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Analytical solutions for turbulent Boussinesq fountains in a linearly stratified environment

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This paper theoretically investigates the initial up-flow of a vertical turbulent fountain (round or plane) in a linearly stratified environment. Conservation equations (volume, momentum and buoyancy) are written under the Boussinesq approximation assuming an entrainment proportional to the vertical velocity of the fountain. Analytical integration leads to exact values of both density and flow rate at the maximal height reached by the fountain. This maximal height is expressed as a function of the release conditions and the stratification strength and plotted from a numerical integration in order to exhibit overall behaviour. Then, analytical expressions for the maximal height are derived from asymptotic analysis and compared to experimental correlations available for forced fountains. For weak fountains, these analytical expressions constitute a new theoretical model. Finally, modified expressions are also proposed in the singular case of an initially non-buoyant vertical release.

Key words: plumes/thermals

1. Introduction

A jet with negative buoyancy, also called a fountain, occurs when a fluid is injected into another one, in such a way that the buoyancy of the release opposes its momentum. One of the pioneering investigations on this topic was by Turner (1966) for a turbulent fountain. From experiments, he had observed that such a fountain extends in its environment as a jet to a maximal height and then falls and forms a down-flow core around the up-flow. The fountain then stabilizes at a final (or steady) height lower than the maximal height due to a modification of the entrainment. From dimensional analysis, he has also suggested that the behaviour of a turbulent fountain is fully governed by the ratio between its initial buoyancy and momentum, i.e. by its initial Froude number.

Fountains are widely met, not only in the laboratory but also in environmental problems and in many industrial applications,such as for instance, explosive volcanic jets (Kaminski, Tait & Carazzo 2005), air conditioning (Baines, Turner & Campbell 1990; Williamson, Armfield & Lin 2011), replenishment of magma chambers (Bloomfield & Kerr 1999). For these reasons, in the literature many studies can be found which consider fountains in different configurations (plane or round, free or close to a wall). Also different kinds of fountains have been studied such

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as immiscible fountains (water in air, Villermaux 1994; Clanet 1998) and laminar miscible fountains (saltwater weakly injected into freshwater, Phillipe *et al.* 2005; Williamson *et al.* 2008). However, most of the studies focus on turbulent miscible fountains (Abraham 1967; McDougall 1981; Mizushina *et al.* 1982; Zhang & Baddour 1998; Pantzlaff & Lueptow 1999; Bloomfield & Kerr 2000; Kaye & Hunt 2006) since they are the most often encountered.

For a round fountain, experiments by Williamson *et al.* (2008) have shown that the flow is fully turbulent as soon as the fountain initial Reynolds number (based on the initial radius) is greater than 2000. Viscous effects then become negligible and the fountain is only governed by its initial buoyancy and momentum. In particular, both maximal and final heights of the fountain can be correlated with the initial Froude number.

Usually, for large Froude number values (release initially dominated by its momentum), the fountain is called a forced fountain, whereas for low Froude number values (release initially dominated by its buoyancy) the fountain is called a weak fountain. However, in contrast with turbulent plumes where two regimes (namely forced and lazy) are strictly separated by a given value of the Froude number (Michaux & Vauquelin 2008), there does not exist any Froude number value which allows separation of forced and weak regimes for fountains.

The modelling of turbulent miscible fountains is generally based on the theory of plumes due to Morton, Taylor & Turner (1956) which assumes that the velocity of the ambient fluid entrained at the edge of the plume (or the fountain) is proportional to the local vertical velocity. The plume is then described by mass, momentum and buoyancy-flux conservation equations with a one-dimensional formalism where viscous effects are neglected. By inverting the sign of the gravitational acceleration, this plume model can be immediately applied to describe the fountain up-flow. Many theoretical contributions based on this theory of plumes can be found in the case of a fountain in an homogeneous environment, from Abraham (1967) to Kaye & Hunt (2006). However, in some situations, fountains develop in stratified environments as in oceans, atmosphere or in large buildings, for instances. In this area, we can refer to some recent papers (Bloomfield & Kerr 1998, 2000; Lin & Armfield 2002; Papanicolaou, Papakonstantis & Christodoulou 2008) based on both theoretical and experimental investigations.

In this study, we investigate theoretically the up-flow of a fountain which develops in a linearly stable stratified environment under the Boussinesq approximation. From an analytical approach, we aim to determine the fountain maximal height and the characteristics of the fountain at this height as a function of the release conditions and the stratification strength. When possible, these analytical results will be compared with the correlations available in the literature.

2. Governing equations

The conservation equations for the starting up-flow of a round fountain are based on the classical approach developed by Morton, Taylor & Turner (1956) for turbulent plumes. Accordingly, interactions between up-flow and down-flow are not taken into account as in the works of Bloomfield & Kerr (1998) or Papanicolaou *et al.* (2008).

As illustrated in figure 1, we consider a fluid of density ρ_i vertically released with the velocity u_i from a circular opening of radius b_i, in a stratified environment whose density $\rho_0(z)$ varies linearly with respect to the vertical coordinate z. The

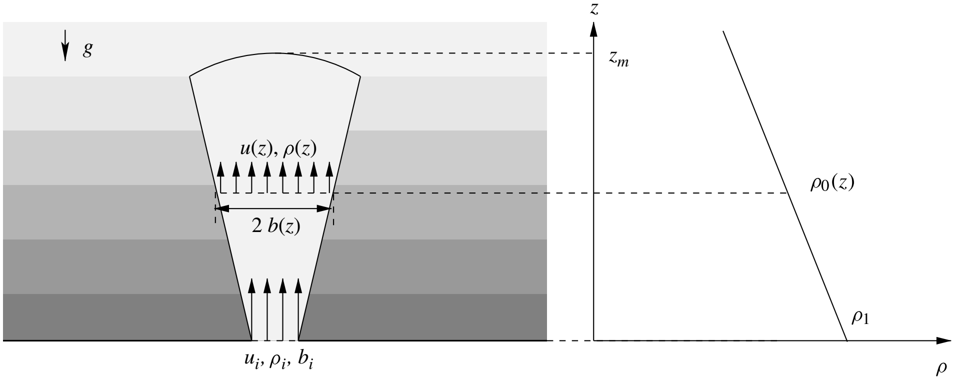

FIGURE 1. Schematic illustration of a round fountain in a stratified environment.

variables $b(z)$, $u(z)$ and $\rho(z)$ denote, respectively, the local radius, the velocity and the density of the fountain. The maximal height reached by the fountain is denoted z_m.

The stratification is quantified by the following constant:

$$N^2 = -\frac{1}{\rho_1}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}, \tag{2.1}$$

where ρ_1 is a reference density, namely the density of the ambient fluid at the source level (i.e. $z=0$). Note that by multiplying this constant with the gravitational acceleration g we obtain the classical Brunt–Väisälä frequency.

Under the Boussinesq approximation and for top-hat profiles, the conservation equations for volume, momentum and buoyancy flux can be expressed as

$$\frac{\mathrm{d}(ub^2)}{\mathrm{d}z} = 2\alpha b u, \quad \frac{\mathrm{d}(u^2b^2)}{\mathrm{d}z} = -g\eta b^2, \quad \frac{\mathrm{d}(\eta u b^2)}{\mathrm{d}z} = b^2 u N^2, \tag{2.2}$$

where the variable $\eta = (\rho - \rho_0)/\rho_1$ introduced can be interpreted as the density deficit. Analytical studies by Hunt & Kaye (2005) in the Boussinesq case, and Michaux & Vauquelin (2008) in the non-Boussinesq case (see also Van den Bremer & Hunt 2010), have shown that these equations can be rewritten in terms of a local dimensionless function defined as follows:

$$\Gamma(z) = \frac{5g\eta b}{8\alpha u^2}, \tag{2.3}$$

which considerably simplifies their integration.

In order to account for the stratified environment, another independent dimensionless function has to be introduced. For later convenience, this function is expressed as

$$\sigma(z) = \frac{u^2N^2}{g\eta^2}. \tag{2.4}$$

From the dimensionless functions Γ and σ and (2.2), governing equations for the fountain primary variables u, b and η can be written as follows:

$$\frac{\mathrm{d}b}{\mathrm{d}z} = \frac{4\alpha}{5}\left(\frac{5}{2}+\Gamma\right), \quad \frac{\mathrm{d}u}{\mathrm{d}z} = -\frac{8}{5}\frac{\alpha u}{b}\left(\frac{5}{4}+\Gamma\right), \quad \frac{\mathrm{d}\eta}{\mathrm{d}z} = \frac{2\alpha\eta}{b}\left(\frac{4}{5}\Gamma\sigma - 1\right). \tag{2.5a,b,c}$$

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In addition, from their definitions, two differential equations for $\Gamma(z)$ and $\sigma(z)$ can be established as

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}z} = \frac{4\alpha\Gamma}{b}\left[1+\left(1+\frac{2}{5}\sigma\right)\Gamma\right] \quad \text{and} \quad \frac{\mathrm{d}\sigma}{\mathrm{d}z} = -\frac{16}{5}\frac{\alpha\sigma\Gamma}{b}(\sigma+1). \tag{2.6a,b}$$

Let us note that when the fountain reaches its maximal height ($z \to z_m$) it turns out that $u(z) \to 0$ and accordingly $\Gamma(z) \to \infty$ whereas $\sigma(z) \to 0$.

3. Analytical integration

The objective is to obtain analytical expressions for the fountain maximal height and the values of the fountain variables at this location.

In a first step, by combining the two equations of (2.6*a,b*), we get

$$\frac{\mathrm{d}\Gamma}{\mathrm{d}\sigma} = -\frac{5}{4}\frac{1}{\sigma(\sigma+1)} - \frac{1}{4}\frac{5+2\sigma}{\sigma(\sigma+1)}\Gamma. \tag{3.1}$$

After integration, this equation provides us with an explicit relation between Γ and σ which is

$$\Gamma = \left[I(\sigma) - I(\sigma_i) + \frac{\Gamma_i\sigma_i^{5/4}}{(\sigma_i+1)^{3/4}}\right]\frac{(\sigma+1)^{3/4}}{\sigma^{5/4}} \tag{3.2}$$

where

$$I(\sigma) = \int_0^\sigma -\frac{5}{4}\frac{t^{1/4}}{(t+1)^{7/4}}\,\mathrm{d}t. \tag{3.3}$$

In a second step, the primary variables are expressed in terms of σ and Γ. For instance if we combine (2.5*a*) with (2.6*a*), we obtain

$$\frac{\mathrm{d}b}{\mathrm{d}\Gamma} = \frac{b}{5}\left[\frac{5}{2}\frac{1}{\Gamma} - \frac{\frac{3}{2}+\sigma}{1+\left(1+\frac{2}{5}\sigma\right)\Gamma}\right]. \tag{3.4}$$

According to (3.1), (3.4) can be re-expressed as

$$\frac{\mathrm{d}b}{b} = \frac{1}{2}\frac{\mathrm{d}\Gamma}{\Gamma} + \frac{3}{8}\frac{\mathrm{d}\sigma}{\sigma} - \frac{1}{8}\frac{\mathrm{d}\sigma}{\sigma+1}, \tag{3.5}$$

and after integration, it becomes

$$\frac{b}{b_i} = \left(\frac{\Gamma}{\Gamma_i}\right)^{1/2}\left(\frac{\sigma}{\sigma_i}\right)^{3/8}\left(\frac{\sigma_i+1}{\sigma+1}\right)^{1/8}. \tag{3.6}$$

For the other primary variables u and η, similar calculations lead to

$$\frac{u}{u_i} = \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{\sigma_i}{\sigma}\right)^{1/8}\left(\frac{\sigma_i+1}{\sigma+1}\right)^{1/8} \quad \text{and} \quad \frac{\eta}{\eta_i} = \left(\frac{\Gamma_i}{\Gamma}\right)^{1/2}\left(\frac{\sigma_i}{\sigma}\right)^{5/8}\left(\frac{\sigma_i+1}{\sigma+1}\right)^{1/8}. \tag{3.7a,b}$$

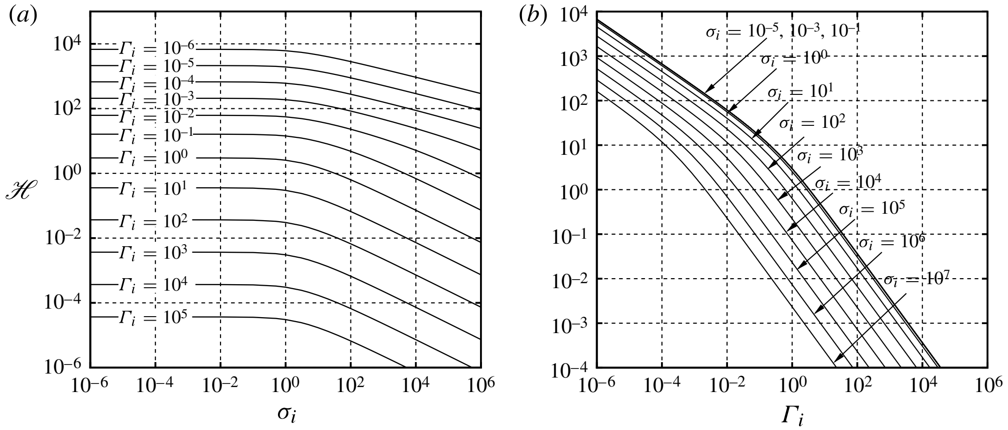

FIGURE 2. (a) Numerical solutions of (3.9) as a function of σ_i and for different values of Γ_i. (b) Numerical solutions of (3.9) as a function of Γ_i and for different values of σ_i.

In a third step, $b(\sigma, \Gamma)$ which is given by (3.6) and then $\Gamma(\sigma)$ which is given by (3.3) are substituted in (2.6b) in order to get a differential equation for $\sigma(z)$:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}z} = -\frac{\Lambda_i}{b_i}\left[I(\sigma) - I(\sigma_i) + \Gamma_i \frac{\sigma_i^{5/4}}{(\sigma_i+1)^{3/4}}\right]^{1/2} (\sigma+1)^{3/2}, \tag{3.8}$$

where the dimensionless constant $\Lambda_i = (16\alpha/5)\Gamma_i^{1/2}\sigma_i^{3/8}/(\sigma_i+1)^{1/8}$ is introduced for the sake of simplicity.

Equation (3.8) governs the evolution of σ with respect to the coordinate z and since the variables u, b, η and Γ depend only on σ, one could find their vertical evolution in the fountain by solving numerically this equation.

3.1. *Fountain maximal height*

Equation (3.8) can also be used to determine the expression for the fountain maximal height (i.e. z_m). Indeed, this maximal height corresponds to the location where the velocity is null, or identically where $\sigma = 0$. By introducing the dimensionless quantity $\mathscr{H} = z_m/b_i$, we have

$$\mathscr{H} = \frac{1}{\Lambda_i}\int_0^{\sigma_i}\left[I(\sigma) - I(\sigma_i) + \Gamma_i \frac{\sigma_i^{5/4}}{(\sigma_i+1)^{3/4}}\right]^{-1/2} (\sigma+1)^{-3/2}\,\mathrm{d}\sigma. \tag{3.9}$$

Equation (3.9) is first solved numerically. Figure 2(a,b) shows the evolution of $\mathscr{H}$ as a function of σ_i and Γ_i respectively. It can be seen on figure 2(a) that when $\sigma_i \ll 1$, $\mathscr{H}$ is almost unaffected by the stratification and then mainly depends on the source conditions, i.e. on the value of Γ_i. On the contrary, when $\sigma_i \gg 1$, $\mathscr{H}$ clearly decreases with σ_i with a slope that increases with Γ_i. This indicates that the vertical development of a weak fountain is much more affected by the stratification than a forced one. A possible explanation for this phenomenon lies in the fact that according to the high initial velocity of a forced fountain, the entrainment process tends to reduce (at least initially) the density deficit and then the buoyancy that opposes the fountain development. In contrast, according to its low initial velocity, a weak fountain does not entrain very much. As a result, its density deficit, which mainly depends in this case on the ambient stratification, increases with respect to the vertical coordinate, enhancing the buoyancy effect. Note that more formally, these considerations can be recovered by analysing (2.5*c*). For instance this equation shows that for a given (moderate) value of σ_i the density deficit η initially decreases if $\Gamma_i\sigma_i \ll 5/4$ and this condition is only met in the case of a forced fountain.

Figure 2(*b*) shows that $\mathscr{H}$ monotonically decreases with Γ_i, exhibiting a slope which is significantly stronger for weak fountains ($\Gamma_i \gg 1$) than for forced fountains ($\Gamma_i \ll 1$). Such a behaviour is observed whatever the value of σ_i and can be again explained by the fact that a weak fountain entrains much less than a forced one.

In § 4, analytical expressions for $\mathscr{H}$ are proposed in these limit cases by means of asymptotic analysis.

3.2. *Fountain variables at the maximal height*

Whatever the values of σ_i and Γ_i, the previous theoretical development allows the fountain variables to be analytically expressed at the maximal height z_m reached by the fountain. Indeed, from (3.3) and (3.6*b*), it can be shown that the density deficit η_m at this height is

$$\eta_m = \eta_i \frac{(\sigma_i+1)^{1/2}}{\left[1 - \dfrac{I(\sigma_i)(\sigma_i+1)^{3/4}}{\Gamma_i \sigma_i^{5/4}}\right]^{1/2}}. \tag{3.10}$$

Moreover, even if the radius and the velocity tend respectively towards ∞ and 0, both the volume flux ($Q = ub^2$) and the buoyancy flux ($B = \eta u b^2$) have finite values which can be expressed as follows:

$$Q_m = Q_i \left[1 - \frac{I(\sigma_i)(\sigma_i+1)^{3/4}}{\Gamma_i \sigma_i^{5/4}}\right]^{1/2}, \quad B_m = B_i(\sigma_i+1)^{1/2}. \tag{3.11}$$

These latter results are useful to investigate the fountain down-flow problem, since η_m, Q_m and B_m correspond to its initial condition.

4. Asymptotic behaviour

We now consider the asymptotic behaviour of $\mathscr{H}$ given by (3.9), for low and strong values of both σ_i and Γ_i.

The case $\sigma_i \ll 1$ corresponds to a low stratification and/or low velocity. In this situation, (3.9) becomes

$$\mathscr{H} \approx \frac{(1+\Gamma_i)^{3/10}}{4\alpha \Gamma_i^{1/2}} \beta\left[(\Gamma_i+1)^{-1}, \tfrac{1}{2}, \tfrac{4}{5}\right], \tag{4.1}$$

where $\beta[x,a,b] = \int_0^x (1-t)^{a-1} t^{b-1}\mathrm{d}t$ is the classical incomplete beta function. It can be observed that σ_i vanishes in this leading-order term which only depends on Γ_i. This result has been previously observed in figure 2 and is also in agreement with the experimental observations of Bloomfield & Kerr (1998) who noticed that for low values of σ_i, the maximal height is only dependent on the initial Froude number Fr (with $1/Fr = 8\alpha\Gamma_i/5$).

In the case of a forced fountain ($\Gamma_i \ll 1$) (4.1) can be further simplified as follows:

$$\mathscr{H} \approx \frac{\beta\left[1, \frac{1}{2}, \frac{4}{5}\right]}{4\alpha\Gamma_i^{1/2}} \approx \frac{0.57}{\alpha}\Gamma_i^{-1/2}, \tag{4.2}$$

where $\beta[1, 1/2, 4/5] \approx 2.299$.

This latter relation is in agreement with the experimental results obtained by Bloomfield & Kerr (2000) who found $\mathscr{H} = (8.37 \pm 0.29)\Gamma_i^{-1/2}$. According to this result, the value for the entrainment coefficient in (4.2) can be estimated: $\alpha = 0.068 \pm 0.002$. This value is lower than, but not significantly different from, the one proposed by Fischer *et al.* (1979) for a jet (0.076 ± 0.004). This corroborates that for a forced fountain $\alpha_{fountain} \lesssim \alpha_{jet} < \alpha_{plume}$ as mentioned and discussed on the basis of physical arguments by Kaye (2008).

In the case of a weak fountain ($\Gamma_i \gg 1$) we obtain from (4.1)

$$\mathscr{H} = \frac{5}{16\alpha\Gamma_i}. \tag{4.3}$$

By substituting Γ_i by its expression (2.3) taken for $z = 0$, we observe that $\mathscr{H}$ becomes independent from the entrainment coefficient α, as already mentioned by Kaye & Hunt (2006) in a theoretical study performed for fountains in a homogeneous environment.

Finally, for $\sigma_i \ll 1$, the results confirm that a low stratification does not affect the maximal height of a turbulent fountain. This has already been observed by Bloomfield & Kerr (1998) for forced fountains, but it can now be theoretically extended for weak fountains.

The case $\sigma_i \gg 1$ corresponds to a strong stratification and/or high velocity and/or a low density deficit. A similar asymptotic analysis as previously can be performed. The relations giving the maximal height tend respectively for $\Gamma_i \ll 1$ and $\Gamma_i \gg 1$ to

$$\mathscr{H} \approx \frac{5\mathscr{A}}{16\alpha}\Gamma_i^{-1/2}\sigma_i^{-1/4} \quad \text{and} \quad \mathscr{H} \approx \frac{5}{8\alpha}\Gamma_i^{-1}\sigma_i^{-1/2}, \tag{4.4a,b}$$

where $\mathscr{A} = \int_0^\infty (I(\sigma) + (5/4)\beta[1, 1/2, 5/4])^{-1/2}(\sigma + 1)^{-3/2}\,\mathrm{d}\sigma \approx 2.5563$.

In both cases, if we replace σ_i and Γ_i by their expressions, it is observed that $\mathscr{H}$ does not depend on the deficit of density at the injection η_i. This has already been observed by Bloomfield & Kerr (1998) from experiments carried out for forced fountains ($\Gamma_i \ll 1$). They have also proposed a correlation similar to (4.4*a*) with a constant of proportionality equal to 11.73 ± 0.63. This experimental value allows the entrainment coefficient to be estimated in (4.4*a*) and here again we recover $\alpha = 0.068 \pm 0.004$. As a result it seems that a highly forced fountain tends to behave like a jet even in a significantly stratified environment.

Finally, let us mention that the results derived in this section (as well as those derived the previous section) cannot be used immediately in the present form to deal with the singular case of a non-buoyant release for which $\eta_i = 0$ and σ_i and Γ_i tend respectively to ∞ and 0. This particular situation is investigated in the next section.

5. Initially non-buoyant fountain

In the particular case of an initially non-buoyant fountain, the released density ρ_i is equal to the ambient density at the source level, similarly to a pure jet. However, due to the decrease of the surrounding density with height, this jet immediately becomes a fountain.

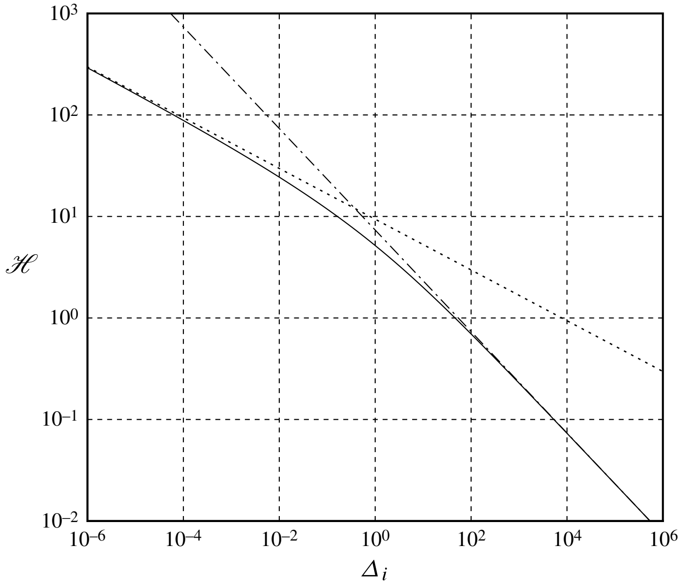

FIGURE 3. Comparison of numerical solution of (5.2) (solid line) and asymptotic behaviour (5.4): $\varDelta_i \ll 1$ (dotted line) and $\varDelta_i \gg 1$ (dash-dotted line).

In order to get around the singularity mentioned at the end of § 4, we introduce the function $\varDelta(z) = \varGamma(z)^2\sigma(z)$, which has a finite value at $z=0$ since

$$\varDelta_i = g\left(\frac{5b_iN}{8\alpha u_i}\right)^2. \tag{5.1}$$

Indeed, it turns out that this function systematically appears by taking the limit as $\eta_i \to 0$ in the results obtained for the general buoyant release case. In particular, it can be shown that the maximal fountain height is now

$$\mathscr{H} = \frac{5}{16\alpha}\varDelta_i^{-1/4}\int_0^\infty \left(I(\sigma)+\varDelta_i^{1/2}+\frac{5}{4}\beta\left[1,\frac{1}{2},\frac{5}{4}\right]\right)^{-1/2}(\sigma+1)^{-3/2}\,\mathrm{d}\sigma. \tag{5.2}$$

At this height, the density deficit, the buoyancy flux and the volume flux are given as follows:

$$\left.\begin{aligned} \frac{\eta_m}{u_iNg^{-1/2}} &= \frac{\varDelta_i^{1/4}}{\left(\varDelta_i^{1/2}+\frac{5}{4}\beta\left[1,\frac{1}{2},\frac{5}{4}\right]\right)^{1/2}}, \\ \frac{Q_m}{Q_i} &= \frac{\left(\varDelta_i^{1/2}+\frac{5}{4}\beta\left[1,\frac{1}{2},\frac{5}{4}\right]\right)^{1/2}}{\varDelta_i^{1/4}}, \\ B_m &= \frac{u_iN}{g^{1/2}}Q_i. \end{aligned}\right\} \tag{5.3}$$

It can be observed that these quantities, as well as the fountain maximal height now only depend on one parameter (i.e. $\varDelta_i$) instead of two as previously.

Figure 3 shows the numerical solution of (5.2). We observe that the initial height varies monotonically with respect to $\varDelta_i$. Though the initial height decreases with $\varDelta_i$, the slope of the curve changes depending on whether the value of Δ_i is small or large. This can be confirmed by performing an asymptotical analysis based on the values taken by Δ_i. For $\Delta_i \ll 1$ and $\Delta_i \gg 1$, (5.2) tends respectively to

$$\mathscr{H} \approx \frac{5\mathscr{A}}{16\alpha}\Delta_i^{-1/4} \quad \text{and} \quad \mathscr{H} \approx \frac{5}{8\alpha}\Delta_i^{-1/2}, \tag{5.4}$$

where $\mathscr{A}$ is the constant already defined in (4.4*a,b*).

These relations (plotted in figure 3) are similar to those obtained in (4.4*a,b*) for $\sigma_i \gg 1$, as expected since a non-buoyant release is an asymptotic case when $\sigma_i \to \infty$.

6. Conclusion

This paper provides an analytical approach based on the model of entrainment of Morton *et al.* (1956) to investigate the initial up-flow of a turbulent fountain growing in a linearly stratified environment under the Boussinesq approximation. It has been shown that the fountain maximal height can be written in the form of an integral which depends on two parameters Γ_i and σ_i. Also, the asymptotic behaviour of the maximal height have been investigated. The main fountain variables at the maximal height have been analytically expressed, which is a significant result useful to tackle the down-flow problem.

As important new physical understanding, it was shown that:

(*a*) For a low stratification and/or a low initial velocity ($\sigma_i \ll 1$), $\mathscr{H}$ does not depend on σ_i. This indicates that the fountain maximal height is only function of the release conditions in this case.

(*b*) For weak fountains ($\Gamma_i \gg 1$) it was shown that $\mathscr{H}$ does not depend on the entrainment coefficient, whatever the value of σ_i. This result suggests that a criterion for discriminating weak fountains from forced fountains could be based on this independence.

(*c*) For a non-buoyant release (fountain with zero buoyancy flux at the source), it has been shown that the whole set of relations can be reformulated by using only one initial parameter, Δ_i, which is a function of the initial momentum flux and of the ambient stratification.

(*d*) In the particular case of a forced fountain ($\Gamma_i \ll 1$), it has been observed that whatever the strength of stratification ($\sigma_i \ll 1$ or $\sigma_i \gg 1$), the relations for the maximal height are in agreement with the scaling laws of Bloomfield & Kerr (1998, 2000) obtained from experiments. By comparison with these experimental results, it has been deduced that the value of the entrainment coefficient α should be around 0.068, independently of the strength of stratification.

(*e*) The entrainment coefficient for a highly forced fountain was found to be close to that of a jet. Consequently, a highly forced fountain exhibits a similar entrainment process to a turbulent jet even for a strong stratification.

All the results obtained for the fountain's maximal height are summarized in table 1. Even though the formalism developed here has been applied only to round fountains, it can be quite straightforwardly extended to plane fountains. Details of calculations are not developed in this paper but, as an extension, the main results obtained for plane fountains are given in table 2.

Initially buoyant release	$\Gamma_i \ll 1$	$\Gamma_i \gg 1$
$\sigma_i \ll 1$	$\mathscr{H} \approx (\beta[1,1/2,4/5]/4\alpha)\Gamma_i^{-1/2}$	$\mathscr{H} \approx (5/16\alpha)\Gamma_i^{-1}$
$\sigma_i \gg 1$	$\mathscr{H} \approx (5\mathscr{A}/16\alpha)\Gamma_i^{-1/2}\sigma_i^{-1/4}$	$\mathscr{H} \approx (5/8\alpha)\Gamma_i^{-1}\sigma_i^{-1/2}$
Initially non-buoyant release	$\Delta_i \ll 1$	$\Delta_i \gg 1$
	$\mathscr{H} \approx (5\mathscr{A}/16\alpha)\Delta_i^{-1/4}$	$\mathscr{H} \approx (5/8\alpha)\Delta_i^{-1/2}$

TABLE 1. Asymptotic behaviour for round fountains. $\Gamma_i = 5g\eta_i b_i/8\alpha u_i^2$, $\sigma_i = u_i^2 N^2/g\eta_i^2$ and $\Delta_i = 5^2 g b_i^2 N^2/8^2\alpha^2 u_i^2$ with $\beta[1,1/2,4/5] \approx 2.3$ and $\mathscr{A} \approx 2.56$. Note that $Fr = 5/(8\alpha\Gamma_i)$.

Initially buoyant release	$\Gamma_i \ll 1$	$\Gamma_i \gg 1$
$\sigma_i \ll 1$	$\mathscr{H} \approx (\beta[1,2/3,2/3]/3\alpha)\Gamma_i^{-2/3}$	$\mathscr{H} \approx (1/2\alpha)\Gamma_i^{-1}$
$\sigma_i \gg 1$	$\mathscr{H} \approx (\mathscr{B}/2\alpha)\Gamma_i^{-2/3}\sigma_i^{-1/3}$	$\mathscr{H} \approx (1/\alpha)\Gamma_i^{-1}\sigma_i^{-1/2}$
Initially non-buoyant release	$\Delta_i \ll 1$	$\Delta_i \gg 1$
	$\mathscr{H} \approx (\mathscr{B}/2\alpha)\Delta_i^{-1/3}$	$\mathscr{H} \approx (1/\alpha)\Delta_i^{-1/2}$

TABLE 2. Asymptotic behaviour for plane fountains. $\Gamma_i = g\eta_i b_i/\alpha u_i^2$, $\sigma_i = u_i^2 N^2/g\eta_i^2$ and $\Delta_i = g b_i^2 N^2/\alpha^2 u_i^2$ with $\mathscr{B} \approx 2.12$ and $\beta[1,2/3,2/3] \approx 2.05$. Note that $Fr = 1/(\alpha\Gamma_i)$.

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