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Accurate refraction–diffraction equations for water waves on a variable-depth rough bottom

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The extended mild-slope equation and the modified mild-slope equation have been used successfully to study refraction–diffraction of linear water waves by steep bottom roughness. Their consistency has been questioned. A systematic derivation of these model equations exposes and illuminates their rationale. Their good performance stems from an accurate representation of (Class I) Bragg resonance. As a benchmark test case, we consider scattering by a sloping bottom with random roughness. The rates of scattering found for the mean field in both of the approximate models agree exactly with the full theory for scattering by small roughness. This greatly improves the limited agreement which was found for the mild-slope equation, and establishes the validity of the above model equations. The study involves operator calculus, a powerful method for simplifying problems with variable coefficients. The augmented mild-slope equation serves to consistently derive accurate model equations.

1. Introduction

Wave propagation problems can be greatly simplified when the geometry of the basin is simple, e.g. constant depth. By separation of variables the dimension of the problem can be reduced, eliminating the vertical coordinate. We consider the linear theory of small-amplitude water waves in water of intermediate depth. More specifically, we consider the scattering due to terms which are linear in the depth variation and its derivatives (we call it Class I scattering). This scattering is typically the most important. The scattering is dominated by Bragg resonance (which is called Class I, in this case; cf. Liu & Yue 1998), in which the wave vectors of the incident wave and of the bottom roughness add up to the wave vector of the scattered field, forming an isosceles triangle in the wave-vector plane. In the case of slowly varying depth, approximate equations for propagation, such as the mild-slope equation (MSE) have been derived (Berkhoff 1972). These equations are in terms of $\varphi(x, y)$, the complex amplitude of the monochromatic velocity potential at the free surface. When studying scattering by steeper roughness, the MSE does not perform well. When the bottom roughness is small, its effect (to first order) can be accounted for (to a good approximation) by the extended MSE (EMSE, cf. Kirby 1986). The modified MSE (MMSE, cf. Chamberlain & Porter 1995) introduces a different correction term to the MSE, proportional to the second derivative of the bottom depth, enhancing its performance. All of these equations are written explicitly in §2. A number of comparisons with experiments and numerical solutions of the full (*linear*) problem have shown that the EMSE and MMSE give very good descriptions of (Class I Bragg resonance) scattering (e.g. Kirby 1986; Chamberlain & Porter 1995; Porter & Staziker 1995; Miles & Chamberlain 1998). This might seem to be due to the inclusion of the correct terms in $|\nabla h|^2$ and in $\nabla^2 h$. Is this the correct explanation?

Miles & Chamberlain (1998) introduce a hierarchy of MSEs in terms of powers of the bottom slope $|\nabla h|$, which is assumed to be small. The MMSE can be derived from a Lagrangian using a trial function for the velocity potential based on the solution on a constant depth, $\phi = \varphi(x, y)F(z, h)$, with $F = \operatorname{sech}(kh)\cosh k(z + h)$ (z is the vertical coordinate, h is the fluid depth and $\nabla = (\partial_x, \partial_y)$ is the horizontal gradient). Discarding a term in $|\nabla h|^2$ and a term in the form $g_2(k, h)\nabla^2 h\varphi$, reduces the MMSE to the MSE. They note that the MMSE is not consistent with their hierarchy – since the trial function has an error which is of order $|\nabla h|$, hence the equation should have errors of order $|\nabla h|^2$ (all the terms are scaled by h). To derive a higher-order equation, they use a trial function which includes the appropriate $O(|\nabla h|)$ term, obtaining a Lagrangian which contains the appropriate $O(|\nabla h|^2)$ terms. However, they neglect terms of order $\nabla^2 h$ in their trial function. These terms are important, at that order (unless $\nabla^2 h$ is negligible compared to $|\nabla h|$). In spite of their criticism of the MMSE, they use it, rather than their new equation, for calculating scattering by a ramp and by a sinusoidal patch. Both these problems are dominated by Class I scattering, for which $O(|\nabla h|^2)$ terms are irrelevant. In contrast to this irrelevance, all the terms in $\nabla^n h$, $n = 1, 2, 3, \ldots$, are equally relevant for Class I scattering by roughness with $O(1)$ steepness. These terms are given in the augmented MSE, in § 3 (cf. Agnon 1999). We focus our attention on these terms rather than on the $O(|\nabla h|^2)$ terms (which we discard). How can one account for an infinite series of terms? The key to this problem is found through considering the operator of detuning from resonance, $\mu = \nabla^2 + k^2$, which operates on the velocity potential ϕ. Miles & Chamberlain (1988) show that φF indeed gives the velocity potential with an $O(\mu)$ error. Thus, the MMSE is correct to (and including) $O(\mu)$, which explains its accuracy in describing (Class I) resonant Bragg scattering. For those components of the wave field which are in exact resonance with the bottom roughness, the infinite series of terms in $\nabla^n h$, $n = 2, 3, \ldots$, adds up exactly to the term $g_2\nabla^2 h\varphi$ in the MMSE (cf. Agnon 1999). This is quite elegant. Thus, it is indeed clear that the $O(|\nabla h|^2)$ term in the MMSE should not be retained. However, it is the term $g_2\nabla^2 h$ that is responsible for the good performance of this equation in Class I Bragg resonance. Hence, for Class I scattering, the relevant hierarchy of equations is in terms of powers of μ. The MSE does not belong in this hierarchy; it is replaced by the MMSE, by maintaining the term $g_2\nabla^2 h\varphi$.

The present work addresses the propagation of small-amplitude surface gravity waves over a statistically rough bottom. We compare the full (linear) problem, the EMSE and the MMSE. The full solution for the case of constant mean depth was addressed in Dyatlov & Pelinovsky (1990) using a Fourier transform method. They included the first-order effects of the bottom roughness, and found that the mean field was attenuated due to scattering into the fluctuation field (this attenuation is not related to friction; it is solely due to transfer into the incoherent field). Examining their result, we realize that it depends only on the components of the bottom roughness which are in exact resonance with the wave field. Pelinovsky, Razin & Sasorova (1998, referred to as PRS hereafter) have applied the MSE to the same problem, in order to test its applicability. Examining their results, we realize again that the attenuation due to their dispersion equation depends only on the resonant interaction with the bottom. However, the agreement of the MSE with the first-order theory was limited to either:

1. the limit of shallow water ($k_0h_0 \ll 1$, where h_0 is the mean depth and k_0 is the

corresponding wavenumber), in which all the waves are nearly resonant, and have the same (uniform) vertical structure; or

2. the limit of large-scale roughness, where the scattered field and the incident field have nearly the same wavenumber, and hence similar vertical structure – thus they are in near resonance with the large-scale bottom roughness.

These results can be understood as follows: the MSE is an approximation which neglects terms in the second (and higher) derivatives of the depth. The effect of these terms diminishes in the above cases 1 (in which small-scale roughness is irrelevant to first-order resonant scattering of long waves) and 2 (in which the second derivative of the depth is small). In the present work we make use of the property of the extended MSE (EMSE) and the modified MSE (MMSE) which give an accurate representation of the wave–bottom interaction in the case of resonance, for any k_0h_0 and for any scale of the bottom roughness (as shown in Agnon 1999, using an accurate, augmented MSE). We apply the mean field method to derive dispersion equations for the mean field in the EMSE and for the MMSE. These models yield exactly the same attenuation as does the full first-order solution. This is our first result. We further extend the analysis to the case of a slowly varying mean depth. Here the Fourier method approach no longer applies and we use operator calculus. The pseudo-differential notation simplifies the analysis, and facilitates the separate (and different) treatment of the slow (but not small) mean-depth variation, and the small (but not necessarily slow) bottom roughness. In §2 we solve the MMSE and EMSE, and in §3 we solve the full (first-order) potential problem, using the augmented MSE. The bottom roughness is assumed to be homogeneous, but not necessarily isotropic. In §4 we discuss the consistent approximation leading to the MMSE. The perfect agreement with the full linear theory found for the damping rate, demonstrates the validity of the EMSE and MMSE for describing scattering of waves by a variable-depth, non-isotropic rough bottom.

2. Solution using the model equations

The problem considered is the propagation of irrotational water waves over a rough bottom. The fluid occupies at rest the domain $-h < z < 0$, where $h(x, y) = h_0(x, y) + \chi(x, y)$ and $\nabla h_0 = O(\epsilon)$, $\chi/h = O(\epsilon)$, $\nabla^2 h_0 = O(\epsilon^2)$, $\epsilon \ll 1$. $\chi(x, y)$ are random depth fluctuations with zero mean. $\nabla = (\partial_x, \partial_y)$ is the horizontal gradient. In the framework of the MSE and its variants, the flow field can be described in terms of the value of the complex velocity potential at the free surface, which is written in the form

$$\varphi(x, y)\exp(\mathrm{i}\omega t)$$

where ω is the wave angular frequency, related to its wavenumber k through the linear dispersion relation

$$\omega^2 = gk\tanh kh \tag{1}$$

g is acceleration due to gravity, $\mathrm{i} = \sqrt{-1}$.

The MSE (and its modification and extensions) were traditionally obtained by approximating the complex velocity potential ϕ in the form

$$\phi = \varphi(x, y)Z(h, z)\exp(\mathrm{i}\omega t), \qquad Z(h, z) \equiv \mathrm{sech}(kh)\cosh(k(z + h)),$$

i.e. the vertical structure is that of free waves. In Agnon (1999), operator calculus is used to show rigorously that this is valid at Class I Bragg resonance. The MSE neglects terms in $\nabla^2 h$, $\nabla^3 h, \ldots$, hence it is only valid on gentle bathymetries (scattering by large-scale roughness). The EMSE and MMSE effectively keep all these terms (at Class I Bragg resonance) and are thus valid (at resonance) for the full range of roughness scales. The EMSE (Kirby 1986) has the form

$$(\nabla^2 + k_0^2)\varphi + uh_0^{-1}\nabla h_0 \cdot \nabla\varphi - vh_0^{-1}\nabla\chi \cdot \nabla\varphi + vk_0^2\chi\varphi = O(\nabla h)^2 \qquad (2)$$

where

$$c_0 c_{g0} u = h_0 \frac{d(c_0 c_{g0})}{dh_0}$$

so

$$u\nabla h_0 = h_0 \frac{\nabla(c_0 c_{g0})}{c_0 c_{g0}}, \quad v = \frac{h_0}{k_0^2}\frac{\partial(k_0^2)}{\partial h_0} = -\frac{\operatorname{sech}^2(k_0 h_0) g h_0}{c_0 c_{g0}}; \qquad (3)$$

$c = \omega/k$ is the wave celerity, $c_g = \partial\omega/\partial k$ its group velocity and the subscript $_0$ stands for the value at h_0. The MMSE (Chamberlain & Porter 1995) has the form

$$(\nabla^2 + k^2)\varphi + uh^{-1}\nabla h \cdot \nabla\varphi = -sh^{-1}(\nabla^2 h)\varphi + O(\nabla h)^2 \qquad (4)$$

where

$$s(k,h) = \frac{gh}{cc_g}\int_{-h}^{0} Z\, Z_h \mathrm{d}z.$$

On a rough bottom, (4) can be expressed as

$$(\nabla^2 + k_0^2)\varphi + uh_0^{-1}\nabla h \cdot \nabla\varphi + h_0^{-1}u\nabla\chi \cdot \nabla\varphi + sh_0^{-1}(\nabla^2\chi)\varphi + vh_0^{-1}k_0^2\chi\varphi = O(\nabla h)^2 \qquad (5)$$

Assuming the vertical structure to be in the form of Z, equation (2) and (4) are equivalent (cf. Chamberlain & Porter 1995). This assumption is valid at Class I Bragg resonance.

In order to simplify the analysis, we perform a Liouville transformation that eliminates the term in ∇h_0, and transforms the left-hand side of (2) into a Helmholtz equation (e.g. Mei 1989). This reduces the problem to one that is effectively equivalent to the case of constant h_0. We obtain

$$(\nabla^2 + k_c^2)\psi = h_0^{-1}v(\nabla\chi \cdot \nabla\psi - k_c^2\chi\psi) + O(\nabla h)^2, \qquad (6)$$

where

$$\psi = (c_0 c_{g0})^{1/2}\varphi, \quad k_c^2 = k_0^2 - \frac{\nabla^2(c_0 c_{g0})^{1/2}}{(c_0 c_{g0})^{1/2}}; \qquad (7)$$

since the difference between k_c and k_0 is $O(\epsilon^2)$, we set $k_c = k_0$.

Similarly, we obtain from (5)

$$(\nabla^2 + k_0^2)\psi = -h_0^{-1}u\nabla\chi \cdot \nabla\psi - h_0^{-1}s(\nabla^2\chi)\psi - h_0^{-1}k_0^2 v\chi\psi + O(\nabla h)^2. \qquad (8)$$

Following PRS we obtain from the MSE

$$(\nabla^2 + k_0^2)\psi = -h_0^{-1}u\nabla\chi \cdot \nabla\psi - h_0^{-1}k_0^2 v\chi\psi + O(\nabla^2 h) + O(\nabla h)^2. \qquad (9)$$

We continue the derivation with the transformed variable, ψ.

To study wave scattering in a basin with weakly rough topography, the mean field method is used (e.g. Howe 1971). This method presents the wave field in the form $\psi = \psi_0 + \psi'$: ψ_0 is the coherent, mean field; ψ' is the incoherent, fluctuating (zero-mean) field, assumed to be weak due to the weak random variability of the bottom depth. After using a perturbation method, the incoherent component can be eliminated and a closed integro-differential equation is obtained for the mean field component. This approach was used to find the attenuation coefficient for a monochromatic wave in the framework of linear potential theory (Dyatlov & Pelinovsky 1990) and in the framework of the MSE (PRS). Here this approach will be applied to (5). The same approach applies to (4).

Taking an ensemble average (as in Howe 1971, denoted by $\langle\rangle$) of the transformed EMSE (5), we obtain

$$(\nabla^2 + k_0^2)\psi_0 - h_0^{-1}v\langle\nabla\chi\cdot\nabla\psi'\rangle + h_0^{-1}vk_0^2\langle\chi\psi'\rangle = 0. \tag{10}$$

Subtracting (9) from (5) we obtain at the leading order the following equation for ψ':

$$(\nabla^2 + k_0^2)\psi' - h_0^{-1}v\langle\nabla\chi\cdot\nabla\psi_0\rangle + h_0^{-1}vk_0^2\langle\chi\psi_0\rangle = 0. \tag{11}$$

We now operate on the last equation with $(\nabla^2 + k_0^2)^{-1}$. We find

$$\psi' = (\nabla^2 + k_0^2)^{-1}h_0^{-1}v(\langle\nabla\chi\cdot\nabla\psi_0\rangle - k_0^2\langle\chi\psi_0\rangle). \tag{12}$$

The operators can be evaluated in Fourier space, since the commutator of ∇ and k_0, ∇k_0, is $O(\epsilon)$ and can be neglected.

We may now follow the analysis of PRS, regarding the classical MSE. However, we replace their u and v by the corresponding coefficients from the EMSE.

The depth fluctuations are assumed to be statistically homogeneous, as in PRS, but not necessarily isotropic, so that

$$\langle\hat{\chi}(\boldsymbol{q}-\boldsymbol{\lambda})\hat{\chi}(\boldsymbol{\lambda}-\boldsymbol{\xi})\rangle = \langle\chi^2\rangle T(|\boldsymbol{q}-\boldsymbol{\lambda}|,\alpha)\delta(\boldsymbol{q}-\boldsymbol{\xi}), \tag{13}$$

where $\hat{\chi}$ is the Fourier transform of χ, and δ is Dirac's delta; T is the two-dimensional spatial spectrum of the correlation coefficient and α is the argument of $\boldsymbol{q}-\boldsymbol{\lambda}$. For the MSE this leads to the following dispersion relation for $\boldsymbol{q}_{MS}$, the wavenumber of the mean field:

$$q_{MS}^2 = k_0^2 + \langle\chi^2\rangle h_0^{-2}\iint_{-\infty}^{+\infty}\frac{(k_0^2(u+v) - u\boldsymbol{\lambda}\cdot\boldsymbol{q})^2}{\lambda^2 - k_0^2}T(|\boldsymbol{q}-\boldsymbol{\lambda}|,\alpha)\mathrm{d}\boldsymbol{\lambda}; \tag{14}$$

$\boldsymbol{\lambda}$ is the wavenumber of the scattered fluctuation field. Using polar coordinates, PRS expressed the (leading-order) attenuation coefficient, Im q_{MS} in terms of the semi-residue of the pole at $\lambda = k_0$ of the integrand in (14):

$$\mathrm{Im}q_{MS} = \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2}k_0^3\int_0^{2\pi}\left(2u\sin^2\tfrac{1}{2}\theta + v\right)^2 T\left(2k_0\sin\tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right)\mathrm{d}\theta, \tag{15}$$

where θ is the argument of $\boldsymbol{\lambda}$ and $\alpha = \frac{1}{2}\theta - \frac{1}{2}\pi$ at resonance ($\lambda = k_0$). Here we have generalized their result by relaxing the condition that the depth fluctuations are isotropic. The attenuation coefficient for the EMSE is found in the same way:

$$\begin{aligned}\mathrm{Im}q_E &= \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2}k_0^3\int_0^{2\pi}\left(2u_E\sin^2\tfrac{1}{2}\theta + v\right)^2 T\left(2k_0\sin\tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right)\mathrm{d}\theta \\ &= \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2}k_0^3v^2\int_0^{2\pi} T\left(2k_0\sin\tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right)\cos^2\theta\mathrm{d}\theta,\end{aligned} \tag{16}$$

where we denoted the coefficient of $\nabla\chi$ in the EMSE by

$$u_E = -v. \tag{17}$$

We see that the value of $\mathrm{Im}q_E$ is identical to the attenuation coefficient γ found by PRS (see the next section), using the full (first-order) potential theory (after accounting for the directionality of T). This is because the mean field evolution is determined completely by the interaction with the components of the bathymetry spatial spectrum (K, α) which are in Class I Bragg resonance with the mean field, which has wavenumber $\boldsymbol{q} \approx (k_0, 0)$. Mathematically, this is due to the value of the integral being determined by the residue at the pole. Physically this is due to the dynamics of stochastic interaction being determined fully by the interaction at exact resonance. The resonance condition is indeed

$$(K, \alpha) = \boldsymbol{\lambda} - (k_0, 0) \quad \text{where} \quad \lambda = k_0,$$

i.e.

$$\left(K = 2k_0 \sin \tfrac{1}{2}\theta, \alpha = \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right). \tag{18}$$

For the MMSE, at resonance, the term $s(\nabla^2\chi)$ can be written as

$$s(\nabla^2\chi) = -sK^2\chi = -s\left(2k_0 \sin \tfrac{1}{2}\theta\right)^2 \chi = -4sk_0^2 \sin^2 \tfrac{1}{2}\theta\chi. \tag{19}$$

Thus we get for the MMSE the attenuation coefficient:

$$\begin{aligned} \mathrm{Im} q_M &= \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2} k_0^3 \int_0^{2\pi} \left(v + (2u - 4s)\sin^2 \tfrac{1}{2}\theta\right)^2 T\left(2k_0 \sin \tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right) \mathrm{d}\theta \\ &= \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2} k_0^3 v^2 \int_0^{2\pi} T\left(2k_0 \sin \tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi\right) \cos^2 \theta \mathrm{d}\theta \end{aligned} \tag{20}$$

since from Leibniz rule we have

$$u = 2s - v \tag{21}$$

Thus, the attenuation coefficient for the MMSE also agrees exactly with the full first-order potential theory. This is again because the MMSE agrees with first-order potential theory at Class I Bragg resonance. In the next section we address the attenuation on a topography with a slowly varying mean depth, using first-order potential theory. We use operator calculus to generalize the dispersion equation given by PRS (cf. also Dyatlov & Pelinovsky 1990).

3. Solution using first-order potential theory

In order to study the scattering using potential flow theory, we use the augmented MSE (cf. Agnon 1999). The idea is to choose a 'locally fixed' reference depth, h_{00}, and expand the bottom boundary condition in $\delta_h \equiv h - h_{00} = \chi + h_0 - h_{00}$, in a small neighbourhood. ∇ and h_{00} commute, which greatly simplifies the analysis.

The velocity potential obeys the Laplace equation:

$$\phi_{zz} = -\nabla^2\phi. \tag{22}$$

The linearized free surface boundary condition is

$$w \equiv \phi_z = \frac{\omega^2}{g}\varphi, \tag{23}$$

and the bottom boundary condition, which we expand (to second order in δ_h) about the reference depth h_{00}, is

$$\phi_z = -\nabla \cdot (\delta_h \nabla\phi), \quad z = -h_{00}. \tag{24}$$

Following Rayleigh (1876) we use the compact notation

$$\cos(z\nabla) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}\nabla^{2n}, \quad \sin(z\nabla) \equiv \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}\nabla^{2n+1},$$

and similar expressions for other trigonometric functions, including $\mathrm{sinc}(z\nabla) \equiv \sin(z\nabla)/(z\nabla)$. These are representations of integral operators in the form of pseudo-differential operators. It is useful to think of these operators, and the non-local operators and kernels that will be introduced in what follows, as operators in Fourier space:

$$FT(OP(\nabla)\phi) = OP(\mathrm{i}\boldsymbol{k})FT(\phi), \tag{25}$$

where FT stands for the Fourier transform (with respect to (x, y)) and OP is some function; $\boldsymbol{k}$ is the Fourier variable (wavenumber).

The potential ϕ which solves the Laplace equation (22) is given by a Taylor series in terms of the still-water-level values φ and w (the vertical velocity, ϕ_z), as follows:

$$\phi = \cos(z\nabla)\varphi + z\,\mathrm{sinc}(z\nabla)w \tag{26}$$

(cf. Miles & Chamberlain 1998). Substituting this expression into (24), we obtain

$$\sin(h_{00}\nabla)\nabla\varphi + \cos(h_{00}\nabla)w = -\nabla\cdot(\delta_h(\cos(h_{00}\nabla)\nabla\varphi - \sin(h_{00}\nabla)w)). \tag{27}$$

The manipulation of operator functions is essentially the same as that of scalar functions and can be checked by applying the addition and multiplication properties of ∇ to the Taylor series, term by term (cf. Courant & Hilbert 1953; Berg 1967, for a simple account of operational calculus). Caution should be exercised regarding the commuting of ∇ and h.

Operating on (27) with $\sec(h_{00}\nabla)$ we obtain, to order ε,

$$(\tan(h_{00}\nabla)\nabla + \omega^2/g)\varphi = -\sec(h_{00}\nabla)\nabla\cdot(\delta_h \sec(h_{00}\nabla)\nabla\varphi). \tag{28}$$

On the left-hand side we now have the operator

$$\left(\frac{\nabla\tan h\nabla}{k\tanh kh} + 1\right)\frac{\omega^2}{g}, \tag{29}$$

where k is the real root of the dispersion relation. In addition, (1) has a series of imaginary roots, $\mathrm{i}k^{(n)}$, which stand for the evanescent modes:

$$gk^{(n)}\tan k^{(n)}h = \omega^2, \qquad n = 1, 2, \ldots.$$

As with an algebraic polynomial, the 'dispersion operator' (29) can be factored into an infinite product:

$$\left(\frac{\nabla^2}{k^2} + 1\right)\prod_{n=1}^{\infty}\left(1 - \frac{\nabla^2}{(k^{(n)})^2}\right)\frac{\omega^2}{g} \equiv \frac{\nabla^2 + k^2}{G(h\nabla)}.$$

In considering propagating waves (i.e. non-evanescent) only the first factor, (∇^2/k^2+1), is small (it is $O(\nabla\delta_h)$ and vanishes for free waves on a flat bottom). The other factors were collected in $1/G(h\nabla)$. G is an operator which is even in $h\nabla$ and in $\kappa \equiv hk$:

$$G(p) \equiv h_{00}^{-1}\frac{p^2 + \kappa^2}{p\tan(p) + \kappa\tanh(\kappa)}.$$

Note that at resonance

$$G(\kappa) = \lim_{\kappa' \to \kappa} G(\kappa') = g \left(\frac{\mathrm{d}(\omega^2)}{\mathrm{d}(k^2)} \right)^{-1} = \frac{g}{cc_g}. \tag{30}$$

In order to get an MSE-type equation, we operate on (28) by the non-singular component, $G(h\nabla)$. This yields a left-hand side in the form of the Helmholtz equation (and of the MSE), appropriate for the restriction to propagating modes, but without further approximations of the vertical structure (as those of the MSE, and its previous linear and nonlinear extensions):

$$(\nabla^2 + k^2)\varphi = -G(h_{00}\nabla)\sec(h_{00}\nabla)\nabla \cdot (\delta_h \sec(h_{00}\nabla)\nabla\varphi). \tag{31}$$

This is the form of the augmented MSE given by Agnon (1999). Now, we separate the contribution of the (slow) variation of h_0, and the (small) variation of χ. We expand $-G(h_{00}\nabla)\sec(h_{00}\nabla)\nabla(h_0 - h_{00})$ to $O(\nabla h)$, with the result

$$(\nabla^2 + k^2)\varphi + h_0^{-1}u\nabla\chi \cdot \nabla\varphi = -G(h_0\nabla)\sec(h_0\nabla)\nabla \cdot (\chi \sec(h_0\nabla)\nabla\varphi). \tag{32}$$

Note that the left-hand side is the classical MSE. We have used the rule

$$\nabla^n(h_0\varphi) = h_0\nabla^n\varphi + n\nabla h_0\nabla^{n-1}\varphi + O(\varepsilon^2). \tag{33}$$

We now make a Liouville transformation (equation (7)). It leads to the augmented MSE:

$$(\nabla^2 + k^2)\psi = -G(h_0\nabla)\sec(h_0\nabla)\nabla \cdot (\chi \sec(h_0\nabla)\nabla\psi). \tag{34}$$

This is a pseudo-differential equation. It serves as a basis for consistent derivation of approximate differential equations, such as the MMSE and EMSE, which accurately describe resonant interaction (see §4 and Agnon 1999). Following the derivation of (10) and (12), we obtain

$$(\nabla^2 + k_0^2)\psi_0 = -\langle G(h_0\nabla)\sec(h_0\nabla)\nabla \cdot (\chi \sec(h_0\nabla)\nabla\psi')\rangle, \tag{35}$$

$$\psi' = -(\nabla^2 + k_0^2)^{-1}\langle G(h_0\nabla)\sec(h_0\nabla)\nabla \cdot (\chi \sec(h_0\nabla)\nabla\psi_0)\rangle. \tag{36}$$

As with the EMSE, the augmented MSE leads this time to the following dispersion relation for $\boldsymbol{q}$, the wavenumber of the mean field:

$$q^2 = k_0^2 + \langle\chi^2\rangle h_0^{-2} \iint_{-\infty}^{+\infty} \frac{(k_0^2(u_A + v_A) - u_A \boldsymbol{\lambda} \cdot \boldsymbol{q})^2}{\lambda^2 - k_0^2} T(|\boldsymbol{q} - \boldsymbol{\lambda}|, \alpha)\mathrm{d}\boldsymbol{\lambda}, \tag{37}$$

where

$$v_A = -u_A = -h_0 G(\mathrm{i}\lambda h_0)\mathrm{sech}(\lambda h_0)\mathrm{sech}(k_0 h_0). \tag{38}$$

We may express the (leading-order) attenuation coefficient, Im_q in terms of the semi-residue of the pole at $\lambda = k_0$ of the integrand in (34). Since at resonance, $\lambda = k_0$,

$$v_A = -h_0 G(\mathrm{i}k_0 h_0)\,\mathrm{sech}\,(k_0 h_0)\,\mathrm{sech}\,(k_0 h_0) = v \tag{39}$$

(cf. (30)), the leading-order result is

$$\mathrm{Im}q = \tfrac{1}{4}\pi\langle\chi^2\rangle h_0^{-2}k_0^3 v^2 \int_0^{2\pi} T(2k_0 \sin\tfrac{1}{2}\theta, \tfrac{1}{2}\theta - \tfrac{1}{2}\pi)\cos^2\theta\mathrm{d}\theta \tag{40}$$

which is the result quoted in §2 (cf. PRS; Dyatlov & Pelinovsky 1990), generalized to the case of uneven mean bottom (as well as relaxing the condition that the depth fluctuations are isotropic). It is quite striking to note (as we have in §2) that

$$\mathrm{Im}q = \mathrm{Im}q_E = \mathrm{Im}q_M. \tag{41}$$

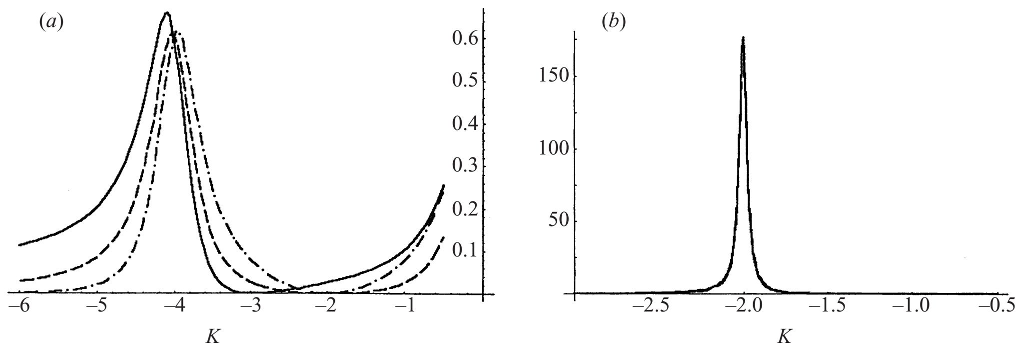

FIGURE 1. The spectrum of the incoherent component ψ' scattered by $\chi(\boldsymbol{K})$, plotted versus K, for the three models: ———, EMSE; –·–·–, AMSE; ———, MMSE. The spectrum is normalized by the amplitude of $\chi(\boldsymbol{K})$ and by the amplitude of the coherent wave. (*a*) $kh = 2$; Im$q = 0.63$; (*b*) $kh = 1$; Im$q = 0.015$.

The wave vector of ψ' is the sum of the wave vectors of χ and of ψ_0. It is interesting to compare the spectrum of ψ', found from the augmented MSE, with the corresponding spectrum for ψ' found from the EMSE, given by (12), as well as the values from the MMSE and MSE. Let us consider incident plane waves, given by $\hat{\psi}_0(\boldsymbol{\lambda}) = a\delta(\boldsymbol{\lambda} - \boldsymbol{k}_{00})$ with a wave vector $\boldsymbol{k}_{00} = (k_{00x}, k_{00y}) = (k_0, 0)$. Since we assumed a statistically homogenous bottom roughness, we obtain for the EMSE

$$\langle|\hat{\psi}'(\boldsymbol{\lambda})|^2\rangle_E = \left(\frac{a(-u\boldsymbol{\lambda}\cdot\boldsymbol{k}_{00})}{h_0(-\lambda^2 + q_E^2)}\right)^2 \langle\chi^2\rangle T(\boldsymbol{k}_{00} - \boldsymbol{\lambda}), \tag{42}$$

(we neglect Re$q - k_0$ which was shown to be very small by PRS). For the MSE

$$\langle|\hat{\psi}'(\boldsymbol{\lambda})|^2\rangle_{MS} = \left(\frac{a(k_0^2 w - u\boldsymbol{\lambda}\cdot\boldsymbol{k}_{00})}{h_0(-\lambda^2 + q_{MS}^2)}\right)^2 \langle\chi^2\rangle T(\boldsymbol{k}_{00} - \boldsymbol{\lambda}), \tag{43}$$

and for the MMSE

$$\langle|\hat{\psi}'(\boldsymbol{\lambda})|^2\rangle_M = \left(\frac{a(k_0^2 w - u\boldsymbol{\lambda}\cdot\boldsymbol{k}_{00} + s(\boldsymbol{k}_{00} - \boldsymbol{\lambda})^2)}{h_0(-\lambda^2 + q_M^2)}\right)^2 \langle\chi^2\rangle T(\boldsymbol{k}_{00} - \boldsymbol{\lambda}). \tag{44}$$

For the augmented MSE we obtain

$$\langle|\hat{\psi}'(\boldsymbol{\lambda})|^2\rangle = \left(\frac{a(-u_A\boldsymbol{\lambda}\cdot\boldsymbol{k}_{00})}{h_0(-\lambda^2 + q^2)}\right)^2 \langle\chi^2\rangle T(\boldsymbol{k}_{00} - \boldsymbol{\lambda}). \tag{45}$$

The spectrum of ψ' is not isotropic, even if the roughness χ is (since ψ_0 is not isotropic).

In figure 1 we plot the form of the coefficient of $a^2\langle\chi^2\rangle$ for the different model equations. We see that the agreement is perfect only for Class I Bragg resonance conditions. However, that is where the scattered wave is most pronounced. It is precisely the values of the scattered wave at these resonance conditions that completely determine the value of the attenuation coefficient of the mean field, Imq. This explains the perfect agreement that is found for Imq. In figure 1(*b*) the value of Imq is smaller and the three models are not graphically distinguishable.

4. Discussion

The limitations of the MSE are well known. It has been closely studied and its validity was found to be limited to cases 1 and 2, described in § 1. The errors in the MSE are $O(\nabla^2 h)$ and $O((\nabla h)^2)$. When considering Class I Bragg resonance of waves in intermediate depth ($k_0 h = O(1)$), by bottom roughness with wavenumbers $|\boldsymbol{q} - \boldsymbol{\lambda}| = O(k_0)$, we have $O(\nabla^2 h) = O(k_0^2 \delta_h) = O(\nabla h)$. Note that higher derivative terms are also of the same order: $O(\nabla^n h) = O(k_0^n \delta_h) = O(\nabla h)$, $n = 3, 4, \ldots$. Thus the MSE is no longer valid and we need to include the effect of all these terms. The EMSE addresses this difficulty by adding the term $-v h_0^{-1} \nabla \cdot (\chi \nabla \varphi)$ to the MSE. The MMSE adds, instead, a different term to the MSE: $s h^{-1} (\nabla^2 h) \varphi$. Both these model equations were tested by a number of studies and compared to solutions of the full equations and to experimental measurements. The agreement found was very good, especially near Class I Bragg resonance, where the scattering is most significant. This agreement is rather surprising. How can a single term account for the effect of all the terms in $\nabla^n h$, $n = 2, 3, \ldots$? How can two such different terms, $-v h_0^{-1} \nabla \cdot (\chi \nabla \varphi)$ in the EMSE and $s h^{-1} (\nabla^2 h) \varphi$ in the MMSE, account for the same effect? The original derivations of the EMSE and MMSE relied on the approximation of the vertical structure of the scattered wave field in the form $\operatorname{sech}(kh) \cosh k(z + h) \varphi$, which is used in deriving the MSE. This approximation no longer holds if we wish to obtain the correct terms of $O(\nabla^2 h)$. The MMSE is derived using an approximate trial function for the vertical structure. The augmented MSE (31) is derived without such assumptions. Indeed, $G(h_0 \nabla) \sec(h_0 \nabla) \nabla \cdot (\chi \sec(h_0 \nabla) \nabla \varphi)$ includes terms in $\nabla^n h$, $n = 1, 2, \ldots$. We understand the success of the EMSE and MMSE in describing scattering, by considering the parameter of detuning from Class I Bragg resonance

$$\mu = -(\boldsymbol{k} + \boldsymbol{K})^2 + k^2, \tag{46}$$

where $\boldsymbol{k}$ is the wave vector of the incident wave potential ψ_0, and $\boldsymbol{K}$ is the wave vector of the depth variation χ. $\boldsymbol{k} + \boldsymbol{K}$ is the wave vector of the scattered wave potential ψ'. At resonance the scattered wave has the same wavenumber, k, as the incident wave. The error in the vertical velocity profile $\operatorname{sech}(kh) \cosh k(z + h) \varphi$ used to derive the EMSE and MMSE is $O(\mu)$, and thus the resulting equations are valid at Class I Bragg resonance, as can also be seen when deriving them from the augmented MSE (31) by setting μ to zero (on the right-hand side). For deterministic wave scattering, resonant interaction plays a dominant role, leading to close agreement of the EMSE and MMSE with measurements and with full (linear) theory. In stochastic scattering, the agreement with the damping rate of the mean field is even more striking: it is perfect. This is because it is completely determined by the resonant interaction.

5. Conclusion

First-order scattering of water waves by stochastic bottom roughness was considered, over variable bathymetry. The bottom roughness is homogeneous, but not necessarily isotropic. The scattering of the mean field into the fluctuating field is completely determined by Class I Bragg resonance with the bottom. Hence, it was shown that both the extended MSE and the modified MSE accurately describe this scattering throughout the full range of scales. This is because these equations faithfully describe resonant interaction, in contrast to the MSE which is an approximation, which holds only for scattering by gentle roughness. One-dimensional bathymetries are a special case of the two-dimensional configuration. The change in the phase speed of the mean field, due to Req, is not accurately reproduced by the model equations. PRS have shown that this change is very small.

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