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Range inclusion and diagonalization of complex symmetric operators

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Abstract. We consider the range inclusion and the diagonalization in the Jordan algebra $\mathcal{S}_C$ of C-symmetric operators, that are, bounded linear operators T satisfying $CTC = T^*$, where C is a conjugation on a separable complex Hilbert space $\mathcal{H}$. For $T \in \mathcal{S}_C$, we aim to describe the set $C_{\mathcal{R}(T)}$ of those operators $A \in \mathcal{S}_C$ satisfying the range inclusion $\mathcal{R}(A) \subset \mathcal{R}(T)$. It is proved that (i) $C_{\mathcal{R}(T)} = T\mathcal{S}_C T$ if and only if $\mathcal{R}(T)$ is closed, (ii) $\overline{C_{\mathcal{R}(T)}} = \overline{T\mathcal{S}_C T}$, and (iii) $C_{\overline{\mathcal{R}(T)}}$ is the closure of $C_{\mathcal{R}(T)}$ in the strong operator topology. Also, we extend the classical Weyl–von Neumann Theorem to $\mathcal{S}_C$, showing that every self-adjoint operator in $\mathcal{S}_C$ is the sum of a diagonal operator in $\mathcal{S}_C$ and a compact operator with arbitrarily small Schatten p-norm for $p \in (1, \infty)$.

1 Introduction

This paper is a continuation of a recent paper [44], and aims to study the range inclusion and the diagonalization of complex symmetric operators. We start by recalling some terminology and basic facts.

1.1 Preliminaries

Throughout the following, $\mathcal{H}$ will always denote a separable, complex Hilbert space with an inner product $\langle \cdot, \cdot \rangle$. We let $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ denote the Banach space of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We shall write $\mathcal{B}(\mathcal{H})$ instead of $\mathcal{B}(\mathcal{H}, \mathcal{H})$. A map $C : \mathcal{H} \to \mathcal{H}$ is called a *conjugation* if:

(i) C is antilinear, i.e., $C(\alpha x + y) = \overline{\alpha} Cx + Cy$ for $x, y \in \mathcal{H}$ and $\alpha \in \mathbb{C}$,
(ii) C is invertible with $C^{-1} = C$, and
(iii) $\langle Cx, Cy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$.

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be *complex symmetric* (c.s.) if $CTC = T^*$ for some conjugation C on $\mathcal{H}$; in this case, T is called C*-symmetric*. The general definition of a C-symmetric operator was first given in I. M. Glazman's paper [18], which combined with [19] marks the foundation of the extension theory of C-symmetric differential operators. Glazman's work was complemented in a series of papers such as [26, 28, 30, 32, 35, 37], most of which were devoted to the existence and concrete

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descriptions of extensions of C-symmetric differential operators. The reader is referred to [47] and the references cited therein for more historical comments. Throughout the following, we let C denote a conjugation on $\mathcal{H}$ unless otherwise stated, and let $\mathcal{S}_C$ denote the collection of all C-symmetric operators. The term "symmetric" stems from the fact that an operator T is complex symmetric if and only if there is an orthonormal basis $\{e_i\}$ of $\mathcal{H}$ such that T can be written as a (complex) symmetric matrix relative to $\{e_i\}$.

Now let us introduce the background of the general study of c.s. operators, initiated in [14, 15]. In the finite-dimensional case, the class of c.s. operators, containing those operators induced by Toeplitz matrices and Hankel matrices, has been studied for many years. In fact, the study of c.s. operators has classical roots in the work on automorphic functions [24], projective geometry [25], quadratic forms [39], symplectic geometry [40], and function theory [42]. In the infinite-dimensional case, it is shown that the class of c.s. operators contains many important special operators, such as normal operators, binormal operators, truncated Toeplitz operators [38], and many integration operators. People's current interests in c.s. operators are greatly inspired by many results of S. Garcia, M. Putinar, and W. Wogen [14–16] as well as their connections to concrete operators [10–12] and applications to mathematical physics [13, 20, 34].

Also, c.s. operators play an important role in the study of JB*-triples, a class of complex Banach spaces with well-behaved algebraic, geometric and holomorphic properties. This is due to the Jordan structure of $\mathcal{S}_C$. In fact, $\mathcal{S}_C$ is a weak operator closed subspace of $\mathcal{B}(\mathcal{H})$, closed under the Jordan product $\circ$, defined by

$$A \circ B = \frac{1}{2}(AB + BA), \quad \forall A, B \in \mathcal{B}(\mathcal{H}).$$

$\mathcal{S}_C$ has been studied under the name of Hermitian type Cartan factors for many years. There are six types of Cartan factors, namely rectangular type, Hermitian type, symplectic type, triple spin factors, and two finite-dimensional exceptional Cartan factors. They originally arose in É. Cartan's classification of finite-dimensional bounded symmetric domains (see [2] or [3, Theorem 2.5.9]) and play an important role in the proof of the Gelfand–Naimark theorem for JB*-triples [9].

The present study is inspired by several interesting results concerning $\mathcal{S}_C$, which suggest a rich structure of $\mathcal{S}_C$. In [11], it was proved that each contraction $T \in \mathcal{S}_C$ is the mean of two unitary operators in $\mathcal{S}_C$. In [44], the authors classified Jordan ideals of $\mathcal{S}_C$, showing that Jordan ideals of $\mathcal{S}_C$ arise by intersection from associative ideals of $\mathcal{B}(\mathcal{H})$ and hence are self-adjoint. Moreover, Jordan automorphisms of $\mathcal{S}_C$ are shown to be induced by certain unitary operators. Also, it is proved that those Jordan invertible ones constitute a dense, path connected subset of $\mathcal{S}_C$. These results provide interesting contrasts between $\mathcal{S}_C$ and $\mathcal{B}(\mathcal{H})$.

1.2 Range inclusion in $\mathcal{S}_C$

The first aim of this paper is to characterize the range inclusion of operators in $\mathcal{S}_C$, that is, given an operator $T \in \mathcal{S}_C$, to characterize all those operators $A \in \mathcal{S}_C$ satisfying $\mathcal{R}(A) \subseteq \mathcal{R}(T)$, where $\mathcal{R}(A)$ denotes the range of A.

In [6], R. Douglas proved the following classical result on the range inclusion for bounded linear operators on Hilbert spaces.

Theorem 1.1 [6] *Let A and B be two bounded linear operators on $\mathcal{H}$. Then the following statements are equivalent:*

(i) $\mathcal{R}(A) \subseteq \mathcal{R}(B)$;
(ii) $AA^* \leq \lambda BB^*$ *for some* $\lambda \geq 0$;
(iii) $A = BX$ *for some operator* $X \in \mathcal{B}(\mathcal{H})$.

The preceding result exhibits a close relationship among the notions of range inclusion, majorization, and factorization for bounded linear operators. In [7], M. R. Embry extended this result to Banach spaces and obtained that $A = XB$ for some bounded operator X on $\mathcal{R}(B)$ if and only if $\mathcal{R}(A') \subseteq \mathcal{R}(B')$. Here, A' and B' denote, respectively, the adjoint of A and the adjoint of B. P. H. Wang and X. Zhang [45] established some range inclusion theorems for non-archimedean Banach spaces over general valued fields.

It is natural to ask whether there is a C-symmetric analogue of Douglas' range inclusion theorem. Given a linear subspace $\mathcal{M}$ of $\mathcal{H}$, we denote

$$C_{\mathcal{M}} = \{X \in \mathcal{S}_C : \mathcal{R}(X) \subset \mathcal{M}\}.$$

Let $T \in \mathcal{S}_C$. We attempt to characterize $C_{\mathcal{R}(T)}$. Note that $\mathcal{S}_C$ is not closed in the usual operator multiplication, that is, given $A, B \in \mathcal{S}_C$, $\mathcal{R}(A) \subseteq \mathcal{R}(B)$ does not imply $B = AX$ for some $X \in \mathcal{S}_C$ (see Example 2.5). So we turn to the *quadratic product* in $\mathcal{S}_C$ given by $(X, Y) \longmapsto XYX$ for $X, Y \in \mathcal{S}_C$. For $T \in \mathcal{S}_C$, it is easy to check that

$$T\mathcal{S}_C T := \{TXT : X \in \mathcal{S}_C\} \subseteq C_{\mathcal{R}(T)}.$$

It is natural to ask whether the converse inclusion holds.

The first result of this paper is the following theorem, which provides a complete answer to the preceding question.

Theorem 1.2 *If $T \in \mathcal{S}_C$, then $C_{\mathcal{R}(T)} = T\mathcal{S}_C T$ if and only if $\mathcal{R}(T)$ is closed.*

The preceding result shows that if $T \in \mathcal{S}_C$ with non-closed range, then $T\mathcal{S}_C T$ is properly contained in $C_{\mathcal{R}(T)}$. However, by the following result, they always have the same norm closure.

Theorem 1.3 *If $T \in \mathcal{S}_C$, then $\overline{C_{\mathcal{R}(T)}} = \overline{T\mathcal{S}_C T}$.*

By the preceding result, each $A \in \mathcal{S}_C$ with $\mathcal{R}(A) \subseteq \mathcal{R}(T)$ is a norm limit of operators with the form TXT for $X \in \mathcal{S}_C$. Thus Theorem 1.3 can be viewed as an approximate range inclusion theorem for $\mathcal{S}_C$.

Given $A, B \in \mathcal{S}_C$, it is interesting to compare the closure of $C_{\mathcal{R}(A)}$ with that of $C_{\mathcal{R}(B)}$. To state our result, we introduce a notation. Let $\mathcal{M}, \mathcal{N}$ be two linear subspaces of $\mathcal{H}$. We write $\mathcal{M} \prec \mathcal{N}$ if for each closed subspace $\mathcal{M}_1$ of $\mathcal{H}$ contained in $\mathcal{M}$ and $\varepsilon > 0$ there exists a closed subspace $\mathcal{N}_1$ of $\mathcal{H}$ contained in $\mathcal{N}$ such that $\|P_{\mathcal{M}_1} - P_{\mathcal{N}_1}\| < \varepsilon$, where $P_{\mathcal{M}_1}$ denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_1$.

Theorem 1.4 *If $A, B \in \mathcal{S}_C$, then $\overline{C_{\mathcal{R}(A)}} \subset \overline{C_{\mathcal{R}(B)}}$ if and only if $\mathcal{R}(A) \prec \mathcal{R}(B)$.*

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Hence, $\overline{C_{\mathcal{R}(A)}} = \overline{C_{\mathcal{R}(B)}}$ if and only if $\mathcal{R}(A) \prec \mathcal{R}(B)$ and $\mathcal{R}(B) \prec \mathcal{R}(A)$. Clearly, $\mathcal{R}(A) = \mathcal{R}(B)$ implies $\overline{C_{\mathcal{R}(A)}} = \overline{C_{\mathcal{R}(B)}}$. In general, the converse does not hold (see Example 2.12).

Inspired by Theorem 1.3, we are interested in characterizing the closure of $T\mathcal{S}_C T$ in several usual topologies for $T \in \mathcal{S}_C$, such as the weak* topology, the strong operator topology (sot) and the weak operator topology (wot) . Given a subset $\mathcal{E}$ of $\mathcal{B}(\mathcal{H})$, we write $\overline{\mathcal{E}}^{w*}$, $\overline{\mathcal{E}}^{\text{SOT}}$ and $\overline{\mathcal{E}}^{\text{WOT}}$ to denote the the weak*-closure, the sot -closure and the wot -closure of $\mathcal{E}$, respectively.

Theorem 1.5 *If $T \in \mathcal{S}_C$, then*

$$\overline{C_{\mathcal{R}(T)}}^{\text{SOT}} = \overline{C_{\mathcal{R}(T)}}^{\text{WOT}} = \overline{C_{\mathcal{R}(T)}}^{w*} = C_{\overline{\mathcal{R}(T)}}.$$

In view of the preceding results, it is natural to classify the inclusion relations among $T\mathcal{S}_C T, C_{\mathcal{R}(T)}, \overline{C_{\mathcal{R}(T)}}$ and $C_{\overline{\mathcal{R}(T)}}$ for $T \in \mathcal{S}_C$. It is obvious that

$$T\mathcal{S}_C T \subseteq C_{\mathcal{R}(T)} \subseteq \overline{C_{\mathcal{R}(T)}} \subseteq C_{\overline{\mathcal{R}(T)}}.$$

In Section 2, we shall show that any two of these sets coincide if and only if T has a closed range (see Proposition 2.15).

1.3 Diagonalization in $\mathcal{S}_C$

The other aim of this paper is to study the extension of the Weyl–von Neumann Theorem to $\mathcal{S}_C$.

The Weyl–von Neumann Theorem, due to H. Weyl and J. von Neumann [33, 46], states that, after the addition of a compact (or even Hilbert–Schmidt) operator of arbitrarily small norm, a self-adjoint operator becomes a diagonal operator. Recall that an operator $T \in \mathcal{B}(\mathcal{H})$ is *diagonal* if there is an orthonormal basis for $\mathcal{H}$ consisting of eigenvectors for T. In 1958, S. T. Kuroda [29] strengthened the result by proving that every self-adjoint operator is the sum of a diagonal operator and a compact operator with arbitrarily small Schatten p-norm for $p \in (1, \infty)$.

We shall prove in Section 3 the following result, which can be viewed as an $\mathcal{S}_C$-analogue of the Weyl–von Neumann Theorem.

Theorem 1.6 *Let $T \in \mathcal{S}_C$ be a self-adjoint operator. If $p > 1$ and $\varepsilon > 0$, then there is a diagonal, self-adjoint operator $D \in \mathcal{S}_C$ such that $T - D$ lies in the Schatten p-class $\mathcal{B}_p(\mathcal{H})$ and $\|T - D\|_p < \varepsilon$.*

Remark 1.7

(i) The proof of Theorem 1.6 is inspired by [4, Theorem 38.1]. The main difficulty stems from the fact that we always have to deal with not only the self-adjoint operator T but also the conjugation C at the same time.
(ii) By [27, Theorem 1], if A is a self-adjoint operator not purely singular, then A is not a trace class perturbation of any diagonal operator. Thus the preceding result does not hold for $p = 1$.
(iii) I. D. Berg [1] and W. Sikonia [41] independently proved that each normal operator T on $\mathcal{H}$ is the sum of a diagonal operator and a compact one with arbitrarily

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small norm. The preceding result is sometimes called the Weyl–von Neumann–Berg Theorem. In [22], P. R. Halmos gave a more perspicuous proof that reduces the normal case to the Hermitian one. As an application of Theorem 1.6, we shall prove that finite commuting normal operators in $\mathcal{S}_C$ can be simultaneously diagonalized (see Theorem 3.3).

(iv) In [43], D. Voiculescu considered the simultaneous diagonalization of commuting self-adjoint operators and showed that if $T_1, \ldots, T_n (n \geq 2)$ are commuting self-adjoint operators and $\varepsilon > 0$, then there exist n commuting self-adjoint diagonal operators $D_1, \ldots, D_n$ such that $\|T_i - D_i\|_n < \varepsilon$ for all $i = 1, \ldots, n$, where $\|\cdot\|_n$ is the Schatten n-norm. We do not know whether Voiculescu's result can be extended to the $\mathcal{S}_C$-setting.

As another application of Theorem 1.6, we shall provide a result concerning the irreducible approximation in $\mathcal{S}_C$, which shows that those irreducible ones in $\mathcal{S}_C$ constitute a dense subset of $\mathcal{S}_C$ (see Corollary 3.6). This is an $\mathcal{S}_C$-analogue of Halmos' irreducible approximation theorem [21].

The proofs of Theorems 1.2–1.5 will be given in Section 2. Section 3 is devoted to the proof of Theorem 1.6.

2 The range inclusion in $\mathcal{S}_C$

The aim of this section is to give the proofs of Theorems 1.2–1.5.

2.1 Proof of Theorem 1.2

Before giving the proof of Theorem 1.2, we first make some preparations.

Given a bounded linear operator A, we denote by $\ker A$ the kernel of A.

Lemma 2.1 *Let $A, B \in \mathcal{S}_C$. If $\mathcal{R}(A) \subset \mathcal{R}(B)$, then*

$$\mathcal{R}(A^*) \subset \mathcal{R}(B^*), \quad \ker B \subset \ker A, \quad \ker B^* \subset \ker A^*.$$

Proof Since $CAC = A^*$ and $CA^*C = A$, one can see that $C(\mathcal{R}(A)) = \mathcal{R}(A^*)$. Likewise, we have $C(\mathcal{R}(B)) = \mathcal{R}(B^*)$. It follows immediately that $\mathcal{R}(A^*) \subset \mathcal{R}(B^*)$. Thus $\mathcal{R}(B^*)^\perp \subset \mathcal{R}(A^*)^\perp$, that is, $\ker B \subset \ker A$.

On the other hand, $\mathcal{R}(A) \subset \mathcal{R}(B)$ implies $\mathcal{R}(B)^\perp \subset \mathcal{R}(A)^\perp$ and, equivalently, $\ker B^* \subset \ker A^*$. ∎

Throughout the following, given a closed subspace $\mathcal{M}$ of $\mathcal{H}$, we denote by $P_{\mathcal{M}}$ the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$.

Proof of Theorem 1.2 "$\Longrightarrow$". Assume that $C_{\mathcal{R}(T)} = T\mathcal{S}_C T$. We shall show that $\mathcal{R}(T)$ is closed.

Assume that $\{x_n\}_{n=1}^\infty \subset \mathcal{H}$ and $Tx_n \to y_0 \in \mathcal{H}$. It suffices to show that $y_0 \in \mathcal{R}(T)$. Note that $T \in C_{\mathcal{R}(T)}$. Thus there exists $A \in \mathcal{S}_C$ such that $T = TAT$. Hence, $TATx_n \to y_0$ and $TATx_n \to TAy_0$, which implies $y_0 = TAy_0 \in \mathcal{R}(T)$.

"$\Longleftarrow$". Now assume that $\mathcal{R}(T)$ is closed. Since $T\mathcal{S}_C T \subseteq C_{\mathcal{R}(T)}$ is clear, it remains to show $C_{\mathcal{R}(T)} \subseteq T\mathcal{S}_C T$.

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Denote $\mathcal{N} = (\ker T)^{\perp}$ and $\mathcal{M} = \mathcal{R}(T)$. For $x \in \mathcal{N}$, define $Fx = Tx$. Since $\mathcal{M}$ is norm-closed, F is an invertible operator from $\mathcal{N}$ into $\mathcal{M}$. Define $S = P_{\mathcal{N}} F^{-1} P_{\mathcal{M}}$. One can verify that $TS = P_{\mathcal{M}}$ and $ST = P_{\mathcal{N}}$.

Since $T \in \mathcal{S}_C$, one can see that $C(\ker T) = \ker T^*$, that is, $C(\mathcal{N}^{\perp}) = \mathcal{M}^{\perp}$. Thus $C(\mathcal{N}) = \mathcal{M}$ and $C(\mathcal{M}) = \mathcal{N}$. Denote $C_1 = C|_{\mathcal{M}}$. Then $C_1 : \mathcal{M} \longrightarrow \mathcal{N}$ is conjugate-linear, invertible, and isometric. One can check that $Cx = C^{-1}x = C_1^{-1}x$ for all $x \in \mathcal{N}$.

Claim 1 $C_1(F^{-1})^* = F^{-1}C_1^{-1}$.

In fact, for any $x \in \mathcal{M}$, note that

$$TCx = TC_1x = FC_1x, \quad CT^*x = CF^*x = C_1^{-1}F^*x.$$

Thus $FC_1 = C_1^{-1}F^*$, $(F^{-1})^* C_1 = C_1^{-1}F^{-1}$ and $C_1(F^{-1})^* = F^{-1}C_1^{-1}$.

Claim 2 $S \in \mathcal{S}_C$.

For any $x \in \mathcal{H}$, there is the unique decomposition $x = x_1 + x_2$, where $x_1 \in \mathcal{N}$ and $x_2 \in \mathcal{N}^{\perp}$. So $Cx_1 \in \mathcal{M}$ and $Cx_2 \in \mathcal{M}^{\perp}$. By Claim 1, we have

$$\begin{aligned}
CSCx &= C(P_{\mathcal{N}}F^{-1}P_{\mathcal{M}})(Cx_1 + Cx_2)\\
&= C(P_{\mathcal{N}}F^{-1}P_{\mathcal{M}})Cx_1\\
&= C(P_{\mathcal{N}}F^{-1}P_{\mathcal{M}})C_1^{-1}x_1\\
&= C(P_{\mathcal{N}}F^{-1}C_1^{-1}x_1)\\
&= C_1^{-1}(F^{-1}C_1^{-1}x_1)\\
&= (C_1^{-1}F^{-1}C_1^{-1})x_1 = (F^{-1})^* x_1\\
&= P_{\mathcal{M}}(F^{-1})^* P_{\mathcal{N}}x = S^*x.
\end{aligned}$$

This proves $S \in \mathcal{S}_C$.

Set $X = SAS$. Thus one can easily check that $X \in \mathcal{S}_C$. We shall prove that $TXT = A$. Note that $\mathcal{R}(A) \subset \mathcal{R}(T) = \mathcal{M}$. It follows that

$$TXT = TSAST = P_{\mathcal{M}}AP_{\mathcal{N}} = AP_{\mathcal{N}}.$$

It suffices to prove that $AP_{\mathcal{N}} = A$, that is, $A(I - P_{\mathcal{N}}) = 0$.

Since $\mathcal{R}(A) \subset \mathcal{R}(T)$, it follows from Lemma 2.1 that $\ker T \subset \ker A$. That is, $\mathcal{N}^{\perp} \subset \ker A$. Therefore we conclude that $A(I - P_{\mathcal{N}}) = 0$. ∎

Corollary 2.2 *Let $T, A \in \mathcal{S}_C$. If either $\mathcal{R}(T)$ or $\mathcal{R}(A)$ is closed, then $\mathcal{R}(A) \subset \mathcal{R}(T)$ if and only if $A = TXT$ for some $X \in \mathcal{S}_C$.*

Proof It suffices to prove the necessity.

"$\Longrightarrow$". By Theorem 1.2, we need only deal with the case that $\mathcal{R}(A)$ is closed.

Using Theorem 1.2 again, we can find $Z \in \mathcal{S}_C$ such that $A = AZA$. Since $\mathcal{R}(A) \subset \mathcal{R}(T)$, by Douglas' range inclusion theorem, $A = TY$ for some $Y \in \mathcal{B}(\mathcal{H})$. Noting that $A, T \in \mathcal{S}_C$, we have

$$\begin{aligned}
A &= CA^*C = C(TY)^*C = CY^*T^*C\\
&= (CY^*C)(CT^*C) = (CY^*C)T.
\end{aligned}$$

Then

$$A = AZA = (TY)Z(CY^*C)T = T[YZ(CY^*C)]T.$$

One can verify that $X := YZ(CY^*C) \in \mathcal{S}_C$, which completes the proof. ∎

Remark 2.3 The result of Corollary 2.2 is sharp. In fact, given any $T \in \mathcal{S}_C$, it holds that $\mathcal{R}(T) \subset \mathcal{R}(T)$; if, in addition, $\mathcal{R}(T)$ is not closed, then, by the proof for the necessity of Theorem 1.2, there exists no $X \in \mathcal{B}(\mathcal{H})$ such that $TXT = T$. Here, we provide a concrete example of complex symmetric operator with non-closed range.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and

$$T = \sum_{n=1}^{\infty} \frac{e_n \otimes e_n}{n}.$$

For $x = \sum_{n=1}^{\infty} \alpha_i e_i$, define $Cx = \sum_{n=1}^{\infty} \overline{\alpha_i} e_i$. Then C is a conjugation on $\mathcal{H}$ and one can check that $T \in \mathcal{S}_C$. Clearly, $\mathcal{R}(T)$ is not closed. If $X \in \mathcal{S}_C$ and $T = TXT$, then

$$\frac{1}{n} = \langle Te_n, e_n \rangle = \langle TXTe_n, e_n \rangle = \langle XTe_n, Te_n \rangle = \frac{1}{n^2} \langle Xe_n, e_n \rangle,$$

that is, $\langle Xe_n, e_n \rangle = n$. Since n is arbitrary, we deduce that X is not bounded, a contradiction.

For $e, f \in \mathcal{H}$, we define an operator $e \otimes f$ on $\mathcal{H}$ as

$$(e \otimes f)(x) = \langle x, f \rangle e, \quad \forall x \in \mathcal{H}.$$

Corollary 2.4 *Let $T \in \mathcal{S}_C$. If $x, y \in \mathcal{R}(T)$, then*

$$x \otimes (Cx) \in T\mathcal{S}_C T \quad \textit{and} \quad x \otimes (Cy) + y \otimes (Cx) \in T\mathcal{S}_C T.$$

Proof Denote $X = x \otimes (Cx)$ and $Y = x \otimes (Cy) + y \otimes (Cx)$. It is easy to verify that $X \in \mathcal{S}_C$ and $\overline{\mathcal{R}(X)} = \mathcal{R}(X) \subset \mathcal{R}(T)$. By Corollary 2.2, we have $X \in T\mathcal{S}_C T$. Likewise we have $Y \in T\mathcal{S}_C T$. ∎

At the end of this subsection, we provide several illustrating examples.

The following example shows that Douglas' range inclusion theorem does not hold for $\mathcal{S}_C$.

Example 2.5 Let D be a conjugation on $\mathcal{H}$ and

$$A = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}, \quad T = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix}, \quad C = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \begin{matrix} \mathcal{H} \\ \mathcal{H} \end{matrix},$$

where I is the identity operator on $\mathcal{H}$. Then it is easy to check that C is a conjugation on $\mathcal{H} \oplus \mathcal{H}$, $A, T \in \mathcal{S}_C$ and $\mathcal{R}(A) \subset \mathcal{R}(T)$.

If $X \in \mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ and $A = TX$, then one can verify that

$$X = T^{-1}A = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

Clearly, $X \notin \mathcal{S}_C$.

The following example shows that Theorem 1.2 can not be extended to $\mathcal{B}(\mathcal{H})$ or general Jordan operator algebras.

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Example 2.6 Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ with $\dim \mathcal{M} = \dim \mathcal{M}^{\perp}$. Set

$$T = \begin{bmatrix} I_0 & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix} \quad \text{and} \quad A = \begin{bmatrix} 0 & B \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

where I_0 is the identity operator on $\mathcal{M}$ and B is an invertible operator from $\mathcal{M}^{\perp}$ onto $\mathcal{M}$. Then $\mathcal{R}(A) \subset \mathcal{R}(T)$.

For any $X \in \mathcal{B}(\mathcal{H})$ with

$$X = \begin{bmatrix} X_{1,1} & X_{1,2} \\ X_{2,1} & X_{2,2} \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix},$$

one can check that

$$TXT = \begin{bmatrix} X_{1,1} & 0 \\ 0 & 0 \end{bmatrix} \begin{matrix} \mathcal{M} \\ \mathcal{M}^{\perp} \end{matrix} \neq A.$$

Hence, $A \neq TXT$ for any $X \in \mathcal{B}(\mathcal{H})$.

2.2 Proof of Theorem 1.3

Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$. A conjugate-linear map J on $\mathcal{H}$ is called a *partial conjugation* supported on $\mathcal{M}$ if $\ker J = \mathcal{M}^{\perp}$ reduces J and $J|_{\mathcal{M}}$ is a conjugation.

Lemma 2.7 [15, Theorem 2] *Let $T \in \mathcal{B}(\mathcal{H})$ and C be a conjugation on $\mathcal{H}$. Then $T \in \mathcal{S}_C$ if and only if $T = CJ|T|$ for some partial conjugation J supported on $\overline{\mathcal{R}(|T|)}$ and commuting with $|T|$.*

Proof of Theorem 1.3 Since it is clear that $T\mathcal{S}_C T \subset C_{\mathcal{R}(T)}$, it suffices to prove that $C_{\mathcal{R}(T)} \subset \overline{T\mathcal{S}_C T}$. Fix an operator A in $C_{\mathcal{R}(T)}$. We shall prove that $A \in \overline{T\mathcal{S}_C T}$. This follows readily from the following claim.

Claim *Given $\varepsilon > 0$, there exists $B \in \mathcal{S}_C$ with $\mathcal{R}(B) = \overline{\mathcal{R}(B)} \subset \mathcal{R}(A)$ such that $\|A - B\| < \varepsilon$.*

In fact, if the preceding claim holds, then, by Corollary 2.2, $B \in T\mathcal{S}_C T$ and $\operatorname{dist}(A, T\mathcal{S}_C T) < \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we deduce that $A \in \overline{T\mathcal{S}_C T}$.

By Lemma 2.7, $A = CJP$, where $P = |A|$ and J is a partial conjugation supported on $\overline{\mathcal{R}(P)}$ with $JP = PJ$.

Now we fix an $\varepsilon > 0$ and define three functions f, g, h on $[0, \|A\|]$ as

$$f(t) = \begin{cases} 0, & t \in [0, \varepsilon/2), \\ t, & t \in [\varepsilon/2, \|A\|], \end{cases} \quad h(t) = \begin{cases} 0, & t \in [0, \varepsilon/2), \\ 1/t, & t \in [\varepsilon/2, \|A\|], \end{cases} \quad g = fh.$$

It is easy to see that:

(i) $f(P), g(P)$, and $h(P)$ are positive operators commuting with J,
(ii) $g(P)$ is an orthogonal projection with $\mathcal{R}(f(P)) = \mathcal{R}(g(P)) = \mathcal{R}(h(P))$, and
(iii) $\mathcal{R}(g(P))$ reduces J.

Define $\tilde{J} = Jg(P)$. Then $\tilde{J}$ is a partial anti-conjugation supported on $\mathcal{R}(h(P))$ commuting with $h(P)$. Set $B = CJf(P)$. Note that $Jf(P) = \tilde{J}P$. It follows that $B = C\tilde{J}f(P)$.

By Lemma 2.7, we have $B \in \mathcal{S}_C$. Note that

$$\overline{\mathcal{R}(f(P))} = \mathcal{R}(f(P)) \subset \mathcal{R}(P).$$

Since J is supported on $\overline{\mathcal{R}(P)}$, it follows that

$$\overline{\mathcal{R}(B)} = \mathcal{R}(B) \subset \mathcal{R}(A).$$

Note that

$$\|A - B\| = \|CJP - CJf(P)\| \leq \|P - f(P)\| < \varepsilon.$$

This completes the proof. ∎

The following result can be seen from the proof of Theorem 1.3.

Corollary 2.8 *Let $T \in \mathcal{S}_C$. Then, given $\varepsilon > 0$, there exists $A \in \mathcal{S}_C$ with $\|T - A\| < \varepsilon$ such that $\mathcal{R}(A) = \overline{\mathcal{R}(A)} \subset \mathcal{R}(T)$.*

2.3 Proof of Theorem 1.4

The aim of this subsection is to prove Theorem 1.4.

A subset $\mathcal{E}$ of $\mathcal{H}$ is called an *operator range* if $\mathcal{E} = \mathcal{R}(A)$ for some $A \in \mathcal{B}(\mathcal{H})$. We denote by $\mathrm{Ran}(\mathcal{H})$ the set of all operator ranges in $\mathcal{H}$. Then, by [8, Section 2], $\mathrm{Ran}(\mathcal{H})$ is a lattice.

The following lemma shows that each operator range can be attained in $\mathcal{S}_C$.

Lemma 2.9 *Given $T \in \mathcal{B}(\mathcal{H})$, there exists $A \in \mathcal{S}_C$ with $\mathcal{R}(A) = \mathcal{R}(T)$.*

Proof Let $T = U|T|$ be the polar decomposition of T, where U is a partial isometry. So $\mathcal{R}(T) = \mathcal{R}(U|T|U^*)$.

Denote $P = CU|T|U^*C$. Clearly, P is positive and hence a complex symmetric operator. Then we can find a conjugation J on $\mathcal{H}$ so that $JP = PJ$. Set $A = CJP$. Then, by [17, Theorem 3.1], $A \in \mathcal{S}_C$ and

$$\mathcal{R}(A) = \mathcal{R}(CJP) = C(\mathcal{R}(JP)) = C(\mathcal{R}(P)) = \mathcal{R}(U|T|U^*C) = \mathcal{R}(T).$$

This ends the proof. ∎

For $A \in \mathcal{B}(\mathcal{H})$, we denote $A^t = CA^*C$. Then it is easy to check the following.

(a) $A \in \mathcal{S}_C$ if and only if $A = A^t$.
(b) $A\mathcal{S}_C A^t \subset \mathcal{S}_C$.

Proposition 2.10 *If $A, B \in \mathcal{B}(\mathcal{H})$, then the following are equivalent:*

(i) *$\mathcal{R}(A) \subset \mathcal{R}(B)$;*
(ii) *$C_{\mathcal{R}(A)} \subset C_{\mathcal{R}(B)}$;*
(iii) *$A\mathcal{S}_C A^t \subset B\mathcal{S}_C B^t$.*

Proof (i)$\Longrightarrow$(ii). This is obvious.

(ii)$\Longrightarrow$(i). For any $x \in \mathcal{R}(A)$, by Corollary 2.4, we have

$$x \otimes (Cx) \in C_{\mathcal{R}(A)} \subset C_{\mathcal{R}(B)},$$

which implies $x \in \mathcal{R}(B)$. Hence $\mathcal{R}(A) \subset \mathcal{R}(B)$.

(i)$\Longrightarrow$(iii). By Douglas' range inclusion theorem, $\mathcal{R}(A) \subset \mathcal{R}(B)$ implies $A = BZ$ for some $Z \in \mathcal{B}(\mathcal{H})$. Note that

$$A^t = CA^*C = C(BZ)^*C = CZ^*B^*C = (CZ^*C)(CB^*C) = Z^tB^t.$$

Thus, for any $X \in \mathcal{S}_C$, we have

$$AXA^t = (BZ)X(Z^tB^t) = B(ZXZ^t)B^t.$$

One can verify that $ZXZ^t \in \mathcal{S}_C$. Thus $AXA^t \in B\mathcal{S}_CB^t$.

(iii)$\Longrightarrow$(i). For $x \in \mathcal{H}$, denote $y = Ax$. We shall prove that $y \in \mathcal{R}(B)$. Note that

$$\begin{aligned} y \otimes (Cy) &= (Ax) \otimes (CAx) = (Ax) \otimes [(CAC)(Cx)] \\ &= A[x \otimes (Cx)](CAC)^* = A[x \otimes (Cx)]A^t \in A\mathcal{S}_CA^t. \end{aligned}$$

So $y \otimes (Cy) \in B\mathcal{S}_CB^t$. Then $y \otimes (Cy) = BXB^t$ for some $X \in \mathcal{S}_C$. It follows that $y \in \mathcal{R}(B)$. ∎

Corollary 2.11 *If $T \in \mathcal{B}(\mathcal{H})$, then $C_{\mathcal{R}(T)} = T\mathcal{S}_CT^t$ if and only if $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$.*

Proof By Lemma 2.9, there exists $A \in \mathcal{S}_C$ with $\mathcal{R}(A) = \mathcal{R}(T)$. Then $C_{\mathcal{R}(A)} = C_{\mathcal{R}(T)}$ and, by Proposition 2.10, we have

$$T\mathcal{S}_CT^t = A\mathcal{S}_CA^t = A\mathcal{S}_CA.$$

Then $C_{\mathcal{R}(T)} = T\mathcal{S}_CT^t$ if and only if $C_{\mathcal{R}(A)} = A\mathcal{S}_CA$. The desired result follows readily from Theorem 1.2. ∎

Proof of Theorem 1.4 Denote $\mathcal{M} = \mathcal{R}(A)$ and $\mathcal{N} = \mathcal{R}(B)$. By Theorem 1.3, $\overline{A\mathcal{S}_CA} = \overline{C_\mathcal{M}}$ and $\overline{B\mathcal{S}_CB} = \overline{C_\mathcal{N}}$.

"$\Longrightarrow$". Choose a closed subspace $\mathcal{M}_1$ of $\mathcal{M}$ and an orthonormal basis $\{e_i\}_{i\in\Lambda}$ of $\mathcal{M}_1$. Define $U = \sum_{i\in\Lambda} e_i \otimes (ce:i)$. Then one can check that $U \in \mathcal{S}_C$ is a partial isometry with $\mathcal{R}(U) = \mathcal{M}_1$, which implies $U \in C_\mathcal{M} \subset \overline{C_\mathcal{N}}$. By the hypothesis, there exist $X_n \in C_\mathcal{N}$, $n \geq 1$, such that $X_n \to U$.

By Corollary 2.8, we can find $\{Y_n\} \subset \mathcal{S}_C$ with $\mathcal{R}(Y_n) = \overline{\mathcal{R}(Y_n)} \subset \mathcal{N}$ such that $\|Y_n - X_n\| < \frac{1}{n}$, $n \geq 1$. Then $Y_n \in C_\mathcal{N}$ and $Y_n \to U$. Furthermore, we have $Y_nY_n^* \to UU^*$.

Note that $UU^* = P_{\mathcal{M}_1}$ and $\mathcal{R}(Y_nY_n^*) = \mathcal{R}(Y_n)$ is closed, $n \geq 1$. There exists $\delta > 0$ such that $\sigma(Y_nY_n^*) \subset \{0\} \cup [\delta, \infty)$. Define a continuous function f on $\{0\} \cup (\delta/2, \infty)$ as

$$f(t) = \begin{cases} 0, & t = 0, \\ 1, & t \in (\delta/2, \infty). \end{cases}$$

Then $f(Y_nY_n^*) \to f(UU^*) = P_{\mathcal{M}_1}$. Note that $f(Y_nY_n^*)$ is an orthogonal projection with

$$\mathcal{R}(f(Y_nY_n^*)) = \mathcal{R}(Y_nY_n^*) = \mathcal{R}(Y_n) \subset \mathcal{N}.$$

This proves the necessity.

"$\Longleftarrow$". Clearly, it suffices to prove $C_\mathcal{M} \subset \overline{C_\mathcal{N}}$. By Corollary 2.8, it suffices to prove that any $X \in \mathcal{S}_C$ with $\mathcal{R}(X) = \overline{\mathcal{R}(X)} \subset \mathcal{M}$ satisfies $X \in \overline{C_\mathcal{N}}$.

Denote $\mathcal{M}_1 = \mathcal{R}(X)$ and, for convenience, we write P for $P_{\mathcal{M}_1}$. By Corollary 2.11, we have $X \in C_{\mathcal{M}_1} = P\mathcal{S}_CP^t$. So $X = PYP^t$ for some $Y \in \mathcal{S}_C$. On the other hand, by the

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hypothesis, we can find orthogonal projections $\{P_n\}$ with $\mathcal{R}(P_n) \subset \mathcal{N}$ such that $P_n \to P$. So $P_nYP_n^t \to PYP^t = X$. Noting that $P_nYP_n^t \in \mathcal{S}_C$ and $\mathcal{R}(P_nYP_n^t) \subset \mathcal{R}(P_n) \subset \mathcal{N}$, we conclude that $P_nYP_n^t \in C_{\mathcal{N}}$ and $X \in \overline{C_{\mathcal{N}}}$. ∎

Next, we determine $\overline{T\mathcal{S}_C T}$ for certain compact operators T.

Example 2.12 Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ such that $Ce_n = e_n, n = 1, 2, \ldots$. We define two diagonal operators on $\mathcal{H}$ as

$$A = \begin{bmatrix} 1 & & & & \\ & \frac{1}{2^2} & & & \\ & & \frac{1}{3^2} & & \\ & & & \frac{1}{4^2} & \\ & & & & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{matrix}, \quad B = \begin{bmatrix} 1 & & & & \\ & \frac{1}{2} & & & \\ & & \frac{1}{3} & & \\ & & & \frac{1}{4} & \\ & & & & \ddots \end{bmatrix} \begin{matrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ \vdots \end{matrix}.$$

It is clear that $A, B \in \mathcal{S}_C$ and $A = B^2$. Thus $\mathcal{R}(A) \subset \mathcal{R}(B)$. Noting that

$$\sum_n \frac{e_n}{n^2} = B\Big(\sum_n \frac{e_n}{n}\Big) \in \mathcal{R}(B), \qquad \sum_n \frac{e_n}{n^2} \notin \mathcal{R}(A),$$

we obtain $\mathcal{R}(A) \subsetneq \mathcal{R}(B)$. Next, we shall show that $\overline{C_{\mathcal{R}(A)}} = \overline{C_{\mathcal{R}(B)}}$ or, equivalently, $\overline{A\mathcal{S}_C A} = \overline{B\mathcal{S}_C B}$.

Since it is clear that $[A\mathcal{S}_C A \cup B\mathcal{S}_C B] \subset [\mathcal{S}_C \cap \mathcal{K}(\mathcal{H})]$, where $\mathcal{K}(\mathcal{H})$ is the ideal of operators in $\mathcal{B}(\mathcal{H})$, it suffices to show $[\mathcal{S}_C \cap \mathcal{K}(\mathcal{H})] \subset [\overline{A\mathcal{S}_C A} \cap \overline{B\mathcal{S}_C B}]$.

Arbitrarily choose an operator $T \in \mathcal{S}_C \cap \mathcal{K}(\mathcal{H})$. For each $n \geq 1$, we let P_n denote the orthogonal projection of $\mathcal{H}$ onto $\vee\{e_1, \ldots, e_n\}$. Note that $P_nTP_n \in \mathcal{S}_C$ with

$$\mathcal{R}(P_nTP_n) = \overline{\mathcal{R}(P_nTP_n)} \subset \mathcal{R}(A) \cap \mathcal{R}(B).$$

By Corollary 2.2, $P_nTP_n \in C_{\mathcal{R}(A)} \cap C_{\mathcal{R}(B)}$. Note that $P_nTP_n \to T$. We obtain $T \in \overline{A\mathcal{S}_C A} \cap \overline{B\mathcal{S}_C B}$. Thus we have shown

$$\overline{A\mathcal{S}_C A} = \overline{B\mathcal{S}_C B} = \mathcal{S}_C \cap \mathcal{K}(\mathcal{H}).$$

Remark 2.13 Let $A, B \in \mathcal{B}(\mathcal{H})$ and C be defined as in the preceding example. Set

$$\widetilde{C} = C \oplus C, \quad R = A \oplus B, \quad T = B \oplus A.$$

Then one can see that $R, T \in \mathcal{S}_{\widetilde{C}}$ with $\mathcal{R}(R) \nsubseteq \mathcal{R}(T)$ and $\mathcal{R}(T) \nsubseteq \mathcal{R}(R)$. However, one can show that $\widetilde{C}_{\mathcal{R}(R)}$ and $\widetilde{C}_{\mathcal{R}(T)}$ have the same norm closure.

2.4 Proof of Theorem 1.5

We first make some preparations.

Lemma 2.14 *Let $\mathcal{M}$ be a subspace of $\mathcal{H}$. Then*

$$C_{\overline{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H}) = \overline{C_{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H}) = \overline{C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})}.$$

Proof One can easily check that $C_{\overline{\mathcal{M}}}$ is closed in the weak operator topology. It is clear that

$$[C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})] \subset [\overline{C_{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H})] \subset [C_{\overline{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H})].$$

So it suffices to prove $[C_{\overline{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H})] \subset \overline{C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})}$.

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Now assume that $A \in C_{\overline{\mathcal{M}}} \cap \mathcal{K}(\mathcal{H})$. We shall prove $A \in \overline{C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})}$. By [15, Theorem 3], A can be written as

$$A = \sum_{n=1}^{\infty} \lambda_n (Ce_n) \otimes e_n,$$

where e_n are certain orthonormal eigenvectors of $|A| = |AA^*|^{1/2}$ and the λ_n are the nonzero eigenvalues of $|A|$, repeated according to multiplicity. Clearly, $\lambda_n \to 0$ and $Ce_n \in \mathcal{R}(A) \subset \overline{\mathcal{M}}$. Then

$$\sum_{n=1}^{m} \lambda_n (Ce_n) \otimes e_n \longrightarrow A \quad (m \to \infty).$$

So it remains to prove that $(Ce_n) \otimes e_n \in \overline{C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})}$ for each $n \geq 1$.

Since $Ce_n \in \overline{\mathcal{M}}$, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subset \mathcal{M}$ converging to Ce_n. Clearly, $x_k \otimes (Cx_k) \in C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})$ for all $k \geq 1$. Note that $\{x_k \otimes (Cx_k)\}$ converges to $(Ce_n) \otimes e_n$ in the norm topology. We deduce that $(Ce_n) \otimes e_n \in \overline{C_{\mathcal{M}} \cap \mathcal{K}(\mathcal{H})}$. Thus we complete the proof. ∎

Proof of Theorem 1.5 We denote $\mathcal{M} = \mathcal{R}(T)$. It is trivial to see that $C_{\overline{\mathcal{M}}}$ is closed in both the weak operator topology and the weak* topology; moreover,

$$\overline{C_{\mathcal{M}}}^{w*} \subset \overline{C_{\mathcal{M}}}^{\text{WOT}} \subset C_{\overline{M}}.$$

On the other hand, since $C_{\mathcal{M}}$ is a subspace of $\mathcal{B}(\mathcal{H})$, it is clear that $\overline{C_{\mathcal{M}}}^{\text{WOT}} = \overline{C_{\mathcal{M}}}^{\text{SOT}}$. Then it suffices to show $C_{\overline{\mathcal{M}}} \subset \overline{C_{\mathcal{M}}}^{w*}$.

We need only consider the case that $\mathcal{M}$ is not closed. In this case, we can choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\overline{\mathcal{M}}$. Define $U \in \mathcal{B}(\mathcal{H})$ as

$$U = \sum_{i=1}^{\infty} e_i \otimes (ce_i).$$

Thus U is a partial isometry lying in $\mathcal{S}_C$, $\mathcal{R}(U) = \overline{\mathcal{M}}$ and $(\ker U)^{\perp} = C(\overline{\mathcal{M}})$. So $U \in C_{\overline{\mathcal{M}}}$ and it follows from Theorem 1.2 that $C_{\overline{\mathcal{M}}} = U\mathcal{S}_C U$.

Fix an operator $A \in \mathcal{S}_C$. Then it suffices to prove $UAU \in \overline{C_{\mathcal{M}}}^{w*}$.

For the conjugation C, there is an orthonormal basis $\{f_i\}$ of $\mathcal{H}$ such that $Cf_i = f_i$ for all i (see [13, Lemma 2.11]). For each $n \geq 1$, set $P_n = \sum_{i=1}^{n} f_i \otimes f_i$. It is easy to see $P_n \in \mathcal{S}_C$. Noting that $P_n A P_n \in \mathcal{S}_C$ and, as $n \to \infty$, $P_n A P_n$ converges to A in the SOT. So $UP_n A P_n U$ converges to UAU in the SOT. Now it remains to prove the following claim.

Claim *$UP_nAP_nU \in \overline{C_{\mathcal{M}}}^{w*}$ for each $n = 1, 2, \ldots$.*

In fact, if the preceding claim holds, then, by [4, Proposition 20.3], $UAU \in \overline{C_{\mathcal{M}}}^{w*}$. Set $a_{j,i} = \langle Af_i, f_j \rangle$ for all $i, j \in \{1, 2, \ldots\}$. Since $A \in \mathcal{S}_C$, we have $a_{i,j} = a_{j,i}$. Then

$$P_n A P_n = \sum_{i=1}^{n} a_{i,i} f_i \otimes f_i + \sum_{1 \leq i < j \leq n} a_{i,j} (f_i \otimes f_j + f_j \otimes f_i)$$

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and

(1)

$$UP_nAP_nU$$

(2)

$$= \sum_{i=1}^{n} a_{i,i}(Uf_i) \otimes (U^* f_i) + \sum_{1 \le i < j \le n} a_{i,j}[(Uf_i) \otimes (U^* f_j) + (Uf_j) \otimes (U^* f_i)]$$

(3)

$$= \sum_{i=1}^{n} a_{i,i}(Uf_i) \otimes (U^* Cf_i) + \sum_{1 \le i < j \le n} a_{i,j}[(Uf_i) \otimes (U^* Cf_j) + (Uf_j) \otimes (U^* Cf_i)]$$

(4)

$$= \sum_{i=1}^{n} a_{i,i}(Uf_i) \otimes (CUf_i) + \sum_{1 \le i < j \le n} a_{i,j}[(Uf_i) \otimes (CUf_j) + (Uf_j) \otimes (CUf_i)].$$

Note that $Uf_i, Uf_j \in \mathcal{R}(U) = \overline{\mathcal{M}}$.

For any $x, y \in \overline{\mathcal{M}}$, it follows from Lemma 2.14 that $x \otimes (Cy) + y \otimes (Cx) \in \overline{C_{\mathcal{M}}}$. In view of (1), we obtain $UP_nAP_nU \in \overline{C_{\mathcal{M}}} \subset \overline{C_{\mathcal{M}}}^{w*}$. This proves Claim and completes the proof. ■

The following result classifies the inclusion relations among $T\mathcal{S}_C T$, $C_{\mathcal{R}(T)}$, $\overline{C_{\mathcal{R}(T)}}$ and $C_{\overline{\mathcal{R}(T)}}$ for $T \in \mathcal{S}_C$.

Proposition 2.15 *For $T \in \mathcal{S}_C$, the following are equivalent:*

(i) $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$;
(ii) $C_{\mathcal{R}(T)} = \overline{C_{\mathcal{R}(T)}}$;
(iii) $C_{\mathcal{R}(T)} = \overline{C_{\mathcal{R}(T)}}^{\text{WOT}}$;
(iv) $\overline{C_{\mathcal{R}(T)}} = C_{\overline{\mathcal{R}(T)}}$;
(v) $C_{\mathcal{R}(T)} = A\mathcal{S}_C A$ *for some* $A \in \mathcal{S}_C$;
(vi) $T\mathcal{S}_C T = C_{\mathcal{M}}$ *for some subspace* $\mathcal{M}$ *of* $\mathcal{H}$;
(vii) $T\mathcal{S}_C T = \overline{T\mathcal{S}_C T}$;
(viii) $T\mathcal{S}_C T = \overline{T\mathcal{S}_C T}^{\text{WOT}}$.

Proof "(i)$\Longrightarrow$(iii), (iv) and (viii)". Since $\mathcal{R}(T)$ is norm-closed, one can easily verify that $C_{\mathcal{R}(T)}$ is closed in the wot and, by Theorem 1.2, we have $C_{\mathcal{R}(T)} = T\mathcal{S}_C T$. Hence $T\mathcal{S}_C T$ is closed in the wot .

"(viii)$\Longrightarrow$ (vii)" and "(iii)$\Longrightarrow$ (ii)". Both are obvious.

"(vii)$\Longrightarrow$ (i)". Since $T\mathcal{S}_C T$ is norm closed, by Theorem 1.3, $T\mathcal{S}_C T = \overline{C_{\mathcal{R}(T)}} \supseteq C_{\mathcal{R}(T)}$. Hence,$T\mathcal{S}_C T = C_{\mathcal{R}(T)}$ and, by Theorem 1.2, $\mathcal{R}(T)$ is closed.

"(ii)$\Longrightarrow$ (i)". Choose an $x \in \overline{\mathcal{R}(T)}$. Then there exists a sequence $\{x_n\} \in \mathcal{R}(T)$ converging to x. Clearly, $x_n \otimes (Cx_n) \in C_{\mathcal{R}(T)}$ for all n. Note that $\{x_n \otimes (Cx_n)\}$ converges to $x \otimes (Cx)$ in norm. Since $C_{\mathcal{R}(T)}$ is norm-closed, we deduce that $x \otimes (Cx) \in C_{\mathcal{R}(T)}$. Thus $x \in \mathcal{R}(T)$. This shows that $\mathcal{R}(T) = \overline{\mathcal{R}(T)}$.

"(i)$\Longrightarrow$ (vi)". By Theorem 1.2, we have $T\mathcal{S}_C T = C_{\mathcal{R}(T)}$.

"(vi)$\Longrightarrow$ (v)". Assume that $T\mathcal{S}_C T = C_{\mathcal{M}}$ for some subspace $\mathcal{M}$ of $\mathcal{H}$. It is clear that $T\mathcal{S}_C T \subset C_{\mathcal{R}(T)}$. So $C_{\mathcal{M}} \subset C_{\mathcal{R}(T)}$. Then, for any $x \in \mathcal{M}$, we have $x \otimes (Cx) \in C_{\mathcal{M}} \subset C_{\mathcal{R}(T)}$ and $x \in \mathcal{R}(T)$. This shows $\mathcal{M} \subset \mathcal{R}(T)$.

On the other hand, if $x \in \mathcal{R}(T)$, then, by Corollary 2.4, $x \otimes (Cx) \in T\mathcal{S}_C T = C_{\mathcal{M}}$, which implies $x \in \mathcal{M}$. Hence $\mathcal{R}(T) \subset \mathcal{M}$. We conclude that $\mathcal{R}(T) = \mathcal{M}$ and $T\mathcal{S}_C T = C_{\mathcal{R}(T)}$.

"(v)$\Longrightarrow$ (i)". Assume that $C_{\mathcal{R}(T)} = A\mathcal{S}_C A$. Since $A\mathcal{S}_C A \subset C_{\mathcal{R}(A)}$, it follows that $C_{\mathcal{R}(T)} \subset C_{\mathcal{R}(A)}$. By Proposition 2.10, we have $\mathcal{R}(T) \subset \mathcal{R}(A)$.

On the other hand, if $x \in \mathcal{R}(A)$, then, by Corollary 2.4, $x \otimes (Cx) \in A\mathcal{S}_C A = C_{\mathcal{R}(T)}$, which implies $x \in \mathcal{R}(T)$. Hence $\mathcal{R}(A) \subset \mathcal{R}(T)$ and $\mathcal{R}(A) = \mathcal{R}(T)$. By Proposition 2.10, we have $A\mathcal{S}_C A = T\mathcal{S}_C T$. So $C_{\mathcal{R}(T)} = T\mathcal{S}_C T$. By Theorem 1.2, we deduce that $\mathcal{R}(T)$ is closed.

"(iv)$\Longrightarrow$ (i)". Denote $\mathcal{M} = \mathcal{R}(T)$. We choose an orthonormal basis $\{e_i\}_{i=1}^{\infty}$ of $\overline{\mathcal{M}}$. Define $U \in \mathcal{B}(\mathcal{H})$ as

$$U = \sum_{i=1}^{\infty} e_i \otimes (ce_i).$$

Thus $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry, $\mathcal{R}(U) = \overline{\mathcal{M}}$ and $\mathcal{R}(U^*) = (\ker U)^{\perp} = C(\overline{\mathcal{M}})$. It is easy to check that $U \in \mathcal{S}_C$. So $U \in C_{\overline{\mathcal{M}}}$.

By the hypothesis, we can find $\{T_n\} \subset C_{\mathcal{M}}$ such that $T_n \longrightarrow U$.

Since $\mathcal{R}(T_n) \subset \mathcal{M} \subset \overline{\mathcal{M}} = \mathcal{R}(U)$, by Lemma 2.1, we have

$$(\ker T_n)^{\perp} = \overline{\mathcal{R}(T_n^*)} \subset \mathcal{R}(U^*) = (\ker U)^{\perp} = C(\overline{\mathcal{M}}).$$

Denote $\widetilde{U} = U|_{C(\overline{\mathcal{M}})}$ and $\widetilde{T}_n = (T_n)|_{C(\overline{\mathcal{M}})}$. Thus $\widetilde{U}, \widetilde{T}_n : C(\overline{\mathcal{M}}) \to \overline{\mathcal{M}}$ are bounded linear operators and $\widetilde{U}$ is invertible.

Since $T_n \to U$, it follows that $\widetilde{T}_n \to \widetilde{U}$. By the stability of invertibility, we deduce that $\widetilde{T}_n$ is invertible for n large enough. This implies that $\mathcal{R}(T_n) = \mathcal{R}(\widetilde{T}_n) = \overline{\mathcal{M}}$ for n large enough. Since $\mathcal{R}(T_n) \subset \mathcal{M}$, we obtain $\overline{\mathcal{M}} = \mathcal{M}$. ∎

Remark 2.16 By the preceding result and Theorem 1.2, if $T \in \mathcal{S}_C$ has a non-closed range, then

$$T\mathcal{S}_C T \subsetneq C_{\mathcal{R}(T)} \subsetneq \overline{C_{\mathcal{R}(T)}} \subsetneq C_{\overline{\mathcal{R}(T)}}.$$

On the other hand, if T has a closed range, then all these sets coincide.

Corollary 2.17 *Let $\mathcal{M}$ be a subspace of $\mathcal{H}$. Then the following are equivalent:*

(i) $C_{\mathcal{M}} = T\mathcal{S}_C T$ *for some* $T \in \mathcal{S}_C$*;*
(ii) $C_{\mathcal{M}}$ *is norm closed;*
(iii) $C_{\mathcal{M}}$ *is wot closed;*
(iv) $\mathcal{M} = \overline{\mathcal{M}}$.

Proof The implications (iv)$\Longrightarrow$(iii)$\Longrightarrow$(ii) is obvious.

"(ii)$\Longrightarrow$(iv)". Using a similar argument as in the proof for (ii)$\Longrightarrow$(i) of Proposition 2.15, one can see that $\mathcal{M}$ is closed.

"(i)$\Longrightarrow$(iv)". Assume that $C_{\mathcal{M}} = T\mathcal{S}_C T$ for some $T \in \mathcal{S}_C$. From the proof for (vi)$\Longrightarrow$(v) in Proposition 2.15, one can see that $T\mathcal{S}_C T = C_{\mathcal{R}(T)}$ and $\mathcal{M} = \mathcal{R}(T)$. By Theorem 1.2, we deduce that $\mathcal{M} = \mathcal{R}(T)$ is closed.

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"(iv)$\Longrightarrow$(i)". By Lemma 2.9, we can choose an operator $T \in \mathcal{S}_C$ with $\mathcal{R}(T) = \mathcal{M}$. Then, by Theorem 1.2, $T\mathcal{S}_C T = C_{\mathcal{R}(T)} = C_{\mathcal{M}}$. ∎

3 The diagonalization in $\mathcal{S}_C$

The aim of this section is to study the diagonalization of normal operators in $\mathcal{S}_C$ and give the proof of Theorem 1.6.

3.1 Proof of Theorem 1.6

We first give a key lemma.

Lemma 3.1 *Let $T \in \mathcal{S}_C$ be a self-adjoint operator and $e \in \mathcal{H}$. Then, given $\varepsilon > 0$ and $p > 1$, there exist a finite-rank orthogonal projection $P \in \mathcal{S}_C$ with $e \in \mathcal{R}(P)$ and a finite-rank self-adjoint operator $K \in \mathcal{S}_C$ with $\|K\|_p < \varepsilon$ such that $P(T+K) = (T+K)P$.*

Proof Let E be the projection-valued spectral measure for T and assume that $\sigma(T) \subset [a,b]$. We fix a positive integer n. Then there exist pairwise disjoint Borel subsets $\Delta_1, \Delta_2, \ldots, \Delta_n$ and $a_1, a_2, \ldots, a_n \in [a,b]$ such that

$$[a,b] = \cup_{i=1}^n \Delta_i \quad \text{and} \quad \Delta_i \subset [a_i - \frac{b-a}{2n}, a_i + \frac{b-a}{2n}].$$

For $i = 1, 2, \ldots, n$, denote $\mathcal{H}_i = \mathcal{R}(E(\Delta_i))$. Then each $\mathcal{H}_i$ reduces T and $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$. Put $T_i = T|_{\mathcal{H}_i}$. Then $T = \oplus_{i=1}^n T_i$. For each i, choose $\lambda_i \in \Delta_i$. Since $\sigma(T_i) \subset \Delta_i^-$, it follows that $\|T_i - \lambda_i\| \le (b-a)/n$.

Claim *For each i with $1 \le i \le n$, $\mathcal{H}_i$ reduces C.*

Fix an i with $i = 1, \ldots, n$. It suffices to prove $CE(\Delta_i)C = E(\Delta_i)$. Note that there exists a sequence $\{p_k\}_{k=1}^\infty$ of polynomials with real coefficients such that $p_k(T) \to E(\Delta_i)$ in the strong operator topology. Since $Cp_k(T)C = p_k(T)$ for all k, it follows immediately that $CE(\Delta_i)C = E(\Delta_i)$. This proves the claim.

For each i, denote $C_i = C|_{\mathcal{H}_i}$. Then C_i is a conjugation on $\mathcal{H}_i$ and, from $CTC = T^* = T$, we obtain $C_i T_i C_i = T_i$.

Since $\mathcal{H} = \oplus_{i=1}^n \mathcal{H}_i$, we may assume that $e = \sum_{i=1}^n e_i$ with $e_i \in \mathcal{H}_i$. Then, by Claim, $Ce_i \in \mathcal{H}_i$. For each i with $1 \le i \le n$, put $\mathcal{M}_i = \vee\{e_i, Ce_i\}$. Then $\mathcal{M}_i$ is a subspace of $\mathcal{H}_i$, reducing C, and $1 \le \dim \mathcal{M}_i \le 2$. Then, relative to the decomposition $\mathcal{H}_i = \mathcal{M}_i \oplus (\mathcal{H}_i \ominus \mathcal{M}_i)$, T_i can be written as

$$T_i = \begin{array}{c} \begin{bmatrix} A_i & F_i \\ G_i & B_i \end{bmatrix} \end{array} \begin{array}{l} \mathcal{M}_i \\ \mathcal{H}_i \ominus \mathcal{M}_i \end{array}.$$

Note that $G_i = F_i^*$, since T_i is self-adjoint.

Since $C_i(\mathcal{M}_i) = \mathcal{M}_i$, relative to the decomposition $\mathcal{H}_i = \mathcal{M}_i \oplus (\mathcal{H}_i \ominus \mathcal{M}_i)$, C_i can be written as

$$C_i = \begin{array}{c} \begin{bmatrix} C_{i,1} & 0 \\ 0 & C_{i,2} \end{bmatrix} \end{array} \begin{array}{l} \mathcal{M}_i \\ \mathcal{H}_i \ominus \mathcal{M}_i \end{array}.$$

Then both $C_{i,1}$ and $C_{i,2}$ are conjugations. It follows readily from $C_iT_iC_i = T_i$ that $C_iK_iC_i = K_i$, where

$$K_i := \begin{array}{c} \begin{bmatrix} 0 & F_i \\ G_i & 0 \end{bmatrix} \end{array} \begin{array}{l} \mathcal{M}_i \\ \mathcal{H}_i \ominus \mathcal{M}_i \end{array}.$$

Clearly, K_i is self-adjoint, $\operatorname{rank} K_i \le 4$ and

$$\|K_i\| = \max\{\|F_i\|, \|G_i\|\} \le \left\| \begin{bmatrix} A_i - \lambda_i & F_i \\ G_i & B_i - \lambda_i \end{bmatrix} \right\| = \|T_i - \lambda_i\| \le (b-a)/n.$$

Set $K = -\oplus_{i=1}^n K_i$. Thus K is a self-adjoint, finite-rank operator, $CKC = K$,

$$T + K = \oplus_{i=1}^n (T_i - K_i) = \oplus_{i=1}^n (A_i \oplus B_i)$$

and

$$\|K\|_p \le (\operatorname{rank} K)^{1/p}\|K\| \le (4n)^{1/p}\frac{b-a}{n} \le \frac{4(b-a)}{n^{1/q}},$$

where $q = 1/(1-1/p)$. Thus, for any $\varepsilon > 0$, there exists n large enough such that $\|K\|_p < \varepsilon$.

Denote by P the orthogonal projection of $\mathcal{H}$ onto $\oplus_{i=1}^n \mathcal{M}_i$. Thus $e \in \mathcal{R}(P)$, $\operatorname{rank} P \le 2n$ and $(T+K)P = P(T+K)$; indeed,

$$(T+K)|_{\mathcal{R}(P)} = \oplus_{i=1}^n A_i.$$

Noting that $CPC = P$ and $CKC = K$, that are, $P, K \in \mathcal{S}_C$, we complete the proof. ∎

Now we are going to prove Theorem 1.6.

Proof of Theorem 1.6 Since C is a conjugation, we can find an orthonormal basis $\{e_n\}_{n=1}^\infty$ of $\mathcal{H}$ such that $Ce_n = e_n$ for all n.

Now fix an $\varepsilon > 0$ and $p > 1$. By Lemma 3.1, we can find a finite-rank, self-adjoint operator $K_1 \in \mathcal{S}_C$ with $\|K_1\|_p < \varepsilon/2$ and a finite-rank orthogonal projection $P_1 \in \mathcal{S}_C$ such that $e_1 \in \mathcal{R}(P_1)$ and $P_1(T + K_1) = (T + K_1)P_1$. Denote $\mathcal{H}_1 = \mathcal{R}(P_1)$ and $\widetilde{\mathcal{H}_1} = \mathcal{H} \ominus \mathcal{H}_1$. Then, with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \widetilde{\mathcal{H}_1}$,

$$T + K_1 = \begin{bmatrix} T_1 & 0 \\ 0 & \widetilde{T_1} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & 0 \\ 0 & \widetilde{C_1} \end{bmatrix}, \tag{5}$$

where $C_1T_1C_1 = T_1$ and $\widetilde{C_1}\widetilde{T_1}\widetilde{C_1} = \widetilde{T_1}$.

Now apply Lemma 3.1 again to the self-adjoint operator $\widetilde{T_1}$ and the vector $(I - P_1)e_2$ to get a finite-rank, self-adjoint operator $\widetilde{K_2} \in \mathcal{S}_{\widetilde{C_1}}$ and two subspaces $\mathcal{H}_2, \widetilde{\mathcal{H}_2}$ of $\widetilde{\mathcal{H}_1}$ such that

$$\|\widetilde{K_2}\|_p < \varepsilon/4, \ \ (I - P_1)e_2 \in \mathcal{H}_2, \ \ \dim \mathcal{H}_2 < \infty, \ \ \widetilde{\mathcal{H}_1} = \mathcal{H}_2 \oplus \widetilde{\mathcal{H}_2}$$

and with respect to which

$$\widetilde{T_1} + \widetilde{K_2} = \begin{bmatrix} T_2 & 0 \\ 0 & \widetilde{T_2} \end{bmatrix}, \quad \widetilde{C_1} = \begin{bmatrix} C_2 & 0 \\ 0 & \widetilde{C_2} \end{bmatrix},$$

where $C_2T_2C_2 = T_2$ and $\widetilde{C_2}\widetilde{T_2}\widetilde{C_2} = \widetilde{T_2}$. In view of (5), we can find a finite-rank, self-adjoint operator $K_2 \in \mathcal{S}_C$ with $\|K_2\|_p < \varepsilon/4$ such that

$$(6) \qquad T + K_1 + K_2 = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & \widetilde{T_2} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2, \\ \widetilde{\mathcal{H}_2} \end{matrix} \quad C = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & C_2 & 0 \\ 0 & 0 & \widetilde{C_2} \end{bmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2. \\ \widetilde{\mathcal{H}_2} \end{matrix}$$

Note that $e_1, e_2 \in \mathcal{H}_1 \oplus \mathcal{H}_2$, $C_iT_iC_i = T_i$ $(i = 1,2)$, and $\widetilde{C_2}\widetilde{T_2}\widetilde{C_2} = \widetilde{T_2}$.

By induction, we can find a sequence of self-adjoint, finite-rank operators $\{K_i\}_{i=1}^{\infty} \subset \mathcal{S}_C$ and a sequence of pairwise orthogonal, finite-dimensional subspaces $\{\mathcal{H}_i\}$ such that for each $n \geq 1$,

(i) $e_1, \ldots, e_n \in \oplus_{i=1}^{n}\mathcal{H}_i$,
(ii) $\|K_n\|_p < \varepsilon/2^n$, and
(iii) relative to the decomposition $\mathcal{H} = (\oplus_{i=1}^{n}\mathcal{H}_i) \oplus \widetilde{\mathcal{H}_n}$,

$$(7) \qquad T + \sum_{i=1}^{n} K_i = (\oplus_{i=1}^{n} T_i) \oplus \widetilde{T_n}, \quad C = (\oplus_{i=1}^{n} C_i) \oplus \widetilde{C_n},$$

where $\widetilde{\mathcal{H}_n} = \mathcal{H} \ominus (\oplus_{i=1}^{n}\mathcal{H}_i)$ and

$$\widetilde{C_n}\widetilde{T_n}\widetilde{C_n} = \widetilde{T_n}, \quad C_iT_iC_i = T_i, \quad i = 1, 2, \ldots, n.$$

Since $\{e_i\}$ is an orthonormal basis of $\mathcal{H}$, it follows from statement (i) that $\mathcal{H} = \oplus_{i=1}^{\infty}\mathcal{H}_i$ and hence $C = \oplus_{i=1}^{\infty}C_i$. Set $K = \sum_{i=1}^{\infty}K_i$. Then $K \in \mathcal{B}_p(\mathcal{H})$ is self-adjoint, $\|K\|_p < \varepsilon$ and $T + K = \oplus_{i=1}^{\infty}T_i$. Note that each T_i is self-adjoint and acting on a finite-dimensional space. Thus $T + K$ is diagonal. Since $C_iT_iC_i = T_i$ for all i, it follows that $C(T + K)C = T + K$. Set $D = T + K$. This completes the proof. ∎

Corollary 3.2 *Let T be a self-adjoint operator in $\mathcal{S}_C$. Then, given $\varepsilon > 0$, there exists a compact operator $K \in \mathcal{S}_C$ with $\|K\| < \varepsilon$ such that $T + K$ is diagonal with $\sigma(T + K) = \sigma(T)$.*

Proof Assume that $\sigma_p(T) = \{\lambda_i : i \in \Lambda\}$. Set

$$\mathcal{H}_0 = \vee_{i\in\Lambda} \ker(T - \lambda_i) \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H}_0^{\perp}.$$

Since T is self-adjoint, we deduce that $\mathcal{H}_0$ reduces T. Then

$$T = \begin{bmatrix} T_0 & 0 \\ 0 & T_1 \end{bmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{matrix}.$$

Clearly, T_0 is diagonal and $\sigma_p(T_1) = \varnothing$. Since $CTC = C$, one can see $C(\mathcal{H}_0) = \mathcal{H}_0$, which implies $C(\mathcal{H}_1) = \mathcal{H}_1$ and

$$C = \begin{bmatrix} C_0 & 0 \\ 0 & C_1 \end{bmatrix} \begin{matrix} \mathcal{H}_0 \\ \mathcal{H}_1 \end{matrix}.$$

It is easy to check that each C_i is a conjugation and $T_i \in \mathcal{S}_{C_i}$.

By Theorem 1.6, we can find $K_1 \in \mathcal{S}_{C_1}$ with $\|K_1\| < \frac{\varepsilon}{2}$ such that $T_1 + K_1$ is diagonal. Moreover, by the upper semi-continuity of spectrum, it can be required that $\sigma(T_1 + K_1) \subset \sigma(T_1) + B(0, \frac{\varepsilon}{2})$.

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Since $T_1 + K_1$ is diagonal, without loss of generality, we may assume that $\sigma_p(T_1 + K_1) = \{z_i : i \geq 1\}$. Denote $\mathcal{K}_i = \ker(T_1 + K_1 - z_i)$ for $i \geq 1$. Then

$$T_1 + K_1 = \oplus_{i=1}^{\infty} z_i I_i$$

relative to the decomposition $\mathcal{H}_1 = \oplus_{i=1}^{\infty} \mathcal{K}_i$, where I_i is the identity operator on $\mathcal{K}_i$. Since $T_1 + K_1 \in \mathcal{S}_C$ is diagonal, it follows that

$$C_1(\ker(T_1 + K_1 - z_i)) = \ker(T_1 + K_1 - z_i), \quad i = 1, 2, 3, \ldots.$$

Then each $\mathcal{K}_i$ reduces C_1. We may assume that $C_1 = \oplus_{i=1}^{\infty} C_i'$, where $C_i' = C_1|_{\mathcal{K}_i}$ is a conjugation on $\mathcal{K}_i$.

Fix an $i \geq 1$. If $z_i \in \sigma(T_1)$, then we set $\widetilde{z_i} = z_i$. If $z_i \notin \sigma(T_1)$, then $z_i \notin \sigma_e(T_1) = \sigma_e(T_1 + K_1)$ and hence $\dim \mathcal{K}_i < \infty$; noting that $z_i \in \sigma(T_1) + B(0, \frac{\varepsilon}{2})$, we can find $\widetilde{z_i} \in \sigma(T_1)$ such that $|z_i - \widetilde{z_i}| = \operatorname{dist}(z_i, \sigma(T_1)) < \varepsilon/2$.

Set $K_2 = \oplus_{i=1}^{\infty} (\widetilde{z_i} - z_i) I_i$. Then $K_2 \in \mathcal{S}_{C_1}$, $\|K_2\| \leq \varepsilon/2$ and

$$T_1 + K_1 + K_2 = \oplus_{i=1}^{\infty} \widetilde{z_i} I_i.$$

Clearly, $\sigma(T_1 + K_1 + K_2) = \{\widetilde{z_i} : i \geq 1\}^- \subseteq \sigma(T_1)$.

Claim *K_2 is compact.*

It suffices to show $\lim_i |z_i - \widetilde{z_i}| = 0$. Otherwise, we can choose a subsequence $\{i_k\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $z_{i_k} \notin \sigma(T_1)$, $z_{i_k} \to z_0$ and

$$\inf_k \operatorname{dist}(z_{i_k}, \sigma(T_1)) > 0.$$

Hence, $\operatorname{dist}(z_0, \sigma(T_1)) > 0$ and

$$z_0 \in \sigma_e(T_1 + K_1) = \sigma_e(T_1) \subset \sigma(T_1),$$

a contradiction.

Set $K = 0 \oplus (K_1 + K_2)$. Then $K \in \mathcal{S}_C$ is compact with $\|K\| < \varepsilon$ and

$$T + K = T_0 \oplus (T_1 + K_1 + K_2).$$

By the preceding discussion, we have

$$\sigma(T + K) = \sigma(T_0) \cup \sigma(T_1 + K_1 + K_2) \subset \sigma(T_0) \cup \sigma(T_1) = \sigma(T).$$

It remains to show $\sigma(T) \subset \sigma(T + K)$.

Note that K is compact. Hence

$$\sigma_e(T) = \sigma_e(T + K) \subset \sigma(T + K).$$

If $z \in \sigma(T) \backslash \sigma_e(T)$, then $z \in \sigma_p(T)$ and $z = \lambda_i \in \sigma(T_0) \subset \sigma(T + K)$ for some $i \in \Lambda$. Thus we have proved that $\sigma(T) \subset \sigma(T + K)$, which implies $\sigma(T) = \sigma(T + K)$. ∎

Theorem 3.3 *If $N_1, \ldots, N_n \in \mathcal{S}_C$ are commuting normal operators and $\varepsilon > 0$, then there are commuting, diagonal operators $D_1, \ldots, D_n \in \mathcal{S}_C$ such that $N_i - D_i \in \mathcal{K}(\mathcal{H})$ and $\|N_i - D_i\| < \varepsilon$ for all $i = 1, 2, \ldots, n$.*

Proof Denote by $\mathcal{A}$ the von Neumann algebra generated by $N_1, \ldots, N_n$. It is easy to see $\mathcal{A} \subset \mathcal{S}_C$. By [5, Lemma II.2.8], we can find a self-adjoint operator $A \in \mathcal{B}(\mathcal{H})$

generating $\mathcal{A}$ and $N_1, \ldots, N_n \in C^*(A)$. Clearly, $A \in \mathcal{S}_C$. We can find continuous functions $f_1, \ldots, f_n$ on $\mathbb{R}$ such that $N_i = f_i(A)$, $i = 1, \ldots, n$.

Now fix an $\varepsilon > 0$. In view of [4, Lemma 39.5], there exists $\delta > 0$ such that $\sup_{1 \le i \le n} \|f_i(X) - f_i(Y)\| < \varepsilon$ for all self-adjoint operators X, Y satisfying $\|X\| \le \|A\|$, $\|Y\| \le \|A\|$ and $\|X - Y\| < \delta$.

By Corollary 3.2, we can find self-adjoint, diagonal $D \in \mathcal{S}_C$ with $\sigma(D) = \sigma(A)$ such that $D - A \in \mathcal{K}(\mathcal{H})$ and $\|D - A\| < \delta$. Then $\|D\| = \|A\|$ and

$$\sup_{1 \le i \le n} \|N_i - f_i(D)\| = \sup_{1 \le i \le n} \|f_i(A) - f_i(D)\| < \varepsilon.$$

Note that $f_1(D), f_2(D), \ldots, f_n(D)$ are commuting diagonal operators. Also, since $D \in \mathcal{S}_C$, it follows that $f_1(D), f_2(D), \ldots, f_n(D) \in \mathcal{S}_C$.

Note that $A - D \in \mathcal{K}(\mathcal{H})$. Using the functional calculus for self-adjoint operators, one can easily prove that $f_i(A) - f_i(D) \in \mathcal{K}(\mathcal{H})$, $i = 1, 2, \ldots, n$. This completes the proof. ∎

Using a similar argument as in the proof of Corollary 3.2, one can prove the following.

Corollary 3.4 *If $T \in \mathcal{S}_C$ is normal and $\varepsilon > 0$, then there exists a diagonal operator $N \in \mathcal{S}_C$ such that $\sigma(N) = \sigma(T)$ and $\|T - N\| < \varepsilon$.*

3.2 Irreducible approximation

We conclude this paper with an application of Theorem 1.6 to the irreducible approximation in $\mathcal{S}_C$.

By a classical approximation result of P. R. Halmos [21], the set of irreducible operators is a dense G_δ set and hence a topologically large subset of $\mathcal{B}(\mathcal{H})$. H. Radjavi and P. Rosenthal [36] gave a short proof of the density of irreducible operators in $\mathcal{B}(\mathcal{H})$. It can be seen from their proof that each operator has an arbitrarily small compact perturbation which is irreducible (see [23, Lemma 4.33]).

In a recent paper [31], T. Liu, J. Y. Zhao, and the last author studied irreducible approximation of c.s. operators and obtained the following result.

Theorem 3.5 [31, Theorem 2.1] *Let $T \in \mathcal{S}_C$ and $T = A + iB$, where A, B are self-adjoint. If A or B is diagonal, then, given $\varepsilon > 0$, there exists $K \in \mathcal{S}_C$ with $\|K\| < \varepsilon$ such that $T + K$ is irreducible.*

In addition, it can be required that K lies in the Schatten p-class and $\|K\|_p < \varepsilon$. Using the preceding result, it was proved in [31] that each c.s. operator has a small perturbation being irreducible. Moreover, the following question was raised:

Is every complex symmetric operator a compact perturbation or a small compact perturbation of irreducible ones?

Using Theorems 1.6 and 3.5, one can see the following result, which combining Halmos' result shows that those irreducible ones in $\mathcal{S}_C$ constitute a dense G_δ subset of $\mathcal{S}_C$.

Corollary 3.6 *Given $T \in \mathcal{S}_C$, $p \in (1, \infty)$ and $\varepsilon > 0$, there exists a compact operator $K \in \mathcal{S}_C$ with $\|K\|_p < \varepsilon$ such that $T + K$ is irreducible.*

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