)$ is the upper bound for the optimal burn-in time $b^*(w)$ for fixed w.

Based on the discussions and results given above, the optimization procedure can be summarized as follows:

Algorithm 1. (*Optimization procedure.*)

Stage 1. Fix the elimination level $w \in (0, \kappa]$ and search for the optimal $b^*(w)$ only in the interval $b \in [0, b_{\mathrm{U}}(w)]$ that satisfies

$$c(b^*(w), w) = \min_{b\in[0, b_{\mathrm{U}}(w)]} c(b, w).$$

Stage 2. Search for w^* in the interval $(0, \kappa]$ that satisfies

$$c(b^*(w^*), w^*) = \min_{w\in(0,\kappa]} c(b^*(w), w).$$

Joint optimal solution. The two-dimensional joint optimal solution is given by $(b^*(w^*), w^*)$.

Alternatively to the cost function (7), we now assume that during the field operation the gain is proportional to the mean time to failure. Therefore, the total average cost function in this case is

$$c(b, w) = \frac{c_0 + c_{sr}[1 - p(b, w)]}{p(b, w)} - K \int_0^\infty \bar{F}_{\mathrm{m}}(t \mid b, w)\,\mathrm{d}t. \tag{12}$$

We will now consider the problem of finding the optimal joint burn-in parameters (b^*, w^*) which minimize $c(b, w)$. In order to find the joint optimal burn-in parameters (b^*, w^*) efficiently, the optimization procedures described in the previous case will be applied. In this case, as before, the existence of a finite upper bound for $b^*(w)$ for each fixed w would make the search for $b^*(w)$ substantially more efficient.

Theorem 3. *Assume that $\lambda_{\mathrm{C}1}(t) \le \lambda_{\mathrm{C}2}(t)$ for all $t \ge 0$, and that $\lambda_{\mathrm{C}1}(t)$ is strictly increasing with $\lambda_{\mathrm{C}1}(\infty) \equiv \lim_{t\to\infty} \lambda_{\mathrm{C}1}(t) = \infty$. Let $\{W_i(t),\ t \ge 0\}$, $i = 1, 2$, be any covariate processes which possess the independent increments property. Suppose that $W_1(t+s) - W_1(t) \le_{\mathrm{st}} W_2(t+s) - W_2(t)$ for all $t, s \ge 0$, $W_1(t) \le_{\mathrm{lr}} W_2(t)$ for all $t \ge 0$, and, further, that $W_1(t_1) \le_{\mathrm{lr}} W_1(t_2)$, $W_1(t_1+s) - W_1(t_1) \le_{\mathrm{st}} W_1(t_2+s) - W_1(t_2)$ for any $t_2 > t_1$ and $s \ge 0$. Then, for each fixed $w \in (0, \kappa]$, the upper bound for $b^*(w)$, which is denoted by $b_{\mathrm{U}}(w)$, is given by the unique solution of the equation:*

$$\psi(b \mid w) \equiv \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t - \int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid 0)\bar{F}_{\mathrm{D}2}(t \mid 0, w)\,\mathrm{d}t = 0,$$

where $\psi(b \mid w)$ is strictly decreasing in b with

$$\psi(0 \mid w) > 0 \quad \textit{and} \quad \psi(\infty \mid w) \equiv \lim_{b\to\infty} \psi(b \mid w) < 0.$$

Proof. Fix $w \in (0, \kappa]$ and let

$$M(b, w) \equiv \pi(b, w) \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t + (1 - \pi(b, w)) \int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid b)\bar{F}_{\mathrm{D}2}(t \mid b, w)\,\mathrm{d}t.$$

As in the proof of Theorem 2, the cost $c_1(b, w)$ during the burn-in procedure is strictly increasing in $b \ge 0$. Thus, it is sufficient to show that $M(0, w) > M(b, w)$ for $b > b_{\mathrm{U}}(w)$. Observe that $M(b, w)$ is of the form of a weighted average of $\int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t$ and $\int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid b)\bar{F}_{\mathrm{D}2}(t \mid b, w)\,\mathrm{d}t$. Then, by similar arguments as those described in the proof of Theorem 2, it can be shown that $\bar{F}_{\mathrm{C}1}(t \mid b)$ is strictly decreasing in b with $\lim_{b\to\infty} \bar{F}_{\mathrm{C}1}(t \mid b) = 0$ for any fixed t and $\bar{F}_{\mathrm{D}1}(t \mid b, w)$ is decreasing in b for any t. This implies that $\int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b) \times \bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t$ is strictly decreasing in b with $\lim_{b\to\infty} \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t = 0$. Therefore, there exists $b_{\mathrm{U}}(w)$ such that

$$\int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid 0)\bar{F}_{\mathrm{D}2}(t \mid 0, w)\,\mathrm{d}t > \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t \quad \text{for all } b > b_{\mathrm{U}}(w).$$

Clearly, $b_{\mathrm{U}}(w)$ is the unique solution of the equation

$$\psi(b \mid w) \equiv \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t - \int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid 0)\bar{F}_{\mathrm{D}2}(t \mid 0, w)\,\mathrm{d}t = 0.$$

Then, similar to the procedure given in the poof of Theorem 2,

$$\begin{aligned}\int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid 0)\bar{F}_{\mathrm{D}1}(t \mid 0, w)\,\mathrm{d}t &\ge \int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid 0)\bar{F}_{\mathrm{D}2}(t \mid 0, w)\,\mathrm{d}t \\ &> \int_0^\infty \bar{F}_{\mathrm{C}1}(t \mid b)\bar{F}_{\mathrm{D}1}(t \mid b, w)\,\mathrm{d}t \\ &\ge \int_0^\infty \bar{F}_{\mathrm{C}2}(t \mid b)\bar{F}_{\mathrm{D}2}(t \mid b, w)\,\mathrm{d}t,\end{aligned}$$

and, thus, $M(0, w) > M(b, w)$ for all $b > b_{\mathrm{U}}(w)$. This inequality implies that any $b \in (b_{\mathrm{U}}(w), \infty)$ cannot be the optimal burn-in time for fixed w. Therefore, $b_{\mathrm{U}}(w)$ is the upper bound for the optimal burn-in time $b^*(w)$ for fixed w.

Note that, as in Corollary 1, the results in Theorem 3 can be extended without difficulty to the more general setting of the eventually increasing failure rate $\lambda_{C1}(t)$, but the details are not stated. Furthermore, based on the results in Theorem 3, the two-dimensional optimization algorithm, which is similar to that described previously, can be employed.

4. Nonhomogeneous gamma process

In this section more detailed results on the joint burn-in are stated, assuming that the conditional covariate processes $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, follow nonhomogeneous gamma processes. In practice, the gamma process is intensively used in degradation modeling. For instance, in Wu *et al.* (2011), a gamma process is employed to model gradual damage monotonically accumulating over time. In Tseng *et al.* (2009) and Tsai *et al.* (2012), optimal testing plans of gamma degradation models are discussed. In Xu and Wang (2012), an adaptive gamma process is used to describe the deteriorating nature of the observed condition indicator. In Pan and Balakrishnan (2011), gamma processes are used to model degradation of products with multiple performance characteristics. An excellent survey on the application of gamma processes in maintenance modeling can be found in van Noortwijk (2009).

For a practical formulation of the setting, we will call the corresponding covariate process the 'degradation process' in the following discussions. For our further discussions, we briefly summarize some properties of the gamma process.

The gamma process (see, e.g. Çinlar (1980)) possesses the property of independent increments stated in Subsection 2.2 and is commonly used to describe degradation phenomena whose growth depends on the age of the system. The widespread use of the gamma process is due to its mathematical tractability and its flexibility, making it suitable to model the growth of wear, fatigue, corrosion, crack, erosion, degrading health index, etc. (see, e.g. van Noortwijk (2009)). Under the gamma process assumption, the PDFs of $W_i(t)$, $i = 1, 2$, are given by

$$f_i(y; t) = \frac{\beta^{\alpha_i(t)} y^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i y), \qquad i = 1, 2,\ y \geq 0, \tag{13}$$

where $\beta_i > 0$ and $\alpha_i(t)$ is monotonically increasing in $t \geq 0$ with $\alpha_i(0) = 0$, $i = 1, 2$. Under the gamma process defined in (13), the cumulative distribution of the degradation level accumulated up to the generic time t of an item from the ith subpopulation, $i = 1, 2$, is given by

$$\mathbb{P}\{W_i(t) \leq y\} = \int_0^y \frac{\beta^{\alpha_i(t)} u^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i u)\,\mathrm{d}u, \qquad i = 1, 2.$$

Thus, the survival probability due to cause II (the degradation failure) is

$$\bar{F}_{\mathrm{D}i}(t) = \mathbb{P}\{T_{\mathrm{D}i} > t\} = \mathbb{P}\{W_i(t) \leq \kappa\} = \int_0^\kappa \frac{\beta^{\alpha_i(t)} u^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i u)\,\mathrm{d}u, \qquad i = 1, 2,$$

and the corresponding failure rate function can be obtained by

$$\begin{aligned}\lambda_{\mathrm{D}i}(t) &= \frac{-\mathrm{d}\ln(\bar{F}_{\mathrm{D}i}(t))}{\mathrm{d}t} \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\bigg[\int_0^\kappa \frac{\beta_i^{\alpha_i(t)} u^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i u)\,\mathrm{d}u\bigg]\bigg[\int_0^\kappa \frac{\beta_i^{\alpha_i(t)} u^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i u)\,\mathrm{d}u\bigg]^{-1}\end{aligned}$$

$$= \left[\int_0^\kappa \frac{\beta_i^{\alpha_i(t)} u^{\alpha_i(t)-1}}{\Gamma(\alpha_i(t))} \exp(-\beta_i u)\,\mathrm{d}u \left(\int_0^\infty v^{\alpha_i(t)-1} \exp(-v)\,\mathrm{d}v \right)^2 \right]^{-1}$$
$$\times \int_0^\kappa \bigg[-(\ln \beta_i + \ln u)\alpha_i'(t)\beta_i^{\alpha_i(t)} u^{\alpha_i(t)-1} \exp(-\beta_i u) \left(\int_0^\infty v^{\alpha_i(t)-1} \exp(-v)\,\mathrm{d}v \right)$$
$$+ \beta_i^{\alpha_i(t)} u^{\alpha_i(t)-1} \exp(-\beta_i u) \left(\int_0^\infty \alpha_i'(t)(\ln v) v^{\alpha_i(t)-1} \exp(-v)\,\mathrm{d}v \right) \bigg] \mathrm{d}u$$

for $i = 1, 2$. On the other hand,

$$f_{W_i(b)\,|\,W_i(b)\le w}(y) = \frac{f_i(y; b)}{\int_0^w f_i(u; b)\,\mathrm{d}u} = \frac{y^{\alpha_i(b)-1} \exp(-\beta_i y)}{\int_0^w u^{\alpha_i(b)-1} \exp(-\beta_i u)\,\mathrm{d}u}, \qquad 0 \le y \le w,$$

and, therefore, the conditional probability that a strong or weak component survives up to time $t + b$ in the presence of only cause II, given that it has survived the joint burn-in, is

$$\bar{F}_{\mathrm{D}i}(t \mid b, w)$$
$$= \int_0^w \mathbb{P}\{W_i(b+t) - W_i(b) < \kappa - y\} f_{W_i(b)\,|\,W_i(b)\le w}(y)\,\mathrm{d}y$$
$$= \frac{\int_0^w \big(\int_0^{\kappa - y} (\beta^{\alpha_i(b,t)} v^{\alpha_i(b,t)-1} / \Gamma(\alpha_i(b,t))) \exp(-\beta_i v)\,\mathrm{d}v \big) y^{\alpha_i(b)-1} \exp(-\beta_i y)\,\mathrm{d}y}{\int_0^w y^{\alpha_i(b)-1} \exp(-\beta_i y)\,\mathrm{d}y},$$

where $\alpha_i(b, t) \equiv \alpha_i(b + t) - \alpha_i(b)$. Furthermore, the proportion of the strong subpopulation after the joint burn-in is:

$$\pi(b, w) = \pi \exp\left(-\int_0^b \lambda_{\mathrm{C}1}(u)\,\mathrm{d}u \right) \int_0^w \frac{\beta_1^{\alpha_1(b)} v^{\alpha_1(b)-1}}{\Gamma(\alpha_1(b))} \exp(-\beta_1 v)\,\mathrm{d}v [p(b, w)]^{-1},$$

where

$$p(b, w) = \pi \exp\left(-\int_0^b \lambda_{\mathrm{C}1}(u)\,\mathrm{d}u \right) \int_0^w \frac{\beta^{\alpha_1(b)} v^{\alpha_1(b)-1}}{\Gamma(\alpha_1(b))} \exp(-\beta_1 v)\,\mathrm{d}v$$
$$+ (1 - \pi) \exp\left(-\int_0^b \lambda_{\mathrm{C}2}(u)\,\mathrm{d}u \right) \int_0^w \frac{\beta^{\alpha_2(b)} v^{\alpha_2(b)-1}}{\Gamma(\alpha_2(b))} \exp(-\beta_2 v)\,\mathrm{d}v.$$

As stated before, in order to justify the consideration of joint burn-in, (3) (CNQD) should hold and, at the same time, the subpopulations should be stochastically ordered. In the following corollary the conditions for general covariate processes $\{W_i(t),\ t \ge 0\}$, $i = 1, 2$, suggested in Theorem 1 will be detailed for the gamma processes.

Corollary 2. *Let the degradation processes $\{W_i(t),\ t \ge 0\}$, $i = 1, 2$, be the nonhomogeneous gamma processes defined in (13). If $\beta_1 \ge \beta_2$ and $\alpha_1(t, s) \le \alpha_2(t, s)$ for all $t, s \ge 0$, then*

(i) *the subpopulations are stochastically ordered in the sense of failure rate ordering:*

$$\lambda_{\mathrm{C}1}(t) + \lambda_{\mathrm{D}1}(t) \le \lambda_{\mathrm{C}2}(t) + \lambda_{\mathrm{D}2}(t) \quad \textit{for all } t \ge 0,$$

(ii) *$\{W(t),\ t \ge 0\}$ satisfies the condition CNQD in (3):*

$$\mathbb{P}\{T > b + t, W(b) \le w \mid T > b\} \ge \mathbb{P}\{T > b + t \mid T > b\}\mathbb{P}\{W(b) \le w \mid T > b\}$$

for all t and w for all fixed b.

Proof. Observe that the PDFs of $W_1(t+s) - W_1(t)$ and $W_2(t+s) - W_2(t)$ are given by

$$\frac{\beta^{\alpha_i(t,s)} y^{\alpha_i(t,s)-1}}{\Gamma(\alpha_i(t,s))} \exp(-\beta_i y), \qquad i = 1, 2,$$

respectively, and their ratio (PDF of $W_1(t+s) - W_1(t)$)/(PDF of $W_2(t+s) - W_2(t)$) is

$$\frac{\Gamma(\alpha_2(t,s))}{\Gamma(\alpha_1(t,s))} \frac{\beta_1^{\alpha_1(t,s)}}{\beta_2^{\alpha_2(t,s)}} y^{\alpha_1(t,s)-\alpha_2(t,s)} \exp\{-(\beta_1 - \beta_2) y\},$$

which is decreasing in $y \geq 0$. This implies that $W_1(t+s) - W_1(t) \leq_{\mathrm{lr}} W_2(t+s) - W_2(t)$ for all $t, s \geq 0$, and, thus, $W_1(t+s) - W_1(t) \leq_{\mathrm{st}} W_2(t+s) - W_2(t)$ for all $t, s \geq 0$, and $W_1(t) \leq_{\mathrm{lr}} W_2(t)$ for all $t \geq 0$. Therefore, all the conditions in Theorem 1 are fulfilled.

Now we consider the problem of minimizing the cost function $c(b, w)$ in (7). In the following corollary, the upper bound for $b^*(w)$ for each fixed w will be given and the conditions in Theorem 2 will be described for the gamma processes.

Corollary 3. *Assume that $\lambda_{\mathrm{C}1}(t) \leq \lambda_{\mathrm{C}2}(t)$ for all $t \geq 0$, and that $\lambda_{\mathrm{C}1}(t)$ is strictly increasing with $\lambda_{\mathrm{C}1}(\infty) \equiv \lim_{t\to\infty} \lambda_{\mathrm{C}1}(t) = \infty$. Let the degradation processes $\{W_i(t),\ t \geq 0\}$, $i = 1, 2$, be the nonhomogeneous gamma processes defined in (13). If $\beta_1 \geq \beta_2$, $\alpha_1(t,s) \leq \alpha_2(t,s)$ for all $t, s \geq 0$, and $\alpha_1(t,s)$ is increasing in t for all $s \geq 0$, then, for each fixed $w \in (0, \kappa]$, the upper bound for $b^*(w)$, which is denoted by $b_{\mathrm{U}}(w)$, is given by the unique solution of the equation:*

$$\begin{aligned}
\psi(b \mid w) \equiv{}& \exp\biggl(-\int_0^\tau \lambda_{\mathrm{C}1}(b+u)\,\mathrm{d}u\biggr) \\
&\times \frac{\int_0^w \bigl(\int_0^{\kappa-y} [\beta_1^{\alpha_1(b,\tau)} v^{\alpha_1(b,\tau)-1} / \Gamma(\alpha_1(b,\tau))] \mathrm{e}^{-\beta_1 v}\,\mathrm{d}v\bigr) y^{\alpha_1(b)-1} \mathrm{e}^{-\beta_1 y}\,\mathrm{d}y}{\int_0^w y^{\alpha_1(b)-1} \mathrm{e}^{-\beta_1 y}\,\mathrm{d}y} \\
&- \exp\biggl(-\int_0^\tau \lambda_{\mathrm{C}2}(u)\,\mathrm{d}u\biggr) \int_0^\kappa \frac{\beta_2^{\alpha_2(\tau)} u^{\alpha_2(\tau)-1}}{\Gamma(\alpha_2(\tau))} \mathrm{e}^{-\beta_2 u}\,\mathrm{d}u \\
={}& 0,
\end{aligned}$$

where $\psi(b \mid w)$ is strictly decreasing in b with

$$\psi(0 \mid w) > 0 \quad \textit{and} \quad \psi(\infty \mid w) \equiv \lim_{b\to\infty} \psi(b \mid w) < 0.$$

Proof. If the condition that $\alpha_1(t,s)$ is increasing in t for all $s \geq 0$ is assumed in addition to the condition that $\alpha_1(t,s) \leq \alpha_2(t,s)$ for all $t, s \geq 0$, then all the conditions stated in Theorem 3 are fulfilled.

Remark 4. The condition that $\alpha_1(t,s) \leq \alpha_2(t,s)$ for all $t, s \geq 0$ in Corollary 2 is satisfied if $\mathrm{d}\alpha_1(t)/\mathrm{d}t \leq \mathrm{d}\alpha_2(t)/\mathrm{d}t$ for all $t \geq 0$. Furthermore, the additional condition that $\alpha_1(t,s)$ is increasing in t for all $s \geq 0$ in Corollary 3 is satisfied if $\alpha_1(t)$ is a convex function.

Now we consider the problem of minimizing the cost function $c(b, w)$ in (12). In the following corollary, the upper bound for $b^*(w)$ for each fixed w will be given and the conditions in Theorem 3 will be described for the gamma processes.

Corollary 4. *For the nonhomogeneous gamma processes of degradation processes* $\{W_i(t), t \geq 0\}$, $i = 1, 2$, *under the same assumptions as those described in Corollary 3, for each fixed* $w \in (0, \kappa]$, *the upper bound* $b_{\mathrm{U}}(w)$ *is given by the unique solution of the equation:*

$$\begin{aligned}
&\psi(b \mid w) \\
&\quad\equiv \frac{\int_0^\infty \exp(-\int_0^t \lambda_{\mathrm{C}1}(b+u)\,\mathrm{d}u)[\int_0^w(\int_0^{\kappa-y}[\beta_1^{\alpha_1(b,t)} v^{\alpha_1(b,t)-1}/\,\Gamma(\alpha_1(b,t))]\mathrm{e}^{-\beta_1 v}\,\mathrm{d}v) y^{\alpha_1(b)-1}\mathrm{e}^{-\beta_1 y}\,\mathrm{d}y]\,\mathrm{d}t}{\int_0^w y^{\alpha_1(b)-1}\mathrm{e}^{-\beta_1 y}\,\mathrm{d}y} \\
&\qquad - \int_0^\infty \exp\biggl(-\int_0^t \lambda_{\mathrm{C}2}(u)\,\mathrm{d}u\biggr)\biggl(\int_0^\kappa \frac{\beta^{\alpha_2(t)}u^{\alpha_2(t)-1}}{\Gamma(\alpha_2(t))}\mathrm{e}^{-\beta_2 u}\,\mathrm{d}u\biggr)\,\mathrm{d}t \\
&\quad= 0,
\end{aligned}$$

where $\psi(b \mid w)$ *is strictly decreasing in b with*

$$\psi(0 \mid w) > 0 \quad \textit{and} \quad \psi(\infty \mid w) \equiv \lim_{b\to\infty} \psi(b \mid w) < 0.$$

Proof. The proof is similar to that of Corollary 3.

Example 1. Let us consider the case of two subpopulations where the proportion of strong items is $\pi = 0.6$, so $1 - \pi = 0.4$. Then, we assume that the failure times (in days) due to catastrophic failure (cause I) of the strong and weak subpopulations both follow a distribution with failure rates equal to $\lambda_{\mathrm{C}1}(t) = 0.01t + 0.01$ days^{-1} and $\lambda_{\mathrm{C}2}(t) = t + 1.5$ days^{-1}, respectively, so:

$$\bar{F}_{\mathrm{C}1}(t) = \exp\biggl[-\biggl(\frac{0.01t^2}{2 + 0.01t}\biggr)\biggr] \quad \text{and} \quad \bar{F}_{\mathrm{C}2}(t) = \exp\biggl[-\biggl(\frac{t^2}{2} + 1.5t\biggr)\biggr].$$

The assumed failure rate functions satisfy the following 'practical' properties that: (a) the failure rate of the weak subpopulation has a relatively large positive value near $t = 0$ and increases very steeply, and (b) the failure rate of the strong subpopulation is initially almost negligible and then increases quite slowly. Examples of linear failure rates can be found, for example, in Bain (1974) and Lawless (2003). Other suitable expressions for the failure rates are, for example, the Gompertz (exponential) failure rates (see, e.g. Bagdonavicius *et al.* (2011), for the application of the Gompertz model to failure data), say $\lambda_{\mathrm{C}1}(t) = 0.01\exp(0.3t)$ days^{-1} and $\lambda_{\mathrm{C}2}(t) = 1.5\exp(0.5t)$ days^{-1}. In both the cases, the suggested failure rates satisfy the inequality $\lambda_{\mathrm{C}1}(t) \leq \lambda_{\mathrm{C}2}(t)$ for all $t \geq 0$.

Again, we assume that the degradation processes (cause II) of strong and weak subpopulations follow nonhomogeneous gamma processes with a power-law shape function $\alpha_1(t) = (t/2)^{1.2}$ and $\alpha_2(t) = (t/0.2)^{1.2}$, and scale parameters $\beta_1 = 1.2$ and $\beta_2 = 0.8$. Both the assumptions that $\beta_1 \geq \beta_2$ and $\alpha_1(t+s) - \alpha_1(t) \leq \alpha_2(t+s) - \alpha_2(t)$ for all $t, s \geq 0$ in Corollary 2, as well as the additional assumption in Corollary 3 that $\alpha_1(t, s)$ is increasing in t for all $s \geq 0$ are then satisfied.

Thus, the subpopulations are stochastically ordered in the sense of failure rate ordering and the degradation processes $\{W_i(t),\ t \geq 0\}$ satisfy the condition CNQD in (3).

Then, let the threshold level be $\kappa = 6.0$ mm and the mission length $\tau = 3.0$ days. Setting $c_{sr} = 1.0$, $c_0 = 0.05$, $C = 10$, and $K = 20$ unit of cost, the optimal joint burn-in parameters (b^*, w^*) and the corresponding cost $c(b^*, w^*)$ were obtained. In Table 1, the parameters and total expected cost of the optimal joint burn-in are compared to the time parameter and the corresponding expected cost of the optimal ordinary burn-in. Also the success probability

TABLE 1: Summary of optimal burn-in procedures.

Burn-in type	b^*	w^*	$c(b^*, w^*)$	$\bar{F}_{\mathrm{m}}(\tau \mid b^*, w^*)$	$M(b^*, w^*)$
Joint	0.475	0.476	−16.29	0.908	7.96
Ordinary	1.168	—	−15.64	0.881	7.28

TABLE 2: Results of the study on misspecification.

Failure rate $\lambda_{\mathrm{C2}}(t)$	Covariate parameter β_2	Burn-in type	b^*	w^*	$c(b^*, w^*)$	$\bar{F}_{\mathrm{m}}(\tau \mid b^*, w^*)$	$M(b^*, w^*)$
$10t + 15$	0.8	Joint	0.298	1.038	−13.72	0.810	7.20
		Ordinary	0.343	—	−9.85	0.671	5.98
$t + 1.5$	0.08	Joint	0.323	0.805	−14.85	0.851	7.54
		Ordinary	0.526	—	−11.86	0.743	6.49
$10t + 15$	0.08	Joint	0.223	1.106	−11.67	0.736	6.61
		Ordinary	0.277	—	−9.17	0.647	5.81

$\bar{F}_{\mathrm{m}}(\tau \mid b^*, w^*)$ and the mean residual life $M(b^*, w^*)$ (in days) for an item that survived the burn-in are compared. We can note that the joint burn-in procedure outperforms the ordinary burn-in procedure because it allows us to reduce the total expected cost and to increase the quality characteristics of the items that are put into field operation.

Finally, as suggested by the referee, we have evaluated the effect of misspecifying the failure rate and/or the covariate process on the optimal burn-in, in particular when the failure rate function $\lambda_{\mathrm{C2}}(t)$ of the weak subpopulation is assumed to be greater than the true function, say $\lambda_{\mathrm{C2}}(t) = 10t + 15.0$ days^{-1} and/or when the gamma process parameter β_2 of the weak subpopulation is assumed to be smaller than the true value, say $\beta_2 = 0.08$ (note that the mean degradation level $E\{W_i(t)\} = \alpha_i(t)/\beta_i$ is inversely proportional to β_i).

In Table 2 we present the results of this study, where the optimal burn-in parameters, say (b^*, w^*) and b^* for the joint and ordinary burn-in, respectively, are those obtained under the misspecified $\lambda_{\mathrm{C2}}(t) = 10t + 15.0$ and/or $\beta_2 = 0.08$, whereas the expected cost $c(b^*, w^*)$, the success probability $\bar{F}_{\mathrm{m}}(\tau \mid b^*, w^*)$, and the mean residual life $M(b^*, w^*)$ are evaluated under the true $\lambda_{\mathrm{C2}}(t)$ function and/or β_2 value, say $\lambda_{\mathrm{C2}}(t) = t + 1.5$ days^{-1} and $\beta_2 = 0.8$. This study is restricted to the weak subpopulation because this subpopulation affects the optimal burn-in parameters more than the strong subpopulation parameters.

From the values in Table 2, we see that even when the parameters that index the failure rate and/or the degradation process of the weak subpopulation are greatly misspecified, the joint burn-in always outperforms the ordinary one.

5. Concluding remarks

In the conventional burn-in procedure, the only information used for the elimination procedure is the corresponding 'lifetime' of the item. In this paper, a new joint burn-in procedure which additionally employs the information of the dependent covariate is proposed. For a proper stochastic formulation of the problem, a new type of positive dependence between a random variable and a stochastic process is defined and employed. It has been shown that the conventional time burn-in corresponds to a special case of the joint burn-in procedure proposed in this paper and that, when the joint burn-in procedure is applicable, it outperforms the ordinary time burn-in. The proposed general methodology has been applied to the case when the covariate process follows the gamma process of degradation.

Throughout the paper, the topic is discussed for the models with increasing covariate process $\{W(t),\ t \geq 0\}$. However, for a decreasing covariate process $\{V(t),\ t \geq 0\}$, the theoretical results obtained in this paper could be straightforwardly modified, as $-V(t) \equiv W(t)$ represents an increasing covariate. Furthermore, we assume that there exists the threshold level κ, which defines the second cause of failure (cause II) of the item. However, by letting $\kappa = \infty$, the burn-in model studied in this paper could be easily modified for the case when there is no second type of cause of failure (Remark 3). Even in this case, the information on the covariate process is usefully used for the elimination process. From these points of view, the burn-in model studied in this paper is a very general model.

Acknowledgements

The authors would like to thank the referee for helpful comments which have improved the presentation of this paper. The work of the first author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (grant number 2011-0017338). The work of the first author was also supported by Priority Research Centers Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (grant number 2009-0093827).

References

Badía, F. G., Berrade, M. D. and Campos, C. A. (2003). Why do failure rates decrease? In *Seventh Conference on Applied Mathematics and Statistics*, University of Zaragoza, Zaragoza-Pau, pp. 97–104.

Bagdonavičius, V., Kruopis, J. and Nikulin, M.S. (2011). *Non-Parametric Tests for Censored Data*. ISTE, London.

Bain, L. J. (1974). Analysis for the linear failure-rate life-testing distribution. *Technometrics* **16,** 551–559.

Barlow, R. E. and Proschan, F. (1981). *Statistical Theory of Reliability and Life Testing: Probability Models*. To Begin With, Silver Spring, MD.

Block, H. W. and Savits, T. H. (1997). Burn-in. *Statist. Sci.* **12,** 1–19.

Bocchetti, D., Giorgio, M., Guida, M. and Pulcini, G. (2009). A competing risk model for the reliability of cylinder liners in marine Diesel engines. *Reliab. Eng. System Safety* **94,** 1299–1307.

Cha, J. H. (2000). On a better burn-in procedure. *J. Appl. Prob.* **37,** 1099–1103.

Cha, J. H. (2001). Burn-in procedures for a generalized model. *J. Appl. Prob.* **38,** 542–553.

Cha, J. H. and Finkelstein, M. (2011). Burn-in and the performance quality measures in heterogeneous populations. *Europ. J. Operat. Res.* **210,** 273–280.

Cha, J. H. and Finkelstein, M. (2012). Shocks as burn-in in heterogeneous populations. *Commun. Statist. Theory Meth.* **41,** 325–340.

Çinlar, E. (1980). On a generalization of gamma processes. *J. Appl. Prob.* **17,** 467–480, 893.

Clarotti, C. A. and Spizzichino, F. (1990). Bayes burn-in decision procedures. *Prob. Eng. Inf. Sci.* **4,** 437–445.

Finkelstein, M. (2008). *Failure Rate Modeling for Reliability and Risk*. Springer, London.

Huang, W. and Askin, R. G. (2003). Reliability analysis of electronic devices with multiple competing failure modes involving performance aging degradation. *Quality Reliab. Eng. Internat.* **19,** 241–254.

Jensen, F. B. and Petersen, N. E. (1982). *Burn-In*. John Wiley, Chichester.

Kececioglu, D. B. and Sun, F.-B. (1997). *Burn-In Testing*. Prentice Hall, Upper Saddle River, NJ.

Kim, K. O. and Kuo, W. (2009). Optimal burn-in for maximizing reliability of repairable non-series systems. *Europ. J. Operat. Res.* **193,** 140–151.

Kuo, W. and Kuo, Y. (1983). Facing the headaches of early failures: A state-of-the-art review of burn-in decisions. *Proc. IEEE* **71,** 1257–1266.

Lawless, J. F. (2003). *Statistical Models and Methods for Lifetime Data*, 2nd edn. John Wiley, Hoboken, NJ.

Lehmann, E. L. (1966). Some concepts of dependence. *Ann. Math. Statist.* **37,** 1137–1153.

Mi, J. (1994). Burn-in and maintenance policies. *Adv. Appl. Prob.* **26,** 207–221.

Mi, J. (2003). Optimal burn-in time and eventually IFR. *J. Chinese Inst. Industrial Eng.* **20,** 533–543.

Nelsen, R. B. (1992). On measures of association as measures of positive dependence. *Statist. Prob. Lett.* **14,** 269–274.

Pan, Z. and Balakrishnan, N. (2011). Reliability modeling of degradation of products with multiple performance characteristics based on gamma processes. *Reliab. Eng. System Safety* **96,** 949–957.

Shaked, M. and Shanthikumar, J. G. (2007). *Stochastic Orders*. Springer, New York.

Shaked, M. and Spizzichino, F. (1998). Positive dependence properties of conditionally independent random lifetimes. *Math. Operat. Res.* **23,** 944–959.

Tsai, C.-C., Tseng, S.-T. and Balakrishnan, N. (2012). Optimal design for degradation tests based on gamma processes with random effects. *IEEE Trans. Reliab.* **61,** 604–613.

Tseng, S.-T., Balakrishnan, N. and Tsai, C.-C. (2009). Optimal step-stress accelerated degradation test plan for gamma degradation processes. *IEEE Trans. Reliab.* **58,** 611–618.

Van Noortwijk, J. M. (2009). A survey of the application of gamma processes in maintenance. *Reliab. Eng. System Safety* **94,** 2–21.

Wu, Y., Xie, L., Wu, N. and Li, J. (2011). Time-dependent reliability model of components with strength degradation based-on gamma process. In *Proc. 2011 9th International Conference on Reliability, Maintainability and Safety (ICRMS)*, IEEE, New York, pp. 363–368.

Xu, W. and Wang, W. (2012). An adaptive gamma process based model for residual useful life prediction. In *Proc. 2012 IEEE Conference on Prognostics and System Health Management (PHM)*, IEEE, New York, pp. 1–4.