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CENTRE OF BANACH ALGEBRA VALUED BEURLING ALGEBRAS

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Abstract

We prove that for a Banach algebra A having a bounded $\mathcal{Z}(A)$-approximate identity and for every **[IN]** group G with a weight w which is either constant on conjugacy classes or satisfies $w \geq 1$, $\mathcal{Z}(L^1_w(G) \otimes^\gamma A) \cong \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A)$. As an application, we discuss the conditions under which $\mathcal{Z}(L^1_\omega(G, A))$ enjoys certain Banach algebraic properties, such as weak amenability or semisimplicity.

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1. Introduction

For two algebras A and B, $\mathcal{Z}(A) \otimes \mathcal{Z}(B) = \mathcal{Z}(A \otimes B)$, where $\mathcal{Z}(C)$ denotes the centre of algebra C. If A and B are Banach algebras, then it is natural to ask whether $\mathcal{Z}(A \otimes^\gamma B)$ is isometrically isomorphic to $\mathcal{Z}(A) \otimes^\gamma \mathcal{Z}(B)$, $\otimes^\gamma$ being the Banach space projective tensor product. This is known to be true if A and B are C^*-algebras (see [7, Theorem 5.1]) and if $A = B = L^1(G)$ for any $\mathbf{[FC]}^-$ group G (see [18, Lemma 2.1]). Note that [18] generalises the results of [1, 2] and in these three papers the major focus is on studying amenability and weak amenability properties. The idea behind the proofs given in [2, 18] is to use a projection from $L^1(G)$ onto $\mathcal{Z}(L^1(G))$. An ingenious construction of one such projection is given in [18] which is somewhat different from the usual averaging technique used when working with $\mathbf{[FIA]}^-$ groups. We used this technique in [8, Theorem 4.13] and gave an affirmative answer to the question about $\mathcal{Z}(A \otimes^\gamma B)$ if A is a unital Banach algebra and $B = L^1(G)$, for specific classes of groups G. Analogues of results on $\mathcal{Z}(L^1(G))$ by Mosak [12, 13] were also obtained in [8] for the centre $\mathcal{Z}(L^1(G, A))$ of generalised group algebras.

In this paper we generalise all the results discussed in the preceding paragraph, giving relatively simpler proofs, by working in the more general setting of A-valued Beurling algebras $L^1_\omega(G, A)$. In particular, we drop some restrictions on G and A as imposed in [8, Lemma 4.4 and Theorems 4.7 and 4.13] and obtain a similar

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description for $\mathcal{Z}(L^1_\omega(G,A))$. After a series of technical and interesting results, we present Theorem 2.10 as the main result of this paper. It claims that $\mathcal{Z}(L^1_w(G) \otimes^\gamma A) \cong \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A)$ when G is an **[IN]** group (a group having a neighbourhood of the identity which is invariant under inner automorphisms of G), A has a bounded $\mathcal{Z}(A)$-approximate identity (see Definition 2.5) and the weight w is either constant on conjugacy classes or greater than or equal to 1 throughout G. As an application, we discuss some structure-theoretic questions for $\mathcal{Z}(L^1_\omega(G,A))$.

2. Different identifications of centre

Let G be a locally compact group with identity e. Then $(G,\mathcal{B},m)$ is a measure space, where $\mathcal{B}$ is the Borel σ-algebra and m is the left Haar measure. A weight w on G is a measurable positive function such that $w(xy) \le w(x)w(y)$ for every $x, y \in G$. In view of [14, Theorem 3.7.5], w can be assumed to be continuous. For any Banach algebra A, consider the set

$$\mathrm{L}^1_w(G,A) = \Big\{ f : G \to A : f \text{ is } \mathcal{B}\text{-measurable and } \int_G \|f(x)\| w(x)\, dx < \infty \Big\}.$$

Let $f, g \in \mathrm{L}^1_w(G,A)$. Then $(f * g)(x) = \int_G f(xy)g(y^{-1})\, dy$ and $\|f\|_{w,A} = \int_G \|f(x)\|w(x)\, dx$ define a multiplication and a seminorm on $\mathrm{L}^1_w(G,A)$, respectively. When $A = \mathbb{C}$, we write $\|f\|_{1,w}$ for $\|f\|_{w,A}$. The set $L^1_\omega(G,A)$ of all equivalence classes determined by this seminorm becomes a Banach algebra known as the A-valued Beurling algebra. As is customary, we will treat the elements of $L^1_\omega(G,A)$ as functions. For any $a \in A$, $x, y \in G$ and $f \in L^1_\omega(G,A)$, define $(x \cdot f)(y) = f(x^{-1}y)$, $(f \cdot x)(y) = f(yx)$, $(fa)(x) = f(x)a$ and $(af)(x) = af(x)$. It is easy to check that all these elements belong to $L^1_\omega(G,A)$.

A part of the following result is proved in [4, Lemma 2.7] and an analogous statement for $L^1(G,A)$ is proved in [8, Lemmas 3.2 and 3.3]. The fact that w is locally bounded [9, Lemma 1.3.3] along with some necessary changes in that proof can be used to obtain the following result.

Lemma 2.1. *Let $f \in L^1_\omega(G,A)$ and $y \in G$.*

(1) *The maps $G \ni x \to x \cdot f, f \cdot x \in L^1_\omega(G,A)$ are continuous.*
(2) $\|y \cdot f\|_{w,A} \le w(y)\|f\|_{w,A}$ *and* $\|f \cdot y\|_{w,A} \le w(y^{-1})\Delta(y^{-1})\|f\|_{w,A}$.

We next provide an analogue of the characterisation of the centre of a convolution algebra as given in [13, Proposition 1.2]. If f is a function from G to A, we define $|f|(x) := \|f(x)\|$ for every $x \in G$.

Lemma 2.2. *Let G be a locally compact group and A be a Banach algebra. Then*

$$\mathcal{Z}(L^1_\omega(G,A)) = \{f \in L^1_\omega(G,A) : \Delta(s^{-1})(f \cdot s^{-1})a = a(s \cdot f) \quad \text{for all } s \in G, a \in A\}.$$

Proof. Let $f \in \mathcal{Z}(L^1_\omega(G,A))$. For any $a \in A$, $s \in G$ and any compact symmetric set $U \in \mathcal{B}$, write $a_{sU} = a\chi_{sU}$. It follows from local boundedness of w that $a_{sU} \in L^1_\omega(G,A)$.

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Let $\epsilon > 0$. It suffices to show that $\|\Delta(s^{-1})(f \cdot s^{-1})a - a(s \cdot f)\|_{w,A} < \epsilon$. Following the calculations in the proof of [8, Theorem 3.4],

$$\begin{aligned}\|m(Us^{-1})(f \cdot s^{-1})a - f * a_{sU}\|_{w,A} &\le \int_{Us^{-1}} \int_G |f \cdot s^{-1}a - (f \cdot x)a|(y)w(y)\,dy\,dx \\ &\le m(Us^{-1}) \sup_{x\in U} \|f \cdot s^{-1} - (f \cdot s^{-1}) \cdot x^{-1}\|_{w,A}\|a\|\end{aligned}$$

and

$$\begin{aligned}\|m(U)a(s \cdot f) - a_{sU} * f\|_{w,A} &\le \Delta(s)\|a\| \int_U \int_G \|(s \cdot f)(y) - f(x^{-1}s^{-1}y)\|w(y)\,dy\,dx \\ &\le m(U)\|a\| \sup_{x\in U} \|s \cdot f - s \cdot (x \cdot f)\|_{w,A} \\ &\le m(U)\|a\|w(s) \sup_{x\in U} \|f - (x \cdot f)\|_{w,A}.\end{aligned}$$

So

$$\begin{aligned}&\Big\|\Delta(s^{-1})(f \cdot s^{-1})a - a(s \cdot f) - \frac{1}{m(U)}(f * a_{sU} - a_{sU} * f)\Big\|_{w,A} \\ &\quad\le \Big(\Delta(s^{-1}) \sup_{x\in U} \|f \cdot s^{-1} - (f \cdot s^{-1}) \cdot x^{-1}\|_{w,A} + w(s) \sup_{x\in U} \|f - x \cdot f\|_{w,A}\Big)\|a\|.\end{aligned}$$

The desired result now follows from Lemma 2.1.

For the converse, it is sufficient to prove that $f * g = g * f$ for every continuous function g with compact support. This can be proved exactly as in [8, Theorem 3.4], by showing that in every neighbourhood of $f * g - g * f$ there is an element which belongs to the set $\operatorname{span}\{\Delta(s)^{-1}(f \cdot s^{-1})a - a(s \cdot f) : a \in A, s \in G\}$. □

With this in hand, the following analogue of [8, Lemma 4.3] can be given with a few adjustments in its proof.

Lemma 2.3. *Let G be a locally compact group and A be a Banach algebra. Then*

$$\mathcal{Z}(L^1_\omega(G, A)) \subseteq L^1_w(G, \mathcal{Z}(A)).$$

Proof. Consider a nonzero element f in $\mathcal{Z}(L^1_\omega(G, A))$. Suppose there exists a Borel set E of positive and finite measure such that $f(x) \notin \mathcal{Z}(A)$ for every $x \in E$.

By Lemma 2.2, $fa = af$ for every $a \in A$. Let $\mathcal{B}_E$ denote the σ-algebra consisting of all Borel sets contained in E. Then, for any $F \in \mathcal{B}_E$,

$$\begin{aligned}\Big(\int_F f(x)w(x)\,dx\Big)a &= \int_F f(x)w(x)a\,dx = \int_F ((fa)w)(x)\,dx \\ &= \int_F ((af)w)(x)\,dx = a\Big(\int_F f(x)w(x)\,dx\Big)\end{aligned}$$

for all $a \in A$, that is, $\int_F f(x)w(x)\,dx \in \mathcal{Z}(A)$ for all $F \in \mathcal{B}_E$. Define $H : \mathcal{B}_E \to \mathcal{Z}(A)$ by $H(F) = \int_F f(x)w(x)\,dx$. Then H is a m-continuous (that is, $\lim_{m(F)\to 0} H(F) = 0$) vector

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measure of bounded variation [5, Theorem II.2.4]. Thus, by [5, Corollary III.2.5], there exists a $g \in L^1(E, \mathcal{Z}(A))$ such that $H(F) = \int_F g(x)\,dx$ for all $F \in \mathcal{B}_E$. This shows that $\int_F (f(x)w(x) - g(x))\,dx = 0$ for every $F \in \mathcal{B}_E$, so that $fw = g$ almost everywhere on E by [5, Corollary II.2.5]. Since $g(E) \subseteq \mathcal{Z}(A)$ and $w(E) \subseteq (0, \infty)$, this contradicts the existence of E. Hence, $f(x) \in \mathcal{Z}(A)$ for almost every $x \in G$. □

Remark 2.4. Unlike $L^1(G)$ [12], it can be seen from Lemma 2.3 that for $\mathcal{Z}(L^1_\omega(G, A))$ to be nontrivial it is not sufficient that G be an [**IN**] group. This is because $\mathcal{Z}(A)$ might be trivial. However, if G is an [**IN**] group and A is a Banach algebra with nontrivial centre, then $\mathcal{Z}(L^1_\omega(G, A)) \neq \{0\}$. To prove this, choose a compact neighbourhood E of e which is invariant under inner automorphisms. Then $\chi_E \in \mathcal{Z}(L^1(G))$ [12]. Since w is locally bounded, we have $\chi_E \in L^1_w(G)$. Using the fact that G is unimodular, we obtain from Lemma 2.2 that $\chi_E a \in \mathcal{Z}(L^1_\omega(G, A))$ for every $a \in \mathcal{Z}(A)$.

We will now present some necessary conditions for $\mathcal{Z}(L^1_\omega(G, A))$ to be nontrivial. Note that G being an [**IN**] group is not a necessity as can be demonstrated by taking A to be a Banach algebra with trivial multiplication, in which case $\mathcal{Z}(L^1_\omega(G, A)) \neq \{0\}$ no matter which group G one takes. To get rid of such pathological examples, it is usual to work with Banach algebras having a bounded approximate identity. However, we only need the following relaxed condition.

Definition 2.5. A $\mathcal{Z}(A)$-approximate identity of a Banach algebra A is a net $\{a_\alpha\}$ in A such that $a_\alpha a \to a$ for every $a \in \mathcal{Z}(A)$. If, in addition, the net $\{a_\alpha\}$ is bounded, then we call it a bounded $\mathcal{Z}(A)$-approximate identity.

Note that a Banach algebra A has a bounded $\mathcal{Z}(A)$-approximate identity if either A or $\mathcal{Z}(A)$ has a left or right bounded approximate identity. From now on, the Banach algebra A will be assumed to have a bounded $\mathcal{Z}(A)$-approximate identity $\{a_\alpha\}$.

Lemma 2.6. *Let G be a locally compact group and let A be a Banach algebra. If $0 \neq f \in L^1_w(G, \mathcal{Z}(A))$, then the net $\{a_\alpha f\}$ in $L^1_\omega(G, A)$ converges to f.*

Proof. Let $\{a_\alpha\}$ be bounded by M. For any $\epsilon > 0$, since $L^1_w(G) \otimes \mathcal{Z}(A)$ is dense in $L^1_w(G, \mathcal{Z}(A))$ [17, Theorem 2.2], there exists $f' = \sum_{i=1}^r f_i \otimes a_i \in L^1_w(G) \otimes \mathcal{Z}(A)$ such that $\|f - f'\|_{w,A} < \epsilon$. As $a_\alpha a_i \to a_i$ for every $1 \leq i \leq r$, we can choose α such that $\|a_\beta a_i - a_i\| \leq \epsilon / (\sum_{i=1}^r \|f_i\|_{1,w})$ for every $1 \leq i \leq r$ and $\beta \geq \alpha$. For every $\beta \geq \alpha$,

$$\|a_\beta f' - f'\|_{w,A} = \Big\| \sum_{i=1}^r f_i \otimes (a_\beta a_i - a_i) \Big\|_{w,A} \leq \sum_{i=1}^r \|f_i\|_{1,w} \|a_\beta a_i - a_i\|$$

$$\leq \sum_{i=1}^r \|f_i\|_{1,w} \Big(\epsilon \Big/ \Big(\sum_{i=1}^r \|f_i\|_{1,w} \Big) \Big) \leq \epsilon.$$

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Hence,

$$\begin{aligned}\|a_\beta f - f\|_{w,A} &\le \|a_\beta f - a_\beta f'\|_{w,A} + \|a_\beta f' - f'\|_{w,A} + \|f' - f\|_{w,A}\\ &\le \|a_\beta\|\epsilon + \epsilon + \epsilon < (M+2)\epsilon.\end{aligned}$$

This proves the result. □

Techniques from [12] are used to prove the latter half of the following result.

LEMMA 2.7. *Let G be a locally compact group with a weight w which is either constant on conjugacy classes or satisfies $w \ge 1$, and let A be a Banach algebra. Then G is an* [**IN**] *group whenever $\mathcal{Z}(L^1_\omega(G,A)) \ne \{0\}$.*

PROOF. Let $0 \ne f \in \mathcal{Z}(L^1_\omega(G,A))$. From Lemma 2.3, $x \cdot f, f \cdot x^{-1} \in L^1_w(G, \mathcal{Z}(A))$ for any $x \in G$. Thus, by Lemma 2.6, $\Delta(x^{-1})(f \cdot x^{-1})a_\alpha \to \Delta(x^{-1})f \cdot x^{-1}$ and $a_\alpha(x \cdot f) \to (x \cdot f)$. Hence, in $L^1_w(G)$,

$$\begin{aligned}|\Delta(x^{-1})(f \cdot x^{-1})a_\alpha| &\to \Delta(x^{-1})|f \cdot x^{-1}| = \Delta(x^{-1})|f| \cdot x^{-1},\\ |a_\alpha(x \cdot f)| &\to |(x \cdot f)| = x \cdot |f|.\end{aligned}$$

Case (i): $w \ge 1$. It follows from Lemma 2.2 that $0 \ne |f| \in \mathcal{Z}(L^1_w(G))$, which in turn implies that G is an [**IN**] group, as $w \ge 1$ (see [11]).

Case (ii): w is constant on conjugacy classes. The proof of [8, Lemma 4.4] works here except for the trivial modifications we now describe. Put $h(x) = \|f(x)w(x)\|^{1/2}$. Then using Lemmas 2.2 and 2.6, we obtain

$$\begin{aligned}h(txt^{-1}) = \|f(txt^{-1})w(txt^{-1})\|^{1/2} &= \|(t^{-1} \cdot f \cdot t^{-1})(x)w(x)\|^{1/2}\\ &= \|\Delta(t)^{1/2} f(x)w(x)\|^{1/2} = \Delta(t)^{1/2}h(x).\end{aligned}$$

Now the continuous function $p(s) = \int_G h(sy)h(y)\,dy$ will give a compact neighbourhood of e which is invariant under inner automorphisms. □

Restrictions on the weight in the previous result are not artificial. In fact, if G is an abelian group with a weight w, then there is an equivalent weight $\tilde{w} \ge 1$ on G such that $L^1_w(G)$ and $L^1_{\tilde{w}}(G)$ are isomorphic as Banach algebras [4, Lemma 3.2]. Also, every weight on an abelian group is trivially constant on conjugacy classes. Moreover, if G is a compact group, then $w \ge 1$ [9, Corollary 1.3.4].

Before presenting our main results, let us derive some consequences of what we have obtained so far. In the rest of the paper, $\mathrm{Inn}(G)$ denotes the group of all inner automorphisms ($\mathrm{Ad}_y(x) = y^{-1}xy$) of G. For any function f on G, define $(\mathrm{Ad}_y \cdot f)(x) = f(yxy^{-1})$ for every $x, y \in G$.

LEMMA 2.8. *Let G be an* [**IN**] *group and A a Banach algebra. Then*

$$\mathcal{Z}(L^1_\omega(G,A)) = \{f \in L^1_\omega(G,A) : \mathrm{Ad}_y \cdot f = f \text{ and } fa = af \quad \text{for all } a \in A, y \in G\}.$$

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PROOF. Note that for every $f \in L^1_\omega(G,A)$ we have $(\mathrm{Ad}_y \cdot f)(x) = (y^{-1} \cdot f \cdot y^{-1})(x)$. Using Lemma 2.2 and the fact that every **[IN]** group is unimodular, we obtain

$$\mathcal{Z}(L^1_\omega(G,A)) = \{f \in L^1_\omega(G,A) : a(\mathrm{Ad}_y \cdot f) = fa \quad \text{for all } a \in A, y \in G\}. \tag{2.1}$$

If $f \in L^1_\omega(G,A)$ is such that $fa = af$ and $\mathrm{Ad}_y \cdot f = f$ for every $y \in G$ and $a \in A$, then it readily follows that $a(\mathrm{Ad}_y \cdot f) = af = fa$ for every $y \in G$ and $a \in A$.

To prove the opposite inclusion, let $0 \neq f \in \mathcal{Z}(L^1_\omega(G,A))$. Taking $y = e$ in (2.1), we obtain $fa = af$ for every $a \in A$. From Lemma 2.3, $f, \mathrm{Ad}_y \cdot f \in L^1_w(G, \mathcal{Z}(A))$ for every $y \in G$. So, by Lemma 2.6, $0 = (a_\alpha(\mathrm{Ad}_y \cdot f) - fa_\alpha) \to (\mathrm{Ad}_y \cdot f) - f$ for every $y \in G$. Hence, $\mathrm{Ad}_y \cdot f = f$ for every $y \in G$. □

COROLLARY 2.9. *Let G be an* **[IN]** *group and A a Banach algebra. Then*

$$\mathcal{Z}(L^1_\omega(G,A)) = \{f \in L^1_w(G, \mathcal{Z}(A)) : Ad_y \cdot f = f \quad \textit{for all } y \in G\}.$$

In particular, if $\mathcal{Z}(A)$ has a bounded approximate identity, then

$$\mathcal{Z}(L^1_\omega(G,A)) = \mathcal{Z}(L^1_w(G, \mathcal{Z}(A))) \quad \textit{and} \quad \mathcal{Z}(L^1_w(G) \otimes^\gamma A) \cong \mathcal{Z}(L^1_w(G) \otimes^\gamma \mathcal{Z}(A)).$$

PROOF. The first statement is a direct consequence of Lemmas 2.3 and 2.8. The second statement follows from Lemma 2.3 and the fact that $\mathcal{Z}(A)$ has a bounded $\mathcal{Z}(\mathcal{Z}(A))$-approximate identity. The third statement is a consequence of the well-known fact that $L^1_w(G) \otimes^\gamma A \cong L^1_\omega(G,A)$ [17, Theorem 2.2]. □

Let G be an **[IN]** group and w a weight which is constant on conjugacy classes. Just as in [8], we consider the σ-subalgebra $\mathcal{B}_{\mathrm{inv}} = \{B \in \mathcal{B} : \mathrm{Ad}_y(B) = B \text{ for all } y \in G\}$ and define the corresponding Banach algebra $L^1_{w,\mathrm{inv}}(G,A)$ arising from the triple $(G, \mathcal{B}_{\mathrm{inv}}, m_{\mathrm{inv}} = m_{|\mathcal{B}_{\mathrm{inv}}})$. If $f \in L^1_{w,\mathrm{inv}}(G,A)$, then f is $\mathcal{B}_{\mathrm{inv}}$-measurable and hence $\mathcal{B}$-measurable [8, Lemma 4.6]. Clearly, $L^1_{w,\mathrm{inv}}(G,A) \subseteq L^1_\omega(G,A)$. If w is a weight such that $w \geq 1$, then we may define

$$L^1_{w,\mathrm{inv}}(G,A) = \{f \in L^1_w(G,A) : f \text{ is } \mathcal{B}_{\mathrm{inv}}\text{-measurable}\}.$$

In both these cases, from [8, Lemma 4.6],

$$L^1_{w,\mathrm{inv}}(G,A) = \{f \in L^1_\omega(G,A) : f \text{ is constant on the conjugacy classes of G}\}.$$

As in [17, Theorem 2.2], $L^1_{w,\mathrm{inv}}(G,A) \cong L^1_{w,\mathrm{inv}}(G) \otimes^\gamma A$ for such weights.

THEOREM 2.10. *Let G be an* **[IN]** *group with a weight w which is either constant on conjugacy classes or satisfies $w \geq 1$, and let A be a Banach algebra. Then*

$$\mathcal{Z}(L^1_\omega(G,A)) \cong \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A).$$

PROOF. We claim that $\mathcal{Z}(L^1_w(G)) = L^1_{w,\mathrm{inv}}(G)$.

Let us first assume that w is constant on conjugacy classes. We know that $\mathcal{Z}(L^1(G))$ has a uniformly bounded approximate identity (see the proof of [10, Corollary 1.6]). Since convolution of an L^1 function with an L^∞ function gives a continuous function, $\mathcal{Z}(L^1(G)) \cap C(G)$ is dense in $\mathcal{Z}(L^1(G))$. From [13], $fw \in \mathcal{Z}(L^1(G))$ for a fixed $f \in \mathcal{Z}(L^1_w(G))$, so fw is approximated in $L^1(G)$ by a sequence $\{g_n\} \in \mathcal{Z}(L^1(G)) \cap C(G)$.

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Since both g_n and w are constant on conjugacy classes, f is constant on conjugacy classes. So, $f \in L^1_{w,\mathrm{inv}}(G)$, proving that $\mathcal{Z}(L^1_w(G)) \subseteq L^1_{w,\mathrm{inv}}(G)$. It now follows from Corollary 2.9 that $\mathcal{Z}(L^1_w(G)) = L^1_{w,\mathrm{inv}}(G)$.

If $w \geq 1$, then $L^1_{w,\mathrm{inv}}(G) \subseteq L^1_w(G) \subseteq L^1(G)$. Thus, being constant on conjugacy classes, every element of $L^1_{w,\mathrm{inv}}(G)$ is contained in $\mathcal{Z}(L^1(G))$. This further implies that $L^1_{w,\mathrm{inv}}(G) \subseteq \mathcal{Z}(L^1_w(G))$. Conversely, if $f \in \mathcal{Z}(L^1_w(G))$, then $f \in L^1_w(G)$ and $f \cdot x = x^{-1} \cdot f$ for every $x \in G$ since G is unimodular. This proves that $f \in \mathcal{Z}(L^1(G))$ and hence f is constant on conjugacy classes, proving the claim.

Thus, in both cases,

$$L^1_{w,\mathrm{inv}}(G, \mathcal{Z}(A)) \cong L^1_{w,\mathrm{inv}}(G) \otimes^{\gamma} \mathcal{Z}(A) \cong \mathcal{Z}(L^1_w(G)) \otimes^{\gamma} \mathcal{Z}(A).$$

From Corollary 2.9, $L^1_{w,\mathrm{inv}}(G, \mathcal{Z}(A)) \subseteq \mathcal{Z}(L^1_\omega(G, A))$, so we only need to check the reverse inclusion. Further, since an arbitrary $f \in \mathcal{Z}(L^1_\omega(G, A))$ is a member of $L^1_w(G, \mathcal{Z}(A))$, it is sufficient to show that f is $\mathcal{B}_{\mathrm{inv}}$-measurable.

As f is $\mathcal{B}$-essentially separably valued [16, Proposition 2.15], there exists $E \in \mathcal{B}$ with zero measure such that $f(E^c)$ is contained in a separable space. We can redefine f to be zero on E and hence f is $\mathcal{B}_{\mathrm{inv}}$-essentially separably valued. In view of Corollary 2.9, for every $\phi \in A^*$ and $y \in G$ we have $Ad_y \cdot (\phi \circ f)(x) = (\phi \circ f)(yxy^{-1}) = \phi(f(yxy^{-1})) = \phi(f(x))$ for almost every $x \in G$. Thus, $\phi \circ f \in \mathcal{Z}(L^1_w(G)) = L^1_{w,\mathrm{inv}}(G)$ and hence f is weakly $\mathcal{B}_{\mathrm{inv}}$-measurable. It now follows from [16, Proposition 2.15] that f is $\mathcal{B}_{\mathrm{inv}}$-measurable. □

The following result generalises [8, Theorem 4.13] (by taking $w \equiv 1$). It also generalises [18, Lemma 2.1] fully and [18, Proposition 2.2] to some extent.

COROLLARY 2.11. *Let G be an* **[IN]** *group and A a Banach algebra. Then*

$$\mathcal{Z}(L^1(G, A)) \cong \mathcal{Z}(L^1(G)) \otimes^{\gamma} \mathcal{Z}(A).$$

We note the following interesting consequences of Theorem 2.10 without giving the definitions of the concepts discussed as they are not being used rigorously. We would also like to point out that (2) and (3) below generalise [18, Proposition 2.3].

COROLLARY 2.12. *Let G be an* **[IN]** *group with a weight $w \geq 1$ and A a Banach algebra.*

(1) *$\mathcal{Z}(L^1_\omega(G, A))$ is semisimple if and only if $\mathcal{Z}(A)$ is semisimple.*
(2) *If $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are weakly amenable, then so is $\mathcal{Z}(L^1_\omega(G, A))$.*
(3) *If $\mathcal{Z}(L^1_\omega(G, A))$ is weakly amenable and semisimple, then both $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are weakly amenable.*
(4) *If $\mathcal{Z}(L^1_\omega(G, A))$ is Tauberian and semisimple, then both $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are Tauberian.*
(5) *If both $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are Tauberian, then so is $\mathcal{Z}(L^1_\omega(G, A))$.*
(6) *$\mathcal{Z}(L^1_\omega(G, A))$ is regular if and only if both $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are regular.*
(7) *$\mathcal{Z}(L^1_\omega(G, A)) = \{f \in L^1_w(G, \mathcal{Z}(A)) : f$ is constant on conjugacy classes$\}$.*

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PROOF. (1) We know that $L^1(G)$ has the approximation property [9, page 325]. So, $L^1_w(G)$ has the approximation property because the map $f \in L^1_w(G) \to fw \in L^1(G)$ is a Banach space isomorphism. Hence, the natural map from $L^1_w(G) \otimes^\gamma A$ to $L^1_w(G) \otimes^\lambda A$, the Banach space injective tensor product, is injective [9, Theorem A.2.12]. The restriction of the natural map from $\mathcal{Z}(L^1_w(G) \otimes^\gamma A) = \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A)$ to $\mathcal{Z}(L^1_w(G)) \otimes^\lambda \mathcal{Z}(A) \subseteq L^1_w(G) \otimes^\lambda A$ is also injective from the injectivity of $\otimes^\lambda$. Since $\mathcal{Z}(L^1_w(G))$ is a semisimple and commutative Banach algebra [15, Corollary 2.3.7], the result now follows from [9, Theorem 2.11.6].

(2) This follows from the fact that the projective tensor product of weakly amenable commutative Banach algebras is weakly amenable (see [3, Proposition 2.8.71]).

(3) Since $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are semisimple, there exist multiplicative linear functionals ϕ_1 and ϕ_2 on $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$, respectively, see [9, Definition 2.1.9]. Then $\phi_1 \otimes^\gamma 1_{\mathcal{Z}(A)} : \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A) \to \mathcal{Z}(A)$ and $1_{\mathcal{Z}(L^1_w(G))} \otimes \phi_2 : \mathcal{Z}(L^1_w(G)) \otimes^\gamma \mathcal{Z}(A) \to \mathcal{Z}(L^1_w(G))$ are surjective homomorphisms and [3, Proposition 2.8.64] gives the result.

(4) This follows from (1) and [19, Lemma 2.1].

(5) Since both $\mathcal{Z}(L^1_w(G))$ and $\mathcal{Z}(A)$ are commutative, the result follows from [6, Theorem 1. P_2].

(6) This follows from Theorem 2.10 and [20, Theorem 3].

(7) This is the same as saying that $\mathcal{Z}(L^1_\omega(G, A)) = L^1_{w,\mathrm{inv}}(G, \mathcal{Z}(A))$, which is proved in Theorem 2.10. □

REMARK 2.13. Note that the hypothesis $w \geq 1$ is merely used to obtain semisimplicity of $\mathcal{Z}(L^1_w(G))$. Results (2), (5), (6) and (7) are true if w is constant on conjugacy classes.

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