BOUNDS FOR THE INTEGRAL OF A NON-NEGATIVE FUNCTION IN TERMS OF ITS FOURIER COEFFICIENTS

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ABSTRACT. The first $2N+1$ Fourier coefficients of an unknown, non-negative function $f(\theta)$ are given, and it is required to find bounds for $\int_E f(\theta)\,d\theta$, where E is some given region of integration. We also wish to find the interval E for which the bounds are most strict, when the width of E is specified. $f(\theta)$ may represent a distribution of energy in the interval $0 \leqslant \theta \leqslant 2\pi$; the object is to determine where the energy is chiefly located.

In the present paper we show that if the energy is located mainly in the neighbourhood of not more than M distinct points, significant lower bounds for $\int_E f(\theta)\,d\theta$ can be found in terms of the first $2M+1$ Fourier coefficients. The effectiveness of the method is illustrated by applying the inequalities to some known functions.

The results have application in determining the direction of propagation of ocean waves and other forms of energy.

1. *Introduction*. The following problem arises in connexion with the analysis of ocean waves (Barber (1)). Let $f(\theta)$ be an unknown, non-negative function of θ, integrable and periodic with period 2π. We are given the first $2N+1$ Fourier coefficients of $f(\theta)$:

$$\left.\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta \quad (n = 0, 1, \ldots, N),\\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta \quad (n = 1, 2, \ldots, N).\end{aligned}\right\} \tag{1·1}$$

Can we find upper and lower bounds for the function

$$F(E) = \int_E f(\theta)\,d\theta, \tag{1·2}$$

where E is some given region of integration?

In practice $f(\theta)$, or a related function, may represent the energy density of ocean waves approaching a recording station from a direction specified by θ. Barber (1) has shown that, if the waves are recorded at m different points, then $a_0, a_1, b_1, \ldots, a_N, b_N$, where $N \leqslant \frac{1}{2}m(m-1)$, can be determined from the correlation coefficients of the components of wave motion at the m points; from this information it is required to find, so far as possible, the angular distribution of the wave energy.

For convenience we shall refer to θ as the 'direction' and to (1·2) as the 'energy' contained in the interval E.

An approximation $f_N(\theta)$ to the required function $f(\theta)$ might be given by summing the first $2N+1$ terms of its Fourier series

$$f_N(\theta) = \tfrac{1}{2}a_0 + \sum_{k=1}^{N}(a_n\cos n\theta + b_n \sin n\theta), \tag{1·3}$$

for, under certain conditions, $f_N(\theta)$ tends to $f(\theta)$ as $N \to \infty$. But this approximation may be inadequate, for it often happens that most of the energy comes from a very restricted range of directions; $f(\theta)$ will then have one or more pronounced maxima which can be only poorly approximated by the smooth function $f_N(\theta)$. In addition, $f_N(\theta)$ may take negative values, while $f(\theta)$ is non-negative. (The Cesàro sums C1, however, are non-negative; see, for example, Zygmund (**12**), p. 46.)

In the present paper we make use of the fact that $f(\theta)$ is non-negative, and it is when the energy is concentrated in one or more narrow ranges of direction that our method yields the most information.

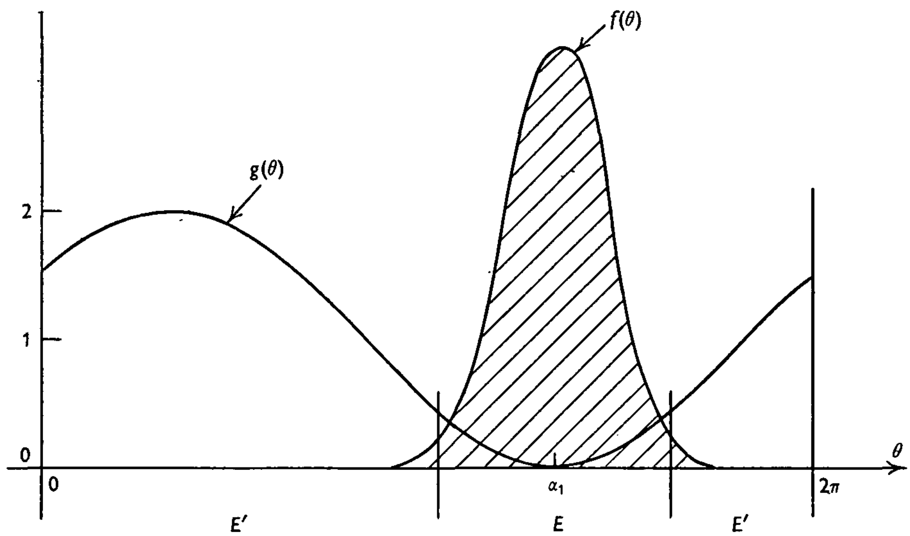

Fig. 1.

The argument is as follows. Let $g(\theta;\, \alpha_1, \alpha_2, \ldots, \alpha_N)$ be a polynomial in $\cos\theta$ and $\sin\theta$ of degree N at most, with coefficients involving $\alpha_1, \ldots, \alpha_N$. Then the integral

$$I(\alpha_1, \ldots, \alpha_N) = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\, g(\theta;\, \alpha_1, \alpha_2, \ldots, \alpha_N)\, d\theta \qquad (1\cdot 4)$$

is expressible in terms of $\alpha_1, \ldots, \alpha_N$ and the first $2N+1$ Fourier coefficients of $f(\theta)$. Suppose that g is always positive, except at $\theta = \alpha_1, \ldots, \alpha_N$, where it vanishes (see Fig. 1 for the case $N = 1$). Then I is never negative, and is small if and only if the energy is nearly all concentrated in the neighbourhood of the points $\alpha_1, \ldots, \alpha_N$. For if E denotes a set of N narrow intervals surrounding $\alpha_1, \ldots, \alpha_N$, the contribution to the integral from within E is small, since g is small there, and the contribution from the remaining regions E' is also small, since the proportion of energy in E' is small. Conversely, if I is small, then the proportion of energy lying outside E is small, for otherwise there would be an appreciable positive contribution to I from the regions E'.

More precisely, let p denote the proportion of energy lying outside E (that is, in E'). Then since the total energy equals πa_0, we have

$$F(E') = p\pi a_0, \quad F(E) = (1-p)\,\pi a_0. \qquad (1\cdot 5)$$

In E we shall have $0 \leqslant g \leqslant G$, say, and in E', $G' < g \leqslant G''$. Thus from (1·4)

$$\frac{I}{a_0} = \frac{1}{\pi a_0}\left[\int_E fg\,d\theta + \int_{E'} fg\,d\theta\right] \tag{1·6}$$

$$\leqslant \frac{1}{\pi a_0}[GF(E) + G''F(E')] \tag{1·7}$$

$$= G(1-p) + G''p. \tag{1·8}$$

When the intervals E are so narrow that $G \ll 1$, and if at the same time $p \ll 1$, then it follows that $I/a_0 \ll 1$. Conversely, since

$$\frac{I}{a_0} \geqslant \frac{1}{\pi a_0}\int_{E'} fg\,d\theta \geqslant \frac{1}{\pi a_0} G'F(E') = G'p, \tag{1·9}$$

we have

$$p \leqslant \frac{I}{G'a_0}. \tag{1·10}$$

Thus if $I/G'a_0$ is small then p is also small, and so nearly all the energy lies within E. In general, a knowledge of $I/G'a_0$ provides an upper bound for p and so a lower bound for $F(E)$ (by (1·5)).

The smaller the value of $I/G'a_0$, the greater the amount of energy known to be contained in E. We therefore seek the values of $\alpha_1, \ldots, \alpha_N$ which make $I(\alpha_1, \ldots, \alpha_N)$ a minimum. (G' depends also on the subsequent choice of E.) The directions $\alpha_1, \ldots, \alpha_N$ which make $I(\alpha_1, \ldots, \alpha_N)$ a minimum will correspond to the predominant directions of the energy, so far as these can be defined. The chief mathematical problem is then to find the minimum of the integral (1·4) and to determine the corresponding directions $\theta = \alpha_1, \ldots, \alpha_N$.

The cases $N = 1$ and $N > 1$ will be considered in §§2 and 3 respectively. In §4 we give some practical examples, where the inequalities are applied to the Fourier coefficients of known functions $f(\theta)$. The tests are found to be reasonably effective.

2. $N = 1$. Let

$$g(\theta;\,\alpha_1) = 2\sin^2\frac{\theta-\alpha_1}{2} \tag{2·1}$$

$$= 1 - \cos\theta\cos\alpha - \sin\theta\sin\alpha_1. \tag{2·2}$$

$g(\theta;\,\alpha_1)$ is positive everywhere except at $\theta = \alpha_1$, where it vanishes (see Fig. 1), and hence it satisfies the conditions stated in §1. Consider then the function

$$I(\alpha_1) = \frac{2}{\pi}\int_0^{2\pi} f(\theta)\sin^2\frac{\theta-\alpha_1}{2}\,d\theta \tag{2·3}$$

$$= a_0 - a_1\cos\alpha_1 - b_1\sin\alpha_1, \tag{2·4}$$

which is a function of α_1 with known coefficients a_0, a_1, b_1. For E we may take the interval of width 2δ having α_1 as mid-point. Everywhere outside E we have

$$g > 2\sin^2\tfrac{1}{2}\delta = G', \tag{2·5}$$

and so the inequality (1·10) becomes

$$p \leqslant \frac{I(\alpha_1)}{2a_0\sin^2\frac{1}{2}\delta}. \tag{2·6}$$

Now from (2·4) the minimum value of $I(\alpha_1)$ is

$$I = a_0 - \surd(a_1^2 + b_1^2), \tag{2·7}$$

and occurs when
$$\cos\alpha_1 : \sin\alpha_1 : 1 = -a_1 : -b_1 : \surd(a_1^2 + b_1^2). \tag{2·8}$$

The best possible inequality (2·6) is therefore

$$p \leqslant \frac{a_0 - \surd(a_1^2 + b_1^2)}{2a_0 \sin^2 \frac{1}{2}\delta}. \tag{2·9}$$

The corresponding direction α_1, given by (2·8), defines the 'predominant' direction of the energy.

We may remark that the maximum and minimum values of $I(\alpha_1)$ are the roots of

$$I^2 - 2a_0 I + (a_0^2 - a_1^2 - b_1^2) = 0; \tag{2·10}$$

also that a necessary and sufficient condition for the energy to be concentrated within a single interval of infinitesimal width is

$$a_0^2 - a_1^2 - b_1^2 = 0. \tag{2·11}$$

3. $N > 1$. Generalizing equation (2·1) we take

$$g(\theta;\, \alpha_1, \alpha_2, \ldots, \alpha_N) = 2^{2N-1} \sin^2 \frac{\theta - \alpha_1}{2} \sin^2 \frac{\theta - \alpha_2}{2} \ldots \sin^2 \frac{\theta - \alpha_N}{2}, \tag{3·1}$$

which is positive everywhere except at the points $\theta = \alpha_1, \alpha_2, \ldots, \alpha_N$, where it vanishes. The integral

$$I(\alpha_1, \ldots, \alpha_N) = \frac{2^{2N-1}}{\pi} \int_0^{2\pi} f(\theta) \sin^2 \frac{\theta - \alpha_1}{2} \ldots \sin^2 \frac{\theta - \alpha_N}{2}\, d\theta \tag{3·2}$$

is expressible in terms of $\alpha_1, \ldots, \alpha_N$ and the first $2N + 1$ Fourier coefficients

$$a_0, a_1, b_1, \ldots, \alpha_N, b_N;$$

we have to investigate the minimum values of $I(\alpha_1, \ldots, \alpha_N)$. We shall now show that under certain conditions the maximum and minimum values of $I(\alpha_1, \ldots, \alpha_N)$ are the roots of the quadratic

$$\Delta_{N-2} I^2 - 2\Delta_{N-1} I + \Delta_N = 0, \tag{3·3}$$

where

$$\Delta_N = \begin{vmatrix} A_0 & A_1 & \ldots & A_N \\ A_1^* & A_0 & \ldots & A_{N-1} \\ \vdots & \vdots & & \vdots \\ A_N^* & A_{N-1}^* & \ldots & A_0 \end{vmatrix}, \tag{3·4}$$

and we have written
$$A_n = a_n - ib_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta)\, e^{-in\theta}\, d\theta. \tag{3·5}$$

(A_n^* denotes the complex conjugate of A_n.) When $N = 1$ equation (3·3) reduces to (2·10) provided we take conventionally $\Delta_{-1} = 1$.

Since

$$\sin^2 \frac{\theta - \alpha_n}{2} = \tfrac{1}{4}(e^{i\theta} - e^{i\alpha_n})(e^{-i\theta} - e^{-i\alpha_n}), \tag{3·6}$$

we have

$$I(\alpha_1, \dots, \alpha_N) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,(e^{i\theta}-e^{i\alpha_1})\dots(e^{i\theta}-e^{i\alpha_N})\,(e^{-i\theta}-e^{-i\alpha_1})\dots(e^{-i\theta}-e^{-i\alpha_N})\,d\theta \quad (3\cdot7)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,(t-x_1)\dots(t-x_N)\,(t^{-1}-x_1^{-1})\dots(t^{-1}-x_N^{-1})\,d\theta, \quad (3\cdot8)$$

where $t = e^{i\theta}$ and $x_n = e^{i\alpha_n}$. Thus

$$x_1 \dots x_N I = \frac{(-1)^N}{2\pi}\int_0^{2\pi} f(\theta)\, t^{-N}(t-x_1)^2 \dots (t-x_N)^2\, d\theta. \quad (3\cdot9)$$

At a stationary value of I, $$\frac{\partial I}{\partial x_n} = -ie^{-i\alpha_n}\frac{\partial I}{\partial \alpha_n} = 0, \quad (3\cdot10)$$

and so on differentiating both sides of (3·9) with respect to x_n,

$$\frac{x_1 \dots x_N}{x_n} I = \frac{(-1)^{N+1}}{\pi}\int_0^{2\pi} f(\theta)\, t^{-N}\frac{(t-x_1)^2\dots(t-x_N)^2}{t-x_n}\,d\theta. \quad (3\cdot11)$$

We shall make use of two lemmas:

LEMMA 1.

$$\begin{vmatrix} x_1^m x_2 \dots x_N & x_1 x_2^m \dots x_N & \dots & x_1 x_2 \dots x_N^m \\ x_1^{N-2} & x_2^{N-2} & \dots & x_N^{N-2} \\ x_1^{N-3} & x_2^{N-3} & \dots & x_N^{N-3} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix} = \begin{cases} (-1)^{N+1} D_N & (m=0), \\ 0 & (m = 1, 2, \dots, N-1), \end{cases} \quad (3\cdot12)$$

where

$$D_N = \begin{vmatrix} x_1^{N-1} & x_2^{N-1} & \dots & x_N^{N-1} \\ x_1^{N-2} & x_2^{N-2} & \dots & x_N^{N-2} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix} = \prod_{n>m}(x_n - x_m). \quad (3\cdot13)$$

For the value of the left-hand determinant is unaltered by dividing the first row by $x_1 x_2, \dots, x_N$ and multiplying the first column by x_1, the second by x_2 and so on. If $m = 0$, the determinant is then identical with D_N except for an interchange of rows. If $m = 1, 2, \dots, N-1$, two rows of the determinant are identical.

LEMMA 2. *When* $m = 0, 1, \dots, N-1$,

$$\begin{vmatrix} \dfrac{x_1^m}{t-x_1} & \dfrac{x_2^m}{t-x_2} & \dots & \dfrac{x_N^m}{t-x_N} \\ x_1^{N-2} & x_2^{N-2} & \dots & x_N^{N-2} \\ x_1^{N-3} & x_2^{N-3} & \dots & x_N^{N-3} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \dots & 1 \end{vmatrix} = \frac{t^m D_N}{(t-x_1)(t-x_2)\dots(t-x_N)}. \quad (3\cdot14)$$

For on multiplying the top row of the left-hand determinant by

$$(t-x_1)(t-x_2)\dots(t-x_N),$$

the first term, for example, becomes

$$x_1^m(t-x_2)\dots(t-x_N) = x_1^m[S_0^{(1)}t^{N-1}-S_1^{(1)}t^{N-2}+\dots+(-1)^{N-1}S_N^{(1)}], \tag{3·15}$$

where $S_n^{(1)}$ denotes the symmetric sum, of degree n, of the roots $x_2, \dots, x_N$, and $S_0^{(1)}$, by convention, equals 1. $S_n^{(1)}$ may be expressed in terms of the symmetric sums S_n of *all* the roots $x_1, x_2, \dots, x_N$ by successive substitution as follows:

$$\left.\begin{aligned} S_1^{(1)} &= S_1 - x_1 S_0^{(1)} = S_1 - x_1 S_0, \\ S_2^{(1)} &= S_2 - x_1 S_1^{(1)} = S_2 - x_1 S_1 + x_1^2 S_0, \\ S_3^{(1)} &= S_3 - x_1 S_2^{(1)} = S_3 - x_1 S_2 + x_1^2 S_1 - x_1^3 S_0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ S_{N-m-1}^{(1)} &= S_{N-m-1} - x_1 S_{N-m-2} + \dots + (-1)^{N-m-1} x_1^{N-m-1} S_0, \end{aligned}\right\} \tag{3·16}$$

all the powers of x_1 on the right-hand side being of degree less than or equal to $(N-m-1)$. For the remaining coefficients we start from the other end:

$$\left.\begin{aligned} S_{N-1}^{(1)} &= x_1^{-1} S_N, \\ S_{N-2}^{(1)} &= x_1^{-1}(S_{N-1} - S_{N-1}^{(1)}) = x_1^{-1} S_{N-1} - x_1^{-2} S_N, \\ S_{N-3}^{(1)} &= x_1^{-1}(S_{N-2} - S_{N-2}^{(1)}) = x_1^{-1} S_{N-2} - x_1^{-2} S_{N-1} + x_1^{-3} S_{N-2}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ S_{N-m}^{(1)} &= x_1^{-1} S_{N-m+1} - x_1^{-2} S_{N-m+2} + \dots + (-1)^{m-1} x_1^{-m} S_N. \end{aligned}\right\} \tag{3·17}$$

On substitution in (3·15) we see that the first term of the top row of the determinant is of the form

$$P_1 x_1^{N-1} + P_2 x_1^{N-2} + \dots + P_N, \tag{3·18}$$

where the P_n are symmetrical expressions in $x_1, x_2, \dots, x_N$ and

$$P_1 = t^m. \tag{3·19}$$

Each of the terms $P_n x_1^{N-m}$ $(m = 2, 3, \dots, N-1)$ can be eliminated by subtracting P_n times the nth row from the first row of the determinant. Only the first term $P_1 x_1^{N-1}$ remains. By (3·19) this proves the lemma.

Now let the nth of equations (3·11) be multiplied by x_n^m times the cofactor of the nth term of the first row of D_N, and let the equations be added. For $m = 0$ this gives

$$(-1)^{N+1} D_N I = \frac{(-1)^{N+1}}{\pi}\int_0^{2\pi} f(\theta)\, t^{-N}(t-x_1)\dots(t-x_N)\, D_N\, d\theta, \tag{3·20}$$

and therefore, if $D_N \neq 0$,

$$\begin{aligned} I &= \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,[1 - S_1 t^{-1} + S_2 t^{-2} - \dots + (-1)^N S_N t^{-N}]\, d\theta \\ &= A_0 S_0 - A_1 S_1 + A_2 S_2 - \dots + (-1)^N A_N S_N. \end{aligned} \tag{3·21}$$

Similarly, for $m = 1, 2, \dots, N-1$, we have

$$0 = A_m^* S_0 - A_{m-1}^* S_1 + \dots + (-1)^N A_{N-m} S_N. \tag{3·22}$$

Finally, we add the equation for $m = N$, which is most conveniently obtained by taking the conjugate of equation (3·21) and using $S_n^* = S_{N-n}/S_N$:

$$I S_N = A_N^* S_0 - A_{N-1}^* S_1 + \dots + (-1)^N A_0 S_N. \tag{3·23}$$

These equations may be written in matrix form:

$$\begin{pmatrix} A_0-I & A_1 & A_2 & \dots & A_N \\ A_1^* & A_0 & A_1 & \dots & A_{N-1} \\ A_2^* & A_1^* & A_0 & \dots & A_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ A_N^* & A_{N-1}^* & A_{N-2}^* & \dots & A_0-I \end{pmatrix} \begin{pmatrix} S_0 \\ -S_1 \\ S_2 \\ \vdots \\ (-1)^N S_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \qquad (3\cdot 24)$$

The diagonal terms of the square matrix are all A_0 except the first and last, which are A_0-I. Since the symmetric sums S_n are not all zero ($S_0 = 1$), it follows that

$$\begin{vmatrix} A_0-I & A_1 & A_2 & \dots & A_N \\ A_1^* & A_0 & A_1 & \dots & A_{N-1} \\ A_2^* & A_1^* & A_0 & \dots & A_{N-2} \\ \vdots & \vdots & \vdots & & \vdots \\ A_N^* & A_{N-1}^* & A_{N-2}^* & \dots & A_0-I \end{vmatrix} = 0, \qquad (3\cdot 25)$$

which on expansion is seen to be identical with (3·3), the result to be proved.

To find the corresponding angles α_n, we first choose the smaller root I of equation (3·25) and then solve any N of equations (3·24) to obtain the N ratios $S_0:S_1:\dots:S_N$. The roots $x_1, x_2, \dots, x_N$ of

$$S_0 t^N - S_1 t^{N-1} + \dots + (-1)^N S_N = 0 \qquad (3\cdot 26)$$

then give the required angles, through the relations $x_n = e^{i\alpha_n}$.

It was assumed in the proof that $D_N \neq 0$, i.e. that all the roots x_n are distinct. Since $I(\alpha_1, \dots, \alpha_N)$ is a bounded function of $\alpha_1, \dots, \alpha_N$, it must always possess at least one maximum and one minimum; but only if these correspond to unequal values of $\alpha_1, \dots, \alpha_N$ does the present theorem necessarily hold.

In one important case, however, the above analysis is certainly valid, namely when the energy is concentrated in N infinitesimal intervals surrounding N distinct directions $\theta_1, \theta_2, \dots, \theta_N$, say. For, when $(\alpha_1, \dots, \alpha_N) = (\theta_1, \dots, \theta_N)$, I vanishes, from (3·2), and further

$$\frac{\partial I}{\partial \alpha_n} = -\frac{2^{2N-1}}{\pi}\int_0^{2\pi} f(\theta)\sin^2\frac{\theta-\alpha_1}{2}\dots\sin(\theta-\alpha_n)\dots\sin^2\frac{\theta-\alpha_N}{2}\,d\theta = 0. \qquad (3\cdot 27)$$

Therefore $(\alpha_1, \dots, \alpha_N) = (\theta_1, \dots, \theta_N)$ is a solution of equation (3·11) and hence also of equations (3·25) and (3·5), with $I = 0$. The determinant (3·25) is of rank N, as will be shown in the appendix, and so the ratios $S_0:S_1:\dots:S_N$ are uniquely determined. Therefore $t = e^{i\theta_n}$ satisfies (3·26). But (3·26) has not more than N roots, which must therefore be identical with the N distinct quantities $e^{i\theta_1}, e^{i\theta_2}, \dots, e^{i\theta_N}$.

When the minimum value of I is small, equation (3·3) shows that it is given by

$$I = \frac{\Delta_N}{2\Delta_{N-1}} \qquad (3\cdot 28)$$

very nearly. Therefore by (1·10) all except a proportion p of the energy is contained in E where

$$p \leqslant \frac{\Delta_N}{2G'\Delta_0\Delta_{N-1}}. \qquad (3\cdot 29)$$

If
$$\frac{\Delta_N}{2\Delta_0\Delta_{N-1}} \ll G', \tag{3·30}$$
then nearly all the energy is contained in E.

Suppose that we take as E the set of N intervals of width 2δ with mid-points $\alpha_1, \ldots, \alpha_N$. If δ is small compared with the distances between successive α's we have, in the nth interval,
$$g(\theta;\ \alpha_1, \alpha_2, \ldots, \alpha_N) = 2^{2N-1}\sin^2\frac{\theta-\alpha_n}{2}\Pi'\sin^2\frac{\alpha_m-\alpha_n}{2} \tag{3·31}$$
very nearly, where in the product m runs from 1 to N excluding n. Thus, outside E,
$$g \geqslant 2^{2N-1}\sin^2\tfrac{1}{2}\delta\min\left(\Pi'\sin^2\frac{\alpha_m-\alpha_n}{2}\right) = G'. \tag{3·32}$$

A rough estimate of G' may be obtained by replacing $\sin\frac{1}{2}\delta$ by $\frac{1}{2}\delta$ and each of the terms $\sin^2\frac{1}{2}(\alpha_m-\alpha_n)$ by a mean value $\frac{1}{2}$. Thus
$$G' \doteqdot 2^{N-2}\delta^2, \tag{3·33}$$
and (3·30) becomes
$$\frac{\Delta_N}{2^{N-1}\Delta_0\Delta_{N-1}} \ll \delta^2. \tag{3·34}$$

If one of the distances $|\alpha_m-\alpha_n|$ is only of order δ or less, then G' will be an order of magnitude less than (3·33). But in that case we may expect that a smaller number of directions α_n would give a significant inequality, for the same value of δ. Therefore a criterion for the energy to be grouped mainly in N separate intervals of width δ is that N shall be the *least* integer for which (3·34) is true.

4. *Applications.* To illustrate the method we shall discuss some examples when the energy distribution $f(\theta)$ has certain simple forms; we shall find how much information about $\int_E f(\theta)\,d\theta$ can be obtained from a knowledge of the first five Fourier coefficients.

Example 1. Suppose that
$$f(\theta) = \begin{cases} \dfrac{\pi}{2\epsilon} & \text{if}\quad \theta_1-\epsilon<\theta<\theta_1+\epsilon, \\ 0 & \text{elsewhere}, \end{cases} \tag{4·1}$$
that is, the energy is evenly distributed in a narrow interval of width 2ϵ and mid-point θ_1 (see Fig. 2a). Then we have
$$A_0 = 1, \quad A_n = e^{in\theta_1}\frac{\sin n\epsilon}{n\epsilon} \quad (n = 1, 2, \ldots, N), \tag{4·2}$$
and so
$$\left.\begin{aligned} \Delta_0 &= 1, \\ \Delta_1 &= 1-\left(\frac{\sin\epsilon}{\epsilon}\right)^2 \doteqdot \tfrac{1}{3}\epsilon^2, \\ \Delta_2 &= \left(1-\frac{\sin 2\epsilon}{2\epsilon}\right)\left(1-2\left(\frac{\sin\epsilon}{\epsilon}\right)^2+\frac{\sin 2\epsilon}{2\epsilon}\right) \doteqdot \tfrac{4}{135}\epsilon^6. \end{aligned}\right\} \tag{4·3}$$

Since Δ_1/Δ_0^2 is of order ϵ^2, we know at once that the energy is mainly grouped in a single interval whose width is of order ϵ.

Let us apply the test when $N = 1$. From (2·8) we find that the 'predominant' direction is given by $\alpha_1 = \theta_1$, and further from (2·9) that

$$p \leqslant \frac{\epsilon^2}{3\delta^2} \tag{4·4}$$

($\sin \frac{1}{2}\delta$ has been replaced by $\frac{1}{2}\delta$). Thus, taking $\delta = \epsilon$, we could tell that not more than one-third of the energy lies outside the original interval, or taking $\delta = 2\epsilon$, that not more than one-twelfth lies outside an interval of twice this width.

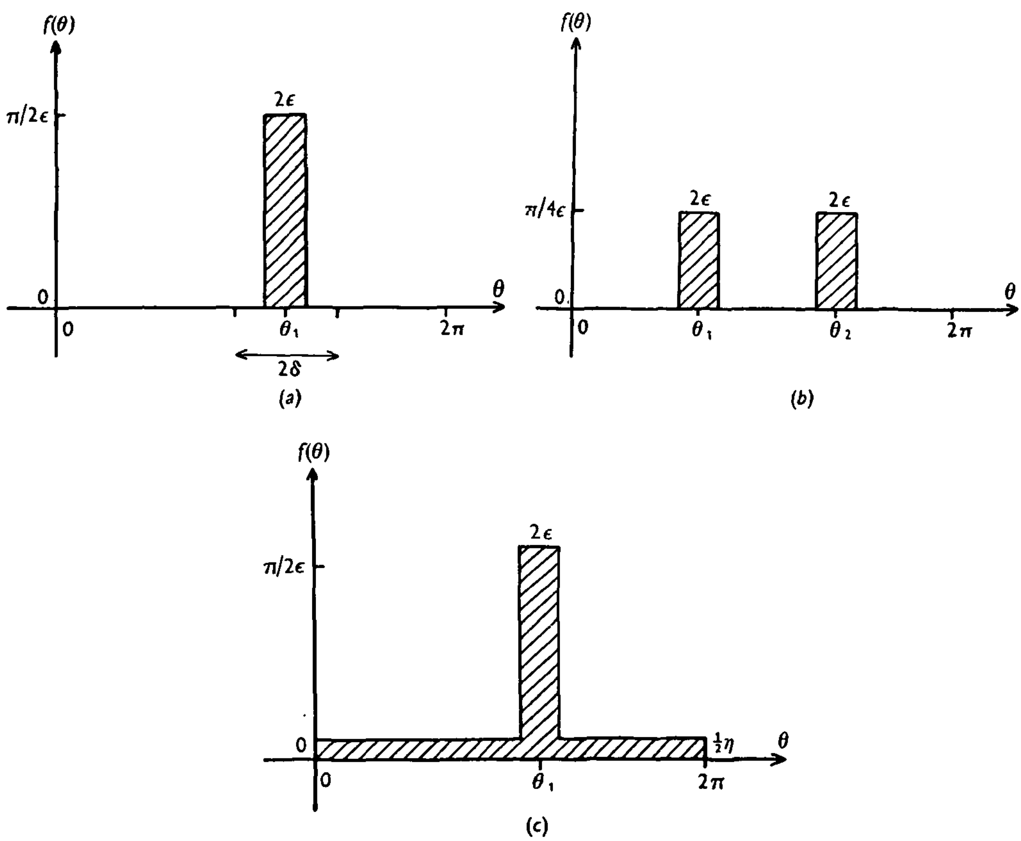

Fig. 2.

To apply the test when $N = 2$ we have to solve

$$\left.\begin{aligned} (1-I)\,S_0 - e^{-i\theta_1}\frac{\sin\epsilon}{\epsilon}S_1 + e^{-2i\theta_1}\frac{\sin 2\epsilon}{2\epsilon}S_2 &= 0,\\ e^{i\theta_1}\frac{\sin\epsilon}{\epsilon}S_0 - S_1 + e^{-i\theta_1}\frac{\sin\epsilon}{\epsilon}S_2 &= 0,\\ e^{2i\theta_1}\frac{\sin 2\epsilon}{2\epsilon}S_0 - e^{i\theta_1}\frac{\sin\epsilon}{\epsilon}S_1 + (1-I)\,S_2 &= 0. \end{aligned}\right\} \tag{4·5}$$

On subtracting the first equation from the third we find

$$S_2 = e^{2i\theta_1}S_0, \tag{4·6}$$

and so from the second
$$S_1 = 2\frac{\sin\epsilon}{\epsilon}e^{i\theta_1}S_0. \tag{4·7}$$

Equation (3·26) then becomes
$$t^2 - 2\frac{\sin\epsilon}{\epsilon}e^{i\theta_1}t + 2e^{2i\theta_1} = 0, \tag{4·8}$$

of which the solution, to order ϵ, is
$$t = e^{i(\theta_1 \pm \epsilon/\sqrt{3})}. \tag{4·9}$$

The predominant directions α_1, α_2 are therefore given by
$$\alpha_1, \alpha_2 = \theta_1 \pm \epsilon/\sqrt{3}. \tag{4·10}$$

The separation of α_1 and α_2 is $2\epsilon/\sqrt{3}$. If we take as E two separate intervals of width 2δ surrounding α_1 and α_2, δ being less than $\epsilon/\sqrt{3}$, we shall obtain a bound G' of order $\delta^2\epsilon^2$. Thus (3·29) will be of order ϵ^2/δ^2, and no advantage is obtained by taking δ much smaller than ϵ (as, indeed, we should expect from the actual form of $f(\theta)$). If, on the other hand, we take for E a single interval $(\theta_1-\delta, \theta_1+\delta)$, where $\delta \geqslant \epsilon/\sqrt{3}$, we have
$$G' = 2^3\sin^2\frac{\delta-\epsilon/\sqrt{3}}{2}\sin^2\frac{\delta+\epsilon/\sqrt{3}}{2} = \frac{(\delta^2-\frac{1}{3}\epsilon^2)^2}{3} \tag{4·11}$$
approximately, and so from (3·29)
$$p \leqslant \frac{4}{5}\frac{\epsilon^4}{(3\delta^2-\epsilon^2)^2}. \tag{4·12}$$

If $\delta = \epsilon$, we have $p \leqslant 1/5$, showing that not more than one-fifth of the energy lies outside the interval (compared with one-third in the previous test). If $\delta = 2\epsilon$, we have $p \leqslant 1/180$, showing that only about 0·5 % of the energy lies outside the interval (compared with one-twelfth previously).

Thus the test for $N = 2$ provides a stricter inequality than the test for $N = 1$, but not by an order of magnitude.

Example 2. Let
$$f(\theta) = \begin{cases} \dfrac{\pi}{4\epsilon} & \text{if} \quad \theta_1-\epsilon < \theta < \theta_1+\epsilon, \\ \dfrac{\pi}{4\epsilon} & \text{if} \quad \theta_2-\epsilon < \theta < \theta_2+\epsilon, \\ 0 & \text{elsewhere,} \end{cases} \tag{4·13}$$

where $2\epsilon < \theta_2-\theta_1 < 2\pi-2\epsilon$, so that the energy is evenly distributed in two non-overlapping intervals of width 2ϵ (see Fig. 2*b*). For simplicity we shall suppose also that $\theta_2-\theta_1 = \frac{1}{2}\pi$, i.e. the average directions for the two intervals are at right angles. Then we have
$$A_0 = 1, \quad A_n = e^{-in\theta_0}\cos\frac{n\pi}{4}\frac{\sin n\epsilon}{n\epsilon} \quad (n = 1, 2, \ldots, N), \tag{4·14}$$
where $\theta_0 = \frac{1}{2}(\theta_1+\theta_2)$, and
$$\left.\begin{aligned} \Delta_0 &= 1, \\ \Delta_1 &= 1-\frac{1}{2}\frac{\sin\epsilon}{\epsilon} \doteqdot \tfrac{1}{2}, \\ \Delta_2 &= 1-\left(\frac{\sin\epsilon}{\epsilon}\right)^2 \doteqdot \tfrac{1}{3}\epsilon^2, \end{aligned}\right\} \tag{4·15}$$

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approximately. Since $\Delta_2/2\Delta_0\Delta_1$ is of order ϵ^2, $\ll 1$, whereas Δ_1/Δ_0^2 is of order unity, we can tell at once that the energy lies mainly within *two* separate intervals whose width is of order ϵ.

In the test for $N = 1$, $I(\alpha_1)$ is a minimum when $\alpha_1 = \theta_0 = \frac{1}{2}(\theta_1 + \theta_2)$. But since Δ_1/Δ_0^2 is of order unity the test gives a barely significant result. If we take, for example, $\delta = \frac{1}{2}\pi$, so that E is the interval of width π and mid-point θ_0, we have from (2·9)

$$p \leqslant 1 - \sqrt{\tfrac{1}{2}} = 0{\cdot}293, \tag{4·16}$$

so that only about seven-tenths of the energy certainly lies in this semicircle.

Now let us apply the test for $N = 2$. We have

$$\left.\begin{aligned} (1-I)\,S_0 - \frac{e^{-i\theta_0}}{\sqrt{2}}\frac{\sin\epsilon}{\epsilon}S_1 \qquad\qquad &= 0,\\ \frac{e^{i\theta_0}}{\sqrt{2}}\frac{\sin\epsilon}{\epsilon}S_0 - \qquad S_1 + \frac{e^{-i\theta_0}}{\sqrt{2}}\frac{\sin\epsilon}{\epsilon}S_2 &= 0,\\ -\frac{e^{i\theta_0}}{\sqrt{2}}\frac{\sin\epsilon}{\epsilon}S_1 + \quad (1-I)\,S_2 &= 0.\end{aligned}\right\} \tag{4·17}$$

Proceeding as before, we find

$$\alpha_1 = \theta_1 + \tfrac{1}{6}\epsilon^2, \quad \alpha_2 = \theta_2 - \tfrac{1}{6}\epsilon^2. \tag{4·18}$$

Thus the two 'predominant' directions differ from the directions θ_1 and θ_2 by quantities of order ϵ^2 only. If E is taken to be two small intervals of width 2δ surrounding α_1 and α_2 we may take

$$G' = 2^3 \sin^2 \tfrac{1}{2}\delta \sin^2 \frac{\alpha_2 - \alpha_1}{2} = \delta^2 \tag{4·19}$$

approximately, and so from (3·29)

$$p \leqslant \frac{\epsilon^2}{3\delta^2}. \tag{4·20}$$

Thus we could tell that at least two-thirds of the energy comes from within two intervals of width 2ϵ almost coinciding with the original intervals, or that eleven-twelfths of the energy comes from within two intervals of twice this width.

As expected, there is a marked improvement in the inequalities obtained from the test for $N = 2$ compared with those obtained from the test for $N = 1$, in this example. However, from the experience of Example 1 we should expect that tests of higher order would give only a smaller improvement.

Example 3. In the two previous examples we assumed that the energy was entirely confined to one or two narrow intervals, that is to say there was no 'background' of energy outside those intervals. To investigate the effect of such a background we may add to the energy distribution of Example 1 a small constant term. Thus

$$f(\theta) = \begin{cases} \frac{1}{2}\eta + \dfrac{\pi}{2\epsilon} & \text{if} \quad \theta_1 - \epsilon < \theta < \theta_1 + \epsilon,\\ \frac{1}{2}\eta & \text{elsewhere,}\end{cases} \tag{4·21}$$

where η is a small quantity (see Fig. 2*c*). The effect of this is to increase A_0 by η but to leave the other Fourier coefficients unaltered:

$$A_0 = 1 + \eta, \quad A_n = e^{-in\theta_1}\frac{\sin n\epsilon}{n\epsilon} \quad (n = 1, 2, \ldots, N). \tag{4·22}$$

Equation (2·9) then gives
$$p \leqslant \frac{2\eta + \frac{1}{3}\epsilon^2}{\delta} \tag{4·23}$$
(higher powers of δ, ϵ, η being neglected).

In order that the background shall be negligible, therefore, we must have $\eta \ll \epsilon^2$, or in other words the background must be an order of magnitude smaller than the square of the width of the interval. It appears, therefore, that the effectiveness of the present tests depends rather critically on the absence of such a background.

APPENDIX

Properties of Δ_N

We now prove the result used in §3. This is part of a more general theorem (Theorem A 9) which was first proved algebraically by Toeplitz (11); other proofs have been given by Fischer (6), Schur (10) and Frobenius (7)†. The proof we now give is more direct than any of those mentioned, and brings to light more clearly the significance of Δ_N.

First we establish the identity
$$\Delta_N \equiv \frac{2^{2N(N+1)}}{(N+1)!\,\pi^{N+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1) \ldots f(\theta_{N+1}) \prod_{m>n} \sin^2 \frac{\theta_m - \theta_n}{2}\, d\theta_1 \ldots d\theta_{N+1}, \tag{A 1}$$
where, in the double product, n runs from 1 to $N+1$ and m from $n+1$ to $N+1$. From (3·4) and (3·5) we have
$$\Delta_N = \frac{1}{\pi^{N+1}} \begin{vmatrix} \int_0^{2\pi} f(\theta_1)\, d\theta_1 & \int_0^{2\pi} f(\theta_2)\, e^{-i\theta_2} d\theta_2 & \cdots & \int_0^{2\pi} f(\theta_{N+1})\, e^{-iN\theta_{N+1}} d\theta_{N+1} \\ \int_0^{2\pi} f(\theta_1)\, e^{i\theta_1} d\theta_1 & \int_0^{2\pi} f(\theta_2)\, d\theta_2 & \cdots & \int_0^{2\pi} f(\theta_{N+1})\, e^{-i(N-1)\theta_{N+1}} d\theta_{N+1} \\ \vdots & \vdots & & \vdots \\ \int_0^{2\pi} f(\theta_1)\, e^{iN\theta_1} d\theta_1 & \int_0^{2\pi} f(\theta_2)\, e^{i(N-1)\theta_2} d\theta_2 & \cdots & \int_0^{2\pi} f(\theta_{N+1})\, d\theta_{N+1} \end{vmatrix}$$
$$= \frac{1}{\pi^{N+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1) \ldots f(\theta_{N+1}) \times \begin{vmatrix} 1 & e^{-i\theta_2} & \cdots & e^{-iN\theta_{N+1}} \\ e^{i\theta_1} & 1 & \cdots & e^{-i(N-1)\theta_{N+1}} \\ \vdots & \vdots & & \vdots \\ e^{iN\theta_1} & e^{i(N-1)\theta_2} & \cdots & 1 \end{vmatrix} d\theta_1 \ldots d\theta_{N+1}$$
$$= \frac{1}{\pi^{N+1}} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1) \ldots f(\theta_{N+1})\, e^{-i\theta_2} e^{-2i\theta_3} \ldots e^{-iN\theta_{N+1}} \times \begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & \cdots & e^{i\theta_{N+1}} \\ \vdots & \vdots & & \vdots \\ e^{iN\theta_1} & e^{iN\theta_2} & \cdots & e^{iN\theta_{N+1}} \end{vmatrix} d\theta_1 \ldots d\theta_{N+1}. \tag{A 2}$$

† Some equivalent geometrical conditions on the Fourier coefficients were given by Carathéodory (2, 3). The equivalence of the algebraic and geometrical conditions was established by Carathéodory and Fejér (4). See also Riesz (8, 9) for related results.

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Interchanging any two of the θ_n does not affect the value of the left-hand side, but interchanges two rows of the determinant on the right-hand side. The two rows can be changed back if at the same time the sign of the right-hand side is changed. Hence, adding all the $(N+1)!$ possible permutations we have

$$(N+1)!\,\Delta_N = \frac{1}{\pi^{N+1}}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1)\dots f(\theta_{N+1})$$

$$\times \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{i\theta_1} & e^{i\theta_2} & \dots & e^{i\theta_{N+1}} \\ \vdots & \vdots & & \vdots \\ e^{iN\theta_1} & e^{iN\theta_2} & \dots & e^{iN\theta_{N+1}} \end{vmatrix} \begin{vmatrix} 1 & 1 & \dots & 1 \\ e^{-i\theta_1} & e^{-i\theta_2} & \dots & e^{-i\theta_{N+1}} \\ \vdots & \vdots & & \vdots \\ e^{-iN\theta_1} & e^{-iN\theta_2} & \dots & e^{-N\theta_{N+1}} \end{vmatrix} d\theta_1\dots d\theta_{N+1}$$

$$= \frac{1}{\pi^{N+1}}\int_0^{2\pi}\cdots\int_0^{2\pi} f(\theta_1)\dots f(\theta_{N+1}) \prod_{m>n} (e^{i\theta_m}-e^{i\theta_n})(e^{-i\theta_m}-e^{-i\theta_n})\, d\theta_1\dots d\theta_{N+1}. \tag{A 3}$$

If equation (3·6) is now used, the identity (A 1) follows immediately.

Since $f(\theta)$ is non-negative, the whole integrand on the right-hand side of equation (A 1) is non-negative, from which it follows that

$$\Delta_N \geqslant 0. \tag{A 4}$$

Suppose now that $f(\theta)$ consists of N 'pulses' of energy, that is, $f(\theta)$ is zero everywhere except near N points $\theta = \theta^{(m)}$, where it becomes infinite in such a way that $\int^\theta f(\theta)\,d\theta$ has a finite discontinuity C_m at this point. In Dirac's notation (5) we may write

$$f(\theta) = \sum_{m=1}^{N} C_m \delta(\theta-\theta_m). \tag{A 5}$$

The function $f(\theta_1)\dots f(\theta_{N+1})$ is zero everywhere except where each θ_n equals some $\theta^{(m)}$. But since there are only N $\theta^{(m)}$, this implies that at least two of the θ_n must be equal. The part of the integrand *under the product sign* then vanishes, and the contribution from the neighbourhood of this point is zero. Hence

$$\textit{If } f(\theta) \textit{ is the sum of } N \textit{ pulses, then } \Delta_N = 0. \tag{A 6}$$

Conversely, if it is assumed that $f(\theta)$ is continuous except possibly for a finite number of pulses, we may show that

$$\textit{If } \Delta_N = 0 \textit{ and if } f(\theta) \geqslant 0, \textit{ then } f(\theta) \textit{ is the sum of at most } N \textit{ pulses.} \tag{A 7}$$

For, suppose that $f(\theta)$ is continuous and positive, or has a positive pulse, at more than N points, say $\theta = \theta^{(1)}, \dots, \theta^{(N+1)}$. When $\theta_1, \dots, \theta_{N+1}$ are in the neighbourhood of these points there will be a positive contribution to the integral (A 1), and since the integrand is never negative Δ_N must be greater than zero, contrary to hypothesis. Therefore $f(\theta)$ cannot be different from zero at more than N distinct points.

The first part (A 6) of the theorem can also be quite simply proved algebraically. For if $f(\theta)$ is given by (A 5) then from (3·5)

$$A_n = \frac{1}{\pi}\sum_{m=1}^{N} C_m e^{-in\theta_m}. \tag{A 8}$$

If these expressions for A_n are substituted in (3·4) it will be found that Δ_N vanishes identically. However, the converse (A 7) is necessarily more difficult to prove, since it depends upon $f(\theta)$ being non-negative.

From (A 7) we deduce that if $f(\theta)$ is the sum of just N pulses, then $\Delta_{N-1} > 0$; for if Δ_{N-1} vanished $f(\theta)$ would consist of no more than $(N-1)$ pulses. Now Δ_{N-1} is a minor of Δ_N, so that Δ_N must be of rank N (which is the result used in §3). Conversely, if $\Delta_N = 0$ but $\Delta_{N-1} > 0$, then $f(\theta)$ consists of just N pulses. Hence

A necessary and sufficient set of conditions for a non-negative function $f(\theta)$ to consist of just N pulses is that

$$\Delta_0 > 0, \quad \Delta_1 > 0, \quad \ldots, \quad \Delta_{N-1} > 0, \quad \Delta_N = 0. \tag{A 9}$$

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