ON SPATIAL MATCHINGS: THE FIRST-IN-FIRST-MATCH CASE

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Abstract

We describe a process where two types of particles, marked *red* and *blue*, arrive in a domain at a constant rate. When a new particle arrives into the domain, if there are particles of the opposite color present within a distance of 1 from the new particle, then, among these particles, it matches to the one with the earliest arrival time, and both particles are removed. Otherwise, the particle is simply added to the system. Additionally, particles may lose patience and depart at a constant rate. We study the existence of a stationary regime for this process, when the domain is either a compact space or a Euclidean space. In the compact setting, we give a product-form characterization of the stationary distribution, and then prove an FKG-type inequality that establishes certain clustering properties of the particles in the steady state.

Keywords: Stochastic processes; FCFS matching; FKG inequality; spatial clustering

2010 Mathematics Subject Classification: Primary 60D05

1. Introduction

Let D be a metric space, with complete metric d, and let λ be a Radon measure defined over it. Suppose for now that D is compact, so that $\lambda(D) < \infty$. We study the time-evolution of a continuous-time stochastic Markov jump process $\{\eta_t\}_{t \geq 0}$ whose state is defined by an ordered *configuration* of two types of points in D. The two types are assigned to be the colors *red* and *blue*, and referred in short by the letters R and B respectively. A configuration here refers to a locally finite collection of points. When D is compact, a configuration consists of a finite number of points.

The process evolves over the space of ordered configurations on $D \times \{$R, B$\}$ as follows. New particles of each type arrive according to an independent Poisson point process on $D \times \mathbb{R}^+$ with intensity $\lambda \otimes \ell$, where ℓ is the Lebesgue measure on $\mathbb{R}^+$. Suppose, for instance, that a red particle arrives at time $t > 0$, at location $x \in D$. In this case, we look for the first blue particle in the ordered sequence η_{t-} whose distance to location x is less than 1. If there is such a particle, $p \in \eta_{t-}$, the new state η_t is obtained by removing the particle p from η_{t-}, while keeping the order of the remaining elements fixed. If there is no such particle, then the new state is obtained by adding a particle, with location x and mark R, to η_{t-}. In this case, the order within the elements of η_{t-} is preserved and the new particle is placed at the end of the sequence η_{t-}. The arrival of a blue particle is handled similarly. Additionally, particles in the configuration are removed at a constant rate $\mu > 0$, while preserving the order of the remaining particles. Note
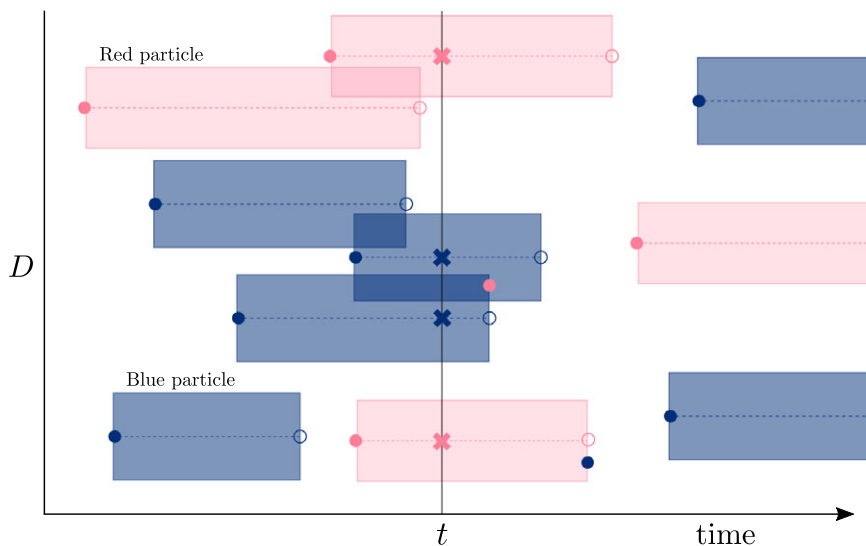
FIGURE 1. An illustration of the FIFM spatial matching process. The vertical dimension represents the set D. The rectangles represent the lifetimes of particles in the system—so, the vertical dimension of a rectangle represents the spatial range of interaction of a particle, the solid disk at the left of the rectangle represent its arrival, and the circle at the right represents its departure. The set of particles present in η_t are marked by crosses; these are the particles whose rectangles intersect the 'vertical line' at time t.

that if η_0 is the empty configuration, then the ordering of particles in η_t is simply the order in which those particles have arrived in the system. We will call this the first-in-first-match (FIFM) spatial matching process.

Figure 1 gives an illustration of the above dynamics.

1.1. Motivation and previous work

The motivation for studying this model comes from modern shared-economy markets, where individuals engage in monetized exchange of goods that are privately owned in a peer-to-peer marketplace. Examples of such marketplaces include ride-sharing networks, such as Uber or Lyft, and renewable energy networks with distributed generation of power. Here, consumers and producers can be viewed as individuals distributed in an abstract space, who engage in transactions with others in close proximity. The abstract space could model, for example, physical location, product preferences, price and willingness to pay, etc. In the example of a ride-sharing network, the position of an individual would correspond to its physical location, while in a renewable energy network, the position could model a combination of location and price. In this study, our goal is to comment on the spatial distribution of individuals in the long run, under a well-defined matching scheme such as the one described in the introduction.

The underlying dynamics in our model can be viewed from a queueing-theoretic perspective. Most queueing-theoretic models study systems where there is an inherent asymmetry between customers and servers. Customers are usually transient agents that arrive with some load, and depart on being processed. Servers meanwhile are present during the whole lifetime of the stochastic process, and serve the customers according to a given policy. In the literature, there are only a few examples of queueing systems where customers and servers are treated

as symmetric agents that serve each other. The double-ended queueing model discussed in Kashyap [15] is a model for a taxi stop where taxis and customers arrive independently according to two Poisson arrivals. If a taxi (or a customer) arrives at the taxi stop and finds a customer (taxi) waiting, then it matches instantly, using say a first-come-first-served (FCFS) policy, and both agents depart. Otherwise, the taxi (customer) waits until it is matched with a customer (taxi) that arrives later.

The FCFS bipartite matching model, which was introduced in Caldentey *et al.* [8] and later studied in some generality in Adan *et al.* [1], is a generalization of Kashyap's model. In this model, the customers and servers belong to finite sets of types, denoted by C and S respectively, which determine whom they can be matched to. The compatibility of matches between the various types of customers and servers is expressed in terms of a bipartite graph $G = (C, S, \mathcal{E})$, where $\mathcal{E} \subset C \times S$. The process, $\{\eta_t\}_{t \in \mathbb{N}}$, is the ordered list of unmatched customers and servers arriving before time $t \in \mathbb{N}$. At each time $t \in \mathbb{N}^+$, one customer, $c_t \in C$, and one server, $s_t \in S$, arrive in the system. Here, $\{c_t\}_{t \in \mathbb{N}^+}$ and $\{s_t\}_{t \in \mathbb{N}^+}$ are independent sequences of independent and identically distributed (i.i.d.) random elements of C and S, with distributions α and β, respectively. The state η_t is obtained from η_{t-1}, c_t, and s_t by instantaneously matching c_t and s_t from elements in η_{t-1}, if possible, using the FCFS policy, and removing the matched pairs. This model is called the *FCFS bipartite matching model*. In the series of works [1,2,8], the authors derive a product-form distribution for the steady state under the so-called *complete resource pooling* condition,

$$\alpha(A) < \sum_{(x,y) \in A \times S \subseteq \mathcal{E}} \beta(y) \qquad \forall A \subsetneq C,$$

or equivalently,

$$\beta(A) < \sum_{(x,y) \in C \times A \subseteq \mathcal{E}} \alpha(x) \qquad \forall A \subsetneq S.$$

The authors also provide expressions for performance measures in the steady state, such as the matching rates between certain types of pairs and the waiting times of agents. These expressions are computationally hard to evaluate, owing to the hardness in computing the normalizing constant in the product-form distribution.

We now briefly discuss variants of the FCFS bipartite matching model that have been considered in the literature. Bušić *et al.* [7] generalize the bipartite matching model by dropping independence of arriving types and considering other matching policies. Büke and Chen [6] study a model where the matching policy is probabilistic. In their model, when a customer (or server) arrives in a system, it looks at the possible matches and, independently of everything else, selects one using a probability distribution. There is also a positive probability of not finding any suitable server (customer), in which case it waits for a compatible server (customer). Büke and Chen [6] also consider models where the users are impatient and may depart if they are not matched by a certain time. An exact analysis of these models becomes quite intractable; Büke and Chen [6] study the fluid and diffusive scaling approximations of these systems.

The model we consider in this paper is essentially a continuous-time and continuous-space version of the model studied by Adan *et al.* [1], with the added feature that particles may also lose patience and depart on their own. This spatial matching model is related to the FCFS bipartite matching model in the following sense. Just as that model has the classes of customers and servers, our model has two classes, the red and blue particles; each particle has a location

in D, which is akin to the type within a class; and two particles are considered compatible in the sense of the FCFS bipartite matching model if they are within a distance of 1 from each other. The added feature that particles depart on their own is necessary since otherwise the system does not possess a stationary regime, even on compact domains. (Indeed, this can easily be seen by observing that the difference between the total number of red and blue particles in the system is a balanced random walk, which is an unstable process.) We refer to this property using the term 'reneging', taking a cue from the queueing theory literature [3], where the term refers to the departure of customers in a queue without receiving any service.

In this paper, in the case when D is compact, we derive a product-form characterization of the steady-state distribution of the process we are considering. The analysis needed to obtain this product-form distribution is an extension of the analysis in Adan *et al.* [1] to the continuum. We guess the reversed process and the steady state, and then check the local balance conditions to get the product-form distribution. This result is stated in Theorem 2.2.

Any exact analysis of the clustering of the steady configuration of the points in the steady state from this product-form distribution is prohibitively hard, as there is no closed-form expression for its normalizing constant. In fact, in discrete systems, it is a $\sharp P$-complete problem to compute the normalizing constant. Instead, we resort to proving qualitative results based on the product form, and show that the point process satisfies a Fortuin–Kasteleyn–Ginibre (FKG) lattice property. The lattice structure is as follows: we say $\tilde{\eta} > \tilde{\eta}'$ if and only if $\tilde{\eta}^{R} \supset \tilde{\eta}'^{R}$ and $\tilde{\eta}^{B} \subset \tilde{\eta}'^{B}$. The resulting positive association inequality can be used in conjunction with the results in Błaszczyszyn and Yogeshwaran [5] to prove that the points of the same type are weakly super-Poissonian. This property is the same as the one exhibited by the Widom–Rowlinson two-particle model (see Section 2 of Chayes *et al.* [9] for a definition).

The two-particle Widom–Rowlinson model (introduced in Widom and Rowlinson [21]) is intriguing as it is the first continuum Markov random process where a phase transition has been rigorously established (see [9,13,19]). It was shown that as the total intensity of particles is increased, symmetry-breaking occurs, i.e., with positive probability, there are more points of one type. A key ingredient in the proof of this (as noticed in Chayes *et al.* [9]) is the existence of an additional FKG inequality in this model with the usual lattice structure from the set inclusion. In our FIFM matching model, we believe that there is a phase transition in our steady state, but in contrast to the case of the two-particle Widom–Rowlinson model, we are unable to prove it using a similar technique i.e., by establishing an FKG inequality for the unordered collection of points.

In this paper, we also consider the same matching dynamics in the infinite Euclidean space $\mathbb{R}^d$. In this regime, using coupling-from-the-past-based arguments, we give a formal definition and a construction of the process, and show that there exists a stationary regime for this process. The existence of the stationary regime is obtained using certain coupling-from-the-past ideas that were developed in Baccelli *et al.* [4].

In the following section, we will discuss the notation required to formally define our model. Other notation will be required as we go along—see Appendix E for a table of notation. We begin Section 2 with the formal definition of the model in a compact domain. In Section 2.1, we give a coupling-based argument that the steady state exists and is unique, and in Section 2.2, we present the product-form distribution for this steady state. Then, in Section 2.3, we introduce and prove the FKG lattice property satisfied by the unordered-version product-form distribution. We then proceed to study the model in an infinite Euclidean space, in Section 3. For the sake of clarity, in each of these sections, we only state the main result; the proofs and auxiliary results are deferred to the appendices.

1.2. Notation

Let D be any metric space, endowed with a Radon measure, $\lambda(\,\cdot\,)$. We use $M(D)$ to denote the space of simple counting measures on D. There is a one-to-one correspondence between $M(D)$ and the space of locally finite collections of points in D. $M(D)$ is equipped with the σ-algebra $\mathcal{F}$ generated by the maps $\gamma \mapsto \gamma(B)$, $B \in \mathcal{B}(D)$, where $\mathcal{B}(D)$ is the Borel σ-algebra on D. In our presentation, we will frequently abuse notation and use the same variable to denote both an element of $M(D)$ and its support, which is a subset of D.

Every particle in our model also carries information about its patience. To encode this, we need the notion of a marked counting measure. A marked simple counting measure on D, with marks in a space K, is a locally finite simple counting measure γ on $D \times K$ such that its projection $\gamma(\,\cdot\,\times K)$ is an element of $M(D)$. We denote the space of all such measures by $M(D, K)$.

We will also require the definition of the space $O(D)$ of locally finite totally ordered collections of points in the space D. For any $\xi \in O(D)$, the order within the elements of ξ will be denoted by $<_\xi$. The order will be used to indicate the priority of the particles when matching with other particles. So, if the state of the system is $\xi \in O(D)$ and an incoming point x can match with either y_1 or y_2, where $y_1 <_\xi y_2$, then it prefers y_1 over y_2. $O(D)$ has a natural projection onto $M(D)$, obtained by dropping the order within its elements—for any $\xi \in O(D)$, the unordered collection is denoted by $\tilde{\xi}$. For compact D, $O(D)$ may be canonically identified with $\bigsqcup_{n=0}^{\infty} \{x \in D^n : x_i \neq x_j, \forall 1 \leq i < j \leq n\}$. Finally, the space of totally ordered marked locally finite collections of points, with marks in K, will be denoted by $O(D, K)$.

For any $\gamma \in M(D)$ (or $O(D)$), we will denote by $|\gamma|$ the number of elements in γ; i.e., $|\gamma| = \gamma(D)$.

As mentioned in the introduction, the symbols $\mathtt{R}$ and $\mathtt{B}$ will be used to denote the types red and blue respectively. Moreover, we will let $C = \{\mathtt{R}, \mathtt{B}\}$, and let a line over a color denote the opposite color; i.e., $\bar{\mathtt{R}} = \mathtt{B}$ and $\bar{\mathtt{B}} = \mathtt{R}$.

Finally, throughout the paper we will let $\mathbb{1}$ denote the indicator function.

2. First-in-first-match matching process on compact domains

In this section, we first give a formal definition of the process on a compact domain. Let D be a compact metric space with a Radon measure λ. The state of the process will contain information about the locations, colors, and order of arrival of the particles present in the system. Thus, the state space will be the set of totally ordered collections of particles with locations in D and with marks in the set $C = \{\mathtt{R}, \mathtt{B}\}$, namely $O(D, C)$. The order represents the order of arrival of particles into the system, and hence represents their priority when two particles are in contention to be matched to the same particle.

We will require the following notation to describe the evolution of the process. For a point $x \in D \times C$, we denote the projection onto D by p_x and the projection onto C by c_x. For any point $x \in D \times C$, we denote the set of incompatible points of opposite color by $N(x) := B(p_x, 1) \times \{\bar{c}_x\}$, where $B(z, r)$ denotes the ball of radius r centered at z. For any subset $A \subseteq D \times C$, we set $N(A) := \cup_{x \in A} N(x)$. Let $\bar{\lambda} := \lambda \otimes m_c$, where m_c is the counting measure on C. For any $\gamma \in O(D, C)$ and $x \in \gamma$, let γ^x be the element of $O(D, C)$ formed by $\{y \in \gamma : y <_\gamma x\}$ ordered as in γ. Further, if γ is represented as a list $(x_1, \ldots, x_n)$, then for any i, $1 \leq i \leq n$, we set $\gamma_1^{i-1} := \gamma^{x_i} = (x_1, \ldots, x_{i-1})$. The region of highest priority of a particle x in γ, denoted by $W_{\gamma, x}$ (or just W_x if the context is clear), is defined to be the set $N(x) \backslash N(\gamma^x)$.

Let us recall the description of the dynamics in terms of the notation developed in the previous section. We consider a Markov jump process $\{\eta_t\}_{t \in \mathbb{R}} \subset O(D, C)$. Suppose that a new particle, $y \in D \times C$, arrives at time t. If the ordered collection $\eta_{t-} \cap N(y)$ is non-empty, then the particle y *matches* to the lowest-ranked particle in this set, and the matched particle is removed

from η_{t-}. Equivalently, y matches to $x \in \eta_{t-}$ if and only if $y \in W_{\eta_{t-},x}$. Otherwise, y is added to the ordered set η_t at the end, so that $y > x$ for all $x \in \eta_t$, while the order among the elements of η_{t-} in η_t is preserved. Additionally, independent of everything else, particles may depart on their own when they lose patience, at rate $\mu > 0$.

This description fixes the form of the generator of the process, which is given by

$$Lf(\eta) := \sum_{x \in \eta} \left(\mu + \bar{\lambda}(W_x) \right) [f(\eta \backslash x) - f(\eta)]$$

$$+ \int_{D \times C} \mathbb{1}(x \notin N(\eta))[f((\eta, x)) - f(\eta)]\bar{\lambda}(dx), \tag{1}$$

where f is a measurable function defined over $O(D, C)$, and $(\eta, x) \in O(D, C)$ represents the list obtained by appending x to η. The form of the generator will be useful in characterizing the stationary distribution. From a function-analytic perspective, to completely define the process in question, we need to further specify the domain of the generator (see [17]). We instead give an explicit construction of the process whose generator satisfies Equation (1). That is, if η_t represents the constructed process, we claim that it is easy to verify that

$$\lim_{t \to 0+} \frac{1}{t} \mathbb{E}[f(\eta_t) - f(\eta_0)|\eta_0] = Lf(\eta_0)$$

for any bounded continuous function f over $O(D, C)$. This constructed process will serve as the formal definition of the process we consider in this paper. As we will see in Section 3, the construction on compact domains can be naturally extended to a construction of an FIFM matching process on (unbounded) Euclidean spaces.

Let Φ be a Poisson point process on $D \times \mathbb{R}^+$, with i.i.d. marks in $C \times \mathbb{R}^+$. The intensity of the point process is $2\lambda \otimes \ell$, where ℓ is the Lebesgue measure on $\mathbb{R}^+$. The two marks are independent; the color is uniformly distributed, and the other mark is an exponential random variable with parameter μ. Let $\eta_0 \in O(D, C)$ be the initial state of the system at time 0. For any $x \in D \times \mathbb{R}^+ \times C \times \mathbb{R}^+$, let p_x, b_x, c_x, and w_x denote the projections of x onto the corresponding four spaces in the product $D \times \mathbb{R}^+ \times C \times \mathbb{R}^+$. Thus, for any $x \in \Phi$, p_x is the spatial position of the point x, b_x is the time of its arrival, c_x is its color, and w_x is its patience. For any subset $T \subset \mathbb{R}$, let Φ_T denote the points that arrive in the set T; i.e., $\Phi_T = \{x \in \Phi : b_x \in T\}$. The following display presents an algorithm for the construction of the process on compact domains.

- **Data:**

    1. Φ: A realization of the arrivals.

    2. η_0: A realization of the initial condition.

    3. $t \in \mathbb{R}^+$: End time of simulation.

- **Result:** η_t: The final state of the system at time t.

1. Let $t_{old} \leftarrow 0$.

2. For each $x \in \eta_0$, assign i.i.d. marks w_x that are exponentially distributed with parameter μ.

(continued)

3. Let $t_{new} \leftarrow \min(\inf\{b_x : x \in \Phi_{(t_{old}, \infty)}\}, \inf\{w_x : x \in \eta_{t_{old}}\})$. If $t_{new} > t$, quit and return $\eta_{t_{old}}$.

4. If t_{new} is due to arrival of a new particle (first infimum):

   - Let the particle be x.
   - If there is a particle of opposite color in $B(p_x, 1)$:

     –Match to the first particle of opposite color in $\eta_{t_{old}} \cap B(p_x, 1)$ and remove that particle. This gives $\eta_{t_{new}}$.

   - Else:

     –Add the particle to the end of $\eta_{t_{old}}$ to give $\eta_{t_{new}}$

5. Else if t_{new} is due to a particle $x \in \eta_{t_{old}}$ losing patience (second infimum), then remove this particle to yield $\eta_{t_{new}}$.

6. Let $t_{old} \leftarrow t_{new}$. Go to Step 3.

2.1. Existence and uniqueness of stationary regime

In this section, we begin analyzing the stationary regime of the process on a compact domain D defined in Section 2. First, the existence and uniqueness of a stationary regime can be established by a simple Lyapunov. For the sake of completeness we give an alternate proof, based on a coupling-from-the-past argument.

Suppose we have a bi-infinite time-ergodic Poisson point process Φ on $D \times \mathbb{R}$, with marks in $C \times \mathbb{R}^+$, where the first coordinate is the color of the particle and the second coordinate is the patience, as in the construction in Section 2. Briefly, in the coupling-from-the-past construction, the steady-state realization is constructed as a limit of a sequence of processes started at increasingly negative times using the same driving process Φ. We state the theorems here and present the proofs later in Appendix A.

Theorem 2.1. *For any $T \in \mathbb{N}$, let $\{\eta_t^T\}_{t \geq -T}$ denote a process started at $-T$ with empty initial conditions and driven using the bi-infinite marked Poisson point process Φ. Then, for any $t \in \mathbb{R}$, the limit $\eta_t = \lim_{T \to \infty} \eta_t^T$ exists almost surely (a.s.). The process η_t is a steady-state version of the FCFS matching process, and moreover, this process is unique in distribution.*

In the next section, we present a product-form characterization of this steady-state distribution. For this, the key step is to construct the reversed process.

2.2. Product-form characterization of the steady state

In this section, we give a product-form characterization of the steady-state distribution, by getting a handle on the time-reversal of the steady-state process η. Let Φ be as in Section 2.1.

Theorem 2.1 implies that there exists a unique bi-infinite spatial matching process that is driven by Φ. We can thus define a (random) matching function, $m : \Phi \to D \times \mathbb{R} \times C$, such that

$$m(x) = \begin{cases} (p_x, b_x + w_x, c_x) & \text{if } x \text{ exits on its own,} \\ (p_y, b_y, c_y) & \text{if } x \text{ matches to } y \in \Phi. \end{cases}$$

Conversely, m stores all the information necessary to build the process $\{\eta_t\}_{t \in \mathbb{R}}$. Indeed, the state of the spatial matching process we are interested in is given by

$$\eta_t = ((p_x, c_x) : x \in \Phi, b_x \leq t < b_{m(x)}),$$

where the list is ordered according to the birth times b_x.

Taking inspiration from Adan *et al.* [1], we will include some additional data in the state of the system that will simplify the description of the reversed process. We shall consider a process that we call the *backward detailed process* generated by Φ and m. This process contains unmatched and matched particles in its state, and we distinguish these types by using the marks 'u' and 'm' respectively. For any particle x in the state, s_x will refer to this mark.

For $t \in \mathbb{R}$, let

$$T_t := \min\{b_x : x \in \Phi, b_x \leq t < b_{m(x)}\}$$

be the time of arrival of the earliest among the unmatched particles at time t. Let

$$\Gamma_{u,t} := \{(p_x, b_x, c_x, u) : x \in \Phi, b_x \leq t < b_{m(x)}\}$$

be the set of (locations, arrival times, and colors of) unmatched particles in $[T_t, t]$. Let

$$\begin{aligned}
\Gamma_{m,t} :=& \{(p_x, b_{m(x)}, c_x, m) : x \in \Phi, b_x \leq t, T_t \leq b_{m(x)} \leq t\} \\
=& \{(p_{m(x)}, b_x, c_{m(x)}, m) : x \in \Phi, b_{m(x)} \leq t, T_t \leq b_x \leq t, c_x \neq c_{m(x)}\} \\
& \cup \{(p_x, b_{m(x)}, c_x, m) : x \in \Phi, b_x \leq t, T_t \leq b_{m(x)} \leq t, c_x = c_{m(x)}\}
\end{aligned}$$

be the set of so-called *matched and exchanged* particles that are present in $[T_t, t]$. In the last expression, the first set of elements corresponds to particles that arrive in the relevant interval, $[T_t, t]$, and are matched by the time t; but instead of recording their positions and types, we record those of their matches. The second set of elements in that expression corresponds to particles that arrive before t and that depart on their own in the time interval $[T_t, t]$; we record the time at which they depart.

Finally, define the backward detailed process, $\hat{\eta}_t$, by the list $((p_x, c_x, s_x) : x \in \Gamma_{u,t} \cup \Gamma_{m,t})$, ordered according to the values of b-projections of the points in $\Gamma_{u,t} \cup \Gamma_{m,t}$. Clearly, the original process η_t can be obtained from $\hat{\eta}_t$ by removing the particles with marks $s_x = m$. Notice that if $|\hat{\eta}_t| > 0$, the first element in $\hat{\eta}_t$, denoted by x_1, always satisfies $s_{x_1} = u$.

The backward detailed process, $\hat{\eta}_t$, is a stationary version of a Markov process. A valid state of this Markov process is any finite list of elements $(x_1, \ldots, x_n)$ from the set $D \times C \times \{u, m\}$ that satisfies the following definition.

Definition 2.1. (Definition of a valid state of $\hat{\eta}_t$) *We say that a finite list of elements $(x_1, \ldots, x_n)$, with $n \in \mathbb{N}$ and $x_i \in D \times C \times \{m, u\}$, is a valid state of $\hat{\eta}_t$ if the following three conditions are satisfied:*

1. $s_{x_1} = u$, *if $n \geq 1$.*

2. *For all $1 \leq i, j \leq n$, $s_{x_i} = s_{x_j} = u$ and $d(p_{x_i}, p_{x_j}) \leq 1$ implies that $c_{x_i} = c_{x_j}$.*

3. *For all $1 \leq i < j \leq n$, $s_{x_i} = u$, $s_{x_j} = m$ and $d(p_{x_i}, p_{x_j}) \leq 1$ implies that $c_{x_i} = c_{x_j}$.*

Condition 2 in the above definition essentially states that there cannot be a compatible unmatched pair in a valid state. This is equivalent to the condition that

$$\{y \in x_1, \ldots, x_n : s_y = u\} \cap N(\{y \in x_1, \ldots, x_n : s_y = u\}) = \emptyset.$$

Condition 3 cannot be violated, since otherwise the particle whose matched and exchanged pair is x_j could instead have matched to some x_i that arrived earlier. This is equivalent to the condition that for all $1 \le j \le n$,

$$s_{x_j} = \mathrm{m} \implies x_j \notin N(\{y \in x_1, \dots, x_j : s_y = \mathrm{u}\}).$$

Any valid state can be achieved by the process $\hat{\eta}_t$ in finite time with positive probability starting from the empty state. Indeed, any valid state $\hat{\eta}$ can result from the empty state if the arrivals occur in the order listed in $\hat{\eta}$, with appropriate patience so that the particles in $\hat{\eta}$ marked u survive until time t, and the particles in $\hat{\eta}$ marked m exit on their own before the next arrival.

Transitions for $\hat{\eta}_t$ occur at the time of arrival of a new particle or at the event of a voluntary departure. At the time of a new arrival, we match and exchange the earliest compatible particle in the list $\hat{\eta}_t$; if there is no possible match we add the arriving particle to the end of the list with mark u. At the time of a voluntary departure, we put the departing particle at the end of the list $\hat{\eta}_t$, while updating the mark to m. Below, we describe the transitions and transition rates of this Markov process in detail.

The transitions and transition rates for $\hat{\eta}_t$ are as follows. Let $\hat{\eta} = (x_1, \dots, x_n)$, $n \in \mathbb{N}$, be a valid state.

1. A particle $x_i \in \hat{\eta}$, with $s_{x_i} = \mathrm{u}$, loses patience: This occurs at rate μ. In this case, the new state is obtained by removing x_i and inserting the point $(p_{x_i}, c_{x_i}, \mathrm{m})$ at the end of the list $\hat{\eta}$. Additionally, we need to prune leading matched and exchanged particles from $\hat{\eta}$ to obtain the new state.

2. A new particle $y = (p_y, c_y)$ arrives and is matched to a particle $x_i \in \hat{\eta}$, with $d(p_{x_i}, p_y) \le 1$ and $c_{x_i} \ne c_y$: This occurs at rate $\bar{\lambda}(dy)\mathbb{1}(y \in W_{x_i})$. The new state is obtained by matching and exchanging the appropriate pair, and then pruning the leading matched and exchanged particles.

3. A new particle y arrives and there is no particle of opposite color within a distance of 1 from it: This occurs at rate $\bar{\lambda}(dy)\mathbb{1}(y \notin N(\hat{\eta}))$. The new state is the one obtained by adding this new particle to the end of the list as an unmatched particle.

We now guess the time-reversed version of the backward detailed process and obtain its transition rates. The following construction will be useful in doing this. Consider a dual process $\check{\eta}_t$, which we call the *forward detailed process*. It is defined as follows: for $t \in \mathbb{R}$ let

$$Y_t := \max\{b_{m(x)} : x \in \Phi, b_x \le t < b_{m(x)}\}$$

be the latest time at which all unmatched particles at time t are matched or exit. Let

$$\begin{aligned}
\Xi_{\mathrm{m},t} :&= \{(p_x, b_{m(x)}, c_x, \mathrm{m}) : x \in \Phi, b_x \le t < b_{m(x)}\} \\
&= \{(p_{m(x)}, b_x, c_{m(x)}, \mathrm{m}) : x \in \Phi, b_{m(x)} \le t < b_x, c_x \ne c_{m(x)}\} \\
&\quad \cup \{(p_x, b_{m(x)}, c_x, \mathrm{m}) : x \in \Phi, b_x \le t < b_{m(x)}, c_x = c_{m(x)}\}
\end{aligned}$$

be the matched and exchanged particles corresponding to the particles that were born before time t, but have not been removed from the system by time t. Let

$$\Xi_{\mathrm{u},t} := \{(p_x, c_x, b_x, \mathrm{u}) : x \in \Phi, t < b_x < Y_t, t < b_{m(x)}\}$$

be the particles in the relevant interval (t, Y_t), whose match arrives after time t. Now, let $\check{\eta}_t$ be the list $((p_x, c_x, s_x) : x \in \Xi_{\mathtt{u},t} \cup \Xi_{\mathtt{m},t})$ ordered according to the values $b_{(\cdot)}$. Thus, the last element in the list, $x_{|\check{\eta}|}$, always has $s_{x_{|\check{\eta}|}} = \mathtt{m}$. For motivations for these definitions, see Adan *et al.* [1].

Under our construction, using the bi-infinite Poisson point process Φ, the process $\{\check{\eta}_t\}_{t \in \mathbb{R}}$ is a stationary process. In fact, it is a stationary version of a Markov process, since all the arrivals and deaths are Markovian. The transitions and transition rates are defined in detail in Appendix C.

The underlying idea in obtaining the product-form distribution is stated in the following lemma.

Lemma 2.1. *For any list of elements γ in $D \times C \times \{\mathtt{u}, \mathtt{m}\}$, let $\mathrm{revx}(\gamma)$ denote the list of elements in γ written in the reverse order, with the marks $\mathtt{u}$ and $\mathtt{m}$ flipped. Then we claim that $\{\mathrm{revx}(\check{\eta}_t)\}_{t \in \mathbb{R}}$ has the same distribution as the time-reversal of the backward detailed process. That is,*

$$\{\hat{\eta}_{-t}\}_{t \in \mathbb{R}} \overset{d}{=} \{\mathrm{revx}(\check{\eta}_t)\}_{t \in \mathbb{R}}.$$

To obtain the product-form expression for the stationary distribution, we need to check certain local balance conditions (see Appendix B.1 for the definition of the local balance conditions). This is the main result of this section. However, before we state the result we need to fix some notation.

For any list $\gamma \in O(D, C \times \{\mathtt{u}, \mathtt{m}\})$ and $i \in \mathbb{N}$, we define $Q_\mathtt{u}^i(\gamma)$ to be the number of unmatched particles *among* the first i particles on γ, and define $Q_\mathtt{m}^i(\gamma)$ to be the number of matched particles *excluding* the first i particles of γ. In this notation, we may drop the reference to γ when the context is clear.

In the following result, we use the operator $\oplus$ to denote the direct sum of two measures, and the operator $\otimes$ to denote the direct product.

Theorem 2.2. *The density of the stationary measure of the backward detailed process $\{\hat{\eta}_t\}_{t \in \mathbb{R}}$, with respect to the measure $\bigoplus_{n=0}^\infty (\lambda \otimes m_c \otimes m_c)^n$ on*

$$O(D, C \times \{\mathtt{u}, \mathtt{m}\}) \subset \sqcup_{n=0}^\infty (D \times C \times \{\mathtt{u}, \mathtt{m}\})^n,$$

is given by

$$\hat{\pi}(\gamma) = K \mathbb{1}(\gamma \, is valid) \prod_{i=0}^{|\gamma|} \frac{1}{2\lambda(D) + Q_\mathtt{u}^i(\gamma)\mu} =: K \mathbb{1}(\gamma \, is \, valid) \hat{\Pi}(\gamma), \tag{2}$$

$$\hat{\pi}(\emptyset) = K, \tag{3}$$

where

$$\hat{\Pi}(\gamma) := \prod_{i=0}^{|\gamma|} \frac{1}{2\lambda(D) + Q_\mathtt{u}^i(\gamma)\mu},$$

and where K is the normalizing constant:

$$K^{-1} = \sum_{n=0}^\infty \int_{(D \times C \times \{\mathtt{u}, \mathtt{m}\})^n} \mathbb{1}(\gamma \, is valid) \prod_{i=0}^n \frac{1}{2\lambda(D) + Q_\mathtt{u}^i(\gamma)\mu} (\lambda \otimes m_c \otimes m_c)^{(n)}(d\gamma).$$

The form of the density in (2) is typical of a product-form distribution seen in the context of queueing theory (see Asmussen et al. [3]). For instance, $2\lambda(D) + Q_\mathtt{u}^{|\gamma|}(\gamma)\mu$ is the total rate of

change when the backward detailed process is in state γ, as $2\lambda(D)$ is the total rate of new arrivals and $Q_u^{|\gamma|}(\gamma)\mu$ is the rate at which unmatched particles depart on their own.

For the stationary distribution π of the original process η_t, we compute the marginals of $\hat{\pi}$. Firstly, a state $\gamma = (x_1, \ldots, x_n) \in O(D, C)$ is a valid state of the process $\{\eta_t\}_{t\in\mathbb{R}}$ if and only if $\{x_1, \ldots, x_n\} \cap N(\gamma) = \emptyset$.

Corollary 2.1. *The density of the stationary distribution π of the process η_t, with respect to the measure $\oplus_{n=0}^{\infty} \bar{\lambda}^n$ on $\sqcup_{n=0}^{\infty}(D \times C)^n$, is given by*

$$\pi(\gamma) = K\mathbb{1}(\gamma \text{ is valid})\prod_{i=1}^{|\gamma|}\frac{1}{\bar{\lambda}(N(\gamma_1^i)) + i\mu}, \tag{4}$$

$$\pi(\emptyset) = K, \tag{5}$$

where K is the same normalizing constant as in Theorem 2.2.

The proofs of Theorem 2.2 and Corollary 2.1 are given in Appendix C.

2.3. Clustering properties and the FKG property

In this section, we focus on the stochastic geometric properties of the steady-state arrangement of the particles in the space D. Hence, we forget the reference to the order of the particles in the steady state. Intuitively, the Janossy density of a point process (see Daley and Vere-Jones [10]) is the relative probability of observing a given configuration of points with respect to a given reference measure. The Janossy density of the steady-state distribution of our point process model, with respect to the Poisson point process on $D \times C$ with intensity $\bar{\lambda}$, is given by dropping the order of particles in Equation (4). That is, the Janossy density is

$$\tilde{\pi}(x_1^n) = K\mathbb{1}(x_1^n \text{ is valid})\tilde{\Pi}(x_1^n),$$

$$\tilde{\Pi}(x_1^n) = \sum_{(X_1^n)\in\mathcal{P}(x_1^n)}\prod_{i=1}^{n}\frac{1}{\bar{\lambda}(N(X_1^i)) + i\mu}, \tag{6}$$

where $\mathcal{P}(x_1^n)$ is the set of all permutations of $x_1, \ldots, x_n$, and K is a normalizing constant.

Let us take a moment to interpret the term $\bar{\lambda}(N(x_1^i))$ that appears in the above expression. This is the sum of the volumes of the union of balls around the red particles in $x_1, \ldots, x_i$ and the union of balls around the blue particles in $x_1, \ldots, x_i$. Since such terms appear in the denominator in Equation (6), we expect that in the steady state the particles of the same color are clumped together.

In a variety of point processes, such as the one-particle Widom–Rowlinson model, or certain Cox processes (Błaszczyszyn and Yogeshwaran [5]), the FKG lattice property is a useful tool for proving stochastic dominance and clustering properties. In the case of the Widom–Rowlinson model, the FKG inequality is also useful in showing the existence of a phase transition (Chayes *et al.* [9]). The FKG lattice property defined on a measure ψ over a finite distributive lattice Ω states that for every $\xi, \gamma \in \Omega$,

$$\psi(\xi \vee \gamma)\psi(\xi \wedge \gamma) \geq \psi(\xi)\psi(\gamma). \tag{7}$$

If ψ satisfies (7), it is said to be log-submodular. The FKG lattice property implies the positive association inequality:

$$\psi(fg) \geq \psi(f)\psi(g) \tag{8}$$

for all increasing functions f and g on Ω, where $\psi(f)$ represents the expectation of f with respect to ψ.

This theorem can also be extended to point processes in the continuum as follows (see [9,12] for details). Let P be a point process on a measurable space S, with Janossy density ψ with respect to a Poisson point process with intensity λ; then the FKG lattice property states that

$$\psi(\xi \cup \gamma)\psi(\xi \cap \gamma) \geq \psi(\xi)\psi(\gamma) \quad \text{for all } \xi, \gamma \in M(S). \tag{9}$$

Under this hypothesis, one can conclude positive association inequalities such as (8), where f and g are now increasing functions on $M(S)$.

Remark 2.1. The FKG lattice property in the continuum point process case can also be stated in terms of the Papangelou conditional intensities: if $\varphi(x, \xi)$ is the Papangelou conditional intensity of a point process with Janossy density ψ, then (9) is equivalent to $\varphi(x, \xi) \geq \varphi(x, \xi')$ for all $x \in S$ and $\xi, \xi' \in M(S)$ with $\xi \supseteq \xi'$ (see Georgii and Küneth [12] for details).

In the following, we describe an FKG lattice property that holds for the steady-state version of our model. The property holds under a specific lattice structure defined on $M(D, C)$. Let $\xi = (\xi^R, \xi^B)$ and $\gamma = (\gamma^R, \gamma^B)$ be two configurations in $M(D, C)$, where ξ^R and γ^R are the red particles and ξ^B and γ^B are the blue particles in the configurations. We say that $\xi > \gamma$ if and only if $\xi^R \supset \gamma^R$ and $\xi^B \subset \gamma^B$.

Theorem 2.3. *For any two finite subsets $\xi = (\xi^R, \xi^B)$ and $\gamma = (\gamma^R, \gamma^B)$ of $D \times C$, we have*

$$\tilde{\Pi}(\xi \vee \gamma)\tilde{\Pi}(\xi \wedge \gamma) \geq \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma), \tag{10}$$

where $\xi \vee \gamma = (\xi^R \cup \gamma^R, \xi^B \cap \gamma^B)$ and $\xi \wedge \gamma = (\xi^R \cap \gamma^R, \xi^B \cup \gamma^B)$.

The proof of this theorem is deferred to Appendix D. We note that a similar FKG lattice property is satisfied in the binary-particle Widom–Rowlinson model with the same lattice structure.

From Theorem 2.3 and the FKG inequality (see Appendix in Chayes et al. [9]), we can conclude that the stationary measure is positively associated; i.e., for any two increasing functions f and g,

$$\mathbb{E}_{\tilde{\eta}} f(\tilde{\eta})\, g(\tilde{\eta}) \geq \mathbb{E}_{\tilde{\eta}} f(\tilde{\eta})\mathbb{E}_{\tilde{\eta}}\, g(\tilde{\eta}), \tag{11}$$

where $\tilde{\eta}$ is a version of the unordered stationary process and has the density $\tilde{\pi}$ with respect to the Poisson point process with intensity $\bar{\lambda}$. The above positive association inequalities also imply that the marginal point processes of the red points (or the blue points) are also positively associated. This can be seen by taking increasing functions f and g that depend only on the red points (or the blue points). From this result and Corollary 3.1 of Błaszczyszyn and Yogeshwaran [5], we can conclude that $\tilde{\eta}^R$ and $\tilde{\eta}^B$ are weakly super-Poissonian. Intuitively, this means that the points are more clustered than the points in a Poisson point process of the same intensity.

2.4. Boundary conditions and monotonicity

In this section, we assume that D is a compact subset of the Euclidean space $\mathbb{R}^d$, for some $d \geq 1$. We will use the FKG lattice property to prove monotonicity of measures under different boundary conditions. To state these results we will require the following notation. Let

$\zeta \subset \mathbb{R}^d \backslash D \times C$ be a valid state, i.e., $N(\zeta) \cap \zeta = \emptyset$. For any such boundary condition, we define a measure on $M(D, C)$ with Janossy density

$$\tilde{\pi}_{D,\zeta}(x_1^n) = K_{D,\zeta} \mathbb{1}(N(x_1^n) \cap (\zeta \cup x_1^n) = \emptyset) \tilde{\Pi}_\zeta(x_1^n),$$

$$\tilde{\Pi}_\zeta(x_1^n) = \sum_{(X_1^n) \in \mathcal{P}(x_1^n)} \prod_{i=1}^n \frac{1}{\bar{\lambda}(N(X_1^i) \cap N(\zeta)^c) + i\mu}. \tag{12}$$

Three important boundary conditions are $\zeta = (\mathbb{R}^d \backslash D) \times \{\text{R}\}$, $\zeta = \emptyset$, and $\zeta = (\mathbb{R}^d \backslash D) \times \{\text{B}\}$. These are respectively termed the *red*, the *free* and the *blue* boundary conditions. We use special notation for the densities with these boundary conditions, namely, $\tilde{\pi}_{S,\text{R}}$, $\tilde{\pi}_S$, and $\tilde{\pi}_{S,\text{B}}$.

The boundary conditions can also be partially ordered: let $\zeta_1 \geq \zeta_2$ if and only if $\zeta_1^{\text{R}} \supset \zeta_2^{\text{R}}$ and $\zeta_1^{\text{B}} \subset \zeta_2^{\text{B}}$. We are now in a position to state the first result of this section.

Theorem 2.4. *Let $D \subset \mathbb{R}^d$ be a compact set. Let $\zeta_1 \geq \zeta_2$ be two boundary conditions on D. Then the measure with density $\tilde{\pi}_{D,\zeta_1}$ stochastically dominates the measure with density $\tilde{\pi}_{D,\zeta_2}$.*

Outline of the proof. By Holley's inequality (see Holley [14]), it is enough to prove that for two states $\eta, \gamma \in M(D, C)$, we have

$$\tilde{\pi}_{D,\zeta_1}(\eta \vee \gamma) \tilde{\pi}_{D,\zeta_2}(\eta \wedge \gamma) \geq \tilde{\pi}_{D,\zeta_1}(\eta) \tilde{\pi}_{D,\zeta_2}(\gamma). \tag{13}$$

We first note that if $N(\eta) \cap (\zeta \cup \eta) = \emptyset$ and $N(\gamma) \cap (\zeta_2 \cup \gamma) = \emptyset$, then

$$N(\eta \vee \gamma) \cap (\zeta_1 \cup (\eta \vee \gamma)) = N(\eta \wedge \gamma) \cap (\zeta_2 \cup ((\eta \wedge \gamma)) = \emptyset.$$

Since the red and blue subsets do not interact by Corollary 8, we only need to show that if $\zeta_2 \subset \zeta_1 \subset (\mathbb{R}^d \backslash D) \times \{\text{R}\}$ and $\eta, \gamma \in M(D, \{\text{R}\})$, then

$$\tilde{\Pi}_{\zeta_1}(\eta \cup \gamma) \tilde{\Pi}_{\zeta_2}(\eta \cap \gamma) \geq \tilde{\Pi}_{\zeta_1}(\eta) \tilde{\Pi}_{\zeta_2}(\gamma).$$

The proof of the last statement follows by a simple modification of the proof of Theorem D.1, presented in Appendix D. Specifically, the inequality (53) in the appendix is modified to

$$\mathbb{E} \prod_{i=1}^{n+m+2k} \begin{bmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\zeta_2)^c) + i\mu \\ \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) - \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) \\ + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}w(i)}) \cap N(\zeta_2)^c) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_2)^c) \end{bmatrix}^{-1}$$

$$\geq \mathbb{E} \prod_{i=1}^{n+m+2k} \begin{bmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\zeta_2)^c) + i\mu \\ \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) - \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) \\ + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}w(i)}) \cap N(\zeta_2)^c) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_2)^c) \end{bmatrix}^{-1}, \tag{14}$$

where the expectation is over a uniformly random choice

$$(\mathfrak{S}, \mathfrak{abc\bar{c}}) \in P(n, m, k, k) \times \mathcal{P}(\eta \backslash \gamma) \times \mathcal{P}(\gamma \backslash \eta) \times \mathcal{P}(\eta \cap \gamma) \times \mathcal{P}(\eta \cap \gamma),$$

$$n = |\eta \backslash \gamma|, \quad m = |\gamma \backslash \eta|, \quad \mp |\eta \cap \gamma|.$$

The rest of the proof follows similar steps to the proof in Appendix D.

Remark 2.2. The above proof can be modified to give the following interesting result. Let $D_1 \subset D_2$ be two compact subsets of $\mathbb{R}^d$. With an abuse of notation, let $\tilde{\pi}_{D_2,\text{R}}(\eta)$ denote the

marginal Janossy density of observing η in D_1, under the measure with density $\tilde{\pi}_{D_2,\mathrm{R}}$. We similarly overload the notation for $\tilde{\pi}_{D_2,\mathrm{B}}$. With this notation, we may prove that $\tilde{\pi}_{D_1,\mathrm{R}} \geq \tilde{\pi}_{D_2,\mathrm{R}}$ and $\tilde{\pi}_{D_1,\mathrm{B}} \leq \tilde{\pi}_{D_2,\mathrm{B}}$. For the proof, we apply Holley's inequality, which requires that the following inequality hold:

$$\tilde{\pi}_{D_1,\mathrm{R}}(\eta \vee \gamma)\tilde{\pi}_{D_2,\mathrm{R}}(\eta \wedge \gamma) \geq \tilde{\pi}_{D_1,\mathrm{R}}(\eta)\tilde{\pi}_{D_2,\mathrm{R}}(\gamma),$$

where $\eta, \gamma \in M(D_1, C)$ are two valid configurations. This follows from the inequality

$$\tilde{\pi}_{D_1,\mathrm{R}}(\eta \vee \gamma)\tilde{\pi}_{D_2,\mathrm{R}}(\xi \cup (\eta \wedge \gamma)) \geq \tilde{\pi}_{D_1,\mathrm{R}}(\eta)\tilde{\pi}_{D_2,\mathrm{R}}(\xi \cup \gamma),$$

where $\xi \in M(D_1 \backslash D_2, C)$ is any configuration such that $\xi \cup \gamma$ is a valid configuration. The latter inequality can be proved using ideas similar to those in the proof of Theorem D.1 in Appendix D. We skip the details of the cumbersome calculations here. We only remark that such a monotonicity of measures allows us to consider the limiting extremal measures

$$\lim_{D_n \nearrow \mathbb{R}^d} \tilde{\pi}_{D_n,\mathrm{R}}, \qquad \lim_{D_n \nearrow \mathbb{R}^d} \tilde{\pi}_{D_n,\mathrm{B}}$$

on the infinite Euclidean space $\mathbb{R}^d$. In the next few sections, we will consider the FIFM process on infinite Euclidean spaces and prove the existence of a stationary regime. We leave the job of exploring of the connection between the stationary measure so obtained and these limiting measures to future work.

3. First-in-first-match matching process on Euclidean spaces

In the next few sections, we extend the definition of the process previously given on a compact space to a non-compact space. We will specifically focus on the Euclidean space $\mathbb{R}^d$, for some $d \geq 1$. The following methodology can be extended to other non-compact spaces that satisfy certain additional assumptions, but we refrain from presenting these results in complete generality. When $D = \mathbb{R}^d$, there are infinitely many arrival and departure events that are triggered in any finite interval of time. So the process cannot be constructed as a jump Markov process using the algorithm presented in Section 2.

The key to the definition and construction of the process on $\mathbb{R}^d$ is the following viewpoint. Let us first understand this viewpoint in the compact setting and see its relation to the algorithm given in Section 2. Let D be a compact space. Let $\Phi \in M(D \times \mathbb{R}^+, C \times \mathbb{R}^+)$ and $\eta_0 \in O(D, C \times \mathbb{R}^+)$ be the driving Poisson point process and the initial condition, respectively, as defined in the Section 2. Here, each particle $x \in \eta_0$ is of the form $x = (p_x, c_x, w_x)$, where p_x, c_x, and w_x are the position, color, and patience of the particle. Similarly, any point $x \in \Phi$ is of the form $x = (p_x, b_x, c_x, w_x)$, where additionally b_x denotes the arrival time in Φ.

We will treat Φ as an element of $O(D \times \mathbb{R}^+, C \times \mathbb{R}^+)$, where the points of Φ are ordered according to their birth times, as in Section 2. We will also treat η_0 as an element of $O(D \times \mathbb{R}^+, C \times \mathbb{R}^+)$, by setting $b_x = 0$ for all $x \in \eta_0$, while preserving the order in η_0. Moreover, we also consider $\Phi \cup \eta_0$ as an element of $O(D \times \mathbb{R}^+, C \times \mathbb{R}^+)$, where all the elements of η_0 are ranked less than the elements of Φ, while preserving the order within each set.

We define a function $\kappa : \Phi \cup \eta_0 \to D \times C \times \mathbb{R}^+ \sqcup \{\diamond\}$, which we call the *killing* function, that is created as the process is built by the algorithm in Section 2. We define $\kappa(x)$ according to the following exhaustive set of rules:

1. If x arrives after y and matches with it, then $\kappa(x) = y$.

2. If x is accepted into the system, then $\kappa(x) = \diamond$.

3. If $x \in \eta_0$, then $\kappa(x) = \diamond$.

According to the description of the process, the function κ satisfies the following recursive property. For any $x \in \Phi$,

$$\kappa(x) = \min \left\{ y \in (\Phi \cup \eta_0) : \begin{matrix} y<x, \ c_y \neq c_x, \ d(p_y, p_x)<1, \ \kappa(y)=\diamond, \ b_y+w_y>b_x, \\ (\forall z, \ y<z<x, \ d(p_y,p_z)<1, \ c_z=c_x, \ \kappa(z) \neq y) \end{matrix} \right\}, \qquad (15)$$

where the minimum above is set equal to $\diamond$ if the above set is empty. In words, the conditions in the definition of the above set select particles of opposite color that arrive before (or are ranked lower), that are accepted when they arrive, whose patience does not run out before x arrives, and that are not matched to any particle arriving before x.

The above recursive property can serve as a definition of the function κ, even in the non-compact case, if we can show that the recursive definition terminates a.s. We note that we could compute the value of $\kappa(x)$ if we knew all values of κ on points in $\Phi \cup \eta_0$ that are within a spatial distance of 2 from x and that arrive before it. It is also enough just to know all the values of κ for points in Φ within a spatial distance of 4 that arrive before x. The following lemma provides the tool needed to verify that the recursive definition terminates.

Lemma 3.1. *Let $\Phi \in O(\mathbb{R}^d \times \mathbb{R}^+, C \times \mathbb{R}^+)$ be a Poisson point process of intensity $\ell \otimes \ell \otimes m_c \otimes \mu e^{-\mu x} \ell(dx)$, where ℓ is the Lebesgue measure on the corresponding Euclidean spaces. Then there is no infinite sequence $\{y_n\}_{n \in \mathbb{N}} \subset \Phi$ such that $b_{y_i} > b_{y_{i+1}}$ and $d(p_{y_i}, p_{y_{i+1}}) \leq 4$ for all $i > 0$.*

Proof. Let $\epsilon > 0$ be fixed. Consider the event $T_\epsilon := \cup_{b \in \mathbb{Q}^+} T_{\epsilon,b}$, where $T_{\epsilon,b}$, $b \in \mathbb{Q}^+$, is the event that the random geometric graph (see Meester and Roy [18]) obtained by joining any two points in $\{y \in \Phi : b \leq b_y \leq b + \epsilon\}$ whose positions are within a distance of 4 from each other does not percolate. From Theorem 3.2 of Meester and Roy [18], we know that $\mathbb{P}(T_{\epsilon,b}) = 1$ for every $b \in \mathbb{Q}^+$ if ϵ is small enough. Hence, $\mathbb{P}(T_\epsilon) = 1$ for small enough ϵ. This precludes the presence of an infinite sequence $y_1, y_2 \ldots \in \Phi$ such that, for all $i \in \mathbb{N}$, $b_{y_i} > b_{y_{i+1}}$ and $d(p_{y_i}, p_{y_{i+1}}) \leq 4$. Indeed, if such a sequence exists, then the limit $b = \lim_{i \to \infty} b_{y_i}$ exists, and by density of $\mathbb{Q}^+$ in $\mathbb{R}^+$, this event belongs to the set T_ϵ^c.

The above lemma ensures that we can obtain the value of $\kappa(x)$ for any $x \in \Phi$ by recursively applying Equation (15). The process in turn can be defined by setting

$$\eta_t = \{(p_y, c_y) : y \in \Phi \cup \eta_0, \ \kappa(y) = \diamond, \ b_y \leq t < b_y + w_y, \ \text{and} \ \kappa(z)=y \implies t < b_z\}$$

for all $t > 0$. On the unbounded domain $\mathbb{R}^d$, this will serve as the definition of the FIFM spatial matching process.

3.1. Construction of stationary regime on Euclidean spaces

The simple coupling-from-the-past argument presented in Section 2 does not hold in the case where $\lambda(D) = \infty$, since we cannot find a sequence of regeneration times going to ∞ in this case. We show, however, that a coupling-from-the-past argument can still be performed locally in space. This is done by first showing that, for a compact subset C of the domain $\mathbb{R}^d$, there exists a time T_C beyond which two simulations agree for all times t beyond time T_C (so, T_C is not a stopping time). A key ingredient in proving the existence of T_C is an analysis of the decay in first-order moment measures of the discrepancies between the two point patterns in simulation. Using this analysis, we are also able to bound the moments of T_C, which enables

the application of ergodic theorems, as in the simple coupling-from-the-past construction in Section 2. In the following sections, we present this coupling-from-the-past argument in detail.

3.2. A coupling of two processes

We will first obtain some results about a coupling of two different processes, $\{\eta_t^1\}_{t \geq 0}$ and $\{\eta_t^2\}_{t \geq 0}$, starting from two different initial conditions, but driven by the same driving process $\Phi \in O(\mathbb{R}^d \times \mathbb{R}, C \times \mathbb{R}^+)$. Let η_0^1 and η_0^2 be the two valid initial conditions ($\eta_0^i \cap N(\eta_0^i) = \emptyset$). At any time $t \geq 0$, there are some particles that are present in both processes. These particles will be called *regular* particles, and denoted by R_t. Particles that are present in η_t^1 and absent in η_t^2 will be called *zombies*, and those that are absent in η_t^1 and present in η_t^2 will be called *antizombies*. We denote zombies and antizombies by Z_t and A_t, respectively. Further, particles in $Z_t \cup A_t$ will be called *special* particles and denoted by S_t.

We now prove that the density of the special particles decays exponentially to zero.

Lemma 3.2. *There exists a constant $c > 0$, $\beta_{S_t} < \beta_{S_0} e^{-ct}$, for all $t > 0$, where β_{S_t} is the intensity of the special points, S_t.*

Proof. Let $K \subset \mathbb{R}^d$ be compact. Define $K^+ = \{y \in \mathbb{R}^d : d(y, K) \leq 1\}$ and $K^- = \{y \in \mathbb{R}^d : d(y, K^c) \leq 1\}^c$, $\partial K^+ = K^+ - K$, and $\partial K^- = K - K^-$. Also, for the sake of brevity, for any $V \subset \mathbb{R}^d$ let V_C denote the set $V \times C$. We will compute the difference $\mathbb{E}[Z_{t+\delta}(K_C) - Z_t(K_C)]$ for small $\delta > 0$, by tracking the changes that may occur in the short time interval $(t, t + \delta)$. Recall that we use the notation W_x^i to denote the domain of influence of the particle $x \in \eta_t^i$, $i = 1, 2$. Also, in the following we have $\bar{\lambda} := \ell \otimes m_c$ on $\mathbb{R}^d \times C$. The following possibilities may occur:

- A zombie in K exits on its own by losing patience. The expected difference is

$$-\mu \delta \mathbb{E} Z_t(K_C) + o(\delta). \tag{16}$$

- With probability $o(\delta)$, two or more particles arrive or depart in K^+. The expected change in $Z_t(K)$, given that this occurs, is $o(\delta)$.

- A zombie in K matches with a particle arriving in K^c, which is accepted in the process η_t^2. This results in a difference of

$$-\delta \mathbb{E} \sum_{x \in Z_t \cap K_C} \bar{\lambda}(W_x^1 \cap K_C^c \cap (N(\eta_t^2))^c) + o(\delta). \tag{17}$$

- A zombie in K matches with a particle arriving in K, which is accepted in the process η_t^2. This new particle is an antizombie. The resulting change is

$$-\delta \mathbb{E} \sum_{x \in Z_t \cap K_C} \bar{\lambda}(W_x^1 \cap K_C \cap (N(\eta_t^2))^c) + o(\delta). \tag{18}$$

- A zombie in K matches with an arriving particle, which also matches with some particle in K^c in the complementary process. This results in an expected change of

$$-\delta \mathbb{E} \sum_{x \in Z_t \cap K_C} \sum_{y \in \eta_t^2 \cap K_C^c} \bar{\lambda}(W_x^1 \cap W_y^2) + o(\delta). \tag{19}$$

- A zombie in K matches with an arriving particle, which also matches with some antizombie in K. This results in an expected change of

$$-\delta\mathbb{E}\sum_{x\in Z_t\cap K_C}\sum_{y\in A_t\cap K_C}\bar{\lambda}(W_x^1\cap W_y^2)+o(\delta). \tag{20}$$

- An antizombie matches with a particle arriving in K that is accepted in the complementary process. This particle becomes a zombie. This results in an expected change of

$$\delta\mathbb{E}\sum_{x\in A_t}\bar{\lambda}(W_x^2\cap K_C\cap(N(\eta_t^1))^c)+o(\delta). \tag{21}$$

- An arriving particle matches with a zombie in K^c and a regular particle in $\eta_t^2\cap K_C$. The regular particle turns into a zombie. This results in a change of

$$\delta\mathbb{E}\sum_{\substack{x\in R_t\cap K_C\\y\in Z_t\cap K_C^c}}\bar{\lambda}(W_x^2\cap W_y^1)+o(\delta). \tag{22}$$

Taking only the contributions from (16), (18), (21), and (22) in the above changes, dividing by δ, and taking the limit as $\delta\to 0$, we obtain

$$\frac{d\mathbb{E}Z_t(K_C)}{dt}\le -\mu\mathbb{E}Z_t(K_C)+\mathbb{E}\left[-\sum_{x\in Z_t\cap K_C}\bar{\lambda}(W_x^1\cap K_C\cap(N(\eta_t^2))^c)\right.$$

$$\left.+\sum_{x\in A_t}\bar{\lambda}(W_x^2\cap K_C\cap(N(\eta_t^1))^c)+\sum_{\substack{x\in R_t\cap T_C\\y\in Z_t\cap T_C^c}}\bar{\lambda}(W_x^2\cap W_y^1)\right].$$

We have similar bounds for derivatives of $\mathbb{E}A_t(K_C)$. Adding these expressions, we obtain

$$\frac{d}{dt}\mathbb{E}S_t(K_C)\le -\mu\mathbb{E}S_t(K_C)+\mathbb{E}\left[\sum_{\substack{x\in R_t\cap K_C\\y\in Z_t\cap K_C^c}}\bar{\lambda}(W_x^2\cap W_y^1)+\sum_{\substack{x\in R_t\cap K_C\\y\in A_t\cap K_C^c}}\bar{\lambda}(W_x^2\cap W_y^1)\right]$$

$$+\mathbb{E}\left[\sum_{x\in A_t\cap K_C^c}\bar{\lambda}(W_x^2\cap K_C\cap N(\eta_t^1)^c)+\sum_{x\in Z_t\cap K_C^c}\bar{\lambda}(W_x^1\cap K_C\cap N(\eta_t^2)^c)\right]$$

$$\le -\mu\mathbb{E}S_t(K_C)+4\bar{\lambda}(\partial K_C^+\cup\partial K_C^-).$$

Taking the limit $K\nearrow\mathbb{R}^d$, by spatial ergodicity of the process S_t, we obtain

$$\frac{d}{dt}\beta_{S_t}\le -\mu\beta_{S_t}, \tag{23}$$

from which we can conclude that $\beta_{S_t}\le\beta_{S_0}e^{-\mu t}$. $\qquad\square$

3.3. The coupling-from-the-past construction

In this section, we present the coupling-from-the-past construction of the stationary regime. Let Φ be a doubly infinite Poisson point process, as in Lemma A.1. That is, Φ is a Poisson

point process defined on $\mathbb{R}^d \times \mathbb{R}$, with i.i.d. marks in $C \times \mathbb{R}^+$, that is time-shift invariant. The intensity of the point process is $2\ell \otimes \ell$, and the marks are independent of each other, with the color uniformly distributed in C and the patience an exponential random variable. Let $\{\theta_t\}_{t \in \mathbb{R}}$ be a sequence of time-shift operators such that $\Phi \circ \theta_t(L \times A) = \Phi(L \times (A - t))$. Let $\{\eta_t^T\}_{t \geq -T}$, $T \in \mathbb{N}$, be a sequence of processes starting at time $-T$ with empty initial conditions and driven by arrivals from Φ. We have $\eta_t^T = \eta_{t+T}^0 \circ \theta_{-T}$.

The processes η_t^1 and η_t^0 are driven by the same Poisson point process Φ beyond time 0. Treating the particles in η_0^1 as the initial conditions, we have a coupling of $\{\eta_t^1\}_{t \geq 0}$ and $\{\eta_t^0\}_{t \geq 0}$ as in Section 3.2. The discrepancies on any bounded set go to zero exponentially fast by Lemma 3.2. In the next lemma, we prove that this exponential rate of convergence is enough to show that the time after which discrepancies never appear in any compact region has finite expectation. For any compact $K \subset D$, define

$$\tau^0(K) := \inf\{t > 0 : \eta_s^1|_K = \eta_s^0|_K, \ s \geq t\}, \tag{24}$$

and in the following, let S_t denote the set of discrepancies, $\eta_t^0 \triangle \eta_t^1$. Note that $\tau^0(K)$ is not a stopping time in our setting, since, first, $S_t \neq \emptyset$ for all $t \geq 0$ a.s. by spatial ergodicity, and second, once discrepancies vanish in K, they can reappear because of interactions with particles from outside of K.

Lemma 3.3. For all compact $K \subset \mathbb{R}^d$, $\mathbb{E}\tau^0(K) < \infty$.

Proof. We view $S_t(K)$, $t \geq 0$, as a birth–death process. Let $S_t(K) = S_0(K) + S^+(0, t] - S^-(0, t]$, where S^+ and S^- are simple counting processes. Since new special particles only result from the interaction of arriving particles with existing special particles, the rate of increase in S^+ is bounded above by

$$\sum_{x \in S_t \cap K} \ell(B(x, 1)) = \ell(B(0, 1))S_t(K).$$

Hence,

$$\mathbb{E}S^+[0, \infty) \leq \ell(B(0, 1)) \int_0^\infty \mathbb{E}S_t(K)dt < \infty,$$

where we use Lemma 3.2 to obtain the last inequality. Since total departures are less than total arrivals, $\mathbb{E}S^-[0, \infty) \leq \mathbb{E}S_0(K) + \mathbb{E}S^+[0, \infty)$. This in particular shows that $S^+[0, \infty)$ and $S^-[0, \infty)$ exist and are finite a.s. Thus, $\lim_{t \to \infty} S_t(K)$ also exists and is finite a.s. By the dominated convergence theorem,

$$\lim_{t \to \infty} \mathbb{E}S_t(K) = \mathbb{E}\lim_{t \to \infty} S_t(K).$$

Thus, by Lemma 3.2, $\lim_{t \to \infty} S_t(K) = 0$ a.s. This shows that $\tau^0(K) < \infty$ a.s.

Since $\tau^0(K) < \infty$ a.s., there is a last exit and we may write $\tau^0(K) \leq \int_0^\infty t S^-(dt)$. We therefore have

$$\mathbb{E}\tau^0(K) \leq \mathbb{E} \int_0^\infty t S^-(dt)$$

$$= \mathbb{E} \int_0^\infty t S^+(dt) - \mathbb{E} \int_0^\infty t(S^+ - S^-)(dt)$$

$$= \mathbb{E} \int_0^\infty t S^+(dt) + E \int_0^\infty (S^+ - S^-)(t)dt,$$

where the last equality follows from a change-of-variables formula. Using the bound of $\ell(B(0, 1))S_t(K)$ on the rate of arrival in S^+, the first expectation can be bounded above by $\ell(B(0, 1)) + \int_0^\infty t\mathbb{E}S_t(K)dt$. We thus have

$$\mathbb{E}\tau^0(K) \le \ell(B(0, 1)) \int_0^\infty t\mathbb{E}S_t(K)dt + \int_0^\infty \mathbb{E}S_t(K)dt$$

$$< \infty,$$

by Lemma 3.2.

We use the above lemma to prove the following existence result, using a coupling-from-the-past argument.

Theorem 3.1. *There exists a stationary version of the FIFM matching dynamics on $\mathbb{R}^d$, $\{\eta_t\}_{t\in\mathbb{R}}$, driven by Φ, such that $\eta_t \circ \theta_s = \eta_{t+s}$ for all $s \in \mathbb{R}$.*

Proof. Let $\tau^T(K)$ be defined as

$$\tau^T(K) := \inf\{t > -T : \eta_s^{T+1}|_K = \eta_s^T|_K, \ s \ge t\};$$

$\tau^T(K)$ denotes the time at which realizations of the processes $\{\eta_t^T\}$ and $\{\eta_t^{T+1}\}$ coincide inside the set K. We have

$$\tau^T(K) = \tau^0(K) \circ \theta_{-T} - T.$$

That is,

$$\tau^T(K) + T = \tau^0(K) \circ \theta_{-T}. \tag{25}$$

Therefore, the sequence $\tau^T(K) + T$ is a stationary and ergodic sequence. By Birkhoff's pointwise ergodic theorem and Lemma 3.3,

$$\lim_{T\to\infty} \sum_{i=0}^T \frac{\tau^i(K) + i}{T} = \mathbb{E}\tau^0(K) < \infty \quad \text{a.s.}$$

Therefore the last term in the summation,

$$\frac{\tau^T(K) + T}{T},$$

goes to 0 as $T \to \infty$. From this we conclude that

$$\lim_{T\to\infty} \tau^T(K) = -\infty \quad \text{a.s.} \tag{26}$$

This implies that for every realization of Φ, any compact set K, and $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that for all $T > k$, $\tau^0(K) \circ \theta_{-T} - T < t$. That is, the executions of all processes $\{\eta_s^T\}$, $T > k$, coincide at time t on the compact set K. Therefore, locally in the sense of total variation, the following limit is well-defined a.s.:

$$\eta_t := \lim_{T\to\infty} \eta_t^T. \tag{27}$$

The process η is $\{\theta_n\}_{n\in\mathbb{Z}}$-compatible, since

$$
\begin{aligned}
\eta_t \circ \theta_1 &= \lim_{T\to\infty} \eta_t^T \circ \theta_1 \\
&= \lim_{T\to\infty} \eta_{T+t}^0 \circ \theta_{-T+1} \\
&= \lim_{T\to\infty} \eta_{t+1+T-1}^0 \circ \theta_{-T+1} \\
&= \eta_{t+1}.
\end{aligned}
$$

Further, the process can also be shown to be $\{\theta_s\}_{s\in\mathbb{R}}$-compatible: for any $s \in \mathbb{R}$, by a similar backward coupling argument, it can be shown that

$$
\lim_{T\to\infty} \eta_t^{T+s} = \lim_{T\to\infty} \eta_t^T \quad \text{a.s.}
$$

Since $\eta_t^{T+s} = \eta_{t+s}^T \circ \theta_{-s}$, we have $\eta_{t+s} = \eta_t \circ \theta_s$. This proves that $\{\eta_t\}_{t\in\mathbb{R}}$ is the stationary regime of the process.

4. Conclusion and future work

In this paper, we focused on a dynamic matching model with a natural policy, under the added assumption that particles may depart without being matched. We were able to find a characterization of the steady-state distribution of the particles. Then using this characterization, we proved the FKG lattice property, which in turn enabled us to conclude that particles of the same type are weakly super-Poissonian. We also proved that there is a stationary regime for the dynamics in the infinite Euclidean space $\mathbb{R}^d$.

The two-particle Widom–Rowlinson model is a simpler model that, like our model, satisfies the FKG property. For this model, the FKG property is used to show the existence of Markov random fields on the infinite Euclidean space $\mathbb{R}^d$. In the future, we would like to see whether this construction works in our setting, and how it relates to the stationary regime constructed on the domain $\mathbb{R}^d$.

The gray version of two-particle Widom–Rowlinson model, which is obtained by removing the reference to the colors of the points, also satisfies an FKG inequality—a result we are unable to prove in our setting. This is a fundamental step in the symmetry-breaking argument of Chayes *et al.* [9], which we were not able to carry over to our setting. A symmetry-breaking argument would show, for certain values of the parameter, that there are more red points than blue points in the steady state, or vice versa. This also has implications for the relaxation times of the Markov process on finite domains. In the future, we would like to explore these models.

Appendices

A. Proof of existence and uniqueness of stationary regime on compact domain

In this section, we provide a proof for Theorem 2.1. Let Φ be a bi-infinite marked Poisson point process on $D \times \mathbb{R}$ with marks in $C \times \mathbb{R}^+$, as defined in Section 2.1. The proof depends on the notion of *regeneration times* of Φ, which we now define.

Definition A.1. *A time $t \in \mathbb{R}$ is called a regeneration time of Φ if*

$$
\forall x \in \Phi, \qquad b_x \leq t \implies b_x + w_x < t.
$$

If t is a regeneration time of Φ, there is no possibility that a particle arriving before time t survives beyond time t. The following lemma provides a sequence of regeneration times that diverge to $-\infty$.

Lemma A.1. *In the setting of this section, there are infinitely many regeneration times in the sequence $0, -1, -2, \ldots$, a.s.*

Proof. Let A_n be the event that $-n$ is a regeneration time. Let us find the probability of the event A_0. We have

$$
\begin{aligned}
\mathbb{P}(A_0) &= \mathbb{P}(w_x < -b_x, \forall x \in \Phi, b_x < 0) \\
&= \mathbb{E} \prod_{x \in \Phi} \mathbb{1}(w_x < -b_x, b_x < 0) \\
&= \mathbb{E} \lim_{s \to \infty} e^{-s \int \mathbb{1}(w_x \geq -b_x)\Phi(dx)} \\
&\geq \limsup_{s \to \infty} \exp\left(2 \int_D \int_{\mathbb{R}^- \times \mathbb{R}^+} \left(e^{-s\mathbb{1}(w \geq -b)} - 1 \right) \mu e^{-\mu w} dw db \lambda(dp) \right) \\
&= \limsup_{s \to \infty} \exp\left(2\lambda(D) \int_{\mathbb{R}^+} (e^{-s} - 1)e^{-\mu b} db \right) \\
&= \limsup_{s \to \infty} \exp\left(2\lambda(D)(e^{-s} - 1)/\mu \right) \\
&= \exp\left(-2\lambda(D)/\mu \right) > 0,
\end{aligned}
$$

where in the fourth equation we have used Fatou's lemma and the Laplace transform formula for Poisson point processes. Now, if θ_t is a time-shift operator, we have $A_n = \theta_{-n}A_0$. By time-ergodicity of Φ, A_n must occur infinitely often, a.s. Thus, there are infinitely many regeneration times in the sequence $\{0, -1, -2, \ldots\}$. $\square$

Using this lemma, we are able to prove Theorem 2.1.

Proof of Theorem 2.1. For any process $\{\eta_s^T\}_{s \geq -T}$ started with empty initial conditions at time $-T$, and driven by the process Φ. We note that $\eta_s^T = \eta_s^{T'}$ for all $s \geq t$ and all regeneration times $t \geq -T$ and $t \geq -T'$.

By Lemma A.1, we know that there is a sequence of regeneration times $\{t_n\}_{n \in \mathbb{N}}$ such that $t_n \to -\infty$ as $n \to \infty$. Thus, we note that the limit $\eta_t = \lim_{T \to \infty} \eta_t^T$ exists a.s. This is an instance of a coupling-from-the-past algorithm (see Chapter 10 of Thorisson [20]).

The uniqueness of a stationary measure can also be shown using a coupling-based argument. We only give an outline of this procedure here. Suppose we consider two stationary measures of the process, $\mathbb{P}^1$ and $\mathbb{P}^2$. Let E_n be the event that $\{\eta_0 \in O(D) : |\eta_0| > n\}$. Fix $\epsilon > 0$. For some large enough n, we must have that $\mathbb{P}^1(E_n) < \epsilon$ and $\mathbb{P}^2(E_n) < \epsilon$. Let Ψ be the set of probability measures on $O(D) \times O(D)$ such that their first and second marginals are equal to $\mathbb{P}^1$ and $\mathbb{P}^2$. Now, the total variation distance between the two measures $\mathbb{P}^1$ and $\mathbb{P}^2$ is

$$
d_{TV}(\mathbb{P}^1, \mathbb{P}^2) = \inf_{\psi \in \Psi} \psi(\eta^{(1)} \neq \eta^{(2)}), \tag{28}
$$

where $(\eta^{(1)}, \eta^{(2)})$ represent the two coordinates of the state space $O(D) \times O(D)$.

For $\psi \in \Psi$, let ψ_t denote the joint probability distribution of the FIFM matching processes sampled at time t, starting at time 0 using a sample from ψ, and using the same driving process

Φ described earlier. Note that $\psi_t \in \Psi$. Using a union bound, $\mathbb{P}_\psi(\{\eta^{(1)} \in E_n\} \cup \{\eta^{(2)} \in E_n\}) < 2\epsilon$. Thus,

$$
\begin{aligned}
d_{TV}(\mathbb{P}^1, \mathbb{P}^2) &\le 2\epsilon + \inf_{\psi \in \Psi} \psi(\eta^{(1)} \ne \eta^{(2)} | \eta^{(1)}, \eta^{(2)} \notin E_n) \\
&\le 2\epsilon + \inf_{\psi \in \Psi} \psi_t(\eta^{(1)} \ne \eta^{(2)} | \eta^{(1)}, \eta^{(2)} \notin E_n).
\end{aligned}
\tag{29}
$$

For simplicity, we can now specialize to a specific starting distribution and let $\psi = \mathbb{P}^1 \otimes \mathbb{P}^2$:

$$
d_{TV}(\mathbb{P}^1, \mathbb{P}^2) \le 2\epsilon + \psi_t(\eta^{(1)} \ne \eta^{(2)} | \eta^{(1)}, \eta^{(2)} \notin E_n).
\tag{30}
$$

Let $\mathbb{P}_\psi$ denote the probability measure of the coupling here, and let $(\eta_t^{(1)}, \eta_t^{(2)})$ denote the coupled processes. Then as a result of the coupling mechanism we can conclude that the two processes couple in a random time T, such that $\mathbb{E}_\psi[T | \eta_0^{(1)}, \eta_0^{(2)} \notin E_n] < \infty$. Loosely speaking, this is true because the T can be bounded from above by T_{sum}, the sum of the patience of all points in $\eta_0^{(1)}$ and $\eta_0^{(2)}$, plus the time to first regeneration of Φ beyond T_{sum}.

Now, using (30) and the coupling inequality, we obtain

$$
d_{TV}(\mathbb{P}^1, \mathbb{P}^2) \le 2\epsilon + \mathbb{E}[T | \eta_0^{(1)}, \eta_0^{(2)} \notin E_n]/t.
$$

Taking $t \to \infty$ and then $n \to \infty$, we may conclude that $d_{TV}(\mathbb{P}^1, \mathbb{P}^2) = 0$. $\square$

B. Characterizing stationary distribution: background discussion

B.1. Dynamic reversibility of Markov processes

In this section we give a brief discussion of a result needed to construct the product-form distribution. This concept will be termed dynamic reversibility of a Markov process, following the terminology in Kelly [16], where the concept was discussed for Markov processes on countable state spaces. We thus define it first on countable state spaces, then on general state spaces.

Let $\{X(t)\}_{t \in \mathbb{R}}$ be a stationary, irreducible continuous-time Markov process with values in a countable state space S. Let $q(j, k)$ denote the transition rate from state $j \in S$ to $k \in S$, and let π be the stationary distribution of the process. In this case, the balance equations are $\sum_{j \in S} \pi(j) q(j, k) = 0$.

The reversed process, $X(-t)$, is also a stationary Markov process, with transition rates

$$
q'(j, k) = \frac{\pi(k) q(k, j)}{\pi(j)}.
$$

The converse of this statement can be used as a characterization of the stationary distribution. We state this result in the following theorem.

Theorem B.1. *Let $X(t)$ be a stationary irreducible Markov process with transition rates $q(j, k)$, $j, k \in S$. If there exists a collection of numbers $q'(j, k)$, $j, k \in S$, and a probability measure π on S such that*

$$
\pi(j) q(j, k) = \pi(k) q'(k, j), \quad j, k \in S,
\tag{31}
$$

and

$$\sum_{k \in S} q'(j, k) = 0, \quad j \in S, \tag{32}$$

then π is the stationary distribution of the process, and q' is the transition rate matrix for the time-reversed process.

Thus, if we can guess the transition rates of the reversed process and a stationary measure, we can verify them by checking a local balance condition of the form (31). See Theorem 1.13 of Kelly [16] for a proof of this result. In practice, finding q' is usually as intractable as finding the stationary distribution directly. However, occasionally we may come across pairs of Markov processes that are reversed versions of each other, perhaps after a transformation of the state space. We state this phenomenon in the next theorem.

Theorem B.2. *Let S and T be two countable spaces. Let X(t) and Y(t) be two stationary irreducible Markov processes with values in S and T, and with transition matrices q and q' respectively. Suppose there is an isomorphism $\phi : S \rightarrow T$ between the two spaces. Also suppose that there is a probability measure π on S such that*

$$\pi(j)q(j, k) = \pi(k)q'(\phi(k), \phi(j)), \quad j, k \in S.$$

Then π is the stationary distribution of X(t) and $\pi(\phi^{-1}(\cdot))$ is the stationary distribution of Y(t).

Theorem B.2 can be extended to a more general setting, as follows.

Theorem B.3. *Let S and T be two locally compact Hausdorff topological spaces. Let X(t) and Y(t) be two stationary Markov jump processes with values in S and T. Suppose the probability semigroup of the process X(t) (resp. Y(t)) is characterized by the generators L_X (resp. L_Y), defined over dom(L_X) (resp. dom(L_Y)), where the domain is a subset of the Banach space of continuous functions over S (resp. T) vanishing at infinity, equipped with the uniform norm topology. Let $\phi : S \rightarrow T$ be a measure space isomorphism such that for all $f \in dom(L_X)$, we have $f \circ \phi^{-1} \in dom(L_Y)$. If π is a probability measure on S such that*

$$\int_S f(x)L_X g(x)\pi(dx) = \int_T L_Y(f \circ \phi^{-1})(y)g \circ \phi^{-1}(y)\phi_*\pi(dy),$$

then π is a stationary distribution for X(t).

Proof. Let g be any element in $dom(L_X)$. Taking a sequence $f_n \in dom(L_Y)$ such that f_n converges pointwise to the constant function 1 as $n \rightarrow \infty$, we have

$$\int_S L_X g(x)\pi(dx) = \int_T L_Y(1)(y)g \circ \phi^{-1}(y)\phi_*\pi(dy) = 0.$$

By standard results from the theory of positive operator semigroups, it is known that for $g \in dom(L_X)$, the map $x \mapsto \mathbb{E}[g(X(t))|X(0) = x]$ belongs to $dom(L_X)$ (see Lemma 1.3 of Engel and Nagel [11], for example). Thus, for $g \in dom(L_X)$, we have

$$\frac{d}{dt} \int_S \mathbb{E}[g(X(t))|X(0) = x]\pi(dx) = \int_S L(\mathbb{E}[g(X_t)| \cdot])(x)\pi(dx) = 0.$$

It is also known that $dom(L_X)$ is dense on the space of continuous functions vanishing at infinity (Theorem 1.4 of Engel and Nagel [11]). This implies that $\mathbb{E}_\pi g(X(t)) = \mathbb{E}_\pi g(X(0))$ for all bounded continuous functions and all $t > 0$, and so π must be a stationary measure of $X(t)$. □

If two processes satisfy the hypotheses of the above theorem, we say that the processes are *dynamically reversible*.

B.2 Additional global notation

In the following few sections we need some notation to define the transitions in the Markov processes. We collect this notation in this section.

Let $\gamma = (x_1, \ldots, x_n)$, $n \in \mathbb{N}$, and x respectively be a list of elements and a particular element belonging to the same abstract space S. We define the following operators:

1. Let $\gamma \blacktriangleleft_i x$, $i = 0, \ldots, n$, denote the insertion of the element x after the ith element in γ; i.e.,
    $$\gamma \blacktriangleleft_i x = (x_1, \ldots, x_i, x, x_{i+1}, \ldots, x_n).$$

2. Let $\gamma \blacktriangleright_i$, $i = 1, \ldots, n$, denote the removal of the ith element of γ; i.e.,
    $$\gamma \blacktriangleright_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

3. Let $\gamma \blacktriangle_i x$ denote the replacement of the ith element in γ with x; i.e.,
    $$\gamma \blacktriangle_i x = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n).$$

In the above notation, we may drop the subscript i if $i = |\gamma|$, i.e., when we are making changes to the last element. Also, when composing these operators, unless clarified using parentheses, the order of application of operations is from left to right.

B.3 Continuous-time first-come-first-served bipartite matching model with reneging

In this section, we illustrate how dynamic reversibility is used in the proof of Theorem 2.2, by working on a countable state space Markov model. This allows us to organize and present the main ideas without the complexity of dealing with measure-valued processes.

Specifically, in this section, we consider the following modified version of the FCFS bipartite matching model considered in Adan *et al.* [1]. Consider two finite sets of types $\mathcal{C} = \{c_1, \ldots, c_I\}$ and $\mathcal{S} = \{s_1, \ldots, s_J\}$ and a bipartite compatibility graph $G = (\mathcal{C}, \mathcal{S}, \mathcal{E})$ with $\mathcal{E} \subset \mathcal{C} \times \mathcal{S}$. Let $\bar{\lambda}$ be a measure on $C \cup S$, and let $\mu > 0$ be a parameter. We say that c and s can be *matched* or are *compatible* if $(c, s) \in \mathcal{E}$ in the compatibility graph E. We define the FCFS bipartite matching model with reneging as a Markov jump process with state space Γ, which is the set of all finite ordered lists of elements from $C \cup S$ such that for every $c \in C$ and $s \in S$ in the list, $(c, s) \notin \mathcal{E}$. Furthermore, given that the state of the process at any time t is $\gamma = (x_1, \ldots, x_n)$, the state is updated with the following transition rates:

1. A new element $x \in C \cup S$ arrives at rate $\lambda(x)$. At the time of the arrival, if there are one or more elements in γ that are compatible with x, then the first such element, x_i, is removed, and we say that x and x_i are matched. If there is no such element, then x is added to the end of the list γ.

2. Each element in the list is removed at rate $\mu > 0$.

The comments and results of Sections 2.1 and 2.2 can be mirrored in this setting. We briefly review them here.

We can simulate the above process using arrivals from a Poisson point process Φ on $(C \cup S) \times \mathbb{R}$, with i.i.d. exponential marks in $\mathbb{R}^+$, and with intensity $\lambda \otimes \ell$. The base of the Poisson point process Φ encodes the arrivals of the agents, and the mark of a point encodes the time each agent is willing to wait (its *patience*), if it is accepted. We will use the following notation: for any point $x \in (C \cup S) \times \mathbb{R} \times \mathbb{R}^+$, c_x will denote its projection onto $C \cup S$, b_x will denote the second coordinate, and w_x will denote the third coordinate.

Standard coupling- or Lyapunov-based arguments can be used to show that this Markov process has a stationary regime. Moreover, a stationary version of the process can be constructed via a coupling-from-the-past scheme that uses an ergodic arrival process, Φ, which is now a Poisson point process on $(C \cup S) \times \mathbb{R}$, with marks as above. To construct the stationary regime, the notion of regeneration time of the system may be defined as follows. We say that $t \in \mathbb{R}$ is a *regeneration time* of Φ if for all $x \in \Phi$ with $b_x < t$, we have $t - b_x > w_x$. The forward-time construction of the process starting from a regeneration time with empty initial conditions is clear. Moreover, if $t_1 < t_2$ are two regeneration times and η^1 and η^2 are such processes starting from t_1 and t_2 respectively, then for $t > t_2$, $\eta_t^1 = \eta_t^2$. Thus, we can construct a bi-infinite stationary version, $\{\eta_t\}_{t \in \mathbb{R}}$, of this process as a factor of Φ, if we can find a sequence of regeneration times going to $-\infty$. Indeed, if $t_1 > t_2 > \cdots$ is such a sequence (with t_0 set to ∞), then the η_t for $t \in [t_i, t_{i-1})$, for some $i \in \mathbb{N}$, is obtained by simulating the process, starting from empty initial conditions, from time t_i until time t. An argument similar to that of Lemma A.1 can be used to show the existence of such a sequence of regeneration times.

This coupling-from-the-past scheme yields the definition of the matching function,

$$m : \Phi \to (C \cup S) \times \mathbb{R},$$

similar to the one defined in Section 2.2. For $x \in \Phi$, let $T \in \mathbb{R}^-$ be a regeneration time before b_x. The value of $m(x)$ can be set by simulating the process using Φ, starting from time T, with empty initial conditions. If x is matched to an agent $y \in \Phi$, then $m(x) = (c_y, b_y)$. Otherwise, if x reneges, then $m(x) = (c_x, b_w + w_x)$.

Given the function m, we can obtain the process η_t, since $\eta_t = (x \in \Phi : b_x \le t < b_{m(x)})$, where the agents in the list are ordered according to their birth times $b_\cdot$. Let $\mathtt{m}$ and $\mathtt{u}$ be additional marks, respectively indicating whether an agent is matched or unmatched. Consider the detailed stochastic process $\{\hat{\eta}_t\}_{t \in \mathbb{R}}$ defined as follows, for $t \in \mathbb{R}$:

1. Let $T_t = \min\{b_x : x \in \Phi, b_x \le t < b_{m(x)}\}$.

2. Let
$$\Gamma_{\mathtt{u},t} = \{(c_x, b_x, \mathtt{u}) : x \in \Phi, T_t \le b_x \le t < b_{m(x)}\}$$
and
$$\Gamma_{\mathtt{m},t} = \{(c_x, b_{m(x)}, \mathtt{m}) \in N : b_x \le t, T_t \le b_{m(x)} \le t\}.$$

3. Define $\hat{\eta}_t := ((c_x, s_x) : x \in \Gamma_{\mathtt{u},t} \cup \Gamma_{\mathtt{m},t})$, where s_y refers to the matched or unmatched status of an agent $y \in \Gamma_{\mathtt{u},t} \cup \Gamma_{\mathtt{m},t}$, and the list is ordered according to the second coordinate, $b_{(\cdot)}$.

Clearly, η_t can be obtained from $\hat{\eta}_t$ by removing the agents with marks $\mathtt{m}$. We call the process $\hat{\eta}_t$ the *backward detailed process*, following the terminology in Adan *et al.* [1].

Since the backward detailed process at time t only depends on the points of Φ before time t, it is a stationary process. Moreover, it is a stationary version of some Markov process, since

for $t < s$, the state at time s can be constructed using the state at time t and the process Φ in the interval $(t, s]$. We describe this Markov process in detail now. A valid state of this Markov process is a finite list of elements $(x_1, \ldots, x_n)$ from the set $(C \cup S) \times \{u, m\}$, such that the following hold:

1. $s_{x_1} = u$ if $n \geq 1$.

2. If $s_{x_i} = s_{x_j} = u$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

3. For all $i < j$, if $s_i = u$ and $s_j = m$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

Below, we utilize the definitions in Appendix B.2. Additionally, mirroring our notation in the continuum setting, for any $x \in C \cup S$ let

$$N(\{x\}) = \{y \in C \cup S : (x, y) \in \mathcal{E} \text{ or } (y, x) \in \mathcal{E}\},$$

and for any $A \subseteq C \cup S$ let $N(A) = \cup_{x \in A} N(\{x\})$. With an abuse of notation, we will let $N(x) := N(\{x\})$ for $x \in C \cup S$. Also, for $\gamma \in O(C \cup S, \{u, m\})$ and any $x \in \gamma$, we will write

$$\gamma^x = \{y \in \gamma : y <_\gamma x\};$$

$$\gamma^u = \{y \in \gamma : s_y = u\}, \qquad \gamma^m = \{y \in \gamma : s_y = m\};$$

$$W_x = \begin{cases} N(c_x) \backslash N(\gamma^{u,x}) & \text{if } s_x = u, \\ \varnothing & \text{otherwise.} \end{cases}$$

The transitions and transition rates for the backward detailed process are the following, given that the state of system is $\hat{\eta} = (x_1, \ldots, x_n)$:

1. Any agent $x_i \in \hat{\eta}$, with $s_{x_i} = u$, loses patience. In this case, the new state is $\hat{\eta}' = \hat{\eta} \blacktriangleright_i \blacktriangleleft (c_{x_i}, m)$, except possibly when $i = 1$, where we need to prune all the leading matched and exchanged elements from $\hat{\eta}'$. We still denote the new state by $\hat{\eta} \blacktriangleright_i \blacktriangleleft (c_{x_i}, m)$, even in this case, keeping in mind that all leading matched terms must be removed. Each such transition occurs at rate μ.

2. A new agent $x = (c_x, u)$, $c_x \in C \cup S$, arrives and is matched to the agent $x_i \in \hat{\eta}$, with $(c_{x_i}, c_x) \in \mathcal{E}$ and $s_{x_i} = u$. In this case, the new state is (a valid pruning of) $\hat{\eta} \blacktriangle_i (c_x, m) \blacktriangleleft (c_{x_i}, m)$. This occurs at rate $\bar{\lambda}(c_x) \mathbb{1}\left(c_x \in W_{x_i}\right)$.

3. A new agent $x = (c_x, u)$, with $c_x \in C \cup S$, arrives and is not matched to any agent. The new state is $\hat{\eta} \blacktriangleleft x$. This occurs at rate $\bar{\lambda}(c_x) \mathbb{1}(c_x \notin N(\hat{\eta}^u))$.

We now define the *forward detailed process*, which is the dual of the process $\{\hat{\eta}_t\}_{t \in \mathbb{R}}$, and which we denote by $\{\check{\eta}_t\}_{t \in \mathbb{R}}$. For $t \in \mathbb{R}$, define the following:

1. Let $Y_t = \max\{b_{m(x)} : x \in \Phi, b_x \leq t < b_{m(x)}\}$.

2. Let
$$\Xi_{m,t} = \{(c_x, b_{m(x)}, m) : x \in \Phi, b_x \leq t < b_{m(x)}\}$$
and
$$\Xi_{u,t} = \{(c_x, b_x, u) : t < b_x < Y_t, t < b_{m(x)}\}.$$

3. Let $\check{\eta}_t = ((c_x, s_x) : x \in \Xi_{u,t} \cup \Xi_{m,t})$, where the elements are ordered according to the second coordinate $b_{(\cdot)}$.

The forward detailed process is also a stationary version of a Markov process. Any valid state $(x_1, \ldots, x_n)$ of this Markov process of the system satisfies the following:

1. $s_{x_n} = \mathtt{m}$ when $n \geq 1$.

2. If $s_{x_i} = s_{x_j} = \mathtt{m}$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

3. If $i < j$, $s_{x_i} = \mathtt{u}$, $s_{x_j} = \mathtt{m}$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

The transitions and the transition rates of the Markov process are as follows: given that the state of the system is $\check{\eta}$, the next jump occurs at rate $\bar{\lambda}(C \cup S) + Q_{\mathtt{m}}^0(\check{\eta})\mu$, where $Q_{\mathtt{m}}^i(\check{\eta})$ is the number of matched elements in $\check{\eta}$ after ith location. Intuitively, this is so because the total rate of new arrivals is $\bar{\lambda}(C \cup S)$ and the total death rate is $Q_{\mathtt{m}}^0\mu$, since $Q_{\mathtt{m}}^0$ is the number of matched agents in the forward process. For the sake of brevity, let us denote $\bar{\lambda}(C \cup S) + n\mu$ by $\rho(n)$, for all $n \in \mathbb{N}$. If $\check{\eta}$ is non-empty, at each jump, to obtain the new state, we need to process the first element, x_1, in $\check{\eta}$. This is done with the following probabilities:

1. If x_1 is matched, then it is removed from $\check{\eta}$. The new state is $\check{\eta} \blacktriangleright_1$.

2. If x_1 is unmatched, then for the next state, we sample a random variable $\tau \in \mathbb{N}$ with distribution

$$\mathbb{P}(\tau = k) = \frac{\mu}{\rho(Q_{\mathtt{m}}^k + 1)} \prod_{i=1}^{k-1} \frac{\rho(Q_{\mathtt{m}}^i)}{\rho(Q_{\mathtt{m}}^i + 1)},$$

and then sample $x_{n+1}, \ldots x_\tau$ i.i.d. random unmatched elements in $\{C \cup S\}$ with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(C \cup S)$.

   (a) If there is an FCFS matching x_i, $2 \leq i \leq \tau$, then we set the new state to $(x_1^{\max(n,\tau)})\blacktriangle_i (c_{x_1}, \mathtt{m}) \blacktriangleright_1$, with the understanding that all the ending unmatched agents are discarded.

   (b) If there is no FCFS matching, we set the new state to $(x_1^{\max(n,\tau)}) \blacktriangleleft_\tau (c_{x_1}, \mathtt{m}) \blacktriangleright_1$.

If the state is $\check{\eta} = \emptyset$, then the next jump occurs at rate $\bar{\lambda}(C \cup S)$. When a jump occurs, a random unmatched point $x_1 \in C \cup S$ is sampled with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(C \cup S)$. The next state is decided as in Step 2 above, with this new $\check{\eta} = x_1$; we ignore the details here.

We have the following theorem.

Theorem B.4. *The two processes $\hat{\eta}_t$ and $\check{\eta}_t$ are dynamically reversible. The concerned isomorphism ϕ is the one that takes a valid state $\hat{\eta}$, reverses the order of its elements, and flips the marks $\mathtt{u}$ and $\mathtt{m}$. The stationary distribution is*

$$\pi(\hat{\eta}) = K \mathbb{1}(\hat{\eta} \text{ is valid}) \prod_{i=1}^{n} \frac{\lambda(c_{x_i})}{\rho(Q_{\mathtt{u}}^i)},$$

$$\Pi(\emptyset) = K,$$

where $\hat{\eta} = (x_1, \ldots, x_n)$ and where K is a normalizing constant.

Outline of the proof. To prove this, we start by looking at the balance equations of the form in Theorem B.2. Let $\hat{\eta}$ be the state of the backward detailed process, and let $\hat{\eta}'$ be a state after a valid transition. In the following, we illustrate the local balance condition (31) for only one type of transition; other types can be handled similarly. Let q and q' be the transition rates of

the backward and forward detailed processes, respectively. Also, let $c_j(\gamma)$ denote the type of the jth element in γ, for any $\gamma \in O(C \cup S)$.

Suppose that $\hat{\eta} = (x_1, \ldots, x_n)$, $n > 0$, and that $\hat{\eta}'$ is obtained from $\hat{\eta}$ when one of the elements x_i, at some location $i > 1$, is matched and exchanged with a new arrival $(c_x, \mathtt{u})$. In this case, $\hat{\eta}' = \hat{\eta} \blacktriangle_i (c_x, \mathtt{m}) \blacktriangleleft (c_{x_i}, \mathtt{m})$, and

$$\frac{\pi(\hat{\eta})}{\pi(\hat{\eta}')} q(\hat{\eta}, \hat{\eta}') = \lambda(c_x) \prod_{j=1}^{|\hat{\eta}|} \frac{\lambda(c_j(\hat{\eta}))}{\rho(Q_{\mathtt{u}}^j(\hat{\eta}))} \prod_{j=1}^{|\hat{\eta}'|} \frac{\rho(Q_{\mathtt{u}}^j(\hat{\eta}'))}{\lambda(c_j(\hat{\eta}'))}. \tag{33}$$

The first $i-1$ elements in $\hat{\eta}'$ are $x_1, \ldots, x_{i-1}$, and $|\hat{\eta}'| = |\hat{\eta}| + 1$. Moreover, $Q_{\mathtt{u}}^j(\hat{\eta}) = Q_{\mathtt{u}}^j(\hat{\eta}') + 1$ for $i \leq j \leq |\hat{\eta}|$. Hence, Equation (33) simplifies to

$$\frac{\pi(\hat{\eta})}{\pi(\hat{\eta}')} q(\hat{\eta}, \hat{\eta}') = \prod_{j=i}^{|\hat{\eta}|} \frac{\rho(Q_{\mathtt{u}}^j(\hat{\eta}'))}{\rho(Q_{\mathtt{u}}^j(\hat{\eta}') + 1)} \times \rho(Q_{\mathtt{u}}^{|\hat{\eta}'|}(\hat{\eta}))$$

$$= \mathbb{P}(\tau_{\phi(\hat{\eta}')} > |\phi(\hat{\eta}')| - i) \times \rho(Q_{\mathtt{m}}^0(\phi(\hat{\eta}')))$$

$$= q'(\phi(\hat{\eta}'), \phi(\hat{\eta})).$$

We claim that local balance equations for other valid transitions can be handled similarly. This completes the proof of this theorem.

C. Characterization of invariant measure on compact domains: the proofs

In this section we present detailed calculations to show that the backward detailed process $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$ and the forward detailed process $\{\check{\eta}_t\}_{t\in\mathbb{R}}$, defined in Section 2.2, are dynamically reversible as jump Markov processes. As a consequence we obtain the proof of Theorem 2.2.

We first define the valid states of the Markov process corresponding to the forward detailed process, and we present the transitions and the transition rates, since these were skipped in the discussion in Section 2.2.

A valid state of the forward detailed process is given by the following rules.

Definition C.1. $(x_1, \ldots, x_n) \in O(D \times C \times \{\mathtt{u}, \mathtt{m}\})$ *is a valid state of the forward detailed process if the following hold:*

1. $s_{x_n} = \mathtt{m}$ *if* $n \geq 1$.

2. *For all* i, j, *if* $s_{x_i} = s_{x_j} = \mathtt{m}$ *and* $d(p_{x_i}, p_{x_j}) \leq 1$, *then* $c_{x_i} = c_{x_j}$.

3. *For all* $i < j$, *if* $s_{x_i} = \mathtt{u}$, $s_{x_j} = \mathtt{m}$ *and* $d(p_{x_i}, p_{x_j}) \leq 1$, *then* $c_{x_i} = c_{x_j}$.

Condition 2 in the above definition essentially states that there cannot be a compatible matched pair in a valid state. This is equivalent to the condition that

$$\{y \in x_1, \ldots x_n : s_y = \mathtt{m}\} \cap N(\{y \in x_1, \ldots x_n : s_y = \mathtt{m}\}) = \emptyset.$$

This is because, if there were a violating pair at time t, such a pair could potentially have been matched to each other before time t, instead of being matched to their present matches. Condition 3 is required since otherwise, for a violating pair x_i and x_j with $i < j$, the particle x_j could instead have been matched with the particle x_i, which arrived before the particle to

which x_j is actually matched. This is equivalent to the condition that for all $1 \leq j \leq n$,

$$s_{x_j} = \mathtt{m} \implies x_j \notin N(\{y \in x_1, \ldots, x_i : s_y = \mathtt{u}\}).$$

The Markov process corresponding to $\{\check{\eta}_t\}$ evolves as follows. Let $\check{\eta} = (x_1, \ldots, x_n)$, $n = |\check{\eta}|$, be the state of the system at some time t. The next jump occurs at rate $\rho(Q_{\mathtt{m}}^0) = 2\lambda(D) + Q_{\mathtt{m}}^0\mu$, where, for the sake of brevity, we use the notation $\rho(n) = 2\lambda(D) + n\mu$ for any $n \in \mathbb{N}$. If $\check{\eta}$ is non-empty, the first element, x_1, in the list $\check{\eta}$ is processed at the next jump according to the following rules:

1. If x_1 is matched, then it is removed from $\check{\eta}$. The new state is $\check{\eta} \blacktriangleright_1$.

2. If x_1 is unmatched, then for the next state, we sample a random variable $\tau \in \mathbb{N}$ with distribution

$$\mathbb{P}(\tau = k) = \frac{\mu}{\rho(Q_{\mathtt{m}}^k + 1)} \prod_{i=1}^{k-1} \frac{\rho(Q_{\mathtt{m}}^i)}{\rho(Q_{\mathtt{m}}^i + 1)},$$

and then sample $x_{n+1}, \ldots x_\tau$ i.i.d. random unmatched points in $D \times C$ with distribution $\lambda \otimes m_c / (2\lambda(D))$.

   (a) If there is an FIFM matching x_i, $2 \leq i \leq \tau$, for x_1, then we set the new state to $(x_1^{\max(n,\tau)}) \blacktriangle_i (p_{x_1}, c_{x_1}, \mathtt{m}) \blacktriangleright_1$, with the understanding that all the unmatched particles at the end of the list are discarded.

   (b) If there is no FIFM matching, then we set the new state to $(x_1^{\max(n,\tau)}) \blacktriangleleft_\tau (p_{x_1} c_{x_1}, \mathtt{m}) \blacktriangleright_1$.

If the state is $\check{\eta} = \emptyset$, then the next jump occurs at rate $\rho(0) = 2\lambda(D)$. A random unmatched particle x_1 is sampled with distribution $\bar{\lambda}/2\lambda(D)$. The next state is decided as in Step 2 above with $\check{\eta} = (x_1)$.

We are now ready to prove Theorem 2.2.

Proof of Lemma 2.1 and Theorem 2.2. To obtain the stationary distribution, we check the conditions of Theorem 6.1. The space $O(D, C \times \{\mathtt{u}, \mathtt{m}\})$ is viewed as a subset of $\sqcup_{n=0}^{\infty}(D \times C \times \{\mathtt{u}, \mathtt{m}\})^n$, and we use the induced topology on $O(D, C \times \{\mathtt{u}, \mathtt{m}\})$. With this topology, $O(D, C \times \{ttu, \mathtt{m}\})$ is a locally compact Hausdorff space. Let $\hat{D} = D \times C \times \{\mathtt{u}, \mathtt{m}\}$, and let $\hat{\lambda}$ be the measure $\lambda \otimes m_c \otimes m_c$ on $\hat{D}$. The probability semigroup of the processes $\hat{\eta}$ acts over the Banach space of continuous functions that vanish at infinity, where we use the uniform norm topology. Moreover, the generator of $\hat{\eta}$ can at least be defined on the space of compactly supported continuous functions, and has the form

$$L_1 f(\hat{\eta}) = \sum_{i=1}^{|\hat{\eta}|} \mathbb{1}(s_{x_i} = \mathtt{u})\mu[f(\hat{\eta} \blacktriangleright_i \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) - f(\hat{\eta})]$$

$$+ \int_{\tilde{D}} \sum_{i=1}^{|\hat{\eta}|} \mathbb{1}(x \in W_{x_i})[f(\hat{\eta} \blacktriangle_i (p_x, c_x, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) - f(\hat{\eta})]\bar{\lambda}(dx) \quad (34)$$

$$+ \int_{\tilde{D}} \mathbb{1}(x \notin N(\hat{\eta}^{\mathtt{u}}))[f(\hat{\eta} \blacktriangleleft (p_x, c_x, \mathtt{u})) - f(\hat{\eta})]\bar{\lambda}(dx).$$

Similarly, it can be seen that the generator of $\check{\eta}$ can also be defined over the space of compactly supported continuous functions, and the value of the generator $L_2 g(\check{\eta})$ is the sum of the following terms (in the order of the transitions listed earlier):

- (1): $\rho(Q_{\mathrm{m}}^0(\check{\eta}))\mathbb{1}(s_{x_1} = \mathrm{m})[g(\check{\eta} \blacktriangleright_1) - g(\check{\eta})]$.

- (2a):

$$
\rho(Q_{\mathrm{m}}^0(\check{\eta}))\mathbb{1}(s_{x_1} = \mathrm{u})
$$
$$
\times \left(\sum_{k=2}^{|\check{\eta}|} \mathbb{P}(\tau(\check{\eta}) > k)\mathbb{1}((p_{x_1}, c_{x_1}) \in W_{x_k})[g(\hat{\eta} \, \blacktriangle_k \, (p_{x_1}, c_{x_1}, \mathrm{m}) \blacktriangleright_1) - g(\hat{\eta})] \right.
$$
$$
+ \sum_{k=|\check{\eta}|+1}^{\infty} \mathbb{P}(\tau(\check{\eta}) > k) \int_{\tilde{D}^{k-|\check{\eta}|}} \left(\mathbb{1}((p_{x_1}, c_{x_1}) \in W_{x_k}) \right.
$$
$$
\left. \left. \times [g((x_1^k)\blacktriangle_k \, (p_{x_1}, c_{x_1}, \mathrm{m}) \blacktriangleright_1) - g(\hat{\eta})] \right) \bar{\lambda}(dx_{|\check{\eta}|+1}^k) \right),
$$

where we have set $s_{x_j} = \mathrm{u}$ for all $j > |\check{\eta}|$.

- (2b):

$$
\rho(Q_{\mathrm{m}}^0(\check{\eta}))\mathbb{1}(s_{x_1} = \mathrm{u})
$$
$$
\times \left(\sum_{k=1}^{|\check{\eta}|} \mathbb{P}(\tau = k)\mathbb{1}((p_{x_1}, c_{x_1}) \notin N(\check{\eta}_1^{k,\mathrm{u}}))[g(\hat{\eta} \, \blacktriangleleft_k \, (p_{x_1}, c_{x_1}, \mathrm{m}) \blacktriangleright_1) - g(\hat{\eta})] \right.
$$
$$
+ \sum_{k=|\check{\eta}|+1}^{\infty} \mathbb{P}(\tau = k) \int_{\tilde{D}^{k-|\check{\eta}|}} \mathbb{1}((p_{x_1}, c_{x_1}) \notin N(x_1^{k,\mathrm{u}}))
$$
$$
\left. \times [g((x_1^k) \blacktriangleleft (p_{x_1}, c_{x_1}, \mathrm{m}) \blacktriangleright_1) - g(\check{\eta})]\bar{\lambda}(dx_{|\check{\eta}|+1}^k) \right),
$$

where $s_{x_j} = \mathrm{u}$ for all $j > |\check{\eta}|$.

When $\check{\eta} = \emptyset$, we have the following terms in $L_2 g(\emptyset)$:

- (2a.):

$$
\rho(0) \int_{\tilde{D}} \sum_{k=2}^{\infty} \mathbb{P}(\tau(\emptyset) > k - 1) \int_{\tilde{D}^{k-1}} \left(\mathbb{1}((p_{x_1}, c_{x_1}) \in W_{x_k}) \right.
$$
$$
\left. \times [g((x_2^{k-1}) \blacktriangleleft (p_{x_1}, c_{x_1}, \mathrm{m})) - g(\emptyset)] \right) \bar{\lambda}(dx_2^k)\bar{\lambda}(dx_1),
$$

where $s_{x_j} = \mathrm{u}$ for all $j > 0$.

- (2b.):

$$
\rho(0) \int_{\tilde{D}} \sum_{k=1}^{\infty} \mathbb{P}(\tau_1 = k) \int_{\tilde{D}_{\mathrm{u}}^{k-1}} \left(\mathbb{1}((p_{x_1}, c_{x_1}) \notin N(x_1^{k,\mathrm{u}})) \right.
$$
$$
\left. \times [g((x_2^k) \blacktriangleleft (p_{x_1}, c_{x_1}, \mathrm{m})) - g(\emptyset)] \right) \bar{\lambda}(dx_2^k)\bar{\lambda}(dx_1),
$$

where $s_{x_j} = \mathrm{u}$ for all $j > 0$.

For the product-form distribution, we check the balance condition of Theorem 6.1, with an appropriate measure space isomorphism ϕ. The isomorphism is given by the function *revx*, defined in Section 2.2, which reverses the order of points and flips the marks u and m of a valid state. In the following, let $\phi := \mathrm{revx}$ denote this function.

Taking any two compactly supported continuous functions f, g, and taking each term of $\int gL_1 f + gf\hat{\pi}(d\hat{\eta})$, we show that it corresponds to a few terms in $\int L_2(g \circ \phi^{-1})(\phi(\hat{\eta}))f(\hat{\eta}) + gf\hat{\pi}(d\hat{\eta})$, so that the sum of these expressions is equal. In particular, the following steps suffice:

1. Take the first summation term of $\int gL_1 f + gf\, d\hat{\pi}$ when it is expanded using Equation (34). Take the ith term, with $i > 1$. Set $\hat{\eta}' = \hat{\eta} \blacktriangleright_i \blacktriangleleft (p_i, c_i, \mathtt{m})$. Let $n := |\hat{\eta}|$, and so $n = |\hat{\eta}'|$. Also, let $\hat{\eta} = (x_1, \ldots, x_n)$ and $\hat{\eta}' = (x_1', \ldots, x_n')$. The corresponding term is

$$
\int \mu \mathbb{1}(s_{x_i} = \mathtt{u}) g(\hat{\eta}) f(\hat{\eta}') \hat{\pi}(d\hat{\eta})
$$

$$
:= \mu \sum_{n=i}^{\infty} \int_{\hat{D}^n} \mathbb{1}(s_{x_i} = \mathtt{u}) \hat{\pi}(\hat{\eta}) g(\hat{\eta}) f(\hat{\eta}') \hat{\lambda}(d\hat{\eta})
$$

$$
= \sum_{n=i}^{\infty} \int_{\hat{D}^n} \hat{\pi}(\hat{\eta}') \frac{\mu \hat{\pi}(\hat{\eta}' \blacktriangleleft_{i-1} (p_{x_n'}, c_{x_n'}, \mathtt{u}) \blacktriangleright)}{\hat{\pi}(\hat{\eta}')} \tag{35}
$$

$$
\times g(\hat{\eta}' \blacktriangleleft_i (p_{x_n'}, c_{x_n'}, \mathtt{u}) \blacktriangleright) f(\hat{\eta}') \hat{\lambda}(d\hat{\eta}')
$$

$$
= \int \mathbb{1}(n \ge i) \mathbb{P}(\tau_{\phi(\hat{\eta}')} = n - i + 1)
$$

$$
\times \rho(Q_{\mathtt{m}}^0(\phi(\hat{\eta}'))) g(\hat{\eta}' \blacktriangleleft_i (p_{x_n'}, c_{x_n}, \mathtt{u}) \blacktriangleright) f(\hat{\eta}') \hat{\pi}(d\hat{\eta}'),
$$

where in the second equality we use that $\hat{\eta} = \hat{\eta}' \blacktriangleleft_{i-1} (p_{x_n'}, c_{x_n'}, \mathtt{u}) \blacktriangleright$, and in the third equality we use that

$$
\frac{\mu \pi(\hat{\eta}' \blacktriangleleft_{i-1} (p_n', c_n', \mathtt{u}) \blacktriangleright)}{\pi(\hat{\eta}')} = \frac{\mu}{\rho(N_{\mathtt{u}}^i(\hat{\eta}))} \prod_{j=i}^{n-1} \frac{\rho(N_{\mathtt{u}}^j(\hat{\eta}'))}{\rho(N_{\mathtt{u}}^{j+1}(\hat{\eta}))} \rho(N_{\mathtt{u}}^n(\hat{\eta}')) \tag{36}
$$

$$
= \mathbb{P}(\tau_{\phi(\hat{\eta}')} = n - i + 1) \rho(N_{\mathtt{m}}^0(\phi(\hat{\eta}'))).
$$

Similarly, in the first summation term of $\int gL_1 f + gf$, taking $i = 1$, and letting $k(\hat{\eta})$ be the maximum element such that $x_2, \ldots, x_k$ are all matched in $\hat{\eta}$, we have

$$
\int \mu g(\hat{\eta}) f(\hat{\eta}') \hat{\pi}(\hat{\eta})
$$

$$
= \mu \sum_{n=2}^{\infty} \int_{\hat{D}^n} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat{\eta}) = j) g(x_1^n) f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) \hat{\pi}(\hat{\eta}) \hat{\lambda}(d\hat{\eta})
$$

$$
+ \mu \sum_{n=1}^{\infty} \int_{\hat{D}^n} \mathbb{1}(k(\hat{\eta}) = n) g(x_1^n) f(\emptyset) \hat{\pi}(\hat{\eta}) \hat{\lambda}(\hat{\eta}),
$$

where we have used the fact that if $k(\hat{\eta}) = |\hat{\eta}|$, then $\hat{\eta}' = \emptyset$. Consider the first term in the above expression. Setting $m = n - j + 1$ and $x_1'^m = (x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m}))$, we obtain

$$
\mu \sum_{n=2}^{\infty} \int_{\hat{D}^n} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat{\eta}) = j) g(x_1^n) f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) \hat{\pi}(\hat{\eta}) \hat{\lambda}(d\hat{\eta})
$$

$$
= \mu \sum_{j=1}^{\infty} \sum_{m=2}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x_m'} = \mathtt{m}) \bar{\lambda}(\tilde{D})^{j-1} \mathbb{E}_{X_2^j} \left[g((p_{x_m'}, c_{x_m'}, \mathtt{u}) X_2^j x_1'^{m-1}) \right.
$$

$$
\left. f(x_1'^m) \hat{\pi}((p_{x_m'}, c_{x_m'}, \mathtt{u}) X_2^j x_1'^{m-1}) \right] \hat{\lambda}(dx_1'^m),
$$

(37)

where X_2^j are i.i.d. particles in $\tilde{D}$ with marks $\mathtt{m}$, with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(\tilde{D})$. Using a computation similar to that of (36), it is easy to see that the right-hand side of (38) is

$$
\sum_{m=2}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x_m'} = \mathtt{m}) \rho(Q_{\mathtt{m}}^0(\phi(x_1'^m))) \sum_{j=1}^{\infty} \mathbb{P}(\tau(\phi(x_1'^m)) = m + j - 1) \hat{\pi}(x_1'^m)
$$

$$
\times \mathbb{E}_{X_2^j} \left[g((p_{x_m'}, c_{x_m'}, \mathtt{u}) X_2^j x_1'^{m-1}) f(x_1'^m) \right] \hat{\lambda}(dx_1'^m).
$$

(38)

Similarly, the second term in (37) is

$$
\mu \sum_{n=1}^{\infty} \int_{\hat{D}^n} \mathbb{1}(k(\hat{\eta}) = n) g(x_1^n) f(\emptyset) \hat{\pi}(\hat{\eta}) \hat{\lambda}(\hat{\eta})
$$

$$
= \hat{\pi}(\emptyset) \rho(0) \sum_{j=1}^{\infty} \mathbb{P}(\tau(\emptyset) = j) \mathbb{E}_{X_1^j} f(\emptyset) g(X_1^j),
$$

(39)

where the X_2^j are as before, and X_1 is an independent sample from $\tilde{D}$ with mark $\mathtt{u}$.

2. Now take the second summation in $\int gL_1 f + gf\hat{\pi}$, and consider the ith term, $i > 1$, in that summation. Using a computation similar to that of the previous item, we obtain

$$
\int \int_{\tilde{D}} \mathbb{1}(s_{x_i} = \mathtt{u}) \mathbb{1}\left((p, c) \in W_{x_i}\right) f(\hat{\eta}') g(\hat{\eta}) \bar{\lambda}(dp, dc) \hat{\pi}(d\hat{\eta})
$$

$$
= \sum_{n=i}^{\infty} \int_{\hat{D}^n} \int_{\tilde{D}} \mathbb{1}(s_{x_i} = \mathtt{u}) \mathbb{1}\left((p, c) \in W_{x_i}\right)
$$

$$
f(x_1^n \blacktriangle_i (p, c, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) g(x_1^n) \hat{\pi}(x_1^n) \tilde{\lambda}(dp, dc) \hat{\lambda}(d\hat{\eta}).
$$

Setting $x_1'^{n+1} = x_1^n \blacktriangle_i (p, c, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})$ and $m = n + 1$, we have that the right-hand side of the above equation is equal to

$$
\sum_{m=i+1}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x_i'} = \mathtt{m} = s_{x_m'}) \mathbb{1}((p_{x_m'}, c_{x_m'}) \in W_{x_i'})
$$

$$
f(x_1'^m) g(x_1'^{m-1} \blacktriangle_i (p_{x_m'}, c_{x_m'}, \mathtt{u})) \hat{\pi}(x_1'^{m-1} \blacktriangle_i (p_{x_m'}, c_{x_m'}, \mathtt{u})) \hat{\lambda}(dx_1'^m)
$$

$$
= \sum_{m=i+1}^{\infty} \int_{\hat{D}^m} \rho(Q_{\mathtt{m}}^0(\phi(x_1'^m))) \mathbb{P}(\tau(\phi(x_1'^m)) > m - i)) \mathbb{1}(s_{x_i'} = \mathtt{m} = s_{x_m'})
$$

$$
\mathbb{1}((p_{x_m'}, c_{x_m'}) \in W_{x_i'}) f(x_1'^m) g(x_1'^{m-1} \blacktriangle_i (p_{x_m'}, c_{x_m'}, \mathtt{u})) \hat{\pi}(x_1'^m) \hat{\lambda}(dx_1'^m).
$$

(40)

Similarly, taking the first term in the second summation, and letting $k(\hat{\eta})$ be as before, we have

$$
\int \int_{\tilde{D}} \mathbb{1}((p, c) \in W_{x_1}) f(\hat{\eta}') g(\hat{\eta}) \tilde{\lambda}(dp, dc) \hat{\pi}(d\hat{\eta})
$$
$$
= \sum_{n=2}^{\infty} \int_{\hat{D}^n} \int_{\tilde{D}} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat{\eta}) = j) \mathbb{1}((p, c) \in W_{x_1})
$$
$$
f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathfrak{m})) g(x_1^n) \hat{\pi}(x_1^n) \tilde{\lambda}(dp, dc) \hat{\lambda}(dx_1^n)
$$
$$
+ \sum_{n=1}^{\infty} \int_{\hat{D}^n} \int_{\tilde{D}} \mathbb{1}(k(\hat{\eta}) = n) \mathbb{1}((p, c) \in W_{x_1}) f(\emptyset) g(x_1^n) \hat{\pi}(x_1^n) \tilde{\lambda}(dp, dc) \hat{\lambda}(dx_1^n). \tag{41}
$$

Computations similar to those in (38) and (39) show that the above is equal to

$$
\sum_{m=2}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x_m'} = \mathfrak{m}) \rho(Q_{\mathfrak{m}}^0(\phi(x_1'^m))) \sum_{j=1}^{\infty} \mathbb{P}(\tau(\phi(x_1'^m)) > m + j - 1)
$$
$$
\times \mathbb{E}_{X_2^{j+1}} \mathbb{1}((p_{x_1}, c_{x_1}) \in W_{X_{j+1}}) \left[g((p_{x_m'}, c_{x_m'}, \mathfrak{u}) X_2^j x_1'^{m-1}) f(x_1'^m) \right]
$$
$$
\times \hat{\pi}(x_1'^m) \hat{\lambda}(dx_1'^m) \tag{42}
$$
$$
+ \hat{\pi}(\emptyset) \rho(0) \sum_{j=1}^{\infty} \mathbb{P}(\tau(\emptyset) > j) \mathbb{E}_{X_1^{j+1}} \mathbb{1}((p_{X_1}, c_{X_1}) \in W_{X_{j+1}}) f(\emptyset) g(X_1^j),
$$

where X_1 and X_{j+1} are i.i.d. with marks $\mathfrak{u}$, and the X_2^j are i.i.d. with marks $\mathfrak{m}$.

3. Finally,

$$
\int \int_{\tilde{D}} \mathbb{1}((p, c) \notin N(\hat{\eta}^{\mathfrak{u}})) g(\hat{\eta}) f(\hat{\eta}') \tilde{\lambda}(dp, dc) \hat{\pi}(d\hat{\eta})
$$
$$
= \sum_{n=0}^{\infty} \int_{\hat{D}^n} \int_{\tilde{D}} \mathbb{1}((p, c) \notin N(x \in x_1^n : s_x = \mathfrak{u}))
$$
$$
g(x_1^n) f(x_1^n \blacktriangleleft (p, c, \mathfrak{u})) \hat{\pi}(x_1^n) \tilde{\lambda}(dp, dc) \hat{\lambda}(dx_1^n).
$$

Setting $m = n + 1$, $x_1'^m = x_1^n \blacktriangleleft (p, c, \mathfrak{u})$, we have

$$
\sum_{m=1}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x_m'} = \mathfrak{u}) g(x_1'^{m-1}) f(x_1'^m) \hat{\pi}(x_1'^{m-1}) \hat{\lambda}(dx_1'^m)
$$
$$
= \sum_{m=1}^{\infty} \int_{\tilde{D}^m} \rho(Q_{\mathfrak{u}}^0(\phi(x_1'^m))) \mathbb{1}(s_{x_m'} = \mathfrak{u}) g(x_1'^{m-1}) f(x_1'^m) \hat{\pi}(x_1'^{m-1}) \hat{\lambda}(dx_1'^m). \tag{43}
$$

Since summing over the right-hand sides of (35)–(43) for all $i \geq 1$ produces $\int fL_2 g + fg\hat{\pi}(d\eta)$, we conclude that the two Markov processes $\hat{\eta}_t$ and $\check{\eta}_t$ are dynamically reversible with respect to $\hat{\pi}$.

Proof of Corollary 2.1. We calculate the marginal distribution of the unmatched particles from the distribution in Theorem 2.2. Given that $\eta = (x_1, \ldots, x_n)$ is in the steady state, let l_i,

$1 \le i \le n$, denote the number of matched particles present between x_i and x_{i+1} in the detailed version of the process. The only restriction that these particles must satisfy is that they must be incompatible with $x_1, \ldots, x_i$. Integrating over the positions of each of the l_i particles gives a factor $\bar{\lambda}(D \times C \backslash N(x_1, \ldots, x_i))$. Thus, we have

$$
\begin{aligned}
\pi(x_1, \ldots, x_n) &= K \mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \sum_{l_i \in \mathbb{N}} \frac{(\bar{\lambda}(D \times C \backslash N(x_1, \ldots, x_i)))^{l_i}}{(2\lambda(D) + i\mu)^{l_i+1}} \\
&= K \mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \frac{1}{2\lambda(D) + i\mu - \bar{\lambda}(D \times C \backslash N(x_1, \ldots, x_i))} \\
&= K \mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \frac{1}{\bar{\lambda}(N(\eta_1^i)) + i\mu}.
\end{aligned}
$$

D. Proof of the FKG inequality

To prove the FKG lattice property in our setting, we need the following auxiliary lemma.

Lemma D.1. *Let $(\alpha_i)_{i=1}^{n}$ and $(\beta_j)_{j=1}^{m}$ be two sets of positive numbers. Let $P(n,m)$ be the set of all increasing paths in the grid $[n] \times [m]$, so that for any $\sigma \in P(n, m)$, we have that $\sigma(0) = (0, 0)$, $\sigma(m + n) = (n, m)$, and $\sigma(i + 1) - \sigma(i)$ is either (1, 0) or (0, 1), for all $0 \le i < m + n$. Then we have*

$$
\sum_{\sigma \in P(n,m)} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \prod_{i=1}^{n} \frac{1}{\alpha_i} \prod_{i=1}^{m} \frac{1}{\beta_i},
$$

where σ_x and σ_y denote the x- and the y-coordinate, respectively.

Proof. The proof is by induction on m. For $m = 1$, we need to show that

$$
\sum_{\sigma \in P(n,1)} \prod_{i=1}^{n+1} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \frac{1}{\beta_1} \prod_{i=1}^{n} \frac{1}{\alpha_i}. \tag{44}
$$

We use induction on n to prove (44). For $n = 1$, this is clear, since

$$
\frac{1}{(\alpha_1)(\alpha_1 + \beta_1)} + \frac{1}{(\beta_1)(\beta_1 + \alpha_1)} = \frac{1}{(\alpha_1)(\beta_1)}.
$$

Let the length of the sequence α be n. Assuming the inductive hypothesis for (44), we have

$$
\begin{aligned}
&\sum_{k=1}^{n+1} \sum_{\substack{\sigma \in P(n,1), \\ \sigma(k) - \sigma(k-1) = (0,1)}} \prod_{i=1}^{n+1} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} \\
&= \left(\sum_{k=1}^{n} \sum_{\substack{\sigma \in P(n,1), \\ \sigma(k) - \sigma(k-1) = (0,1)}} \prod_{i=1}^{n} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} + \prod_{i=1}^{n} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1}
\end{aligned}
$$

$$= \left(\sum_{\sigma \in P(n-1,1)} \prod_{i=1}^{n} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} + \prod_{i=1}^{n} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1}$$

$$= \left(\prod_{i=1}^{n-1} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1} \left(\frac{1}{\beta_1} + \frac{1}{\alpha_n} \right)$$

$$= \frac{1}{\beta_1} \left(\prod_{i=1}^{n} \frac{1}{\alpha_i} \right),$$

where in the first step we have grouped the first n terms together. This finishes the proof of the base case (Equation (44)) for the induction on m.

Now, suppose that the length of the sequence β is equal to m, $m > 1$. Let $P_k(n, m-1)$, $0 \le k \le n$, denote the set of paths in $P(n, m-1)$ where the first $(0,1)$ jump is at location k. We have the following decomposition of the summation:

$$\sum_{\sigma \in P(n,m)} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \sum_{k=1}^{n+1} \sum_{P_k(n,m-1)} \sum_{r=0}^{k-1} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x^r(i)} + \beta_{\sigma_y^r(i)}},$$

where σ^r denotes the path obtained by inserting a $+(0, 1)$ jump in the rth location of σ. Thus,

$$\sum_{\sigma \in P(n,m)} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \sum_{\substack{k=1,\ldots,n+1; \\ \sigma \in P_k(n,m-1)}} \sum_{r=0}^{k-1} \prod_{i=1}^{k} \frac{1}{\alpha_{\sigma_x^r(i)} + \beta_{\sigma_y^r(i)}} \prod_{i=k+1}^{n+m} \frac{1}{\alpha_{\sigma_x^r(i)} + \beta_{\sigma_y^r(i)}}.$$

The innermost summation in the above expression is the $(n, 1)$ case of this lemma, and therefore, by the induction hypothesis, we get that

$$\prod_{i=k+1}^{n+m} \frac{1}{\alpha_{\sigma_x^k(i)} + \beta_{\sigma_y^k(i)}} \times \frac{1}{\beta_1} \prod_{i=1}^{k-1} \frac{1}{\alpha_i} = \frac{1}{\beta_1} \prod_{i=1}^{n+m-1} \frac{1}{\alpha_{\sigma_x(i)} + \beta'_{\sigma_y(i)}},$$

where β' is the sequence of length $m - 1$ with $\beta'_i = \beta_{i+1}$, for $1 \le i \le m - 1$. Therefore,

$$\sum_{\sigma \in P(n,m)} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \frac{1}{\beta_1} \sum_{k=0}^{n} \sum_{P_k(n,m-1)} \prod_{i=1}^{n+m-1} \frac{1}{\alpha_{\sigma_x(i)} + \beta'_{\sigma_y(i)}}$$

$$= \frac{1}{\beta_1} \sum_{\sigma \in P(n,m-1)} \prod_{i=1}^{n+m-1} \frac{1}{\alpha_{\sigma_x(i)} + \beta'_{\sigma_y(i)}}$$

$$= \prod_{i=1}^{n} \frac{1}{\alpha_i} \prod_{i=1}^{m} \frac{1}{\beta_i},$$

by the induction hypothesis. This completes the proof the lemma. $\square$

We are now ready to prove a weak form of the FKG lattice property for the Janossy density $\tilde{\pi}$.

Lemma D.2. *Let ξ and γ be disjoint finite subsets of $D \times C$ such that the set $\xi \cup \gamma$ is valid. Then we have*

$$\tilde{\pi}(\xi \cup \gamma)\tilde{\pi}(\emptyset) \geq \tilde{\pi}(\xi)\tilde{\pi}(\gamma). \tag{45}$$

Moreover, if ξ and γ are such that $N(\xi) \cap N(\gamma) = \emptyset$, then equality holds.

Proof. Let $|\xi| = n$ and $|\gamma| = m$. There is a canonical bijection between $\mathcal{P}(\xi \cup \gamma)$ and $\mathcal{P}(\xi) \times \mathcal{P}(\gamma) \times P(n, m)$. For $(a, b, \sigma) \in \mathcal{P}(\xi) \times \mathcal{P}(\gamma) \times P(n, m)$, we denote by (σ, ab) the corresponding element in $\mathcal{P}(\xi \cup \gamma)$. Also, for any sets $A, C \subset D \times C$, let $N_C(A) = N(A \cap C)$.

We have

$$\tilde{\Pi}(\xi \cup \gamma) = \sum_{\substack{a \in \mathcal{P}(\xi), b \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N((\sigma, ab)_1^l)) + l\mu}$$

$$\geq \sum_{\substack{a \in \mathcal{P}(\xi), b \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N_\xi((\sigma, ab)_1^l)) + \bar{\lambda}(N_\gamma((\sigma, ab)_1^l)) + l\mu} \tag{46}$$

$$= \sum_{\substack{a \in \mathcal{P}(\xi), b \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N(a_1^{\sigma_x(l)})) + \bar{\lambda}(N(b_1^{\sigma_y(l)})) + l\mu}$$

$$= \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma),$$

where in the second step we use that

$$\bar{\lambda}(N((\sigma, ab)_1^l)) \leq \bar{\lambda}(N_\xi((\sigma, ab)_1^l)) + \bar{\lambda}(N_\gamma((\sigma, ab)_1^l)),$$

and in the last step we use Lemma D.1 with $\alpha_i = \lambda(N(X_1^i)) + i\mu$ and $\beta_i = \lambda(N(Y_1^i)) + i\mu$. Also, $\sigma_x(l)$ and $\sigma_y(l)$ are the two coordinates of the point $\sigma(l) \in \mathbb{Z}^2$ on the path σ. The result now follows, since $K = \tilde{\pi}(\emptyset)$. Note also that if $N(\xi) \cap N(\gamma) = \emptyset$, then equality holds in (46). $\qquad\square$

The above theorem is useful in the proof of Theorem 2.3 only through the following corollary.

Corollary D.1. *Suppose $\gamma = \gamma^{\mathrm{R}} \cup \gamma^{\mathrm{B}}$ is a valid configuration (i.e., $\gamma \cap N(\gamma) = \emptyset$), where γ^{R} and γ^{B} are the red and blue particles, respectively, in γ. Then $\tilde{\pi}(\gamma) = \frac{1}{K}\tilde{\pi}(\gamma^{\mathrm{R}})\tilde{\pi}(\gamma^{\mathrm{B}})$, or equivalently, $\tilde{\Pi}(\gamma) = \tilde{\Pi}(\gamma^{\mathrm{R}})\tilde{\Pi}(\gamma^{\mathrm{B}})$.*

We now state and prove an FKG lattice property for the usual subset ordering for the same type of particles.

Theorem D.1. *Let ξ and γ be finite subsets of $D \times \{\mathrm{R}\}$. Then*

$$\tilde{\Pi}(\xi \cup \gamma)\tilde{\Pi}(\xi \cap \gamma) \geq \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma). \tag{47}$$

A similar property holds when ξ and γ are finite subsets of $D \times \{\mathrm{B}\}$.

Proof. Let $A = \xi \setminus \gamma$, $B = \gamma \setminus \xi$, and $C = \xi \cap \gamma$. The statement of the theorem is equivalent to showing that

$$\tilde{\Pi}(A \cup B \cup C)\tilde{\Pi}(C) \geq \tilde{\Pi}(A \cup C)\tilde{\Pi}(B \cup C). \tag{48}$$

Let $\bar{C}$ denote another copy of C, where we add overlines to the particles to distinguish them from particles of C. Also, for any E, $\gamma \subseteq D \times C$, let $N_E(\gamma) = N(\gamma \cap E)$.

Using the auxiliary Lemma 8, the left-hand side of the inequality above can be expressed as

$$
\begin{aligned}
& \tilde{\Pi}(A \cup B \cup C)\tilde{\Pi}(\bar{C}) \\
&= \sum_{\substack{e \in \mathcal{P}(A \cup B \cup C) \\ d \in \mathcal{P}(\bar{C})}} \prod_{i=1}^{n+m+k} \frac{1}{\bar{\lambda}(N(e_1^i)) + i\mu} \prod_{j=1}^{k} \frac{1}{\bar{\lambda}(N(d_1^j)) + j\mu} \\
&= \sum_{\substack{e \in \mathcal{P}(A \cup B \cup C) \\ d \in \mathcal{P}(\bar{C})}} \sum_{\sigma \in P(n+m+k,k)} \prod_{i=1}^{n+m+2k} \frac{1}{\bar{\lambda}(N(e_1^{\sigma_x(i)})) + \bar{\lambda}(N(d_1^{\sigma_y(i)})) + i\mu},
\end{aligned}
\tag{49}
$$

where we use $\sigma_x(i)$, $\sigma_y(i)$ to represent the first and second coordinates of $\sigma(i)$.

Since

$$\bar{\lambda}(N_{A \cup B \cup C}(\gamma)) \leq \begin{pmatrix} \bar{\lambda}(N_A(\gamma)) + \bar{\lambda}(N_B(\gamma)) + \bar{\lambda}(N_C(\gamma)) \\ -\bar{\lambda}(N_A(\gamma) \cap N_C(\gamma)) - \bar{\lambda}(N_B(\gamma) \cap N_C(\gamma)) \end{pmatrix}, \tag{50}$$

we claim that (49) is greater than

$$\sum_{\substack{e \in \mathcal{P}(A \cup B \cup C) \\ d \in \mathcal{P}(\bar{C}) \\ \sigma \in P(n+m+k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N_A(e_1^{\sigma_x(i)})) + \bar{\lambda}(N_B(e_1^{\sigma_x(i)})) + \bar{\lambda}(N_C(e_1^{\sigma_x(i)})) + \bar{\lambda}(N_{\bar{C}}(d_1^{\sigma_y(i)})) \\ -\bar{\lambda}(N_A(e_1^{\sigma_x(i)}) \cap N_C(e_1^{\sigma_x(i)})) - \bar{\lambda}(N_B(e_1^{\sigma_x(i)}) \cap N_C(e_1^{\sigma_x(i)})) + i\mu \end{pmatrix}^{-1}. \tag{51}$$

Note that in the last two expressions, and in expressions below, the terms within the parentheses are stacked to suitably render the long expressions—these are not to be confused with other notation used in combinatorics.

Let $P(n, m, k, k)$ be the set of all increasing paths from vertex $(0,0,0,0)$ to vertex (n, m, k, k) in $\mathbb{Z}^4$; $\sigma(0) = (0, 0, 0, 0)$ and $\sigma(n + m + 2k) = (n, k, k, k)$, for all $\sigma \in P(n, m, k, k)$. We denote the coordinates of $\sigma \in \mathbb{Z}^4$ by $(\sigma_x, \sigma_y, \sigma_z, \sigma_w)$. Using the canonical bijection between

$$\mathcal{P}(A \cup B \cup C \cup \bar{C}) \times P(n + m + k, k) \quad \text{and}$$
$$\mathcal{P}(A) \times \mathcal{P}(B) \times \mathcal{P}(C) \times \mathcal{P}(\bar{C}) \times P(n, m, k, k),$$

we see that (51) is equal to

$$\sum_{\substack{a \in \mathcal{P}(A), b \in \mathcal{P}(B) \\ c \in \mathcal{P}(B), \bar{c} \in \mathcal{P}(\bar{C}) \\ \sigma \in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(a_1^{\sigma_x(i)})) + \bar{\lambda}(N(b_1^{\sigma_y(i)})) + \bar{\lambda}(N(c_1^{\sigma_z(i)})) + \bar{\lambda}(N(\bar{c}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(a_1^{\sigma_x(i)}) \cap N(c_1^{\sigma_z(i)})) - \bar{\lambda}(N(b_1^{\sigma_y(i)}) \cap N(c_1^{\sigma_z(i)})) + i\mu \end{pmatrix}^{-1}. \tag{52}$$

Applying similar reductions to the right-hand side of Equation (48), we note that the result follows if we prove

$$
\sum_{\substack{a \in \mathcal{P}(A), b \in \mathcal{P}(B) \\ c \in \mathcal{P}(B), \bar{c} \in \mathcal{P}(\bar{C}) \\ \sigma \in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(a_1^{\sigma_x(i)})) + \bar{\lambda}(N(b_1^{\sigma_y(i)})) + \bar{\lambda}(N(c_1^{\sigma_z(i)})) + \bar{\lambda}(N(\bar{c}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(a_1^{\sigma_x(i)}) \cap N(c_1^{\sigma_z(i)})) - \bar{\lambda}(N(b_1^{\sigma_y(i)}) \cap N(c_1^{\sigma_z(i)})) + i\mu \end{array} \right)^{-1}
$$

$$
\geq \sum_{\substack{a \in \mathcal{P}(A), b \in \mathcal{P}(B) \\ c \in \mathcal{P}(B), \bar{c} \in \mathcal{P}(\bar{C}) \\ \sigma \in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(a_1^{\sigma_x(i)})) + \bar{\lambda}(N(b_1^{\sigma_y(i)})) + \bar{\lambda}(N(c_1^{\sigma_z(i)})) + \bar{\lambda}(N(\bar{c}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(a_1^{\sigma_x(i)}) \cap N(c_1^{\sigma_z(i)})) - \bar{\lambda}(N(b_1^{\sigma_y(i)}) \cap N(\bar{c}_1^{\sigma_w(i)})) + i\mu \end{array} \right)^{-1}.
\tag{53}
$$

Note that the left and the right side of the last inequality differ only in the terms $\bar{\lambda}\left(N\left(b_1^{\sigma_y(i)} \right) \cap N\left(\bar{c}_1^{\sigma_z(i)} \right) \right)$ and $\bar{\lambda}\left(N\left(b_1^{\sigma_y(i)} \right) \cap N\left(\bar{c}_1^{\sigma_w(i)} \right) \right)$.

The inequality 53 can be expressed in the following equivalent way:

$$
\mathbb{E}_{\mathfrak{S} \mathfrak{a} \mathfrak{b} \mathfrak{c} \bar{\mathfrak{c}}} \prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + i\mu \end{array} \right)^{-1}
$$

$$
\geq \mathbb{E}_{\mathfrak{S} \mathfrak{a} \mathfrak{b} \mathfrak{c} \bar{\mathfrak{c}}} \prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}) \cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) + i\mu \end{array} \right)^{-1},
\tag{54}
$$

where the expectation is over a uniformly random element $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \bar{\mathfrak{c}}, \mathfrak{S})$ of the set $\mathcal{P}(A) \times \mathcal{P}(B) \times \mathcal{P}(C) \times \mathcal{P}(\bar{C}) \times P(n,m,k,k) = \mathcal{P}(A \cup B \cup C \cup \bar{C})$. In the following, we will simply write $\mathbb{E}$ in place of the symbol $\mathbb{E}_{\mathfrak{S} \mathfrak{a} \mathfrak{b} \mathfrak{c} \bar{\mathfrak{c}}}$.

To prove (54), we first prove it on a smaller σ-algebra. We say that two permutations $\gamma_1, \gamma_2 \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$ are equivalent if by dropping the overline marks of the particles in C in both γ_1 and γ_2, we obtain the same sequence of elements. Let $\mathcal{F}$ denote the σ-algebra generated by this equivalence relation. We show that for any $(\sigma, a, b, c, \bar{c}) \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$,

$$
\mathbb{E}\left[\prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + i\mu \end{array} \right)^{-1} \Bigg| \mathcal{F} \right] (\sigma, a, b, c, \bar{c})
$$

$$
\geq \mathbb{E}\left[\prod_{i=1}^{n+m+2k} \left(\begin{array}{c} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)})) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}) \cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) + i\mu \end{array} \right)^{-1} \Bigg| \mathcal{F} \right] (\sigma, a, b, c, \bar{c}).
\tag{55}
$$

Fix $(\sigma, abc\bar{c}) \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$. We can express $\bar{c}$ as the composition of a permutation $\tau \in \mathcal{P}([k])$ and c, so that $\bar{c}_i = c_{\tau(i)}$. Since all permutations in the equivalence class of $(\sigma, a, b, c, \bar{c})$ are equally likely, each conditional expectation in (55) can be expressed as an expectation over auxilliary i.i.d. Bernoulli(1/2) random variables $\{\beta_i\}_{i=1}^{k}$. To make this precise, let

$$
S_{i,j} = \mathbb{1}(j \leq \sigma_z(i)),
$$

$$
T_{i,j} = \mathbb{1}(\tau^{-1}(j) \leq \sigma_w(i)),
$$

$$
p_i = i\mu + \bar{\lambda}(N(a_1^{\sigma_x(i)})) + \bar{\lambda}(N(b_1^{\sigma_y(i)})),
$$

$$
\bar{\beta}_j = 1 - \beta_j
$$

for all $1 \leq i \leq n+m+2k$ and $1 \leq j \leq k$. Also, let

$$c^{\beta\sigma(i)} = \{c_j \in C : S_{i,j} = 1, \beta_j = 1\} \cup \{c_j \in C : T_{i,j} = 1, \beta_j = 0\}$$

and similarly

$$c^{\bar\beta\sigma(i)} = \{c_j \in C : S_{i,j} = 1, \bar\beta_j = 1\} \cup \{c_j \in C : T_{i,j} = 1, \bar\beta_j = 0\}.$$

We have

$$\mathbb{E}_\beta \left[\prod_{i=1}^{n+m+2k} \left(\begin{array}{c} p_i + \bar\lambda(N(c^{\beta\sigma(i)})) + \bar\lambda(N(c^{\bar\beta\sigma(i)})) \\ -\bar\lambda(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) - \bar\lambda(N(c^{\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \end{array} \right)^{-1} \right.$$

$$\left. - \prod_{i=1}^{n+m+2k} \left(\begin{array}{c} p_i + \bar\lambda(N(c^{\beta\sigma(i)})) + \bar\lambda(N(c^{\bar\beta\sigma(i)})) \\ -\bar\lambda(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) - \bar\lambda(N(c^{\bar\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \end{array} \right)^{-1} \right] \geq 0. \tag{56}$$

Using the fact that $\int_0^\infty e^{-cx} dx = \frac{1}{c}$ for any $c > 0$, we may write the above inequality as

$$\int_{\mathbb{R}^{+n+m+2k}} \mathbb{E}_\beta \left[\exp \left(-\sum_{i=1}^{n+m+2k} x_i \left(\begin{array}{c} p_i + \bar\lambda(N(c^{\beta\sigma(i)})) + \bar\lambda(N(c^{\bar\beta\sigma(i)})) \\ -\bar\lambda(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) - \bar\lambda(N(c^{\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \end{array} \right) \right) \right.$$

$$\left. - \exp \left(-\sum_{i=1}^{n+m+2k} x_i \left(\begin{array}{c} p_i + \bar\lambda(N(c^{\beta\sigma(i)})) + \bar\lambda(N(c^{\bar\beta\sigma(i)})) \\ -\bar\lambda(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) - \bar\lambda(N(c^{\bar\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \end{array} \right) \right) \right] dx_1^{n+m+2k} \geq 0. \tag{57}$$

It is enough to prove that the integrand in positive for every x_1^{n+m+2k}. Symmetrizing the expression by replacing β_j with $\bar\beta_j$, we obtain the following equivalent expression:

$$0 \leq \mathbb{E}_\beta \left[\exp \left(-\sum_{i=1}^{n+m+2k} x_i \bar\lambda(N(c^{\beta\sigma(i)})) + x_i \bar\lambda(N(c^{\bar\beta\sigma(i)})) \right) \right.$$

$$\times \left(\exp\left(\sum_i x_i \bar\lambda(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) \right) - \exp\left(\sum_i x_i \bar\lambda(N(c^{\bar\beta\sigma(i)}) \cap N(a_1^{\sigma x(i)})) \right) \right)$$

$$\left. \times \left(\exp\left(\sum_i x_i \bar\lambda(N(c^{\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \right) - \exp\left(\sum_i x_i \bar\lambda(N(c^{\bar\beta\sigma(i)}) \cap N(b_1^{\sigma y(i)})) \right) \right) \right]. \tag{58}$$

To prove this, we use the FKG inequality on the lattice $\{0, 1\}^{n+m+2k}$ with measure

$$\nu(\beta) = \exp \left(-\sum_{i=1}^{n+m+2k} x_i \bar\lambda(N(c^{\beta\sigma(i)})) + x_i \bar\lambda(N(c^{\bar\beta\sigma(i)})) \right).$$

Claim D.2. *The measure ν is log-submodular.*

Proof. Let $\beta, \gamma \in \{0, 1\}^{n+m+2k}$. Then

$$\left(\begin{array}{c} \bar\lambda(N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\} \cup \{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})) - \bar\lambda(N(\{c_j : S_{i,j}=1, \beta_j=1\} \cup \{c_j : T_{i,j}=1, \beta_j=0\})) \\ -\bar\lambda(N(\{c_j : S_{i,j}=1, \gamma_j=1\} \cup \{c_j : T_{i,j}=1, \gamma_j=0\})) + \bar\lambda(N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\} \cup \{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=0\})) \end{array} \right)$$

$$= \left(\begin{array}{c} \bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\})] - \bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j=1\})] \\ -\bar\lambda[N(\{c_j : S_{i,j}=1, \gamma_j=1\})] + \bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\})] \end{array} \right)$$

$$+ \left(\begin{array}{c} \bar\lambda[N(\{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})] - \bar\lambda[N(\{c_j : T_{i,j}=1, \beta_j=0\})] \\ -\bar\lambda[N(\{c_j : T_{i,j}=1, \gamma_j=0\})] + \bar\lambda[N(\{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=0\})] \end{array} \right) \tag{59}$$

$$- \left(\begin{array}{c} \bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})] \\ -\bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j=0\})] \\ -\bar\lambda[N(\{c_j : S_{i,j}=1, \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \gamma_j=0\})] \\ +\bar\lambda[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=0\})] \end{array} \right).$$

Let us look at the first term in (59). We have

$$\begin{pmatrix} \bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\})] - \bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j=1\})] \\ -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \gamma_j=1\})] + \bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\})] \end{pmatrix}$$

$$= \begin{pmatrix} \bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\})] \\ -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j=1\}) \cap N(\{c_j : S_{i,j}=1, \gamma_j=1\})] \end{pmatrix},$$

which is non-positive since $N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\})$ is contained in both $N(\{c_j : S_{i,j}=1, \beta_j=1\})$ and $N(\{c_j : S_{i,j}=1, \gamma_j=1\})$.

Similarly, we may prove that the second term in (59) is non-positive. For the third term in that equation, we have

$$- \begin{pmatrix} -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \gamma_j=0\})] \\ -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j=0\})] \\ +\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})] \\ +\bar{\lambda}N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=0\}) \end{pmatrix}$$

$$\leq - \begin{pmatrix} -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})] \\ -\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j=0\})] \\ +\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=0\})] \\ +\bar{\lambda}[N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=1\}) \cap N(\{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=0\})] \end{pmatrix}$$

$$\leq 0.$$

By symmetry,

$$\begin{pmatrix} \bar{\lambda}N(\{c_j : S_{i,j}=1, \beta_j \vee \gamma_j=0\} \cup \{c_j : T_{i,j}=1, \beta_j \vee \gamma_j=1\}) - \bar{\lambda}N(\{c_j : S_{i,j}=1, \beta_j=0\} \cup \{c_j : T_{i,j}=1, \beta_j=1\}) \\ -\bar{\lambda}N(\{c_j : S_{i,j}=1, \gamma_j=0\} \cup \{c_j : T_{i,j}=1, \gamma_j=1\}) + \bar{\lambda}N(\{c_j : S_{i,j}=1, \beta_j \wedge \gamma_j=0\} \cup \{c_j : T_{i,j}=1, \beta_j \wedge \gamma_j=1\}) \end{pmatrix} \leq 0.$$

Therefore,

$$\sum_i -x_i \bar{\lambda}(N(c^{\beta \vee \gamma \sigma(i)})) - x_i \bar{\lambda}(N(c^{\bar{\beta} \vee \gamma \sigma(i)})) - \sum_i x_i \bar{\lambda}(N(c^{\beta \wedge \gamma \sigma(i)})) + x_i \bar{\lambda}(N(c^{\beta \bar{\wedge} \gamma \sigma(i)}))$$

$$+ \sum_i x_i \bar{\lambda}(N(c^{\beta \sigma(i)})) + x_i \bar{\lambda}(N(c^{\bar{\beta} \sigma(i)})) + \sum_i x_i \bar{\lambda}(N(c^{\gamma \sigma(i)})) + x_i \bar{\lambda}(N(c^{\bar{\gamma} \sigma(i)}))$$

$$\geq 0.$$

Consequently, $\nu(\boldsymbol{\beta} \vee \boldsymbol{\gamma})\nu(\boldsymbol{\beta} \wedge \boldsymbol{\gamma}) \geq \nu(\boldsymbol{\beta})\nu(\boldsymbol{\gamma})$.   □

Now we show that the two relevant functions in (58) are increasing in β.

Claim D.3. *The functions*

$$f(\boldsymbol{\beta}) = \exp\left(\sum_i x_i \bar{\lambda}(N(c^{\beta \sigma(i)}) \cap N(a_1^{\sigma_x(i)})) \right)$$

$$- \exp\left(\sum_i x_i \bar{\lambda}(N(c^{\bar{\beta} \sigma(i)}) \cap N(a_1^{\sigma_x(i)})) \right),$$

$$g(\boldsymbol{\beta}) = \exp\left(\sum_i x_i \bar{\lambda}(N(c^{\beta \sigma(i)}) \cap N(b_1^{\sigma_y(i)})) \right)$$

$$- \exp\left(\sum_i x_i \bar{\lambda}(N(c^{\bar{\beta} \sigma(i)}) \cap N(b_1^{\sigma_y(i)})) \right)$$

are increasing in β.

Proof. Let

$$h(\boldsymbol{\beta}) = \sum_i x_i \bar{\lambda}(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma_x(i)})).$$

For any $J \subseteq [k]$, let

$$q_{i,J} = \bar{\lambda}(\cap_{j \in J} N(c_j) \cap N(a_1^{\sigma_x(i)})).$$

Using the inclusion–exclusion formula, we may write

$$h(\boldsymbol{\beta}) = \sum_i x_i \sum_{J \subseteq [k]} (-1)^{|J|-1} q_{i,J} \prod_{j \in J} (\beta_j S_{i,j} + \bar{\beta}_j T_{i,j}).$$

Now, let $\beta_1, \ldots, \beta_k$ be given. Fix $l \in [k]$. Fixing all β_j, $j \neq l$, and taking the difference of the values of h when $\beta_l = 1$ and when $\beta_l = 0$, we obtain

$$h(\boldsymbol{\beta}, \beta_l = 1) - h(\boldsymbol{\beta}, \beta_l = 0)$$

$$= \sum_{i=1}^{n+m+2k} x_i (S_{i,l} - T_{i,l}) \sum_{J \subseteq [k-1]} (-1)^{|J|} q_{i,\{Jl\}} \prod_{j \in J} (\beta_j S_{i,j} + \bar{\beta}_j T_{i,j})$$

$$= \sum_{i=1}^{n+m+2k} x_i (S_{i,l} - T_{i,l}) \bar{\lambda}[(N(c_l) \cap N(a_1^{\sigma_x(i)})) \backslash N(c_1^{\sigma_z(i)} \backslash c_l)]$$

$$\geq 0,$$

since we have assumed that $S_{i,l} \geq T_{i,l}$. Similarly, taking

$$h'(x) = \sum_i x_i \bar{\lambda}(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma_x(i)}))$$

$$= \sum_i x_i \sum_{J \subseteq [k]} (-1)^{|J|-1} q_{i,J} \prod_{j \in J} (\bar{\beta}_j S_{i,j} + \beta_j T_{i,j}),$$

we have

$$h'(\boldsymbol{\beta}, \beta_l = 1) - h'(\boldsymbol{\beta}, \beta_l = 0)$$

$$= \sum_{i=1}^{n+m+2k} x_i (T_{i,l} - S_{i,l}) \bar{\lambda}[(N(c_l) \cap N(a_1^{\sigma_x(i)})) \backslash N(c_1^{\sigma_w(i)} \backslash c_l)]$$

$$\leq 0.$$

Thus, $f(\boldsymbol{\beta}, \beta_l = 1) - f(\boldsymbol{\beta}, \beta_l = 0) \geq 0$. By symmetry in the problem, this is also true for g. $\square$

We are now in a position to apply the FKG theorem to the right-hand side of (58). Since

$$\mathbb{E}_\beta \left[e^{-\sum_{i=1}^{n+m+2k} x_i \bar{\lambda}(N(c^{\beta\sigma(i)})) + x_i \bar{\lambda}(N(c^{\bar{\beta}\sigma(i)}))} \right.$$

$$\left. \times \left(e^{\sum_i x_i \bar{\lambda}(N(c^{\beta\sigma(i)}) \cap N(a_1^{\sigma_x(i)}))} - e^{\sum_i x_i \bar{\lambda}(N(c^{\bar{\beta}\sigma(i)}) \cap N(a_1^{\sigma_x(i)}))} \right) \right]$$

$$= 0,$$

this completes the proof of Theorem D.1.

The statement of the previous theorem is combinatorial in nature. However, its proof is interesting, since we were able to use probabilistic tools by introducing artificial randomness.

Having proved Corollary D.1 and Theorem D.1, we conclude with the proof of Theorem 2.3.

Proof of Theorem 2.3. By Corollary 8 we have

$$\tilde{\Pi}(\xi)\tilde{\Pi}(\gamma) = \tilde{\Pi}(\xi^R)\tilde{\Pi}(\xi^B)\tilde{\Pi}(\gamma^R)\tilde{\Pi}(\gamma^B).$$

By Theorem 8, we have

$$\tilde{\Pi}(\xi^R)\tilde{\Pi}(\gamma^R)\tilde{\Pi}(\xi^B)\tilde{\Pi}(\gamma^B) \le \tilde{\Pi}(\xi^R \cup \gamma^R)\tilde{\Pi}(\xi^B \cap \gamma^B)\tilde{\Pi}(\xi^R \cap \gamma^R)\tilde{\Pi}(\xi^R \cup \gamma^B).$$

Now, since $\xi \vee \gamma$ and $\xi \wedge \gamma$ are valid configurations, the proof follows from using Corollary D.1 again. □

E. Table of notation

Notation	Definition
D	Domain of interaction of particles (a metric space)
$C := \{R, B\}$	The set of types of particles, *reds* and *blues*
$\bar{R} := B, \bar{B} := R$	Opposite color
λ	Radon measure on D
m_C	Counting measure on C
$\bar{\lambda}$	$\bar{\lambda} := \lambda \otimes m_C$
μ	The parameter of the exponential random variables describing patience of particles
$M(D, K)$	Space of simple Radon counting measures on D, with marks in K
$O(D, K)$	Space of simple locally finite ordered subsets of D, with marks in K
$\|\gamma\|, \gamma \in O(D, K)$	Number of elements in γ
γ^x	The set $\{y \in \gamma : y <_\gamma x\}$ ordered as in γ
$p_x, b_x, c_x, w_x,$ $x \in D \times C$	Position, birth time, color, and patience of x; i.e., $x = (p_x, c_x)$
$N(A), A \subset D \times C$	$N(A) := \cup_{x \in A} \{y \in D \times C : c_y \ne c_x, d(p_y, p_x) < 1\}$
W_x	Region of maximum priority of $x \in \gamma$: $W_x = N(x) \backslash N(\gamma_x)$
$\eta_t \in O(D, C)$	Ordered collection of particles present in the system at time t
Φ	Poisson arrival process used in the construction of the process—a random element of $M(D \times \mathbb{R}^+, C \times \mathbb{R}^+)$
S_t	Set of discrepancies $\eta_t^0 \triangle \eta_t^1$ in the coupling-from-the-past construction
κ	Killing function
m	Matching function
u, m	Marks to indicate whether a particle is matched or unmatched in the detailed processes
$\hat{\eta}_t$	Backward detailed process
$\check{\eta}_t$	Forward detailed process

Notation	Definition
$Q_{\mathrm{u}}^i(\gamma)$	For $\gamma \in O(D, C \times \{\mathrm{u}, \mathrm{m}\})$, the number of unmatched particles among the first i particles of γ
$Q_{\mathrm{m}}^i(\gamma)$	For $\gamma \in O(D, C \times \{\mathrm{u}, \mathrm{m}\})$, the number of matched particles excluding the first i particles of γ
$\hat{\pi}$	Density of the stationary measure of the backward detailed process
π	Density of the stationary measure of the process η_t
$\tilde{\pi}$	Janossy density of stationary version of the point process η_0
$\mathcal{P}(C)$	The set of all permutations of the elements of a finite set C
$P(m, n)$	The set of all paths in a square lattice from $(0, 0)$ to (m, n)
(σ, X_1^n, Y_1^m)	A representation of the map that gives the canonical bijection between $P(n, m) \times \mathcal{P}(x_1^n) \times \mathcal{P}(y_1^m)$ and $\mathcal{P}(x_1^n, y_1^m)$.

Acknowledgements

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