ON THE SHARPNESS OF TIAN'S CRITERION FOR K-STABILITY

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Abstract. Tian's criterion for K-stability states that a Fano variety of dimension n whose alpha invariant is greater than $n/(n+1)$ is K-stable. We show that this criterion is sharp by constructing n-dimensional singular Fano varieties with alpha invariants $n/(n+1)$ that are not K-polystable for sufficiently large n. We also construct K-unstable Fano varieties with alpha invariants $(n-1)/n$.

§1. Introduction

Given a complex Fano manifold X, Tian [Tia87] introduced the α-*invariant* $\alpha(X)$ of X which measures the integrability of exponentials of plurisubharmonic functions. In the same paper, Tian proved that X admits a Kähler–Einstein metric if $\alpha(X) > n/(n+1)$ where $n = \dim X$, known as Tian's criterion. In this article, we will use the algebraic interpretation of α-invariants in terms of singularities of pairs due to Demailly [CS08, Appendix A] (see also [Shi10, Appendix A]).

DEFINITION 1.1. ([Tia87, CS08]) Let X be a $\mathbb{Q}$-Fano variety, that is, X is a normal projective variety over $\mathbb{C}$, $-K_X$ is $\mathbb{Q}$-Cartier and ample, and X has klt singularities. The α-invariant $\alpha(X)$ is defined as

$$\alpha(X) := \inf\{\operatorname{lct}(X;D) \mid D \text{ is an effective } \mathbb{Q}\text{-divisor and } D \sim_{\mathbb{Q}} -K_X\}.$$

Tian's criterion [Tia87] has been generalized and complemented by Demailly–Kollár [DK01], Odaka–Sano [OS12] and Fujita [Fuj19a]. The following theorem summarizes their results in the language of K-stability.

THEOREM 1.2. ([Tia87, DK01, OS12, Fuj19a]) *Let X be an n-dimensional $\mathbb{Q}$-Fano variety. Then,*

(1) *X is K-stable if either $\alpha(X) > n/(n+1)$ or $\alpha(X) = n/(n+1)$, $n \geq 2$ and X is smooth;*
(2) *X is K-semistable if $\alpha(X) \geq n/(n+1)$.*

The purpose of this article is to study the sharpness of the assumptions in Theorem 1.2. Our main result goes as follows.

THEOREM 1.3. *For any $n \gg 0$ (more precisely, $n = 4$ or $n \geq 7$ for X, and $n \geq 14$ for Y), there exist a hypersurface $X \subset \mathbb{P}^{n+1}$ of degree $n+1$, and a complete intersection $Y \subset \mathbb{P}^{n+2}$ of a hyperquadric and a degree n hypersurface, such that the following properties hold:*

(1) *Both X and Y are Gorenstein canonical with only one singular point;*
(2) *$\alpha(X) = n/(n+1)$ and X is not K-polystable;*
(3) *$\alpha(Y) = (n-1)/n$ and Y is K-unstable.*

These examples show that the assumptions of Theorem 1.2 are almost sharp. More precisely, in Theorem 1.2.1, the smoothness assumption cannot be removed, and the lower bound of $\alpha(X)$ in Theorem 1.2.2 cannot be smaller than $(n-1)/n$. We remark here that using different methods, Fujita constructed a log Fano hyperplane arrangement (X, Δ) that is not K-polystable and $\alpha(X, \Delta) = n/(n+1)$ (see [Fuj17, Section 9]).

Our construction is motivated by the work [Fuj19a]. It follows from the argument of [Fuj19a] that if X is a $\mathbb{Q}$-Fano variety of dimension n such that $\alpha(X) = n/(n+1)$ but X is not K-stable, then X has a weakly exceptional singularity whose corresponding Kollár component has log discrepancy n. Candidates of such singularities have been proposed by Kudryavtsev [Kud01] and Prokhorov [Pro00], although both of their constructions contain a gap (see Remark 4.12). Therefore, a large part of our argument is devoted to rectifying their construction, which eventually reduces to estimating the global log canonical thresholds of general hypersurfaces or complete intersections and relies heavily on Pukhlikov's technique of hypertangent divisors (see, e.g., [Puk13, Section §3]). As a consequence of this analysis, we also prove the following:

THEOREM 1.4.   *Fix $\epsilon > 0$. Then for $n \gg 0$, we have $\mathrm{lct}(X; |H|_{\mathbb{Q}}) = 1$ where $X \subseteq \mathbb{P}^{n+1}$ is a general hypersurface of degree $d \leq (2-\epsilon)n$ and H is the hyperplane class.*

If X is a non-K-stable $\mathbb{Q}$-Fano variety of dimension n with $\alpha(X) = n/(n+1)$, then Fujita's characterization [Fuj19a] also implies that X specially degenerates to a K-semistable $\mathbb{Q}$-Fano variety X_0 with $\alpha(X_0) = 1/(n+1)$ (see Corollary 3.2). Indeed X_0 has the smallest α-invariant among all K-semistable $\mathbb{Q}$-Fano varieties by [FO18, Theorem 3.5]. These $\mathbb{Q}$-Fano varieties have been studied by Jiang [Jia17b] where he showed that $\mathbb{P}^n$ is the only K-semistable Fano manifold with the smallest α-invariant. We provide a full characterization of such $\mathbb{Q}$-Fano varieties to complement Jiang's results (see Theorem 3.8).

The following result is inspired by the work of Blum and Xu in [BX19] where it is shown that the moduli functor of uniformly K-stable $\mathbb{Q}$-Fano varieties with fixed dimension and volume is represented by a separated Deligne–Mumford stack of finite type.

THEOREM 1.5.   *Let n be a positive integer and V be a positive rational number. Then the moduli functor of $\mathbb{Q}$-Fano varieties X satisfying $\alpha(X) > \frac{1}{2}$ of dimension n and volume V is represented by a separated Deligne–Mumford stack of finite type, which has a coarse moduli space that is a separated algebraic space. In particular, $\mathrm{Aut}(X)$ is finite for a $\mathbb{Q}$-Fano variety X satisfying $\alpha(X) > \frac{1}{2}$.*

We ask the following question about the sharpness of Tian's criterion and Jiang's conjecture [Jia17b, Conjecture 1.6].

QUESTION 1.6.   *Let $n = \dim X \geq 2$ be an integer.*

(1) *Does there exist a K-semistable $\mathbb{Q}$-Fano variety X such that $1/(n+1) < \alpha(X) < \frac{1}{n}$?*
(2) *Does there exist a K-unstable $\mathbb{Q}$-Fano variety X such that $(n-1)/n < \alpha(X) < n/(n+1)$?*

This paper is organized as follows. In Section 2, we collect some preliminary materials on K-stability, weakly exceptional singularities, Kollár components, and orbifold cones. In Section 3, we recall Fujita's characterization on non-K-stable $\mathbb{Q}$-Fano varieties X of dimension n with $\alpha(X) = n/(n+1)$. In Section 3, we prove Theorem 1.5. We also provide

a generalization of [Jia17b, Theorem 1.2] in Theorem 3.8. In Section 4, we estimate the global log canonical thresholds of general hypersurfaces or complete intersections based on Pukhlikov's work [Puk05, Puk13, Puk18]. Finally in Section 5, we present our main constructions to prove Theorem 1.3.

§2. Preliminary

2.1 Notation and conventions

We work over the complex numbers. We follow the terminologies in [KM98]. A projective variety X is $\mathbb{Q}$-*Fano* if X has klt singularities and $-K_X$ is ample. A pair (X, Δ) is *log Fano* if X is projective, $-K_X - \Delta$ is $\mathbb{Q}$-Cartier ample and (X, Δ) is klt. If (X, Δ) is a klt pair, X is projective and H is an ample $\mathbb{Q}$-divisor on X, then the *global log canonical threshold* $\mathrm{lct}(X, \Delta; |H|_\mathbb{Q})$ is defined as the largest $t > 0$ such that $(X, \Delta + tD)$ is lc for every effective $\mathbb{Q}$-divisor $D \sim_\mathbb{Q} H$. If $\Delta = 0$, we simply write $\mathrm{lct}(X; |H|_\mathbb{Q})$. The α-*invariant* of a log Fano pair (X, Δ) is defined as $\alpha(X, \Delta) := \mathrm{lct}(X, \Delta; |-K_X - \Delta|_\mathbb{Q})$.

2.2 K-stability

DEFINITION 2.1. ([Tia97, Don02, LX14, Li15, OS15]) Let (X, Δ) be an n-dimensional log Fano pair. Let L be an ample line bundle on X such that $L \sim_\mathbb{Q} -l(K_X + \Delta)$ for some $l \in \mathbb{Q}_{>0}$.

(1) A *normal test configuration* $(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L})/\mathbb{A}^1$ of $(X, \Delta; L)$ consists of the following data:

- a normal variety $\mathcal{X}$, an effective $\mathbb{Q}$-divisor Δ_tc on $\mathcal{X}$, together with a flat projective morphism $\pi : (\mathcal{X}, \mathrm{Supp}(\Delta_\mathrm{tc})) \to \mathbb{A}^1$;
- a π-ample line bundle $\mathcal{L}$ on $\mathcal{X}$;
- a $\mathbb{G}_m$-action on $(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L})$ such that π is $\mathbb{G}_m$-equivariant with respect to the standard action of $\mathbb{G}_m$ on $\mathbb{A}^1$ via multiplication;
- $(\mathcal{X} \setminus \mathcal{X}_0, \Delta_\mathrm{tc}|_{\mathcal{X} \setminus \mathcal{X}_0}; \mathcal{L}|_{\mathcal{X} \setminus \mathcal{X}_0})$ is $\mathbb{G}_m$-equivariantly isomorphic to $(X, \Delta; L) \times (\mathbb{A}^1 \setminus \{0\})$.

  A normal test configuration is called a *product* test configuration if
  $$(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L}) \cong (X \times \mathbb{A}^1, \Delta \times \mathbb{A}^1; \mathrm{pr}_1^* L \otimes \mathrm{pr}_2^* \mathcal{O}_{\mathbb{A}^1}(k \cdot 0))$$
  for some $k \in \mathbb{Z}$. A product test configuration is called a *trivial* test configuration if the above isomorphism is $\mathbb{G}_m$-equivariant with respect to the trivial $\mathbb{G}_m$-action on X and the standard $\mathbb{G}_m$-action on $\mathbb{A}^1$ via multiplication.

  A normal test configuration $(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L})$ is called a *special test configuration* if $\mathcal{L} \sim_\mathbb{Q} -l(K_{\mathcal{X}/\mathbb{A}^1} + \Delta_\mathrm{tc})$ and $(\mathcal{X}, \mathcal{X}_0 + \Delta_\mathrm{tc})$ is plt. In this case, we say that (X, Δ) *specially degenerates* to $(\mathcal{X}_0, \Delta_{\mathrm{tc},0})$ which is necessarily a log Fano pair.

(2) Assume $\pi : (\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L}) \to \mathbb{A}^1$ is a normal test configuration of $(X, \Delta; L)$. Let $\bar{\pi} : (\overline{\mathcal{X}}, \overline{\Delta}_\mathrm{tc}; \overline{\mathcal{L}}) \to \mathbb{P}^1$ be the natural $\mathbb{G}_m$-equivariant compactification of π. The *generalized Futaki invariant* of $(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L})$ is defined by the intersection formula
$$\mathrm{Fut}(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L}) := \frac{1}{(-K_X - \Delta)^n} \left(\frac{n}{n+1} \cdot \frac{(\overline{\mathcal{L}}^{n+1})}{l^{n+1}} + \frac{(\overline{\mathcal{L}}^n \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\Delta}_\mathrm{tc}))}{l^n} \right).$$

(3) - The log Fano pair (X, Δ) is said to be *K-semistable* if $\mathrm{Fut}(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L}) \geq 0$ for any normal test configuration $(\mathcal{X}, \Delta_\mathrm{tc}; \mathcal{L})/\mathbb{A}^1$ and any $l \in \mathbb{Q}_{>0}$ such that L is Cartier.

- The log Fano pair (X, Δ) is said to be *K-stable* if it is K-semistable and $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) = 0$ for a normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})/\mathbb{A}^1$ if and only if it is a trivial test configuration.
- The log Fano pair (X, Δ) is said to be *K-polystable* if it is K-semistable and $\mathrm{Fut}(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L}) = 0$ for a normal test configuration $(\mathcal{X}, \Delta_{\mathrm{tc}}; \mathcal{L})/\mathbb{A}^1$ if and only if it is a product test configuration.
- The log Fano pair (X, Δ) is said to be *strictly K-semistable* if it is K-semistable but not K-polystable.
- The log Fano pair (X, Δ) is said to be *K-unstable* if it is not K-semistable.

By the work of Li and Xu [LX14], to test K-(poly/semi)stability of a log Fano pair (X, Δ) it suffices to test on special test configurations (see [Fuj19b, Section 6] for precise statements).

Next we recall the valuative criterion of K-stability due to Fujita [Fuj17] and Li [Li17].

DEFINITION 2.2. ([Fuj19b, Definition 1.1 and 1.3]) Let X be a $\mathbb{Q}$-Fano variety of dimension n. Let F be a prime divisor over X, that is, there exists a projective birational morphism $\pi : Y \to X$ with Y normal such that F is a prime divisor on Y.

(1) For any $t \geq 0$, we define $\mathrm{vol}_X(-K_X - tF) := \mathrm{vol}_Y(-\pi^* K_X - tF)$.
(2) The *pseudo-effective threshold* $\tau(F)$ of F with respect to $-K_X$ is defined as

$$\tau(F) := \sup\{\tau > 0 \,|\, \mathrm{vol}_X(-K_X - \tau F) > 0\}.$$

(3) Let $A_X(F)$ be the log discrepancy of F with respect to X. We set

$$\beta(F) := A_X(F) \cdot ((-K_X)^n) - \int_0^{\tau(F)} \mathrm{vol}_X(-K_X - tF) \mathrm{d}t.$$

(4) The prime divisor F over X is said to be *dreamy* if the $\mathbb{Z}_{\geq 0}^2$-graded algebra

$$\bigoplus_{k,j \in \mathbb{Z}_{\geq 0}} H^0(X, \mathcal{O}_X(-klK_X - jF))$$

is finitely generated for some $l \in \mathbb{Z}_{>0}$ with $-lK_X$ Cartier. Note that this definition does not depend on the choice of l.

The following theorem summarizes results from [Fuj19b, Theorems 1.3 and 1.4], [Li17, Theorem 3.7], and [BX19, Corollary 4.3].

THEOREM 2.3. ([Fuj19b, Li17, BX19]) *Let X be a $\mathbb{Q}$-Fano variety. Then the following are equivalent:*

(1) *X is K-stable (resp. K-semistable);*
(2) *$\beta(F) > 0$ (resp. $\beta(F) \geq 0$) holds for every prime divisor F over X;*
(3) *$\beta(F) > 0$ (resp. $\beta(F) \geq 0$) holds for every dreamy prime divisor F over X.*

2.3 Weakly exceptional singularities and Kollár components

DEFINITION 2.4. Let $x \in X$ be a klt singularity. We say that a proper birational morphism $\sigma : Y \to X$ provides a *Kollár component* F, if σ is an isomorphism over $X \setminus \{x\}$ and $\sigma^{-1}(x)$ is a prime divisor F on Y such that (Y, F) is plt and $-F$ is $\mathbb{Q}$-Cartier σ-ample. We denote by Δ_F the different $\mathbb{Q}$-divisor on F, that is, $K_F + \Delta_F = (K_Y + F)|_F$.

DEFINITION 2.5. Let X be a $\mathbb{Q}$-Fano variety. Let $x \in X$ be a closed point. Suppose $\sigma: Y \to X$ provides a Kollár component F over $x \in X$. Then the *Seshadri constant* $\epsilon(F)$ of F with respect to $-K_X$ is defined as

$$\epsilon(F) := \sup\{\epsilon > 0 \,|\, \sigma^*(-K_X) - \epsilon F \text{ is ample}\}.$$

The notion of weakly exceptional singularities in the sense of Shokurov is crucial in our construction.

DEFINITION 2.6. ([Pro00, Definition 4.1]) A klt singularity $x \in X$ is said to be *weakly exceptional* if there exists only one Kollár component over it.

The following criterion connects weakly exceptional singularities with α-invariant of Kollár components. Its present form first appeared in [CS11, Theorem 3.10] which essentially follows from [Pro00, Theorem 4.3] and [Kud01, Theorem 2.1].

THEOREM 2.7. ([CS11, Theorem 3.10]) *A klt singularity $x \in X$ is weakly exceptional if and only if there exists a Kollár component F over $x \in X$ satisfying $\alpha(F, \Delta_F) \geq 1$.*

2.4 Orbifold cones

DEFINITION 2.8. Let V be a normal projective variety. Let M be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V.

(1) The *affine orbifold cone* $C_a(V, M)$ is defined as

$$C_a(V, M) := \operatorname{Spec} \bigoplus_{m=0}^{\infty} H^0(V, \mathcal{O}_V(\lfloor mM \rfloor)).$$

(2) The *projective orbifold cone* $C_p(V, M)$ is defined as

$$C_p(V, M) := \operatorname{Proj} \bigoplus_{m=0}^{\infty} \bigoplus_{i=0}^{\infty} H^0(V, \mathcal{O}_V(\lfloor mM \rfloor)) \cdot s^i,$$

where the grading of $H^0(V, \mathcal{O}_V(\lfloor mM \rfloor))$ and s are m and 1, respectively.

For the projective orbifold cone $C_p(V, M)$, we denote the $\mathbb{Q}$-Cartier divisor corresponding to $(s = 0)$ by V_∞. If $\{M\} = \sum_{i=1}^k \frac{a_i}{b_i} M_i$ for some prime divisors M_i on V and $0 < a_i < b_i$ coprime integers, we denote $\Delta_M := \sum_{i=1}^k \frac{b_i - 1}{b_i} M_i$. The pair $(V_\infty, \operatorname{Diff}_{V_\infty}(0))$ obtained by taking adjunction of the pair $(C_p(V, M), V_\infty)$ is isomorphic to (V, Δ_M).

Let us illustrate some basic properties of orbifold cones from [Kol04] and [LL19, Section 3.3.1].

PROPOSITION 2.9. *Let V be a normal projective variety. Let M be an ample $\mathbb{Q}$-Cartier divisor on V.*

(1) *Both $C_a(V, M)$ and $C_p(V, M)$ are normal.*
(2) *The following conditions are equivalent:*

    (i)   *$C_a(V, M)$ is klt;*
    (ii) *$(C_p(V, M), V_\infty)$ is plt;*
    (iii) *(V, Δ_M) is a log Fano pair, and $M \sim_{\mathbb{Q}} -r^{-1}(K_V + \Delta_M)$ for some $r \in \mathbb{Q}_{>0}$.*

(3) *If the conditions in (2) are satisfied, then $K_{C_p(V, M)} \sim_{\mathbb{Q}} -(1 + r)V_\infty$.*

The following proposition has appeared in [LX20, Section 2.4]. Here we provide a proof without using stack constructions.

PROPOSITION 2.10. *Let $x \in (X, \Delta)$ be a klt singularity. Suppose $\sigma : Y \to X$ provides a Kollár component F over x. Denote by $\mathfrak{a}_j := \{ f \in \mathcal{O}_{X,x} \mid \mathrm{ord}_F(f) \geq j \}$ for $j \in \mathbb{Z}$. Then we have an isomorphism of graded rings*

$$\bigoplus_{j=0}^{\infty} \mathfrak{a}_j / \mathfrak{a}_{j+1} \cong \bigoplus_{j=0}^{\infty} H^0(F, \mathcal{O}_F(\lfloor -jF|_F \rfloor)),$$

where $F|_F$ is the $\mathbb{Q}$-divisor class on F defined in [HLS19, Definition A.4].

Proof. Since X is normal, we have $\mathfrak{a}_j = \sigma_* \mathcal{O}_Y(-jF)$. Thus we have the following exact sequence

$$0 \to \mathfrak{a}_{j+1} \to \mathfrak{a}_j \to \sigma_*(\mathcal{O}_Y(-jF)/\mathcal{O}_Y(-(j+1)F)) \to R^1\sigma_* \mathcal{O}_Y(-(j+1)F).$$

Since

$$-(j+1)F - (K_Y + \sigma_*^{-1}\Delta) = -\sigma^*(K_X + \Delta) - (j + A_{(X,\Delta)}(\mathrm{ord}_F))F,$$

the divisor $-(j+1)F - (K_Y + \sigma_*^{-1}\Delta)$ is σ-ample whenever $j \geq 0$. Hence $R^1\sigma_* \mathcal{O}_Y(-(j+1)F) = 0$ whenever $j \geq 0$. Thus for every $j \geq 0$ we obtain a canonical isomorphism

$$\mathfrak{a}_j / \mathfrak{a}_{j+1} \cong \sigma_*(\mathcal{O}_Y(-jF)/\mathcal{O}_Y(-(j+1)F)).$$

Since $(Y, F + \sigma_*^{-1}\Delta)$ is plt and F is $\mathbb{Q}$-Cartier on Y, the sheaf $\mathcal{O}_Y(-jF)$ is Cohen–Macaulay by [KM98, Corollary 5.25]. Hence the sheaf $\mathcal{O}_Y(-jF)/\mathcal{O}_Y(-(j+1)F)$ is also Cohen–Macaulay whose support is F. By [HLS19, Lemma A.3], we know that over an open set F° of F with $\mathrm{codim}_F F \setminus F^\circ \geq 2$, there is a canonical isomorphism between $(\mathcal{O}_Y(-jF)/\mathcal{O}_Y(-(j+1)F))|_{F^\circ}$ and $\mathcal{O}_{F^\circ}(\lfloor -jF|_F \rfloor)$. Thus this isomorphism extends to a canonical isomorphism

$$\mathcal{O}_Y(-jF)/\mathcal{O}_Y(-(j+1)F) \cong \mathcal{O}_F(\lfloor -jF|_F \rfloor).$$

This implies $\mathfrak{a}_j / \mathfrak{a}_{j+1} \cong H^0(F, \mathcal{O}_F(\lfloor -jF|_F \rfloor))$ for every $j \geq 0$. Since all the isomorphisms above are canonical, they are all compatible with the obvious product structure of the graded ring. Hence we finish the proof.

The next result is a slight generalization of [LX20, Proposition 5.3] (see [LL19, Proposition 3.5] for an analogous result in terms of conical Kähler–Einstein metrics).

PROPOSITION 2.11. *Let (V, Δ) be an $(n-1)$-dimensional log Fano pair where Δ has standard coefficients, that is, its coefficients belong to $\{1 - \frac{1}{m} \mid m \in \mathbb{Z}_{>0}\}$. Let M be an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V satisfying $M \sim_{\mathbb{Q}} -r^{-1}(K_V + \Delta)$ for some $0 < r \leq n$ and $\Delta_M = \Delta$. Denote by $X := C_p(V, M)$. Then*

(1) *the pair $(X, (1 - \frac{r}{n})V_\infty)$ is K-semistable if and only if (V, Δ) is K-semistable;*
(2) *the pair $(X, (1 - \frac{r}{n})V_\infty)$ is K-polystable if and only if (V, Δ) is K-polystable.*

Proof. For the "if" part of (1), we follow the proof of [LX20, Proposition 5.3]. Let $\pi : (\mathcal{X}, (1 - \frac{r}{n})\mathcal{V}; \mathcal{L}) \to \mathbb{A}^1$ be a special test configuration of $(X, (1 - \frac{r}{n})V_\infty; L)$ where

$L \sim_{\mathbb{Q}} -l(K_X + (1 - \frac{r}{n})V_\infty)$ is an ample line bundle on X. By definition, we have

$$\mathrm{Fut}(\mathcal{X}, (1-\tfrac{r}{n})\mathcal{V}; \mathcal{L}) = -\frac{(-K_{\overline{\mathcal{X}}/\mathbb{P}^1} - (1-\frac{r}{n})\overline{\mathcal{V}})^{n+1}}{(n+1)(-K_X - (1-\frac{r}{n})V_\infty)^n}.$$

Since $(1+r)V_\infty \sim_{\mathbb{Q}} -K_X$, we know that there exists $k \in \mathbb{Q}$ such that

$$(1+r)\overline{\mathcal{V}} \sim_{\mathbb{Q}} -K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \pi^* \mathcal{O}_{\mathbb{P}^1}(k).$$

Therefore,

$$\mathrm{Fut}(\mathcal{X}, (1-\tfrac{r}{n})\mathcal{V}; \mathcal{L}) = -\frac{(\frac{n+1}{n}r\overline{\mathcal{V}} - \pi^* \mathcal{O}_{\mathbb{P}^1}(k))^{n+1}}{(n+1)(\frac{n+1}{n}rV_\infty)^n} = k - \frac{r(\overline{\mathcal{V}}^{n+1})}{n(V_\infty^n)}.$$

Denote by $\mathcal{V}^\nu$ and $\overline{\mathcal{V}}^\nu$ be the normalization of $\mathcal{V}$ and $\overline{\mathcal{V}}$, respectively. Then by adjunction, we have $(K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\mathcal{V}})|_{\overline{\mathcal{V}}^\nu} = K_{\overline{\mathcal{V}}^\nu/\mathbb{P}^1} + \Delta_{\overline{\mathcal{V}}^\nu}$ where $\Delta_{\overline{\mathcal{V}}^\nu} \geq 0$ is the different. Denote by $\Delta_{\mathcal{V}^\nu} := \Delta_{\overline{\mathcal{V}}^\nu}|_{\mathcal{V}^\nu}$. Since $(\overline{\mathcal{X}}, \overline{\mathcal{V}}) \times_{\mathbb{P}^1} (\mathbb{P}^1 \setminus \{0\}) \cong (X, V) \times (\mathbb{P}^1 \setminus \{0\})$, we know that $\Delta_{\overline{\mathcal{V}}^\nu}|_{\overline{\mathcal{V}}^\nu \setminus \overline{\mathcal{V}}_0^\nu}$ corresponds to $\Delta \times (\mathbb{P}^1 \setminus \{0\})$. Denote by Δ_{tc} the Zariski closure of $\Delta \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{V}^\nu$. Then clearly $\Delta_{\mathcal{V}^\nu} \geq \Delta_{\mathrm{tc}}$ and $(\mathcal{V}^\nu, \Delta_{\mathrm{tc}})/\mathbb{A}^1$ is a normal test configuration of (V, Δ). It is clear that $L|_{V_\infty} \sim_{\mathbb{Q}} \frac{(n+1)l}{n}(-K_V - \Delta)$. Denote by $l' := \frac{(n+1)l}{n}$. Then

$$\mathrm{Fut}(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu}) = \frac{1}{(-K_V - \Delta)^{n-1}} \left(\frac{n-1}{n} \cdot \frac{(\overline{\mathcal{L}}|_{\overline{\mathcal{V}}^\nu}^n)}{l'^n} + \frac{(\overline{\mathcal{L}}|_{\overline{\mathcal{V}}^\nu}^{n-1} \cdot (K_{\overline{\mathcal{V}}^\nu/\mathbb{P}^1} + \overline{\Delta}_{\mathrm{tc}}))}{l'^{n-1}} \right)$$

$$\leq \frac{1}{(-K_V - \Delta)^{n-1}} \left(\frac{n-1}{n} \cdot \frac{(\overline{\mathcal{L}}|_{\overline{\mathcal{V}}^\nu}^n)}{l'^n} + \frac{(\overline{\mathcal{L}}|_{\overline{\mathcal{V}}^\nu}^{n-1} \cdot (K_{\overline{\mathcal{V}}^\nu/\mathbb{P}^1} + \Delta_{\overline{\mathcal{V}}^\nu}))}{l'^{n-1}} \right)$$

$$= \frac{1}{(-K_V - \Delta)^{n-1}} \left(\frac{n-1}{n} \cdot \frac{(\overline{\mathcal{L}}^n \cdot \overline{\mathcal{V}})}{l'^n} + \frac{(\overline{\mathcal{L}}^{n-1} \cdot \overline{\mathcal{V}} \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\mathcal{V}}))}{l'^{n-1}} \right).$$

Since $-K_{\overline{\mathcal{X}}/\mathbb{P}^1} \sim_{\mathbb{Q}} (1+r)\overline{\mathcal{V}} - \pi^* \mathcal{O}_{\mathbb{P}^1}(k)$ and $\overline{\mathcal{L}} \sim_{\mathbb{Q}} l'r\overline{\mathcal{V}} - \pi^* \mathcal{O}_{\mathbb{P}^1}(lk)$, we have

$$\frac{(\overline{\mathcal{L}}^n \cdot \overline{\mathcal{V}})}{l'^n} = r^n(\overline{\mathcal{V}}^{n+1}) - \frac{n^2}{n+1}r^{n-1}(\overline{\mathcal{V}}^n \cdot \pi^* \mathcal{O}_{\mathbb{P}^1}(k))$$

and

$$\frac{(\overline{\mathcal{L}}^{n-1} \cdot \overline{\mathcal{V}} \cdot (K_{\overline{\mathcal{X}}/\mathbb{P}^1} + \overline{\mathcal{V}}))}{l'^{n-1}} = -r^n(\overline{\mathcal{V}}^{n+1}) + \frac{n^2+1}{n+1}r^{n-1}(\overline{\mathcal{V}}^n \cdot \pi^* \mathcal{O}_{\mathbb{P}^1}(k)).$$

Since $(V_\infty^n) = (M^{n-1}) = (-K_V - \Delta)^{n-1}/r^{n-1}$ and $(\overline{\mathcal{V}}^n \cdot \mathcal{O}_{\mathbb{P}^1}(k)) = k(V_\infty^n)$, we have

$$\mathrm{Fut}(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu}) \leq \frac{1}{r^{n-1}(V_\infty^n)} \left(-\frac{r^n}{n}(\overline{\mathcal{V}}^{n+1}) + r^{n-1}k(V_\infty^n) \right) = k - \frac{r(\overline{\mathcal{V}}^{n+1})}{n(V_\infty^n)}.$$

In particular, $\mathrm{Fut}(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu}) \leq \mathrm{Fut}(\mathcal{X}, (1-\frac{r}{n})\mathcal{V}; \mathcal{L})$ with equality if and only if $\Delta_{\mathcal{V}^\nu} = \Delta_{\mathrm{tc}}$. Thus we finish proving the "if" part of (1).

For the "if" part of (2), let us take the special test configuration $(\mathcal{X}, (1-\frac{r}{n})\mathcal{V}); \mathcal{L})/\mathbb{A}^1$ such that $(\mathcal{X}_0, (1-\frac{r}{n})\mathcal{V}_0)$ is K-polystable and $\mathrm{Fut}(\mathcal{X}, (1-\frac{r}{n})\mathcal{V}; \mathcal{L}) = 0$. Such a special test configuration always exists due to [LWX18, Theorem 1.3]. Since (V, Δ) is K-polystable and $\mathrm{Fut}(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu}) \leq \mathrm{Fut}(\mathcal{X}, (1-\frac{r}{n})\mathcal{V}; \mathcal{L})$, we know that $\mathrm{Fut}(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu}) = 0$. This implies that $(\mathcal{V}^\nu, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}^\nu})$ is a product test configuration of (V, Δ) and $\Delta_{\mathcal{V}^\nu} = \Delta_{\mathrm{tc}}$. By inversion

of adjunction, the pair $(\mathcal{X}, \mathcal{V})$ is plt and $\mathcal{V}^\nu = \mathcal{V}$. In particular, $(\mathcal{X}_0, \mathcal{V}_0)$ is plt with different divisor $\Delta_{\mathrm{tc},0}$ and $(\mathcal{V}_0, \Delta_{\mathrm{tc},0}) \cong (V, \Delta)$. Let us take a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor $\mathcal{M} \sim \mathcal{V}|_{\mathcal{V}}$ on $\mathcal{V}$ whose support does not contain $\mathcal{V}_0$ such that $\mathcal{M}|_{\mathcal{V} \setminus \mathcal{V}_0} \cong M \times (\mathbb{A}^1 \setminus \{0\})$. Thus $\mathcal{M} \cong M \times \mathbb{A}^1$ which implies $(\mathcal{V}_0, \mathcal{M}_0) \cong (V, M)$, where $\mathcal{M}_0 \sim \mathcal{V}_0|_{\mathcal{V}_0}$. Then by Lemma 2.12, we know that there exists a special test configuration of $(\mathcal{X}_0, (1 - \frac{r}{n})\mathcal{V}_0)$ with central fiber isomorphic to $(X, (1 - \frac{r}{n})V_\infty)$ and vanishing generalized Futaki invariant. Since $(\mathcal{X}_0, (1 - \frac{r}{n})\mathcal{V}_0)$ is K-polystable, by definition we have $(\mathcal{X}_0, (1 - \frac{r}{n})\mathcal{V}_0) \cong (X, (1 - \frac{r}{n})V_\infty)$ which finishes proving the "if" part of (2).

For the "only if" parts of (1) and (2), suppose $(\mathcal{V}, \Delta_{\mathrm{tc}}; \mathcal{L}_{\mathcal{V}})$ is a special test configuration of (V, Δ). We will construct a special test configuration $(\mathcal{X}, (1 - \frac{r}{n})\mathcal{V}; \mathcal{L})/\mathbb{A}^1$ with the same generalized Futaki invariant. The construction goes as follows. First, let us take $\mathcal{M}$ to be the closure of $M \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{V}$. Since $\mathcal{V}_0$ is irreducible and Cartier on $\mathcal{V}$, we know that $\mathcal{M}$ is $\mathbb{Q}$-Cartier and $\mathbb{Q}$-linearly equivalent to $-r^{-1}(K_{\mathcal{V}/\mathbb{A}^1} + \Delta_{\mathrm{tc}})$. Denote by $\pi : \mathcal{V} \to \mathbb{A}^1$. Then let us consider the following variety

$$\mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1} \bigoplus_{m=0}^\infty \bigoplus_{i=0}^\infty \pi_* \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \cdot s^i,$$

where the grading of $\pi_* \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor)$ and s are m and 1, respectively. We denote the divisor on $\mathcal{X}$ corresponding to $(s = 0)$ by $\mathcal{V}_\infty$. It is clear that $\mathcal{X}$ is normal. We will show that $\mathcal{V}_\infty \cong \mathcal{V}$ and $\mathcal{X}_0 \cong C_p(\mathcal{V}_0, \mathcal{M}_0)$.

Since $\mathcal{V}_\infty$ is defined by $(s = 0)$, we have $\mathcal{V}_\infty = \mathrm{Proj}_{\mathbb{A}^1} \oplus_{i=0}^\infty \pi_* \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \cdot s^0 \cong \mathcal{V}$. Consider $\{M\} = \sum_{j=1}^k \frac{a_j}{b_j} M_j$ where M_j are distinct prime divisors on V and $0 < a_j < b_j$ are coprime integers. Then $\Delta = \Delta_M = \sum_{j=1}^k \frac{b_j - 1}{b_j} M_j$. In particular, all nonzero coefficients of Δ are at least $\frac{1}{2}$. Let $\mathcal{M}_j$ be the Zariski closure of M_j in $\mathcal{V}$. Since $(\mathcal{V}_0, \Delta_{\mathrm{tc},0})$ is klt, we know that $\mathcal{M}_{j,0}$ and $\mathcal{M}_{j',0}$ do not any have component in common for $j \neq j'$. As a result, we have $\lfloor m\mathcal{M} \rfloor|_{\mathcal{V}_0} = \lfloor m\mathcal{M}_0 \rfloor$ as $\mathbb{Z}$-divisors on $\mathcal{V}_0$. Let us consider the homomorphisms

$$\pi_* \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \otimes \kappa(0) \xrightarrow{g_m} H^0(\mathcal{V}_0, \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \otimes \mathcal{O}_{\mathcal{V}_0}) \xrightarrow{h_m} H^0(\mathcal{V}_0, \mathcal{O}_{\mathcal{V}_0}(\lfloor m\mathcal{M}_0 \rfloor)).$$

Denote the composition by $f_m := h_m \circ g_m$. Since $\mathcal{M}$ is π-ample, by Serre vanishing there exists $m_0 \in \mathbb{Z}_{>0}$ such that g_m is an isomorphism for any $m \geq m_0$. We also know that $\mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \otimes \mathcal{O}_{\mathcal{V}_0}$ and $\mathcal{O}_{\mathcal{V}_0}(\lfloor m\mathcal{M}_0 \rfloor)$ agree over the smooth locus of $\mathcal{V}_0$. Let coherent sheaves $\mathcal{G}_m$ and $\mathcal{G}'_m$ on $\mathcal{V}_0$ be the kernel and cokernel of the morphism $\mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \otimes \mathcal{O}_{\mathcal{V}_0} \to \mathcal{O}_{\mathcal{V}_0}(\lfloor m\mathcal{M}_0 \rfloor)$. Since $\mathcal{V}_0$ is normal, we know that both supports of $\mathcal{G}_m$ and $\mathcal{G}'_m$ have dimension at most $n - 3$. Meanwhile, if $m_1 \mathcal{M}$ is Cartier then for any $k \in \mathbb{Z}_{>0}$, we have

$$\mathcal{G}_{m+km_1} = \mathcal{G}_m \otimes \mathcal{O}_{\mathcal{V}_0}(km_1 \mathcal{M}_0), \quad \mathcal{G}'_{m+km_1} = \mathcal{G}'_m \otimes \mathcal{O}_{\mathcal{V}_0}(km_1 \mathcal{M}_0).$$

Thus we have $\dim \ker(f_m) = O(m^{n-3})$ and $\dim \mathrm{coker}(f_m) = O(m^{n-3})$. It is also clear that f_m is an isomorphism whenever $m \geq m_0$ and $m_1 \mid m$. For simplicity, we may assume $m_1 \geq m_0$.

Next, we show that (f_m) induces a morphism $\phi : C_p(\mathcal{V}_0, \mathcal{M}_0) \to \mathcal{X}_0$ that is finite, birational, and isomorphic outside a codimension 2 subset. For simplicity, denote by $P_m := \pi_* \mathcal{O}_{\mathcal{V}}(\lfloor m\mathcal{M} \rfloor) \otimes \kappa(0)$ and $P'_m := H^0(\mathcal{V}_0, \mathcal{O}_{\mathcal{V}_0}(\lfloor m\mathcal{M}_0 \rfloor))$. Then we know that

$$\mathcal{X}_0 = \mathrm{Proj} \bigoplus_{m=0}^\infty \bigoplus_{i=0}^\infty P_m \cdot s^i, \qquad C_p(\mathcal{V}_0, \mathcal{M}_0) = \mathrm{Proj} \bigoplus_{m=0}^\infty \bigoplus_{i=0}^\infty P'_m \cdot s^i.$$

Here the gradings of P_m, P'_m, and s are m, m, and 1, respectively. From the definition of $f_m : P_m \to P'_m$, we see that it induces ring homomorphism $\psi : \oplus_{(m,i) \in \mathbb{Z}^2_{\geq 0}} P_m \cdot s^i \to \oplus_{(m,i) \in \mathbb{Z}^2_{\geq 0}} P'_m \cdot s^i$ which preserves the $\mathbb{Z}^2_{\geq 0}$-grading. Since f_m is an isomorphism for $m_1 \mid m$, we see that ψ induces an isomorphism for the subrings graded by $\{(m,i) \in \mathbb{Z}^2_{\geq 0} : m_1 \mid m\}$. This implies that ψ induces a finite morphism $\phi : C_p(\mathcal{V}_0, \mathcal{M}_0) \to \mathcal{X}_0$. Moreover, since both $\ker(f_m)$ and $\mathrm{coker}(f_m)$ have dimension $O(m^{n-3})$, both kernel and cokernel of $\psi_l = \oplus^l_{m=0} f_m \cdot s^{l-m}$ have dimension $O(l^{n-2})$. Hence the Hilbert polynomials of both kernel and cokernel of the structure morphism $\phi^\# : \mathcal{O}_{\mathcal{X}_0} \to \phi_* \mathcal{O}_{C_p(\mathcal{V}_0, \mathcal{M}_0)}$ have degree at most $n-2$. This implies that ϕ is an isomorphism outside a codimension 2 subset. In particular, ϕ is birational.

From the above properties of ϕ, we have that $C_p(\mathcal{V}_0, \mathcal{M}_0)$ is the normalization of $\mathcal{X}_0$. Since $\mathcal{X}_0$ is Cartier in $\mathcal{X}$ and $C_p(\mathcal{V}_0, \mathcal{M}_0)$ is klt, by inversion of adjunction we know that $(\mathcal{X}, \mathcal{X}_0)$ is plt. But then $\mathcal{X}_0$ itself is normal (see, e.g., [KM98, Proposition 5.51]) and hence ϕ is an isomorphism. Since $\mathcal{V}_{\infty,0}$ is the infinity section of $\mathcal{X}_0$, we know that $\mathcal{X}_0$ (and hence $\mathcal{X}$) is smooth at the generic point of $\mathcal{V}_{\infty,0}$. Therefore, the different divisor of $\mathcal{V}_\infty$ in $\mathcal{X}$ has no component of $\mathcal{V}_{\infty,0}$. In particular, $(K_{\mathcal{X}/\mathbb{A}^1} + \mathcal{V}_\infty)|_{\mathcal{V}_\infty} = K_{\mathcal{V}_\infty/\mathbb{A}^1} + \Delta_{\mathrm{tc}}$. Hence from the discussion of the "if" parts, we know $\mathrm{Fut}(\mathcal{X}, (1 - \frac{r}{n})\mathcal{V}_\infty; \mathcal{L}) = \mathrm{Fut}(\mathcal{V}, \Delta_{\mathrm{tc}}; \mathcal{L}_{\mathcal{V}})$. Hence K-semistability (resp. K-polystability) of $(X, (1 - \frac{r}{n})V_\infty)$ implies K-semistability (resp. K-polystability) of (V, Δ). We finish the proof.

LEMMA 2.12. *Let X be an n-dimensional normal projective variety. Let V be a prime divisor on X. Assume that (X, V) is plt and $-K_X \sim_{\mathbb{Q}} (1 + r)V$ is ample for some $0 < r \leq n$. Let $M \sim V|_V$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V. Then there exists a special test configuration $(\mathcal{X}, (1 - \frac{r}{n})\mathcal{V}; \mathcal{L})$ of $(X, (1 - \frac{r}{n})V)$ such that the central fiber $(\mathcal{X}_0, \mathcal{V}_0)$ is isomorphic to $(C_p(V, M), V_\infty)$ and $\mathrm{Fut}(\mathcal{X}, (1 - \frac{r}{n})\mathcal{V}; \mathcal{L}) = 0$.*

Proof. The construction is obtained by degeneration to normal cone. We will use the filtration language [BX19, Section 2.3] to construct $\mathcal{X}$. Let $R := \oplus^\infty_{m=0} H^0(X, \mathcal{O}_X(mV))$ be the graded ring of X. Let R_m be the mth graded piece of R. Consider the $\mathbb{N}$-filtration $\mathcal{F}$ of R defined as

$$\mathcal{F}^\lambda R_m = H^0(X, \mathcal{O}_X((m-\lambda)V)) \subset H^0(X, \mathcal{O}_X(mV)) = R_m \quad \text{for any } \lambda \in \mathbb{Z}_{\geq 0}.$$

Note that $\mathcal{F}^\lambda R_m = R_m$ for any nonpositive integer λ. The Rees algebra of $\mathcal{F}$ is defined as

$$\mathrm{Rees}(\mathcal{F}) := \bigoplus^\infty_{m=0} \bigoplus^\infty_{\lambda=-\infty} t^{-\lambda} \mathcal{F}^\lambda R_m \subset R[t, t^{-1}].$$

The associated graded ring of $\mathcal{F}$ is defined as

$$\mathrm{gr}_\mathcal{F} R := \bigoplus^\infty_{m=0} \bigoplus^\infty_{\lambda=-\infty} \mathrm{gr}^\lambda_\mathcal{F} R_m, \quad \text{where } \mathrm{gr}^\lambda_\mathcal{F} R_m := \frac{\mathcal{F}^\lambda R_m}{\mathcal{F}^{\lambda+1} R_m}.$$

Note that the gradings of $\mathrm{Rees}(\mathcal{F})$ and $\mathrm{gr}_\mathcal{F} R$ are both given by m. In our set-up, it is easy to see that both $\mathrm{Rees}(\mathcal{F})$ and $\mathrm{gr}_\mathcal{F} R$ are finitely generated. Let $\mathcal{X} := \mathrm{Proj}_{\mathbb{A}^1} \mathrm{Rees}(\mathcal{F})$. Then from [BX19, Section 2.3.1] we know that $\mathcal{X}_0 \cong \mathrm{Proj}\, \mathrm{gr}_\mathcal{F} R$. We will show that $\mathcal{X}_0 \cong C_p(V, M)$.

When $\lambda \leq -1$, we know $\mathcal{F}^\lambda R_m = \mathcal{F}^{\lambda+1} R_m = R_m$ so $\mathrm{gr}^\lambda_\mathcal{F} R_m = 0$. When $\lambda \geq m+1$, we know $\mathcal{F}^\lambda R_m = \mathcal{F}^{\lambda+1} R_m = 0$ so $\mathrm{gr}^\lambda_\mathcal{F} R_m = 0$. Let us assume $\lambda \in [0, m] \cap \mathbb{Z}$. By Kawamata–Viehweg

vanishing, we know $H^1(X, \mathcal{O}_X((m-\lambda-1)V)) = 0$ since $(m-\lambda-1)V - K_X = (m-\lambda+r)V$ is ample. Hence from the proof of Proposition 2.10, we have a canonical isomorphism

$$\mathrm{gr}_{\mathcal{F}}^{\lambda} R_m = H^0(X, \mathcal{O}_X((m-\lambda)V))/H^0(X, \mathcal{O}_X((m-\lambda-1)V)) \cong H^0(V, \mathcal{O}_V(\lfloor (m-\lambda)M \rfloor)).$$

Thus $\mathrm{gr}_{\mathcal{F}} R \cong \oplus_{m=0}^{\infty} \oplus_{i=0}^{\infty} H^0(V, \mathcal{O}_V(\lfloor mM \rfloor)) \cdot s^i$ as graded rings, so $\mathcal{X}_0 \cong C_p(V, M)$.

So far we have constructed a variety $\mathcal{X}$ which provides a special degeneration from X to $\mathcal{X}_0 \cong C_p(V, M)$. Let $\mathcal{V}$ be the Zariski closure of $V \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{X}$. Denote the graded ideal of V in X by $I_V \subset R$. Then it is clear that $I_{V,m} = H^0(X, \mathcal{O}_X((m-1)V))$. By [BX19, Section 2.3.1], the graded ideal of the scheme-theoretic central fiber $\mathcal{V}_0$ is given by the initial ideal $\mathrm{in}(I_V)$ where

$$\mathrm{in}(I_V) := \bigoplus_{m=0}^{\infty} \bigoplus_{\lambda=-\infty}^{\infty} \mathrm{im}(\mathcal{F}^{\lambda} R_m \cap I_V \to \mathrm{gr}_{\mathcal{F}}^{\lambda} R_m) \subset \mathrm{gr}_{\mathcal{F}} R.$$

We may restrict to $\lambda \in [0, m] \cap \mathbb{Z}$. Then it is easy to see that

$$\mathrm{im}(\mathcal{F}^{\lambda} R_m \cap I_V \to \mathrm{gr}_{\mathcal{F}}^{\lambda} R_m) = \begin{cases} \mathrm{gr}_{\mathcal{F}}^{\lambda} R_m & \text{if } \lambda \geq 1, \\ 0 & \text{if } \lambda \leq 0. \end{cases}$$

Thus $\mathrm{in}(I_V) = \oplus_{m=0}^{\infty} \oplus_{\lambda=1}^{m} \mathrm{gr}_{\mathcal{F}}^{\lambda} R_m$. This implies that $\mathcal{V}_0 \cong V_{\infty}$ under the isomorphism $\mathcal{X}_0 \cong C_p(V, M)$. Therefore, $(\mathcal{X}, (1-\frac{r}{n})\mathcal{V}; \mathcal{L})$ is a special test configuration of $(X, V; L)$ where $\mathcal{L} := -l(K_{\mathcal{X}/\mathbb{A}^1} + (1-\frac{r}{n})\mathcal{V})$ is Cartier for some $l \in \mathbb{Z}_{>0}$. Since the λ-grading on $\mathrm{gr}_{\mathcal{F}} R/\mathrm{in}(I_V)$ is trivial, we know that $(\mathcal{V}, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}})$ is a trivial test configuration of $(V, \Delta; L|_V)$ where Δ_{tc} is the Zariski closure of $\Delta \times (\mathbb{A}^1 \setminus \{0\})$ in $\mathcal{V}$. Since $\mathcal{X}_0$ is normal and Cartier in $\mathcal{X}$, we know that $\mathcal{X}$ is smooth at the generic point of $\mathcal{V}_0$. Then by normality of $\mathcal{V}$, we know $(K_{\mathcal{X}/\mathbb{A}^1} + \mathcal{V})|_{\mathcal{V}} = K_{\mathcal{V}/\mathbb{A}^1} + \Delta_{\mathrm{tc}}$. Hence, by exactly the same computation in the proof of "if" part of Proposition 2.11(1), we know

$$\mathrm{Fut}(\mathcal{X}, (1-\tfrac{r}{n})\mathcal{V}; \mathcal{L}) = \mathrm{Fut}(\mathcal{V}, \Delta_{\mathrm{tc}}; \mathcal{L}|_{\mathcal{V}}) = 0.$$

The proof is finished.

§3. Fujita's characterization

In this section, we make a few observations on the restrictions on $\mathbb{Q}$-Fano varieties X with $\alpha(X) = \frac{\dim(X)}{\dim(X)+1}$ that are not K-stable. We first have the following characterization, essentially due to Fujita.

THEOREM 3.1. (cf. [Fuj19a]) *Let X be an n-dimensional $\mathbb{Q}$-Fano variety. Assume that $\alpha(X) = n/(n+1)$ and X is not K-stable. Then there exists a birational morphism $\sigma : (F \subset Y) \to (x \in X)$ extracting a Kollár component F over $x \in X$, such that the following properties hold:*

(1) $A_X(F) = n$, $\tau(F) = \epsilon(F) = n+1$.
(2) *The divisor $\sigma^*(-K_X) - (n+1)F$ is semiample but not big. The ample model $\pi : Y \to Z$ of $\sigma^*(-K_X) - (n+1)F$ satisfies that all fibers of π are one-dimensional and a generic fiber is $\mathbb{P}^1$. Moreover, $\pi|_F : F \to Z$ is an isomorphism.*
(3) $\alpha(F, \Delta_F) \geq 1$ *where Δ_F is the different. Moreover, ord_F is a minimizer of the normalized volume function $\widehat{\mathrm{vol}}$ on $\mathrm{Val}_{X,x}$.*

In particular, $x \in X$ is a weakly exceptional singularity.

Proof. Since X is not K-stable, by Theorem 2.3, there exists a dreamy prime divisor F over X such that $\beta(F) \leq 0$. Meanwhile, [FO18, Lemma 3.3] implies that $A_X(F) \geq \alpha(X)\tau(F) = n/(n+1)\tau(F)$. Then by [Fuj19a, Lemma 3.1, Proposition 3.2 and 3.3] we know that there exists a proper birational morphism $\sigma : Y \to X$ with Y normal such that $F = \text{Exc}(\sigma)$ is a prime divisor on Y, $-F$ is σ-ample, $\sigma(F)$ is a closed point $x \in X$, and parts (1) and (2) hold. Moreover, part (3) on $\alpha(F, \Delta_F) \geq 1$ implies that (F, Δ_F) is klt, that is, F is a Kollár component. Thus it suffices to show that part (3) holds.

Finally we prove part (3). From part (2), we know that $F \cong Z$ are both normal. Denote by Δ_F, the different divisor of F in Y. Let $B \sim_{\mathbb{Q}} -K_F - \Delta_F$ be any effective $\mathbb{Q}$-divisor on F. We denote by $B_Z := (\pi|_F)_* B$. From parts (1) and (2), we know that

$$(-K_Y - 2F)|_F = (\sigma^*(-K_X) - (n+1)F)|_F = -(n+1)F|_F.$$

Hence $-K_F - \Delta_F = (-K_Y - F)|_F = -nF|_F$. Since $\pi|_F : F \to Z$ is an isomorphism, we know that

$$\sigma^*(-K_X) - (n+1)F \sim_{\mathbb{Q}} \frac{n+1}{n} \pi^*(\pi|_F)_*(-K_F - \Delta_F) \sim_{\mathbb{Q}} \frac{n+1}{n} \pi^* B_Z.$$

It is clear that $\pi^* B_Z$ does not contain F as a component. Thus $B_X := \sigma_* \pi^* B_Z$ satisfies that $B_X \sim_{\mathbb{Q}} -n/(n+1)K_X$ and $\text{ord}_F(B_X) = n$. Since $\alpha(X) = n/(n+1)$, the pair (X, B_X) is log canonical. We know

$$\sigma^*(K_X + B_X) = K_Y - (n-1)F + \pi^* B_Z + nF = K_Y + F + \pi^* B_Z.$$

Hence $(Y, F + \pi^* B_Z)$ is log canonical. By adjunction, $(F, \Delta_F + B)$ is log canonical since $B = (\pi^* B_Z)|_F$. By choosing B whose support contains a log canonical center of (F, Δ_F), we see that (F, Δ_F) is klt. Hence inversion of adjunction implies that F is a Kollár component over $x \in X$ and $\alpha(F, \Delta_F) \geq 1$.

By part (2) we know that

$$(\sigma^*(-K_X) - (n+1)F)^n = (-K_X)^n - (n+1)^n(-1)^{n-1}(F^n) = 0.$$

It is clear that $\text{vol}_{X,x}(\text{ord}_F) = (-1)^{n-1}(F^n)$. From part (1), we know that $A_X(F) = n$. Thus we get $(-K_X)^n = (1 + \frac{1}{n})^n \widehat{\text{vol}}_{X,x}(\text{ord}_F)$. Since $\alpha(X) = n/(n+1)$, Theorem 1.2 implies that X is K-semistable. Hence [Liu18, Theorem 21] implies that ord_F is a divisorial minimizer of the normalized volume function $\widehat{\text{vol}}$ on $\text{Val}_{X,x}$. Thus the proof of part (3) is finished. $\square$

A quick consequence of the proof of Theorem 3.1 is that every $\mathbb{Q}$-Fano variety X satisfying $\alpha(X) = n/(n+1)$ that is not K-stable always specially degenerates to a K-polystable $\mathbb{Q}$-Fano variety $\mathcal{X}_0$ with $\alpha(\mathcal{X}_0) = 1/(n+1)$.

COROLLARY 3.2. *Let X be an n-dimensional $\mathbb{Q}$-Fano variety. Assume that $\alpha(X) = n/(n+1)$ and X is not K-stable. Then there exists a special test configuration $(\mathcal{X}, \mathcal{L})$ of X such that $\text{Fut}(\mathcal{X}; \mathcal{L}) = 0$. Moreover, $\mathcal{X}_0$ is K-polystable and $\alpha(\mathcal{X}_0) = 1/(n+1)$. In particular, if $n \geq 2$ then X is not K-polystable.*

Proof. Denote by $\mathfrak{a}_j := \{f \in \mathcal{O}_{X,x} \mid \text{ord}_F(f) \geq j\}$ for $j \in \mathbb{Z}$. From the proof of [Liu18, Lemma 33], we know that there exists a special test configuration $(\mathcal{X}; \mathcal{L})$ such that

$\mathrm{Fut}(\mathcal{X};\mathcal{L}) = 0$ and $\mathcal{X}_0 \cong \mathrm{Proj}(\oplus_{j=0}^{\infty}\mathfrak{a}_j/\mathfrak{a}_{j+1})[s]$. Since F is a Kollár component over $x \in X$, by adjunction of plt pairs we know that $F|_F$ is a $\mathbb{Q}$-divisor on F that is well-defined up to $\mathbb{Z}$-linear equivalence. Moreover, if $\{F|_F\} = \sum_i \frac{a_i}{b_i}B_i$ for $a_i < b_i$ coprime positive integers and B_i prime divisors on F, then (using the same notation as in Theorem 3.1) $\Delta_F = \frac{b_i-1}{b_i}B_i$ (see, e.g., [K$^+$ 92, Section 16]). By [LX20, Section 2.4] or Proposition 2.10, we know that $\oplus_{j=0}^{\infty}\mathfrak{a}_j/\mathfrak{a}_{j+1} \cong \oplus_{m=0}^{\infty}H^0(F,\mathcal{O}_F(\lfloor -mF|_F \rfloor))$ as graded rings which implies $\mathcal{X}_0 \cong C_p(F,(-F)|_F)$. Since $\alpha(F,\Delta_F) \geq 1$, Theorem 1.2 implies that (F,Δ_F) is K-stable. Hence Proposition 2.11 implies that $\mathcal{X}_0$ is K-polystable since $-K_F - \Delta_F \sim_{\mathbb{Q}} -nF|_F$. To show $\alpha(\mathcal{X}_0) = 1/(n+1)$, we notice that $\alpha(\mathcal{X}_0) \geq 1/(n+1)$ by [FO18, Theorem 3.5], and $\alpha(\mathcal{X}_0) \leq 1/(n+1)$ since $-K_{\mathcal{X}_0} \sim_{\mathbb{Q}} (n+1)F_{\infty}$. Hence the proof is finished. $\qquad\square$

The equality $\alpha(X) + \alpha(\mathcal{X}_0) = 1$ here is not a coincidence, as can be seen by the following statement:

PROPOSITION 3.3. *Let* $f : X \to C$, $g : Y \to C$ *be two* $\mathbb{Q}$-*Gorenstein flat families of* $\mathbb{Q}$-*Fano varieties (i.e., all geometric fibers are integral, normal and* $\mathbb{Q}$-*Fano,* $K_{X/C}$ *and* $K_{Y/C}$ *are both* $\mathbb{Q}$-*Cartier) over a smooth pointed curve* $0 \in C$. *Let* $X_0 = f^{-1}(0)$ *and* $Y_0 = g^{-1}(0)$. *Assume that there exists an isomorphism* $\rho : X \backslash X_0 \xrightarrow{\cong} Y \backslash Y_0$ *over the punctured curve* $C \backslash 0$ *and* $\alpha(X_0) + \alpha(Y_0) > 1$. *Then* ρ *induces an isomorphism* $X \cong Y$ *over* C.

Proof. By assumption X is birational to Y over C. Let m be a sufficiently large and divisible integer and let $D_1 \in |-mK_{X_0}|$, $D_2 \in |-mK_{Y_0}|$ be general divisors in the corresponding linear systems. Choose effective divisors $D_{X,1} \sim_{\mathbb{Q}} -mK_X$, $D_{Y,2} \sim_{\mathbb{Q}} -mK_Y$ not containing X_0 or Y_0 such that $D_{X,1}|_{X_0} = D_1$ and $D_{Y,2}|_{Y_0} = D_2$. Let $D_{Y,1}$ and $D_{X,2}$ be their strict transforms to the other family. Let $D_X = \frac{\alpha(Y_0)}{\alpha(X_0)+\alpha(Y_0)}D_{X,1} + \frac{\alpha(X_0)}{\alpha(X_0)+\alpha(Y_0)}D_{X,2}$ and similar $D_Y = \frac{\alpha(Y_0)}{\alpha(X_0)+\alpha(Y_0)}D_{Y,1} + \frac{\alpha(X_0)}{\alpha(X_0)+\alpha(Y_0)}D_{Y,2}$. Apparently D_Y is the strict transform of D_X. As D_1 is a general member of a very ample linear system and $D_{X,2}|_{X_0} \sim_{\mathbb{Q}} -mK_{X_0}$, by the definition of alpha invariant we see that $(X_0, \frac{1}{m}D_X|_{X_0})$ is klt. Similarly $(Y_0, \frac{1}{m}D_Y|_{Y_0})$ is klt. The result then follows from the next lemma. $\qquad\square$

LEMMA 3.4. *Let* $f : X \to C$, $g : Y \to C$ *be* $\mathbb{Q}$-*Gorenstein flat families of* $\mathbb{Q}$-*Fano varieties over a smooth pointed curve* $0 \in C$ *with central fibers* X_0 *and* Y_0. *Let* $D_X \sim_{\mathbb{Q}} -K_X$, $D_Y \sim_{\mathbb{Q}} -K_Y$ *be effective* $\mathbb{Q}$-*divisors not containing* X_0 *or* Y_0. *Assume that there exists an isomorphism*

$$\rho : (X, D_X) \times_C C^{\circ} \cong (Y, D_Y) \times_C C^{\circ}$$

over $C^{\circ} = C \backslash 0$, *that* $(X_0, D_X|_{X_0})$ *is klt and* $(Y_0, D_Y|_{Y_0})$ *is lc. Then* ρ *extends to an isomorphism* $(X, D_X) \cong (Y, D_Y)$.

Proof. This follows from the exact same proof of [LWX19, Theorem 5.2] (see also [BX19, Proposition 3.2]). $\qquad\square$

REMARK 3.5. When the total spaces have terminal singularities, Proposition 3.3 is proved by Cheltsov [Che09, Theorem 1.5].

Proof of Theorem 1.5. Boundedness follows from [Jia17a]; openness follows from [BL18b, Theorem B]; separatedness follows from Proposition 3.3; the existence of coarse moduli space is due to Keel and Mori [KM97].

We believe such moduli spaces are interesting and deserve further investigation. See [BX19, Corollary 1.4] for an analogous result on moduli of uniformly K-stable $\mathbb{Q}$-Fano varieties.

We also give an alternative proof of the finiteness of automorphism group in Theorem 1.5 without using stacks.

COROLLARY 3.6. *Let X be a $\mathbb{Q}$-Fano variety such that $\alpha(X) > \frac{1}{2}$, then $\mathrm{Aut}(X)$ is finite.*

Proof. Assume that $\mathrm{Aut}(X)$ is not finite, then it contains $\mathbb{G}_m$ or $\mathbb{G}_a$. Let $\mathcal{X}_1 = \mathcal{X}_2 = X \times \mathbb{A}^1$. If $\mathbb{G}_m \subseteq \mathrm{Aut}(X)$, then it induces an isomorphism $\mathcal{X}_1 \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\sim} \mathcal{X}_2 \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus \{0\}) = X \times (\mathbb{A}^1 \setminus \{0\})$ through the diagonal action of $\mathbb{G}_m$. Since $\alpha(X) > \frac{1}{2}$, this map extends to an isomorphism of $\mathcal{X}_1$ and $\mathcal{X}_2$ over $\mathbb{A}^1$ by Proposition 3.3. Thus the map $\mathbb{G}_m \to \mathrm{Aut}(X)$ extends to a map $\mathbb{A}^1 \to \mathrm{Aut}(X)$, a contradiction. Similarly, if $\mathbb{G}_a \subseteq \mathrm{Aut}(X)$, then the inclusion $\mathbb{A}^1 \setminus \{0\} \subseteq \mathbb{G}_a$ given by $t \mapsto t^{-1}$ induces an automorphism of $X \times (\mathbb{A}^1 \setminus \{0\})$ over $\mathbb{A}^1 \setminus \{0\}$ which extends to an automorphism over $\mathbb{A}^1$, hence $\mathbb{G}_a \to \mathrm{Aut}(X)$ extends to a map $\mathbb{P}^1 \to \mathrm{Aut}(X)$, again a contradiction.

Another quick consequence of Theorem 3.1 is a refinement of Tian's criterion in small dimension. Note that the dimension 2 case was proved by Fujita [Fuj].

COROLLARY 3.7. *Let X be a $\mathbb{Q}$-Fano variety such that $\alpha(X) = n/(n+1)$ where $n = \dim X$. Assume that either $n = 2$ or $n = 3$ and X has terminal singularities. Then X is K-stable.*

Proof. By Theorem 3.1, it suffices to show that under the given assumptions, X cannot have a weakly exceptional singularity whose corresponding Kollár component has log discrepancy n. For 3-fold terminal singularities, this follows from [Pro00, Corollary 4.8]. In the surface case, it is well known (and not hard to verify) that a klt surface singularity is weakly exceptional if and only if the dual graph of the exceptional divisors on the minimal resolution has a fork, in which case the Kollár component lives on the minimal resolution and corresponds to the central fork. In particular, the log discrepancy of the Kollár component is at most 1 and does not satisfy our requirement.

From Corollary 3.2, we see that K-semistable $\mathbb{Q}$-Fano varieties with smallest α-invariant naturally occur in studying non-K-stable $\mathbb{Q}$-Fano varieties with largest α-invariant. In [Jia17b, Theorem 1.2], Jiang showed that a K-semistable Fano manifold X with $\alpha(X) = \frac{1}{\dim(X)+1}$ is isomorphic to projective space. The following result provides a characterization of possibly singular such X.

THEOREM 3.8. *Let X be an n-dimensional K-semistable $\mathbb{Q}$-Fano variety satisfying that $\alpha(X) = 1/(n+1)$. Then there exists an $(n-1)$-dimensional K-polystable log Fano pair (V, Δ) where Δ has standard coefficients, and M an ample $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V satisfying $nM \sim_{\mathbb{Q}} -K_V - \Delta$ and $\Delta_M = \Delta$, such that $C_p(V, M)$ is the unique K-polystable special degeneration of X. In addition, $\alpha(C_p(V, M)) = 1/(n+1)$.*

Proof. Let X' be the unique K-polystable special degeneration of X whose existence is proved in [LWX18, Theorem 1.3]. Then we have $\alpha(X') \leq \alpha(X) = 1/(n+1)$ by [BL18b, Theorem B] and $\alpha(X') \geq 1/(n+1)$ by [FO18, Theorem 3.5]. Thus $\alpha(X') = 1/(n+1)$. Then by [Jia17b, Proposition 3.1], there exists a prime divisor V on X' such that (X', V) is plt and $-K_{X'} \sim_{\mathbb{Q}} (n+1)V$. Let $M \sim V|_V$ be a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V. Thus Lemma 2.12

implies that there exists a special test configuration $(\mathcal{X}'; \mathcal{L}')$ of X' such that the central fiber $\mathcal{X}'_0$ is isomorphic to $C_p(V, M)$ and $\mathrm{Fut}(\mathcal{X}'; \mathcal{L}') = 0$. Since X' is K-polystable, we have $X' \cong \mathcal{X}'_0 \cong C_p(V, M)$. Denote $\Delta := \mathrm{Diff}_V(0)$, then it is clear that Δ has standard coefficients, $\Delta = \Delta_M$, and

$$-K_V - \Delta = (-K_X - V)|_V \sim_{\mathbb{Q}} nV|_V = nM.$$

Since $C_p(V, M) \cong X'$ is K-polystable, the log Fano pair (V, Δ) is also K-polystable by Proposition 2.11. Hence we finish the proof.

The next result shows that in Theorem 3.1 the different divisor Δ_F never vanishes. Note that Proposition 3.9 and Lemma 3.10 are not needed in the rest of the paper.

PROPOSITION 3.9. *Assume $n \geq 2$. Then with the notation of Theorem 3.1, we have* $\Delta_F \neq 0$.

Proof. Assume to the contrary that $\Delta_F = 0$. Then by Theorem 3.1 and Lemma 3.10, we know that $X \cong C_p(F, \mathcal{O}_F(-F))$ and $\alpha(X) \leq 1/(n+1)$ which contradicts to $\alpha(X) = n/(n+1)$. $\square$

LEMMA 3.10. *Let X be a $\mathbb{Q}$-Fano variety of dimension $n \geq 2$. Let $\sigma : Y \to X$ be a proper birational morphism that provides a Kollár component F with $x = \sigma(F)$. Assume that $\epsilon(F) = \tau(F) > A_X(F)$. If $\mathcal{O}_Y(F)$ is locally free in codimension 2, then $X \cong C_p(F, \mathcal{O}_F(-F))$. In particular, $\epsilon(F) = A_X(F) + 1$ and $\alpha(X) \leq \frac{1}{A_X(F)+1}$.*

Proof. Let $D := \sigma^*(-K_X) - \epsilon(F) \cdot F$. By definition we know that D is a nef $\mathbb{R}$-divisor that is not big. Since $\epsilon(F) > A_X(F) > 0$, $-K_Y - F = \sigma^*(-K_X) - A_X(F) \cdot F$ is ample. Hence the cone theorem implies that $\overline{NE}(Y)$ is polyhedral, so $\epsilon(F) \in \mathbb{Q}$. By Shokurov's basepoint-free theorem and Kawamata–Viehweg vanishing theorem, the $\mathbb{Q}$-divisor D is semiample and defines an algebraic fiber space $\pi : Y \to Z$. Since $-K_Y - F = D + (\epsilon(F) - A_X(F))F$ is ample, F is a π-ample $\mathbb{Q}$-Cartier divisor. Following the same argument as in [Fuj19a, Proposition 3.3], the fibers of π are one-dimensional, π is a generic $\mathbb{P}^1$-fibration, and $\pi|_F : F \to Z$ is an isomorphism. In particular, $-K_Y - 2F$ has zero intersection number with a generic fiber of π, hence $-K_Y - 2F = D$ which yields $\epsilon(F) = A_X(F) + 1$.

Let F^0 be the biggest open subset of the smooth locus F_{reg} of F such that $\mathcal{O}_Y(F)$ is locally free along F^0. By assumption we have $\mathrm{codim}_F F \setminus F^0 \geq 2$. Denote by $Y^0 := Y \setminus (F \setminus F^0)$ with $i : Y^0 \hookrightarrow Y$ and $j : F^0 \hookrightarrow F$ the open immersions, hence $\mathrm{codim}_Y Y \setminus Y^0 \geq 3$. Denote by $Z^0 := \pi(F^0)$. Since $\mathcal{O}_Y(F)$ is locally free along F^0, we have the short exact sequence

$$0 \to \mathcal{O}_{Y^0}((m-1)F^0) \to \mathcal{O}_{Y^0}(mF^0) \to \mathcal{O}_{F^0}(mF^0|_{F^0}) \to 0. \tag{3.1}$$

By taking i_*, we get an exact sequence

$$0 \to i_*\mathcal{O}_{Y^0}((m-1)F^0) \to i_*\mathcal{O}_{Y^0}(mF^0) \to j_*\mathcal{O}_{F^0}(mF^0|_{F^0}) \to R^1 i_*\mathcal{O}_{Y^0}((m-1)F^0). \tag{3.2}$$

Since Y^0 and F^0 are open subsets of Y and F, respectively, whose complements have codimension at least 2, we have these natural isomorphisms

$$i_*\mathcal{O}_{Y^0}(mF^0) \cong \mathcal{O}_Y(mF), \quad j_*\mathcal{O}_{F^0}(mF^0|_{F^0}) \cong \mathcal{O}_F(mF|_F).$$

From the assumption that F is Cartier in codimension 2, we have that $\mathcal{O}_F(mF|_F)$ is a well-defined $\mathbb{Q}$-Cartier Weil divisorial sheaf on F satisfying $\mathcal{O}_F(mF|_F) \cong \mathcal{O}_F(F|_F)^{[m]}$. By

[KM98, Corollary 5.25] we have that $\mathcal{O}_Y(mF)$ is Cohen–Macaulay for any $m \in \mathbb{Z}$. Since $\operatorname{codim}_Y Y \setminus Y^0 \geq 3$, the local cohomology long exact sequence implies that

$$R^1 i_* \mathcal{O}_{Y^0}((m-1)F^0) \cong \mathcal{H}^2_{Y \setminus Y^0}(Y, \mathcal{O}_Y((m-1)F)) = 0.$$

Hence the exact sequence (3.2) becomes

$$0 \to \mathcal{O}_Y((m-1)F) \to \mathcal{O}_Y(mF) \to \mathcal{O}_F(mF|_F) \to 0. \tag{3.3}$$

Let M be a $\mathbb{Q}$-Cartier Weil divisor on F representing $\mathcal{O}_F(-F)$, then $M|_{F^0}$ is Cartier on F^0. Denote by $M_Z := (\pi|_F)_* M$. After tensoring (3.3) by $\mathcal{O}_Y(m \cdot \pi^* M_Z)$ and taking the reflexive hull, we get an exact sequence

$$0 \to \mathcal{O}_Y((m-1)F + m \cdot \pi^* M_Z) \to \mathcal{O}_Y(m(F + \pi^* M_Z)) \to \mathcal{O}_F \to 0. \tag{3.4}$$

The reason that (3.4) is exact can be deduced as follows: we first tensor (3.1) by $\mathcal{O}_{Y^0}(m \cdot \pi^* M_Z)$ (since $\mathcal{O}_{Y^0}(m \cdot \pi^* M_Z)$ is locally free along F^0, the exactness remains), then taking i_* (since $\pi^* M_Z$ is a $\mathbb{Q}$-Cartier Weil divisor on Y, we again use [KM98, Corollary 5.25] to show that $\mathcal{O}_Y((m-1)F + m \cdot \pi^* M_Z)$ is Cohen–Macaulay); then the argument above works in a similar way.

Now applying π_* to the exact sequence (3.4) when $m = 1$, we get an exact sequence

$$0 \to \pi_* \mathcal{O}_Y(\pi^* M_Z) \to \pi_* \mathcal{O}_Y(F + \pi^* M_Z) \to \mathcal{O}_Z \to R^1 \pi_* \mathcal{O}_Y(\pi^* M_Z). \tag{3.5}$$

It is clear that $K_Y \sim_{\mathbb{Q},\pi} -2F$ is π-anti-ample, hence Kawamata–Viehweg vanishing implies that $R^1 \pi_* \mathcal{O}_Y(\pi^* M_Z) = 0$. The projection formula yields $(\pi_* \mathcal{O}_Y(\pi^* M_Z))|_{Z^0} \cong \mathcal{O}_{Z^0}(M_Z)$. As abuse of notation we also denote the open immersion $Z^0 \hookrightarrow Z$ by j. Hence

$$\pi_* \mathcal{O}_Y(\pi^* M_Z) \cong j_* ((\pi_* \mathcal{O}_Y(\pi^* M_Z))|_{Z^0}) \cong j_* \mathcal{O}_{Z^0}(M_Z) \cong \mathcal{O}_Z(M_Z),$$

where the first equality follows from the fact that $\mathcal{O}_Y(\pi^* M_Z)$ satisfies S_2-condition and π is flat. As a result, (3.5) becomes

$$0 \to \mathcal{O}_Z(M_Z) \to \pi_* \mathcal{O}_Y(F + \pi^* M_Z) \to \mathcal{O}_Z \to 0. \tag{3.6}$$

Hence $\pi_* \mathcal{O}_Y(F + \pi^* M_Z)$ is an extension of $\mathcal{O}_Z$ by $\mathcal{O}_Z(M_Z)$. It is clear that

$$\operatorname{Ext}^1(\mathcal{O}_Z, \mathcal{O}_Z(M_Z)) \cong H^1(Z, \mathcal{O}_Z(M_Z)) \cong H^1(F, \mathcal{O}_F(M)).$$

By assumption we have that

$$K_F = (K_Y + F)|_F \sim_{\mathbb{Q}} (\sigma^* K_X + A_X(F) \cdot F)|_F = -A_X(F) \cdot M.$$

Hence $M - K_F \sim_{\mathbb{Q}} (A_X(F) + 1)M$ is ample. Then Kawamata–Viehweg vanishing yields $H^1(F, \mathcal{O}_F(M)) = 0$. So the exact sequence (3.6) splits, that is, $\pi_* \mathcal{O}_Y(F + \pi^* M_Z) \cong \mathcal{O}_Z(M_Z) \oplus \mathcal{O}_Z$. By restricting to Z^0, tensoring with $\mathcal{O}_{Z^0}(-M_Z)$ and taking i_*, we have

$$\pi_* \mathcal{O}_Y(F) \cong \mathcal{O}_Z \oplus \mathcal{O}_Z(-M_Z).$$

Denote by $Y' := \pi^{-1}(Z^0)$. From [Fuj19a, Proposition 3.3] and [LZ18, Lemma 5] we know that $\pi^0 = \pi|_{Y'} : Y' \to Z^0$ is a smooth $\mathbb{P}^1$-fibration. Since π^0 admits a section, it is a $\mathbb{P}^1$-bundle. As a consequence,

$$\bigoplus_{m=0}^{\infty} (\pi^0)_* \mathcal{O}_{Y'}(mF^0) \cong \bigoplus_{m=0}^{\infty} \operatorname{Sym}^m \left((\pi^0)_* \mathcal{O}_{Y'}(F^0) \right). \tag{3.7}$$

Applying j_* to (3.7) yields

$$
\bigoplus_{m=0}^{\infty} \pi_* \mathcal{O}_Y(mF) \cong \bigoplus_{m=0}^{\infty} j_* \operatorname{Sym}^m\big((\pi^0)_* \mathcal{O}_{Y'}(F^0)\big)
$$
$$
\cong \bigoplus_{m=0}^{\infty} j_*\left(\bigoplus_{k=0}^{m} \mathcal{O}_{Z^0}(-kM_Z)\right)
$$
$$
\cong \bigoplus_{m=0}^{\infty} \bigoplus_{k=0}^{m} \mathcal{O}_Z(-kM_Z).
$$

Since F is π-ample, we have

$$
Y \cong \operatorname{Proj}_Z \bigoplus_{m=0}^{\infty} \pi_* \mathcal{O}_Y(mF) \cong \operatorname{Proj}_Z \bigoplus_{m=0}^{\infty} \bigoplus_{k=0}^{m} \mathcal{O}_Z(-kM_Z).
$$

Hence (Y,F) is isomorphic to the canonical blow-up of the projective orbifold cone $C_p(F, \mathcal{O}_F(M))$ at the vertex, which implies $X \cong C_p(F, \mathcal{O}_F(M))$.

Since $-K_F \sim_{\mathbb{Q}} A_X(F) \cdot M$, we have that $-K_X \sim_{\mathbb{Q}} (A_X(F)+1)F_\infty$ where F_∞ is the section of the projective orbifold cone at infinity. Hence $\alpha(X) \le \frac{1}{A_X(F)+1}$.

§4. Log canonical thresholds on general complete intersections

Let $r \ge 1$, $n \ge 3$ and $d_1, \dots, d_r \ge 2$ be integers. In this section, we study the (global) log canonical threshold of the hyperplane class H on a general smooth complete intersection $X = F_1 \cap \cdots \cap F_r \subseteq \mathbb{P}^{n+r}$ of codimension r and dimension n, where each F_i is a hypersurface of degree d_i. In particular, we construct (Corollary 4.11) weakly exceptional singularities with the properties described in Theorem 3.1 and also prove Theorem 1.4. First let us specify the generality condition we will consider. For this we recall the regularity condition introduced by Pukhlikov (see, e.g., [Puk18] or [Puk13, Section 3.2]). Let $x \in X$ and let f_i be the defining equation of F_i. Choose a system of affine linear coordinates $z_* = (z_1, \dots, z_{n+r})$ with origin at x and write f_i as

$$
f_i = q_{i,1} + q_{i,2} + \cdots + q_{i,d_i},
$$

where $q_{i,j} = q_{i,j}(z_*)$ is homogeneous of degree j. We rearrange the $d = \sum_{i=1}^{r} d_i$ polynomials $q_{i,j}$ ($1 \le i \le r$, $1 \le j \le d_i$) into a sequence $q_1, \dots, q_d$ such that $q_{i,j}$ precedes $q_{i',j'}$ if and only if $j < j'$ or $j = j'$ and $i < i'$. In particular, $\deg q_1 \le \cdots \le \deg q_d$. Since X is smooth, the linear subspace $\Sigma_x = (q_1 = \cdots = q_r = 0)$ has codimension r and can be identified with the tangent space of X at x. By the following lemma, we may assume that $d \ge n+1$ throughout the remaining part of the section.

LEMMA 4.1. *Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Fano complete intersection of codimension r, dimension n and index s(i.e., $-K_X \sim sH$). Assume that $s \ge r+1$, then $\operatorname{lct}(X; |H|_{\mathbb{Q}}) = 1$ where H is the hyperplane class.*

Proof. The proof is similar to the hypersurface case in [Che01]. Let $D \sim_{\mathbb{Q}} H$ be an effective divisor on X. By [Suz17, Proposition 2.1], there exists a closed subset $Z \subseteq X$ of dimension at most $r-1$ such that $\operatorname{mult}_x D \le 1$ for all $x \in X \backslash Z$. In particular, (X, D) is lc outside Z. Let $W \subseteq X$ be a linear space section of X of codimension r that is disjoint from Z. Let $H_1, \dots, H_{r+1} \subseteq X$ be general hyperplane sections containing W and let

$\Delta = (1-\epsilon)D + \frac{r}{r+1}(H_1 + \cdots + H_{r+1})$ where $0 < \epsilon \ll 1$. Then (X, Δ) is klt outside $Z \cup W$ by construction. As $-(K_X + \Delta) \sim_{\mathbb{Q}} (s - r - 1 + \epsilon)H$ is ample, $\mathrm{Nklt}(X, \Delta)$ is connected by Kollár–Shokurov's connectedness lemma. Since $\mathrm{mult}_W \Delta \geq r$, W is already a non-klt center of (X, Δ); hence since Z is disjoint from W, we deduce that (X, Δ) is klt along Z and it follows that $(X, (1-\epsilon)D)$ is also klt along Z. Hence (X, D) is lc and $\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq 1$. The other direction of the inequality is obvious.

DEFINITION 4.2. Let $k = \min\{d, n + r - 2\}$. The complete intersection X is said to be *Pukhlikov regular* (or simply *P-regular*) at x, if for any linear form $h = h(z_*)$ not in the span of $q_1, \ldots, q_r$, the sequence $h, q_1, \ldots, q_k$ is a regular sequence in $\mathcal{O}_{\mathbb{P}^{n+r}, x}$. We say that X is *P-regular* if it is P-regular at every $x \in X$.

Note that when X is a Fano complete intersection of index one, this is exactly the P-regularity condition introduced by Pukhlikov [Puk18]. We now state the generality condition we need.

DEFINITION 4.3. Let $0 \leq m < n$ be an integer. We say that X is *m-strongly P-regular* if it is P-regular at x and for any linear form $h = h(z_*)$ not in the span of $q_1, \ldots, q_r$, the algebraic set $Z = (h = q_1 = \cdots = q_m = 0) \cap X$ is irreducible and reduced. We say that X is *m- strongly P-regular* if it is *m*-strongly P-regular at every $x \in X$.

This can be viewed as a generalization of the conditions introduced in [Puk05, Section 2.1]. Clearly if X is *m*-strongly P-regular then it is also m'-strongly P-regular for all $m' < m$. Note also that if $n \geq 2r + 3$ and X is P-regular, then X is at least r-strongly P-regular as by Lefschetz hyperplane theorem, the cycle $Z = (h = q_1 = \cdots = q_r = 0) \cap X$ is automatically irreducible and reduced.

Assume that the complete intersection X is P-regular at x. Following Pukhlikov, we further introduce the following setup. For each $1 \leq i \leq r$ and $1 \leq j \leq d_i$, let

$$f_{i,j} = q_{i,1} + \cdots + q_{i,j}$$

be the truncated equation of the hypersurface F_i at x. Let $\bar{f}_1, \ldots, \bar{f}_d$ be the rearrangement of all the $f_{i,j}$ corresponding to $q_1, \ldots, q_d$. Let $e_i = \deg q_i = \deg \bar{f}_i$. Let $1 \leq \ell \leq k := \min\{d, n + r - 2\}$ be an integer and let u, v be such that $q_\ell = q_{u,v}$. We define the ℓth slope β_ℓ to be $\frac{v+1}{v}$ (note that $v = e_\ell$) if $v < d_u$ and $q_{u,v+1}$ belongs to the sequence $q_1, \ldots, q_k$; otherwise (including when $\ell > k$) we set $\beta_\ell = 1$. We also call the linear system of divisors

$$\Lambda_\ell = \left\{ \sum_{1 \leq i \leq \ell, \beta_i > 1} \bar{f}_i s_i \,\middle|\, s_i = s_i(z_*) \text{ is homogeneous}, \deg s_i = e_\ell - e_i \right\} \subseteq |\mathcal{O}_X(e_\ell)|$$

the ℓth hypertangent linear system of X at x. If L is a hyperplane section of X at x that does not contain the tangent space Σ_x of X, we also refer to $\Lambda_{L,\ell} := \Lambda_\ell|_L$ as the ℓth hypertangent linear system of L at x. By the P-regularity condition, we have $\mathrm{mult}_x \Lambda_{L,\ell} = e_\ell + 1$ when $\beta_\ell > 1$ and in this case it is not hard to see that as $\mathrm{Bs}(\Lambda_{L,\ell})$ is the intersection of L with all $(q_i = 0)$ for which $i \leq \ell$ and $\beta_i > 1$, the tangent cone of $\mathrm{Bs}(\Lambda_{L,\ell})$ is a complete intersection of codimension $\lambda_\ell := \#\{i \leq \ell \,|\, \beta_i > 1\}$ in $\mathbb{P}(L \cap \Sigma_x)$. Therefore $\mathrm{codim}_L \mathrm{Bs}(\Lambda_{L,\ell}) = \lambda_\ell$ in a neighborhood of x and

$$\mathrm{mult}_x \mathrm{Bs}(\Lambda_{L,\ell}) = \prod_{1 \leq i \leq \ell, \beta_i > 1} (e_i + 1) = \prod_{1 \leq i \leq \ell, \beta_i > 1} e_i \beta_i$$

(for more details, see [Puk13, Section §3]). If in addition X is ℓ-strongly P-regular at x, then we also have $\mathrm{Bs}(\Lambda_{L,\ell}) = L \cap (\bar{f}_1 = \cdots = \bar{f}_\ell = 0)$ and

$$\deg \mathrm{Bs}(\Lambda_{L,\ell}) = \deg X \cdot \prod_{1 \le i \le \ell, \beta_i > 1} e_i$$

and hence

$$\frac{\mathrm{mult}_x}{\deg} \mathrm{Bs}(\Lambda_{L,\ell}) = \frac{1}{\deg X} \prod_{1 \le i \le \ell, \beta_i > 1} \beta_i = \frac{1}{\deg X} \prod_{1 \le i \le \ell} \beta_i. \tag{4.1}$$

It is also clear that $\lambda_k = k - r$.

We can then state the lower bound of log canonical threshold on m-strongly P-regular complete intersections.

LEMMA 4.4. *Let $m \ge 1$ be an integer such that $\beta_m > 1$ and let $X \subseteq \mathbb{P}^{n+r}$ be an $(m-1)$-strongly P-regular complete intersection, then with the above notations, we have*

$$\mathrm{lct}(X; |H|_{\mathbb{Q}}) \ge \min \left\{ 1, \frac{2}{\deg X} \prod_{i \ge 1, i \ne m} \beta_i \right\}.$$

Proof. The proof is similar to that of [Puk05, Theorem 2], based on the technique of hypertangent divisors. Denote by c the right hand side of the above inequality. Let $D \sim_{\mathbb{Q}} H$ be an effective divisor on X. It suffices to show that (X, cD) is lc. Since $c \le 1$, the non-lc center of the pair (X, cD) has codimension at least 2. Let $x \in X$ be a point in the non-lc center of (X, cD). Let $\pi : \tilde{X} \to X$ be the blowup of x, let E be the exceptional divisor and let $\tilde{D}$ be the strict transform of D. Then by (the proof of) [Puk13, Section 7, Proposition 2.3] (applied to a general surface section of (X, cD)), there exists a hyperplane $\Pi \subseteq E$ such that

$$\mathrm{mult}_x(cD) + \mathrm{mult}_\Pi(c\tilde{D}) > 2. \tag{4.2}$$

Let $\mathcal{L}$ be the linear system of hyperplane sections of X whose strict transform contains Π and let $L \in \mathcal{L}$ be a general member. Note that $\mathcal{L}$ is spanned by the tangent hyperplanes of X at x and another hyperplane. Thus by the P-regularity condition, $\mathrm{Bs}(\mathcal{L})$ has codimension at least $r + 1 \ge 2$ in X and the divisor $L \cdot D \subseteq L$ is well defined. By (4.2), we have $\mathrm{mult}_x(cD \cdot L) > 2$ or

$$\frac{\mathrm{mult}_x}{\deg}(D \cdot L) > \frac{2}{c \cdot \deg X}. \tag{4.3}$$

Let $\Lambda_{L,\ell}$ be the ℓth hypertangent linear system of L at x. Since X is $(m-1)$-strongly P-regular, the cycle $Z_\ell = \mathrm{Bs}(\Lambda_{L,\ell})$ is irreducible and reduced when $1 \le \ell \le m-1$.

By (4.3), there exists an irreducible component W_1 of $D \cdot L$ such that

$$\frac{\mathrm{mult}_x}{\deg} W_1 \ge \frac{\mathrm{mult}_x}{\deg}(D \cdot L) > \frac{2}{c \cdot \deg X} \ge \frac{2}{\deg X}$$

and hence by (4.1) with $\ell = 1$, we have $W_1 \ne Z_1$ (recall that $\beta_1 = 2$ and indeed we have $\beta_\ell \le 2$ for all ℓ). It follows that W_1 is not contained in the divisor $H_1 = \Lambda_{L,1}$ (assuming

$m - 1 \geq 1$) and we have a well-defined codimension 2 cycle $W_1 \cdot H_1$ in L with

$$\frac{\mathrm{mult}_x}{\deg}(W_1 \cdot H_1) \geq \frac{2}{1} \cdot \frac{\mathrm{mult}_x}{\deg} W_1 = \beta_1 \cdot \frac{\mathrm{mult}_x}{\deg} W_1$$

as $\mathrm{mult}_x H_1 = 2$ and $H_1 \sim H$. Let W_2 be an irreducible component of $W_1 \cdot H_1$ such that

$$\frac{\mathrm{mult}_x}{\deg} W_2 \geq \frac{\mathrm{mult}_x}{\deg}(W_1 \cdot H_1) > \frac{2\beta_1}{c \cdot \deg X} \geq \frac{2\beta_1}{\deg X}.$$

Assume that $\beta_2 > 1$, then by (4.1) again we have $W_2 \neq Z_2$ (assuming $m - 1 \geq 2$). Since W_2 is already contained in H_1 and L we deduce that W_2 is not contained in a general divisor $H_2 \in \Lambda_{L,2}$, and in this case we get a codimension 3 cycle W_3 in L as an irreducible component of $W_2 \cdot H_2$ with

$$\frac{\mathrm{mult}_x}{\deg} W_3 \geq \frac{\mathrm{mult}_x}{\deg}(W_2 \cdot H_2) \geq \beta_2 \cdot \frac{\mathrm{mult}_x}{\deg} W_2 > \frac{2\beta_1\beta_2}{c \cdot \deg X}. \tag{4.4}$$

If $\beta_2 = 1$, we simply set $W_3 = W_2$ and the inequality (4.4) clearly still holds. Iterating this process, we find an irreducible cycle W_m of codimension $\lambda_m = \lambda_{m-1} + 1$ (where $\lambda_\ell := \#\{i \leq \ell \,|\, \beta_i > 1\}$) in L with

$$\frac{\mathrm{mult}_x}{\deg} W_m > \frac{2\beta_1\beta_2\cdots\beta_{m-1}}{c \cdot \deg X}.$$

By the P-regularity condition, Z_ℓ has codimension λ_ℓ in a neighbourhood of x, hence if $\beta_{m+1} > 1$ (so that $\lambda_{m+1} = \lambda_m + 1$) then W_m is not contained in $\mathrm{Bs}(\Lambda_{L,m+1})$ and we may repeat the construction above to get an irreducible cycle W_{m+1} (of codimension λ_{m+1} in L) in the support of $W_m \cdot H_{m+1}$ (where H_{m+1} is a general divisor in $\Lambda_{L,m+1}$) such that

$$\frac{\mathrm{mult}_x}{\deg} W_{m+1} > \frac{2\beta_1\beta_2\cdots\beta_{m-1}\beta_{m+1}}{c \cdot \deg X}.$$

If $\beta_{m+1} = 1$ then we simply set $W_{m+1} = W_m$ and the above inequality still holds for obvious reason. Iterate this process again and eventually we obtain an irreducible cycle W_k (where $k = \min\{d, n + r - 2\}$) such that

$$\frac{\mathrm{mult}_x}{\deg} W_k > \frac{2\beta_1\beta_2\cdots\beta_{m-1}\beta_{m+1}\cdots\beta_k}{c \cdot \deg X}.$$

But we always have $\frac{\mathrm{mult}_x}{\deg} W \leq 1$ for any subvarieties $Z \subseteq X$, hence

$$c > \frac{2}{\deg X} \prod_{i \geq 1, i \neq m} \beta_i$$

(recall that $\beta_i = 1$ when $i > k$), a contradiction. Therefore, (X, cD) is lc at every $x \in X$.

Our remaining task is to show that a general complete intersection of given degree is m-strongly P-regular for suitable m so that we can apply Lemma 4.4. First we deal with the P-regularity condition.

LEMMA 4.5. *Let $r \geq 1$, $n \geq 2r + 3$ and $d_1, \ldots, d_r \geq 2$ be integers. Then a general complete intersection $X \subseteq \mathbb{P}^{n+r}$ of hypersurfaces of degrees $d_1, \ldots, d_r$ is P-regular.*

Proof. This should be well known to experts and is essentially a direct consequence of the work of Pukhlikov [Puk18, Section 4]. For reader's convenience we sketch the proof. Let

$\mathbb{P} = \mathbb{P}^{n+r}$. Let $U \to B \subseteq \prod_{i=1}^{r} \mathbb{P}H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d_i))$ be the universal family of smooth complete intersections of given type and $U \to \mathbb{P}$ be the natural projection. It suffices to show that for a smooth complete intersection containing x to violate the P-regularity at x imposes at least $n+1$ conditions on the coefficients of $q_1, \ldots, q_d$ for any $x \in \mathbb{P}$, since then the locus of irregular points has codimension at least $n+1$ in U and cannot dominate B. For this we may fix the coefficients of $q_1, \ldots, q_r$ and further assume that $d \geq n+r-2$ by introducing auxiliary terms in the sequence $q_1, \ldots, q_d$.

Let $p_1, \ldots, p_{n-2}$ be the sequence obtained by restricting $q_{r+1}, \ldots, q_{n+r-2}$ to $T_x X = (q_1 = \cdots = q_r = 0) \cap T_x \mathbb{P}$ and let $m_i = \deg p_i \geq 2$. Let $\Pi = \mathbb{P}(T_x X)$. Then the P-regularity of X at x is equivalent to saying that every hyperplane section of $W = (p_1 = \cdots = p_{n-2} = 0) \subseteq \Pi$ is zero dimensional. There are now two cases to consider. If all the p_i vanish on a line, it would require at least

$$\sum_{i=1}^{n-2} (m_i + 1) - 2(n-2) = \sum_{i=1}^{r} \frac{a_i(a_i+1)}{2} - (n+r-2) \tag{4.5}$$

conditions on the coefficients of p_i (where a_i is the largest subscript j such that $q_{i,j}$ appears in $q_1, \ldots, q_{n+r-2}$). As $\sum_{i=1}^{r} a_i = n+r-2$, an elementary argument shows that if $n \geq 2r+3$ then this number is $\geq n+1$. On the other hand, for a fixed hyperplane $\Sigma \subseteq \Pi$ we claim that for $W \cap \Sigma$ to contain a component of positive dimension that is not a line would require at least $2n$ conditions on the coefficients of p_i. As the hyperplane Σ varies in an $(n-1)$-dimensional family, this will prove the statement we want.

Let $p_i' = p_i|_{\Sigma}$ and let $1 \leq \ell \leq n-2$. To verify the claim, it suffices to consider the case where $p_1', \ldots, p_{\ell-1}'$ form a regular sequence and $p_\ell'|_B = 0$ for some component B of $(p_1' = \cdots = p_{\ell-1}' = 0) \subseteq \Sigma$ whose linear span $\langle B \rangle$ is different from a line. If $m_\ell = 2$, then $\ell \leq r$ and as $\dim B = n - \ell - 1$, we see that (see, e.g., Lemma 4.13) this already imposes at least

$$\binom{n-\ell+1}{2} \geq \binom{n-r+1}{2} \geq 2n \tag{4.6}$$

conditions on p_ℓ' (given that $n \geq 2r+3$). Similarly if $m_\ell = 3$ and $\ell \leq r+1$, then the vanishing of p_ℓ' on B imposes at least

$$\binom{n-\ell+2}{3} \geq \binom{n-r+1}{3} \geq 2n \tag{4.7}$$

conditions on p_ℓ'. Thus we may assume that either $m_\ell \geq 4$ or $\ell \geq r+2$; in particular, this implies $\sum_{i=1}^{\ell} m_i \geq 2\ell + 2$. Since any nonzero product $\prod_{j=1}^{m_i} \ell_j$ (where each ℓ_j is a linear form on $\langle B \rangle$) does not vanish on B and the set of such products has dimension $m_i(n-2-b)+1$ (where $b = \operatorname{codim}_{\Sigma} \langle B \rangle$) in the space of degree m_i homogeneous forms on $\langle B \rangle$, we deduce that our assumption on $p_1', \ldots, p_\ell'$ imposes at least

$$\left(\sum_{i=1}^{\ell} m_i \right)(n-2-b) + \ell - \dim \operatorname{Gr}(b, n-1) \geq (2\ell+2)(n-2-b) + \ell - b(n-1-b) \tag{4.8}$$

conditions on their coefficients. Since $\operatorname{codim} B \leq \ell-1$ and $\langle B \rangle$ is not a line, we have $b \leq \ell-1$ and $b \leq n-4$ when $\ell = n-2$. Therefore, it is straightforward to check that the right hand side of (4.8), as a quadratic function in b, in minimized at $b = \ell-1$ when $\ell \leq n-3$ or at

$b = n - 4$ when $\ell = n - 2$. It then follows from another elementary argument that the right hand side of (4.8) is $\geq 2n$.

For later applications we also need the following slightly more general version.

LEMMA 4.6.   *Let $n, r \geq 1$, $0 \leq s < r$ and $d_1, \ldots, d_r \geq 2$ be integers such that*

$$n \geq 2r + 3 + \max \left\{ 2 \sum_{i=1}^{s} d_i, \frac{1}{2} \sum_{i=1}^{s} d_i(d_i+1) \right\}.$$

Let $y \in \mathbb{P}^{n+r}$, let $Q_1, \ldots, Q_s$ be general hypersurfaces of degrees $d_1, \ldots, d_s$ with a cone singularity at y(i.e., $\mathrm{mult}_y Q_i = d_i$) and let $Q_{s+1}, \ldots, Q_r$ be general hypersurfaces of degrees $d_{s+1}, \ldots, d_r$. Then the complete intersection $X = Q_1 \cap \cdots \cap Q_r \subseteq \mathbb{P}^{n+r}$ is P-regular.

Proof.   As before we assume $d = \sum_{i=1}^{r} d_i \geq n + r - 2$. Since $s < r$, a general such complete intersection is smooth and does not contain y. For each $x \neq y \in \mathbb{P}^{n+r}$, we may choose affine linear coordinates $z_* = (z_1, \ldots, z_{n+r})$ with origin at x such that a hypersurface has a cone singularity at y if and only if its equation only involves the variables $z_2, \ldots, z_{n+r}$. After a further change of variable, we may also assume that $T_x X$ (throughout the proof we use the same notations as in the proof of Lemma 4.5) is defined by the vanishing of some of the z_i's. As before, it suffices to show that violation of P-regularity condition imposes at least $n + 1$ conditions on the coefficients of the p_i. If z_1 vanishes on $T_x X$, then the restricted expression p_i can be an arbitrary degree m_i homogeneous polynomial and the same calculation in Lemma 4.5 applies. If z_1 does not vanish on $T_x X$, the only restriction is that when p_i comes from some Q_j with $j \leq s$, it can only vary among the equations of (arbitrary) degree m_i hypersurfaces with a cone singularity at $w = [1 : 0 : \cdots : 0] \in \Pi$.

The proof then proceeds as before. For instance (this is the most complicated case), if $\Sigma \subseteq \Pi$ is a hyperplane containing w and $B \subseteq \Sigma$ is a positive dimensional subvariety whose linear span $\langle B \rangle$ also contains w but is different from a line, then requiring $p'_1, \ldots, p'_\ell$ ($m_\ell \geq 3$) to vanish on B imposes at least

$$\left(\sum_{i=1}^{\ell} m_i \right)(n - 2 - b) + \ell - \dim \mathrm{Gr}(b, n-2) - {\sum}' m_i \geq 3n - 5 - \sum_{i=1}^{s} \frac{1}{2} d_i(d_i+1) \geq 2n$$

conditions on the coefficients of $p'_1, \ldots, p'_\ell$ (where b is the codimension of $\langle B \rangle$ in Σ and $\sum'$ sums over all $1 \leq i \leq \ell$ such that p_i comes from some Q_j with $1 \leq j \leq s$; clearly $\sum' m_i \leq \sum_{i=1}^{s} \frac{1}{2} d_i(d_i+1)$). On the other hand, requiring all p_i to contain a line passing through w imposes at least $m_i + 1$ (resp. 1) on coefficients of p_i if p_i comes from the equation of Q_j for some $j > s$ (resp. $j \leq s$), and we get a total of at least

$$\sum_{i=s+1}^{r} \frac{1}{2} a_i(a_i+1) - (r-s) \geq \sum_{i=s+1}^{r} \frac{1}{2} a_i(a_i+1) - r$$

$$\geq 2 \sum_{i=s+1}^{r} a_i - r \geq 2 \left(n + r - 2 - \sum_{i=1}^{s} d_i \right) - r \geq n + 1$$

conditions (to see the second inequality, note that $\sum_{i=s+1}^{r} a_i \geq n + r - 2 - \sum_{i=1}^{s} d_i \geq 3r$ and the quadratic form $\sum a_i(a_i+1)$ is minimized when the differences between the a_i's are at

most 1, hence we may assume $a_i \geq 3$ for all $s + 1 \leq i \leq r$). The remaining cases can be treated in a similar and easier fashion as in Lemma 4.5.

Recall that a smooth P-regular complete intersection $X \subseteq \mathbb{P}^{n+r}$ of dimension $n \geq 2r + 3$ is automatically r-strongly P-regular, we obtain the following immediate corollary in the spirit of [Puk18].

COROLLARY 4.7. *Let $X \subseteq \mathbb{P}^{n+r}$ be a smooth Calabi-Yau complete intersection of dimension n by hypersurfaces of degrees $d_1 \leq \cdots \leq d_r$. Assume that $n \geq 2r + 3$, $d_r \geq 12$ and X is P-regular. Then $\mathrm{lct}(X; |H|_{\mathbb{Q}}) = 1$ where H is the hyperplane class.*

Proof. Since X is CY, $d = \sum_{i=1}^r d_i = n + r + 1$, hence by the definition of β_i, the product $\prod_{i \geq 1} \beta_i = \beta_1 \beta_2 \cdots \beta_{n+r-2}$ is obtained by removing three smallest factors (and then adding r factors of 1) from the following expression

$$\frac{2}{1} \cdot \frac{3}{2} \cdot \cdots \cdot \frac{d_1}{d_1 - 1} \cdot \frac{2}{1} \cdot \cdots \cdot \frac{d_2}{d_2 - 1} \cdot \cdots \cdot \frac{2}{1} \cdot \cdots \cdot \frac{d_r}{d_r - 1},$$

which equals $\deg X = d_1 d_2 \cdots d_r$. As $d_r \geq 12$, we get

$$\prod_{i \geq 1} \beta_i \geq \frac{d_r - 3}{d_r - 2} \cdot \frac{d_r - 2}{d_r - 1} \cdot \frac{d_r - 1}{d_r} \cdot \deg X \geq \frac{3}{4} \deg X. \tag{4.9}$$

By rearranging the sequence $q_1, \ldots, q_d$ we may assume that the first quadratic term q_{r+1} comes from the degree d_r equation and hence $\beta_{r+1} = \frac{3}{2} > 1$. As X is r-strongly P-regular under our assumptions, we have

$$\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq \min \left\{ 1, \frac{2}{\beta_{r+1} \deg X} \prod_{i \geq 1} \beta_i \right\} = \min \left\{ 1, \frac{4}{3 \deg X} \prod_{i \geq 1} \beta_i \right\} \geq 1$$

by Lemma 4.4 and (4.9).

For the construction of K-unstable Fano varieties, we will need the special case of codimension 2 complete intersections where one of the defining hypersurfaces has a cone singularity.

COROLLARY 4.8. *Let $e \geq 2$, $n \geq 8 + \max\{2e, \frac{1}{2} e(e+1)\}$ be integers and let $f(x_0, \ldots, x_n)$ and $g(x_0, \ldots, x_{n+1})$ be homogeneous polynomials of degrees e and $n + 2 - e$, respectively. Assume that the coefficients of f and g are general and let $X = (f = g = 0) \subseteq \mathbb{P}^{n+1}$, then $\mathrm{lct}(X; |H|_{\mathbb{Q}}) = 1$ where H is the hyperplane class.*

Proof. By Lemma 4.6, X is P-regular. By assumption, $n + 2 - e \geq 12$ and X is Calabi–Yau. Hence the result follows directly from Corollary 4.7.

When the dimension n is not too small, Pukhlikov essentially proves that a general hypersurface of degree $d \geq n + 1$ is 2-strongly P-regular. Combining this with Lemma 4.4, we obtain:

COROLLARY 4.9. *Let $n \geq 5$ and let $X \subseteq \mathbb{P}^{n+1}$ be a general hypersurface of degree $d \geq n + 1$, then $\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq \frac{n+1}{d}$.*

Proof. Consider the following variant of the m-strongly P-regular condition on complete intersections (we use the same notation as in Definition 4.3):

(I_m) For any linear form $h = h(z_*)$ not in the span of $q_1, \ldots, q_r$, the algebraic set $Z = (h = q_1 = \cdots = q_m = 0) \cap X$ is equidimensional and for any irreducible component Z_i of Z, we have

$$\frac{\mathrm{mult}_x}{\deg}(Z) = \frac{\mathrm{mult}_x}{\deg}(Z_i).$$

When $m = 2$ and X is a hypersurface, this condition is implied by the conditions (R1.2)–(R1.3) of [Puk05, Section §2.1], thus by Lemma 4.5 and [Puk05, Proposition 4], a general hypersurface of degree $d \geq n+1$ is P-regular and satisfies the condition (I_2) when $n \geq 5$ (although [Puk05, Proposition 4] is only stated for degree $n+1$ hypersurfaces, the same proof works in general). It is also not hard to see that the proof of Lemma 4.4 applies without change to P-regular complete intersections that satisfy (I_{m-1}). In particular, taking $m = 3$ in Lemma 4.4, we have $\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq 3(n-1)/2d \geq (n+1)/d$ as desired.

Again, the case when X is a Calabi–Yau hypersurface is of particular importance in our construction of weakly exceptional singularities and strictly K-semistable Fano varieties with largest alpha invariant.

COROLLARY 4.10. *Let $X \subseteq \mathbb{P}^{n-1}$ be a general hypersurface of degree n where $n = 4$ or $n \geq 7$, then*

$$\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq \frac{n-1}{n}.$$

Proof. When $n = 4$, the result follows from [ACS18, Theorem 1.2 and Lemma 3.2]. If $n \geq 7$, this is a special case of Corollary 4.9.

COROLLARY 4.11. *Let $f = f(x_1, \ldots, x_n)$ be a general homogeneous polynomials of degree n where $n = 4$ or $n \geq 7$, then the singularity $0 \in (f(x_1, \ldots, x_n) + x_{n+1}^{n+1} = 0) \subseteq \mathbb{A}^{n+1}$ is weakly exceptional.*

Proof. Denote the given singularity by $(0 \in X)$. The weighted blow-up of $\mathbb{A}^{n+1}$ with weights $(n+1, \ldots, n+1, n)$ induces a plt blow-up $\pi : (E \subseteq Y) \to (0 \in X)$ with Kollár component E. The corresponding log Fano pair $(E, \mathrm{Diff}_E(0))$ is isomorphic to (c.f. [Kud01, Example 2.4])

$$\left((f(x_1, \ldots, x_n) + x_{n+1} = 0) \subseteq \mathbb{P}(1, \ldots, 1, n), \frac{n}{n+1}(x_{n+1} = 0) \right) = \left(\mathbb{P}^{n-1}, \frac{n}{n+1}S \right),$$

where $S \subseteq \mathbb{P}^{n-1}$ is the hypersurface defined by $f(x_1, \ldots, x_n)$. By Theorem 2.7, our statement is equivalent to

$$\alpha\left(\mathbb{P}^{n-1}, \frac{n}{n+1}S \right) \geq 1. \tag{4.10}$$

To see this, let $D \subseteq \mathbb{P}^{n-1}$ be an effective divisor such that $D \sim_{\mathbb{Q}} H \sim_{\mathbb{Q}} -\frac{n+1}{n}(K_{\mathbb{P}^{n-1}} + n/(n+1)S)$, we need to show that $(\mathbb{P}^{n-1}, n/(n+1)(S+D))$ is lc. We may assume that D is irreducible since being lc is preserved under convex linear combinations. If D is supported on S then $D = \frac{1}{n}S$ and the result is clear. If D is not supported on S then by Corollary 4.10, the pair $(S, (n-1)/nD|_S)$ is lc, hence by inversion of adjunction, $(\mathbb{P}^{n-1}, S + (n-1)/nD)$ is lc. Since $\mathrm{lct}(\mathbb{P}^{n-1}; |H|_{\mathbb{Q}}) = 1$, the pair $(\mathbb{P}^{n-1}, D)$ is also lc. By taking convex linear combinations,

we see that $(\mathbb{P}^{n-1}, n/(n+1)(S+D))$ is lc as well since

$$\frac{n}{n+1}(S+D) = \frac{n}{n+1}\left(S+\frac{n-1}{n}D\right) + \frac{1}{n+1}D.$$

This finishes the proof.

REMARK 4.12. It is not true that $\alpha\left(\mathbb{P}^{n-1}, n/(n+1)S\right) \geq 1$ for every smooth hypersurface $S \subseteq \mathbb{P}^{n-1}$ of degree n. For example, let S be a hypersurface with an Eckardt point at $x = [1:0:\cdots:0]$, for example, $S = (x_2 g(x_1,\ldots,x_n) + h(x_3,\ldots,x_n) = 0)$ (where $x_1,\ldots,x_n$ are the homogeneous coordinates of $\mathbb{P}^{n-1}$) and let $D = (x_2 = 0)$, then it is not hard to see that $(\mathbb{P}^{n-1}, n/(n+1)(S+D))$ is not lc by considering the weighted blow-up at x with weights $(w_2,\ldots,w_n) = (n,1,\ldots,1)$. In particular, the examples of [Kud01, Example 2.4] and [Pro00, Example 4.19] (which are claimed to be weakly exceptional there) are actually not weakly exceptional.

Our next goal is to show that a general complete intersection (of given codimension) is m-strongly P-regular when $n \gg 0$ so that we can apply Lemma 4.4 to a larger range of complete intersections. This intuitively clear result is not needed in the rest of the paper but might be of independent interest. Readers who are mainly interested in the construction of strictly K-semistable and K-unstable Fano varieties with suitably large alpha invariants may skip this part and proceed to the next section.

The proof of m-strong P-regularity is done in several steps, spreading from Lemma 4.13 to 4.16. The idea is to introduce a stronger condition by requiring the complete intersection $(h = q_1 = \cdots = q_m = q_{1,d_1} = \cdots = q_{r,d_r} = 0)$ (recall that each q_i or $q_{i,j}$ is a homogeneous term of the local expression of one of the defining equations of X) to be normal (i.e., smooth in codimension 1). We then show that the number of conditions imposed by the failure of this property at a given point $x \in \mathbb{P}^{n+r}$ is at least quadratic in n, thus since the point x only varies in an $(n+r)$-dimensional family, a general complete intersection will satisfy the strong P-regularity as we want.

LEMMA 4.13. *Let $X \subseteq \mathbb{P}^n$ be a quasi-projective variety of dimension m. Let*

$$\mathcal{S} = \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$$

be the space of all hypersurface of degree d and $\mathcal{S}_X$ the subset of hypersurfaces containing X, then

$$\mathrm{codim}_{\mathcal{S}}\, \mathcal{S}_X \geq \binom{m+d}{d}.$$

Proof. Clearly $\mathcal{S}_X$ is a linear subspace of $\mathcal{S}$. Let $\phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ be a general linear projection whose restriction to X has dense image, and let $\mathcal{N} = \phi^{-1}|\mathbb{P}^m, \mathcal{O}_{\mathbb{P}^m}(d)| \subseteq \mathcal{S}$, then by construction none of the hypersurfaces in $\mathcal{N}$ contain X, thus $\mathcal{N} \cap \mathcal{S}_X = \emptyset$ and we have $\mathrm{codim}_{\mathcal{S}}\mathcal{S}_X \geq \dim\mathcal{N} + 1 = \binom{m+d}{d}$.

LEMMA 4.14. *Let $c \leq q$ and $d_1 \leq \cdots \leq d_q$ be positive integers. Let $X \subseteq \mathbb{A}^n_{x_1,\ldots,x_n}$ be a quasi-projective variety (i.e., it's locally closed in $\mathbb{A}^n$) of dimension $m \geq c$ and let $f_1,\ldots,f_q$ be regular functions on X. Let $\mathcal{T}$ be the set of $g_i \in k[x_1,\ldots,x_n]_{\leq d_i}(i = 1,\ldots,q)$ such that*

$Z = Z(f_1 + g_1, \ldots, f_q + g_q)$ *has codimension at most c in X, then*

$$\operatorname{codim}_{\mathcal{S}} \mathcal{T} \geq \sum_{i=1}^{q-c} \binom{m-c+d_i}{d_i},$$

where $\mathcal{S} = \prod_{i=1}^{q} k[x_1, \ldots, x_n]_{\leq d_i}$.

Proof. Let $\mathcal{T}_j \subseteq \mathcal{S}$ be the set of $(g_i)_{1 \leq i \leq q}$ such that Z has codimension exactly j. By induction on c it suffices to show that

$$\operatorname{codim}_{\mathcal{S}} \mathcal{T}_c \geq \sum_{i=1}^{q-c} \binom{m-c+d_i}{d_i}. \tag{4.11}$$

Let $\sigma = (i_1, \ldots, i_c) \subseteq \{1, 2, \ldots, q\}$ be an index set and let $\mathcal{T}_{c,\sigma} \subseteq \mathcal{S}$ be the set of (g_i) such that $Z_\sigma = \cap_{i \in \sigma} Z(f_i + g_i)$ has codimension c in X, then clearly

$$\mathcal{T}_c \subseteq \bigcup_{|\sigma|=c} \mathcal{T}_{c,\sigma}. \tag{4.12}$$

Fix an index set $\sigma = (i_1, \ldots, i_c)$ and a subsequence $(g_j)_{j \in \sigma}$ such that $Z_\sigma = \cap_{i \in \sigma} Z(f_i + g_i)$ has codimension c in X, then $(g_1, \ldots, g_q) \in \mathcal{T}_c$ if and only if $f_i + g_i$ vanishes on some irreducible component of Z_σ for each $i \notin \sigma$. Then set of such $(g_i)_{i \notin \sigma}$ is either empty or (after translating by an element in this set) isomorphic to the set of g_i that vanishes on some irreducible component of Z_σ. As Z_σ is a complete intersection of dimension $m - c$, this imposes at least $\binom{m-c+d_i}{d_i}$ conditions on g_i by Lemma 4.13 (we identify $k[x_1, \ldots, x_n]_{\leq d}$ with $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$). Since this holds for every $(g_j)_{j \in \sigma}$, we get

$$\operatorname{codim}_{\mathcal{S}} (\mathcal{T}_{c,\sigma} \cap \mathcal{T}_c) \geq \sum_{i \notin \sigma} \binom{m-c+d_i}{d_i}.$$

Combining this with (4.12) and $d_1 \leq \cdots \leq d_q$, we obtain the desired inequality (4.11).

LEMMA 4.15. *Given integers $m, d, c \geq 1$, then for any integer $n \geq m + c$ and any smooth quasi-projective variety $X \subseteq \mathbb{P}^n$ of codimension m we have $\operatorname{codim}_{\mathcal{S}}(\mathcal{S} \setminus \mathcal{T}) \geq p(n)$, where*

$$p(n) = \min\left\{ \binom{n-m+d}{d}, \left(\frac{n-m-1}{2} - c \right) \binom{n-m-c+d-1}{d-1} \right\}$$

and $\mathcal{S} = \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$ is the space of all hypersurface of degree d and $\mathcal{T}$ is the subset of degree d hypersurfaces D such that $X \cap D \subsetneq X$ is smooth in codimension c.

Note that $p(n)$ grows like Cn^d for some constant $C > 0$ when $n \gg 0$.

Proof. Let $x \in X$. Since X is smooth, we can find linear affine coordinates $x_1, \ldots, x_n$ at x such that $\mathrm{d}x_i$ $(i = m+1, \ldots, n)$ generate Ω_X^1 in a neighbourhood of x. We may cover X by finitely many open subset with this property and it suffices to prove the lemma for each of these. Thus we may assume $X \subseteq \mathbb{A}^n$ is locally closed with coordinates $x_1, \ldots, x_n$ chosen as above. Note that up to projectivization, $\mathcal{S}$ can be identified with $k[x_1, \ldots, x_n]_{\leq d}$, the space of degree at most d polynomials. Also note that we may assume $n \gg 0$ in proving the statement. By Lemma 4.13, the set of hypersurfaces D in $\mathcal{S}$ that contains X has codimension at least $\binom{n-m+d}{d} \geq p(n)$. Thus it suffices to find $p(t)$ such that requiring $D \cap X$ to be singular in codimension c imposes at least $p(n)$ conditions on the defining equation of D.

Now let $q = \lfloor \frac{n-m}{2} \rfloor$ and consider

$$f = f_0 + \sum_{i=1}^{q} x_{m+i} g_i \in k[x_1, \ldots, x_n]_{\leq d},$$

where $f_0 \in k[x_1, \ldots, x_n]_{\leq d}$ and $g_i \in k[x_{n-q+1}, \ldots, x_n]_{\leq d-1}$. If we can show that requiring $Z(f) \cap X$ to be singular in codimension c imposes at least $p(n)$ conditions on the $(q+1)$-tuple $(f_0, g_1, \ldots, g_q)$ for some degree d polynomial p, then we are done as then for each choice of $g_1, \ldots, g_q$ we would need the same number of $\geq p(n)$ conditions on f_0. For this it suffices to show that for each f_0, requiring $Z(f) \cap X$ to be singular in codimension c imposes at least $p(n)$ conditions on $(g_1, \ldots, g_q)$.

By the Jacobian criterion, $Z(f) \cap X$ is singular at $x \in X$ if and only if $\mathrm{d}f(x) = 0 \in \Omega_X^1 \otimes k(x)$. Since $\mathrm{d}x_i$ ($i = m+1, \ldots, n$) is a global basis of Ω_X^1, we may rewrite $\mathrm{d}f = \mathrm{d}f_0 + \sum_{i=1}^{q}(g_i \mathrm{d}x_{m+i} + x_{m+i}\mathrm{d}g_i)$ as a linear combinations of $\mathrm{d}x_i$ ($i = m+1, \ldots, n$):

$$\mathrm{d}f = \sum_{i=1}^{q}(f_i + g_i)\mathrm{d}x_{m+i} + \sum_{j=m+q+1}^{n} h_j \mathrm{d}x_j,$$

where f_i and h_j are regular functions on X and as $\frac{\partial g_i}{\partial x_j} = 0$ when $1 \leq i \leq q$ and $1 \leq j \leq m+q$, the f_i's do not depend on g_i (they only depend on $\mathrm{d}f_0$ and the linear relations among $\mathrm{d}x_i$). It then follows that the singular locus of $Z(f) \cap X$ is contained in the zero locus of $f_i + g_i$ ($i = 1, \ldots, q$), hence $Z(f_1 + g_1, \ldots, f_q + g_q) \subseteq X$ would have codimension at most c if $Z(f) \cap X$ is singular in codimension c. By Lemma 4.14, this imposes at least $(q - c)\binom{n-m-c+d-1}{d-1} \geq \left(\frac{n-m-1}{2} - c\right)\binom{n-m-c+d-1}{d-1}$ conditions on $(g_1, \ldots, g_q)$.

LEMMA 4.16. *Given integers $m, r \geq 1$ and let $n \gg 0$. Then for any $2 \leq d_1 \leq \cdots \leq d_r$, a general complete intersection $X \subseteq \mathbb{P}^{n+r}$ by hypersurfaces of degrees $d_1, \ldots, d_r$ is m-strongly P-regular.*

Proof. By Lemma 4.5, a general such complete intersection is P-regular, hence we only need to verify the irreducibility and reducedness of $Z = (h = q_1 = \cdots = q_m = 0) \cap X$ in Definition 4.3. As noted before, we may assume $m > r$ (otherwise by Lefschetz hyperplane theorem it suffices to take $n \geq 2r+3$). As in the proof of P-regularity, it suffices to show that for any $x \in \mathbb{P}^{n+r}$ and any choice of the linear terms $h, q_1, \ldots, q_r$, the failure of the irreducibility or reducedness of Z imposes at least $2n$ conditions on the coefficients of the remaining q_i. Let $p_1, \ldots, p_\ell$ ($\ell \leq m$) be the sequence obtained by removing terms that appear more than once from the sequence $q_{r+1}, \ldots, q_m, q_{1,d_1}, \ldots, q_{r,d_r}$ and then restricting to the linear subspace $\Sigma = (h = q_1 = \cdots = q_r = 0) = (h = 0) \cap \mathbb{P}(T_x X) \cong \mathbb{P}^{n-2}$. Note that $W = (p_1 = \cdots = p_\ell = 0) \cap \Sigma$ can be identified with $Z \cap H_\infty$ where H_∞ is the hyperplane at infinity and we may assume that W has codimension ℓ by a similar proof of P-regularity, hence if W is irreducible and reduced, then the same holds for Z. Therefore, it suffices to show that for W to be reducible or nonreduced imposes at least $2n$ conditions on the coefficients of $p_1, \ldots, p_\ell$. Let $m_i = \deg p_i \geq 2$.

Assume from now on $n \gg 0$. Apply Lemma 4.15 to $W_0 = \Sigma$, we see that requiring $W_1 = W_0 \cap (p_1 = 0)$ to be singular in codimension m imposes more than $2n$ conditions on p_1 (the number of conditions is at least a polynomial of degree 2 in n). Fix a choice of p_1 such that W_1 is smooth in codimension m. By Lemma 4.15 applied to the smooth locus of W_1, we again need more than $2n$ conditions on p_2 for $W_2 = W_1 \cap (p_2 = 0)$ to be singular in

codimension $m-1$. Continue in this way, we see that requiring $W = W_i$ ($1 \leq i \leq \ell$) to be singular in codimension $m+1-i \geq 1$ imposes more than $2n$ conditions on the coefficients of $p_1, \ldots, p_i$. Since each W_i is already a complete intersection, they are also normal if they are smooth in codimension 1; an easy induction using [Har77, Corollary 7.9] then shows that all the W_i are connected, normal and therefore integral. In particular, $W = W_\ell$ is integral. In other words, asking W to be reducible or nonreduced imposed more than $2n$ conditions on coefficients of $p_1, \ldots, p_\ell$ and we are done.

Combining this with Lemma 4.4 in the hypersurface case, we can now give the proof of Theorem 1.4.

Proof of Theorem 1.4. When $d \leq n$, the result follows from [Che01, Theorem 1.3]. If $d \geq n+1$, then choose sufficiently large integer m such that $\frac{2(m+1)}{(2-\epsilon)(m+2)} > 1$, by Lemma 4.16 a general hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d is m-strongly P-regular when $n \gg 0$, thus by Lemma 4.4, we have

$$\mathrm{lct}(X; |H|_{\mathbb{Q}}) \geq \min\left\{1, \frac{m+1}{m+2} \cdot \frac{2(n-1)}{d}\right\} \geq \min\left\{1, \frac{2(m+1)}{(2-\epsilon)(m+2)} \cdot \frac{n-1}{n}\right\} \geq 1$$

and the result follows since we always have $\mathrm{lct}(X; |H|_{\mathbb{Q}}) \leq 1$.

§5. Main constructions

5.1 Strictly K-semistable example

In this section, we construct singular Fano varieties X of dimension $n \gg 1$ that are strictly K-semistable and $\alpha(X) = n/(n+1)$.

By Theorem 3.1, every such Fano variety has at least one weakly exceptional singularity x whose corresponding (unique) Kollár component has log discrepancy n. One such weakly exceptional singularity is given by Corollary 4.11, with corresponding Kollár component $F \cong \mathbb{P}^{n-1}$ and different $\Delta_F = (1 - 1/(n+1))S$ where $S = (f(x_1, \ldots, x_n) = 0)$ is a general smooth hypersurface of degree n in $\mathbb{P}^{n-1}$. We first construct the K-polystable degeneration as in Corollary 3.2. Let $M := n/(n+1)S - (n-1)H$ be a $\mathbb{Q}$-divisor on $\mathbb{P}^{n-1}$ where H is a hyperplane in $\mathbb{P}^{n-1}$. Let X_0 be the projective orbifold cone $C_p(\mathbb{P}^{n-1}, M)$.

LEMMA 5.1. *Let $n = 4$ or $n \geq 7$. Then the projective orbifold cone X_0 is a K-polystable Fano variety. Moreover, X_0 is isomorphic to the hypersurface $(x_0 f(x_1, \ldots, x_n) + x_{n+1}^{n+1} = 0)$ in $\mathbb{P}^{n+1}$ and under this isomorphism, $\mathbb{P}_\infty^{n-1}$ can be identified with the locus $(x_0 = x_{n+1} = 0)$.*

Proof. Since $\alpha(\mathbb{P}^{n-1}, (1 - 1/(n+1))S) \geq 1$ for a general smooth hypersurface $S \subset \mathbb{P}^{n-1}$ of degree n by Corollary 4.11, we know that $(\mathbb{P}^{n-1}, (1 - 1/(n+1))S)$ is K-stable. Hence the K-polystability of $X_0 = C_p(\mathbb{P}^{n-1}, M)$ follows from Proposition 2.11 and the fact that $-K_{\mathbb{P}^{n-1}} - (1 - 1/(n+1))S \sim_{\mathbb{Q}} nM$. By definition, we have

$$X_0 = \mathrm{Proj} \bigoplus_{m=0}^{\infty} \bigoplus_{i=0}^{\infty} H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(\lfloor mM \rfloor)) \cdot s^i.$$

Denote by $R_{m,i} := H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(\lfloor mM \rfloor)) \cdot s^i$ and $R := \oplus_{m=0}^{\infty} \oplus_{i=0}^{\infty} R_{m,i}$. Let L be the coherent sheaf $\mathcal{O}(n+1)$ on X_0 with respect to the grading $R_{m,i} \mapsto m+i$. Then $L \cong \mathcal{O}_{X_0}((n+1)\mathbb{P}_\infty^{n-1})$; since $(n+1)M$ is Cartier on $\mathbb{P}^{n-1}$, we also see that L is an ample line bundle on X_0. We will choose sections $s_0, s_1, \ldots, s_{n+1}$ of L to obtain the desired embedding from X_0 to $\mathbb{P}^{n+1}$.

To begin with, let us choose $s_0 := 1 \cdot s^{n+1} \in R_{0,n+1}$. We also fix a generator z of $H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(S - nH))$. Since $\lfloor nM \rfloor = (n-1)(S - nH)$, we choose $s_{n+1} := z^{n-1} \cdot s \in R_{n,1}$. Let us choose sections $y_1, \ldots, y_n \in H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(H))$ that correspond to projective coordinates $x_1, \ldots, x_n$ of $\mathbb{P}^{n-1}$. Then $y_i z^n$ is a section of $\mathcal{O}_{\mathbb{P}^{n-1}}(nS - (n^2 - 1)H) = \mathcal{O}_{\mathbb{P}^{n-1}}(\lfloor (n+1)M \rfloor)$. For each $1 \le i \le n$, we choose $s_i := y_i z^n \cdot 1 \in R_{n+1,0}$.

To prove that $[s_0 : \cdots : s_{n+1}]$ induces the embedding from X_0 to $\mathbb{P}^{n+1}$, we first show that the linear system generated by $\{s_0, \ldots, s_{n+1}\}$ is base point free. Let $I \subset R$ be the graded ideal generated by $\{s_0, \ldots, s_{n+1}\}$. Thus $s \in \sqrt{I}$ since $s_0 = s^{n+1} \in I$. Thus $R_{m,i} = R_{m,0} \cdot s^i \subset \sqrt{I}$ whenever $i > 0$. If $m > 0$ and $i = 0$, then any element in $R_{(n+1)m,0}$ is a polynomial in $s_1, \ldots, s_n$. Thus $R_{(n+1)m,0} \subset I$ which implies $R_{m,0} \subset \sqrt{I}$ for any $m > 0$. As a result, the base point freeness of the linear system generated by $\{s_0, \ldots, s_{n+1}\}$ follows from $R_+ = \sqrt{I}$. Let us denote the induced morphism by $\phi : X_0 \to \mathbb{P}^{n+1}$.

Let X_0' be the hypersurface in $\mathbb{P}^{n+1}$ defined by $(x_0 f(x_1, \ldots, x_n) + x_{n+1}^{n+1} = 0)$. We will show that the image $\phi(X_0)$ is contained in X_0', that is, $s_0 f(s_1, \ldots, s_n) + s_{n+1}^{n+1} = 0$. It is clear that

$$s_{n+1}^{n+1} = z^{n^2-1} \cdot s^{n+1}, \quad s_0 f(s_1, \ldots, s_n) = f(y_1, \ldots, y_n) z^{n^2} \cdot s^{n+1}.$$

Both terms are considered inside $R_{n(n+1),n} = H^0(\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(n(n+1)M)) \cdot s^{n+1}$. Meanwhile, z^{n^2-1} is identified with a section of $\mathcal{O}_{\mathbb{P}^{n-1}}((n+1)\lfloor nM \rfloor)$ where $n(n+1)M = (n+1)\lfloor nM \rfloor + S$. Hence we may assume that $-f(y_1, \ldots, y_n)z$ is the canonical section of $\mathcal{O}_{\mathbb{P}^{n-1}}(S)$ using which the product structure of R is defined. Therefore,

$$s_{n+1}^{n+1} = z^{n^2-1}(-f(y_1, \ldots, y_n)z) \cdot s^{n+1} = -s_0 f(s_1, \ldots, s_n).$$

So far we have constructed a morphism $\phi : X_0 \to X_0'$. It remains to show that ϕ is an isomorphism. Since L is ample, ϕ is finite. We also know that $(\mathcal{O}_{X_0'}(1)^n) = n+1$ and

$$(L^n) = (n+1)^n (\mathbb{P}_\infty^{n-1})^n = (n+1)^n (M^{n-1}) = n+1,$$

hence ϕ is birational. It is easy to check that the singular locus of X_0' is the point $[1 : 0 : \cdots : 0]$ union $(x_0 = x_{n+1} = f = 0)$, hence X_0' is normal. Thus the Zariski main theorem implies that ϕ is an isomorphism. It is clear from the construction that $\phi^*(x_0 = x_{n+1} = 0) = \mathbb{P}_\infty^{n-1}$ as $\phi^*(x_0 = 0) = (s^{n+1} = 0) = (n+1)\mathbb{P}_\infty^{n-1}$. We finish the proof.

The strictly K-semistable examples as referred to in Theorem 1.3 are now obtained by deforming the above K-polystable examples.

PROPOSITION 5.2. *Let $n = 4$ or $n \ge 7$. Let $f(x_1, \ldots, x_n)$ and $g(x_1, \ldots, x_{n+1})$ be general homogeneous polynomials of degree n and $n+1$, respectively. Let $X \subseteq \mathbb{P}^{n+1}$ be the hypersurface defined by the equation $(x_0 f + g = 0)$. Then $\alpha(X) = n/(n+1)$ and X is strictly K-semistable. In addition, X has a unique singular point at $[1 : 0 : \cdots : 0]$.*

Proof. Since f and g are general, X has a unique isolated singularity at $x = [1 : 0 : \cdots : 0]$ and we may assume that the coefficient of x_{n+1}^{n+1} is nonzero in g. Using the one parameter subgroup $[x_0 : x_1 : \cdots : x_n : x_{n+1}] \mapsto [x_0 : t^{n+1}x_1 : \cdots : t^{n+1}x_n : t^n x_{n+1}]$ $(t \in \mathbb{C}^*)$ of $\mathrm{Aut}(\mathbb{P}^{n+1})$, it is not hard to see that X specially degenerates to the K-polystable Fano variety X_0 as in Lemma 5.1. Denote by $(\mathcal{X}; \mathcal{L})$ this special test configuration of X. Since X_0 is K-polystable, we know that $\mathrm{Fut}(\mathcal{X}; \mathcal{L}) = 0$ (it can be viewed as the Futaki invariant of the induced $\mathbb{G}_m$-action on X_0) and X is K-semistable by [BL18a, Corollary 4]. As X is not isomorphic to X_0, we conclude that X is strictly K-semistable.

We next show that $\alpha(X) \geq n/(n+1)$. It then follows from [OS12, Theorem 1.4] and the above consideration that $\alpha(X) = n/(n+1)$ and X is strictly K-semistable. Let $V = (x_0 = x_{n+1} = 0) \subseteq X_0$ and let $D \sim_{\mathbb{Q}} -K_X \sim H$ be an effective $\mathbb{Q}$-divisor on X (where H is the hyperplane class). Assume that D degenerates to $D_0 \subseteq X_0$ under the above special degeneration of X to X_0. Write $D_0 = tV + D'$ where $V \nsubseteq \mathrm{Supp}(D')$. We now separate into two cases.

Assume first that $t \leq 1$. We claim that $(X_0, n/(n+1)D_0)$ is lc and it then follows from inversion of adjunction (or the lower semicontinuity of lct) that $(X, n/(n+1)D)$ is also lc. As D_0 is $\mathbb{G}_m$-invariant by construction, it suffices to check that $(X_0, n/(n+1)D_0)$ is lc at x and along V (any non-lc center is $\mathbb{G}_m$-invariant and therefore either contains x or is contained in V). Let $\pi : Y_0 \to X_0$ be the weighted blowup as above that extracts the Kollár component F over x. Note that $-K_X \sim H$ and the linear system $|\pi^* H - (n+1)F|$ is base point free (its members include the strict transform of $(x_i = 0)$ for $i = 1, \ldots, n$), hence $\epsilon(F) \geq n+1$; on the other hand, since $(\pi^* H - (n+1)F)^n = 0$, we also have $\epsilon(F) \leq \tau(F) \leq n+1$ by the inequality

$$\left((\pi^* H - (n+1)F)^{n-1} \cdot (\pi^* H - \tau(F)F)\right) \geq 0,$$

hence $\epsilon(F) = \tau(F) = n+1$. As $A_{X_0}(F) = n$, we see that $a(F; X_0, n/(n+1)D_0) \geq -1$, hence by the following Lemma 5.3, $(X_0, n/(n+1)D_0)$ is lc at x. Similarly, as $K_{X_0} + V + n/(n+1)D_0 \sim_{\mathbb{Q}} 0$, $K_V + \Delta_V + n/(n+1)D'|_V = (K_{X_0} + V + n/(n+1)D')|_V$ is anti-nef (where Δ_V is the different), hence since $(V, \Delta_V) \cong (F, \Delta_F)$, the pair $(X_0, V + n/(n+1)D')$ is lc along V by the exact same proof of Lemma 5.3. Therefore as $t \leq 1$, $(X_0, n/(n+1)D_0)$ is also lc along V, proving our claim.

On the other hand, if $t > 1$, then $D' \sim_{\mathbb{Q}} cH$ for some $c \leq n/(n+1)$ and by the argument above, (X_0, D') is lc at x. Note that $x \notin V$, it follows that (X_0, D_0) is also lc at x and by inversion of adjunction (X, D) is lc at x as well. If Z is a positive dimensional non-lc center of (X, D), then Z is contained in the smooth locus of X and we have $\mathrm{mult}_Z D > 1$ by [Kol97, 3.14.1]. But by [Puk02, Proposition 5], $\mathrm{mult}_Z D \leq 1$, a contradiction. Thus the non-lc locus of (X, D) is a union of isolated smooth points. By [Zhu20, Theorem 1.6] (with $\Delta = L = 0$), we immediately obtain $\mathrm{lct}(X; D) \geq n/(n+1)$. Therefore, in both cases $(X, n/(n+1)D)$ is lc. As D is arbitrary, our proof is complete.

The following lemma is used in the above proof.

LEMMA 5.3. *Let (X, D) be pair and $x \in X$ a weakly exceptional singularity. Let $\pi : Y \to X$ be the plt blowup that extracts the unique Kollár component F over x. Assume that $a(F; X, D) \geq -1$, then (X, D) is lc at x.*

Proof. By assumption, we have $K_Y + \Gamma + \lambda F = \pi^*(K_X + D)$ where $\lambda = -a(F; X, D) \leq 1$ and Γ is the strict transform of D. Let Δ_F be the different on F. Since $x \in X$ is weakly exceptional we have $\alpha(F, \Delta_F) \geq 1$. Therefore as $K_F + \Delta_F + \Gamma|_F = (K_Y + \Gamma + F)|_F \sim_{\mathbb{Q}} (1 - \lambda)F|_F$ is anti-nef, we see that $(F, \Delta_F + \Gamma|_F)$ is lc and then by inversion of adjunction $(Y, \Gamma + F)$ is also lc along F. Hence (X, D) is lc at x.

5.2 K-unstable example

Modifying the previous strictly K-semistable example, we can also construct, for each fix integer $e \geq 2$, K-unstable Fano varieties Y of dimension $n \gg 0$ such that $\alpha(Y) = 1 - \frac{1}{n+2-e}$.

Let $e \geq 2$ and let $f(x_0, \ldots, x_n)$, $g(x_0, \ldots, x_{n+1})$, and $h(x_0, \ldots, x_n)$ be general homogeneous polynomials of degrees e, $n+2-e$, and $n+1-e$, respectively. Let $Y \subseteq \mathbb{P}^{n+2}$ be the complete intersection defined by the equation $(f = g + x_{n+2}h = 0)$. We now show that Y is K-unstable with alpha invariant $\alpha(Y) = 1 - \frac{1}{n+2-e}$ in several steps. Let $x = [0 : \cdots : 0 : 1] \in \mathbb{P}^{n+2}$.

LEMMA 5.4.   *Assume that $n \geq 11 + e$. Then with the above notations, $x \in Y$ is a weakly exceptional singularity.*

Proof.   In local coordinates, $x \in Y$ is given by the equation $(f = g + h = 0) \subseteq \mathbb{A}^{n+2}_{x_0, \ldots, x_{n+1}}$. Let $\pi : W \to Y$ be the weighted blowup at x with weights $(w_0, \ldots, w_{n+1}) = (n+2-e, \ldots, n+2-e, n+1-e)$ and let F be the exceptional divisor. Also let Δ_F be the different. Since g is general, we may assume that the coefficient of x_{n+1}^{n+2-e} is nonzero in g. We then have

$$F = \big((f = h + x_{n+1} = 0) \subseteq \mathbb{P}(1^{n+1}, n+1-e)\big) \cong \big((f = 0) \subseteq \mathbb{P}^n\big)$$

and $\Delta_F = (1 - \frac{1}{n+2-e})S$ where $S = (f = h = 0) \subseteq \mathbb{P}^n$ and it suffices to prove that $\alpha(F, \Delta_F) \geq 1$. To see this, let $D \sim_{\mathbb{Q}} -(K_F + \Delta_F) \sim_{\mathbb{Q}} (1 - \frac{1}{n+2-e})H$ be an effective divisor on F (where H is the hyperplane class). Similar to the proof of Corollary 4.11, we may assume that D is irreducible and not supported on S. Note that S is a Calabi-Yau complete intersection of codimension 2, hence by Lemma 4.5, Corollary 4.7, and our assumption that $n \geq 11 + e$, we have $(S, D|_S)$ is lc. As (F, D) is lc by [Che01, Theorem 1.3], we see that $(F, S + D)$ is also lc by inversion of adjunction. It follows that $(F, \Delta_F + D)$ is lc as well. Since D is arbitrary, we obtain $\alpha(F, \Delta_F) \geq 1$.

PROPOSITION 5.5.   *Notation as above. Then Y is not K-semistable and $\alpha(Y) = 1 - \frac{1}{n+2-e}$ when $n \geq 10 + e^2$.*

Proof.   Let $\pi : W \to Y$ be the weighted blowup that extracts the Kollár component F over x as in the previous proof. It is not hard to see that the linear system $|\pi^*(-K_Y) - (n+2-e)F|$ is base point free (consider strict transforms of the hyperplanes $(x_i = 0)$ for $0 \leq i \leq n$) and $(\pi^* H - (n+2-e)F)^n) = 0$, hence $\epsilon(F) = \tau(F) = n+2-e$ by a similar argument as in Proposition 5.2. We then have $\mathrm{vol}_Y(-\pi^* K_Y - tF) = ((-K_Y)^n) + (-t)^n(F^n)$ for all $0 \leq t \leq \tau(F)$ and a straightforward computation yields

$$\beta(F) = A_Y(F) - \frac{1}{((-K_Y)^n)} \int_0^{\tau(F)} \mathrm{vol}(-\pi^* K_Y - tF) \mathrm{d}t$$

$$= (n+1-e) - \frac{n}{n+1} \cdot (n+2-e) < 0$$

as $e \geq 2$, hence Y is not K-semistable by [Fuj19b, Corollary 1.5] or [Li17, Theorem 3.7].

Let $\alpha = 1 - \frac{1}{n+2-e}$ and assume that $n \geq 10 + e^2 > 11 + e$. We now show that $\alpha(Y) = \alpha$. Let F be the Kollár component over x. From the previous proof we have $\tau(F) = n+2-e$ and $A(F) = n+1-e$, hence $\alpha(Y) \leq \alpha$. It remains to show that $(Y, \alpha D)$ is lc for any effective divisor $D \sim_{\mathbb{Q}} -K_Y$. For this we may assume that D doesn't contain the hyperplane section $Z = (x_{n+2} = 0) \cap Y = (f = g = 0) \subseteq \mathbb{P}^{n+1}$ in its support (since (Y, Z) is lc and being lc is preserved under convex linear combination). Since f and g are general and $n \geq 10 + e^2$, by Lemma 4.6 and Corollary 4.8 we have $\mathrm{lct}(Z; |H|_{\mathbb{Q}}) = 1$ and in particular, $(Z, D|_Z)$ ic lc. By inversion of adjunction, $(Y, Z + D)$ is lc in a neighbourhood of Z and hence the non-lc center of (Y, D) is a finite union of isolated point. Therefore by [Zhu20, Theorem 1.6] (with $\Delta = L = 0$) we see that $(Y, n/(n+1)D)$ is lc over the smooth locus of Y. Note that $\alpha < n/(n+1)$,

so it remains to check that $(Y, \alpha D)$ is lc at x. As $A_{Y,\alpha D}(Y) \geq A_Y(F) - \alpha \cdot \tau(F) = 0$, this follows from Lemma 5.4 and Lemma 5.3 and we are done.

REMARK 5.6. Here we describe a K-polystable replacement of Y. Assume $n \geq 20$. Let $\pi : \mathcal{X} \to C$ be a family of codimension two complete intersections of $\mathbb{P}^{n+2}$ of degree $(e, n+2-e)$ over a smooth pointed curve $0 \in C$, such that the special fiber $\mathcal{X}_0 \cong Y$ and π is smooth over $C \setminus \{0\}$. Then by [Zhu20, Theorem 1.3] we know that $\mathcal{X}_t$ is a K-stable Fano manifold for any $t \in C \setminus \{0\}$. Thus by [DS14, CDS15, Tia15], after a finite base change $(0' \in C') \to (0 \in C)$ if necessary, there exists a $\mathbb{Q}$-Gorenstein flat projective family $\mathcal{X}'_{C'} \to C'$ such that $\mathcal{X}'_{0'}$ is a K-polystable $\mathbb{Q}$-Fano variety and $\mathcal{X}'_{C'} \setminus \mathcal{X}'_{0'} \cong \mathcal{X}_{C'} \setminus \mathcal{X}_{0'}$ where $\mathcal{X}_{C'} := \mathcal{X} \times_C C'$. Since $\mathcal{X}_{0'} \cong Y$ is K-unstable, we know that $\mathcal{X}'_{0'}$ is not isomorphic to $\mathcal{X}_{0'}$. Hence Proposition 3.3 and Proposition 5.5 implies that $\alpha(\mathcal{X}'_{0'}) \leq 1 - \alpha(\mathcal{X}_{0'}) = \frac{1}{n+2-e}$. In particular, when $e = 2$ we obtain a K-polystable $\mathbb{Q}$-Fano variety $\mathcal{X}'_{0'}$ satisfying $\alpha(\mathcal{X}'_{0'}) \leq \frac{1}{n}$.

Proof of Theorem 1.3. The existence of X follows from Proposition 5.2 and the existence of Y follows from Proposition 5.5 with $e = 2$. The fact that X and Y are both Gorenstein canonical and have a unique singular point follows from a straightforward computation.

We conclude with the following question (cf. [Jia17b, Conjecture 1.6]).

QUESTION 5.7. Let $n = \dim X \geq 2$ be an integer.

(1) Does there exist a K-semistable $\mathbb{Q}$-Fano variety X such that $1/(n+1) < \alpha(X) < \frac{1}{n}$?
(2) Does there exist a K-unstable $\mathbb{Q}$-Fano variety X such that $(n-1)/n < \alpha(X) < n/(n+1)$?

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