Operator valued analogues of multidimensional Bohr's inequality

Vasudevarao Allu and Himadri Halder

Abstract. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. In this paper, we first establish several sharp improved and refined versions of the Bohr's inequality for the functions in the class $H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ of bounded analytic functions from the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into $\mathcal{B}(\mathcal{H})$. For the complete circular domain $Q \subset \mathbb{C}^n$, we prove the multidimensional analogues of the operator valued Bohr-type inequality which can be viewed as a special case of the result by G. Popescu [Adv. Math. 347 (2019), 1002–1053] for free holomorphic functions on polyballs. Finally, we establish the multidimensional analogues of several improved Bohr's inequalities for operator valued functions in Q.

1 Introduction and some basic questions

Let $H^\infty(\mathbb{D}, X)$ be the space of bounded analytic functions from the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into a complex Banach space X with $\|f\|_{H^\infty(\mathbb{D}, X)} := \sup_{|z|<1} \|f(z)\|$. Let $\mathcal{B}(\mathbb{D}, X)$ be the class of functions f in $H^\infty(\mathbb{D}, X)$ with $\|f\|_{H^\infty(\mathbb{D}, X)} \leq 1$. The Bohr radius $R(X)$ for the class $\mathcal{B}(\mathbb{D}, X)$ is defined by (see [12])

$$R(X) := \sup \left\{ r \in (0,1) : M_r(f) \leq 1 \text{ for all } f(z) = \sum_{k=0}^\infty x_k z^k \in \mathcal{B}(\mathbb{D}, X), z \in \mathbb{D} \right\},$$

where $M_r(f) = \sum_{k=0}^\infty \|x_k\| r^k$ is the associated majorant series of $f \in H^\infty(\mathbb{D}, X)$. The remarkable theorem of Harald Bohr [14] (in improved form) states that $R(X) = 1/3$ for $X = \mathbb{C}$, where the norm of X is the usual modulus of complex numbers. The interest in the Bohr's theorem has been revived when Dixon [18] used it to answer a long-standing question on the characterization of the Banach algebras satisfying the nonunital von Neumann inequality. For the last two decades, there has been an extensive research carried out to the extensions of analytic functions of several complex variables, to planar harmonic mappings, to polynomials, to solutions of elliptic partial differential equations, and to more abstract settings. In 1997, Boas and Khavinson [13] introduced the n-dimensional Bohr radius K_n for the Hardy space of bounded analytic functions on the unit polydisk, and obtained the upper and lower bounds of K_n. In 2006, an improved version of the lower estimate of K_n was obtained by Defant and Frerick [16]. Further estimation of K_n has been obtained by

Defant et al. [17] by using the hypercontractivity of the polynomial Bohnenblust-Hille inequality. In 2014, Bayart et al. [9] obtained the exact asymptotic behaviour of K_n. In 2019, Popescu [25] extended the Bohr inequality for free holomorphic functions to polyballs. For more interesting aspects and generalization of multidimensional Bohr's inequality, we refer to [3–5, 8, 19, 22]. In 2004, Paulsen and Singh [24] extended Bohr's theorem to Banach algebras by finding a general version of Bohr inequality which is valid in the context of uniform algebras. In 2021, Bhowmik and Das [11] extensively studied Bohr inequality for operator valued functions. Further results on Bohr radius, we refer to [10, 21].

For $p \in [1, \infty)$, let $H^p(\mathbb{D}, X)$ be the space of analytic functions from $\mathbb{D}$ into a complex Banach space X such that

$$(1.1) \qquad \|f\|_{H^p(\mathbb{D}, X)} = \sup_{0 < r < 1} \left(\int_0^{2\pi} \|f(re^{it})\|^p \, \frac{dt}{2\pi} \right)^{1/p} < \infty.$$

In [10], Bénéteau et al. have shown that there is no Bohr phenomenon in the Hardy spaces $\|f\|_{H^p(\mathbb{D}, \mathbb{C})}$ for $1 \le p < \infty$. In fact, they have shown that there is no Bohr phenomenon in complex valued Hardy spaces H^q for $q \in (0, \infty)$. In [19], Djakov and Ramanujan have extensively studied the p-Bohr inequality for the power series of the form $\sum_{k=0}^{\infty} |a_k|^p r^k$ for $p \in [1, \infty)$, where $f(z) = \sum_{k=0}^{\infty} a_k z^k$ is a bounded analytic function in $\mathbb{D}$. In [19], the notion of p-Bohr inequality has been extended to the analytic functions of several variables.

1.1 Bohr theorem for operator valued analytic function

One of the main aims of the present paper is to study Bohr inequality in the setting of operator valued analytic functions in the unit disk $\mathbb{D}$, to be more specific, for functions in $H^{\infty}(\mathbb{D}, X)$, where $X = \mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For the rest of our discussion on this, we need to fix some basic notations. For $T \in \mathcal{B}(\mathcal{H})$, $\|T\|$ denotes the operator norm of T. The adjoint operator $T^* : \mathcal{H} \to \mathcal{H}$ of T is defined by $\langle Tx, y \rangle = \langle x, T^*y \rangle$ for all $x, y \in \mathcal{H}$. The operator T is said to be normal if $T^*T = TT^*$, self-adjoint if $T^* = T$, and positive if $\langle Tx, x \rangle \ge 0$ for all $x \in \mathcal{H}$. The absolute value of T is defined by $|T| := (T^*T)^{1/2}$, while $S^{1/2}$ denotes the unique positive square root of a positive operator S. Let I be the identity operator on $\mathcal{H}$. Let $f \in H^{\infty}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ be a bounded analytic function with the expansion

$$(1.2) \qquad f(z) = \sum_{k=0}^{\infty} A_k z^k \quad \text{for} \ \ z \in \mathbb{D},$$

where $A_k \in \mathcal{B}(\mathcal{H})$ for all $k \in \mathbb{N} \cup \{0\}$. For each $f \in H^{\infty}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ of the form (1.2), the function $r \mapsto M_r(f)$ is an increasing function in $[0, 1)$ with $M_0(f) = \|A_0\| \le 1$, where $M_r(f)$ is the associated majorant series of f defined by $M_r(f) = \sum_{k=0}^{\infty} \|A_k\| r^k$ for $r \in [0, 1)$. For each fixed $z \in \mathbb{D}$, we denote $\mathcal{G}_z := \left\{ f(z) = \sum_{k=0}^{\infty} A_k z^k : f \in H^{\infty}(\mathbb{D}, \mathcal{B}(\mathcal{H})) \right\}$. In [6], Allu and Halder have proved that the space $\mathcal{G}_z$ with norm

M_r constitutes a Banach algebra and have shown that functions in $\mathcal{G}_z$ satisfy a von Neumann type inequality. Set $\chi = \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$. By the similar definition as in [15] for the complex valued functions, we define

$$(1.3) \qquad m(\chi, r) := \sup_{f \in \chi} M_r(f) \quad \text{for} \ \ r \in [0, 1).$$

Clearly, $m(\chi, 0) = 1$. It is worth mentioning that $m(\chi, r)$ is an increasing function of r and hence $m(\chi, r) \geq 1$ for $r \in [0, 1)$. For the arbitrary functions f in $H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$, not necessarily $\|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))} \leq 1$, we have $m(\chi, r) \leq M_r(f) \|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))}$. Thus, finding or estimating $m(\chi, r)$, it is relevance to know $\|f(z)\|$ i.e., to understand the rate of the growth of functions in $H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$. However, a precise value for $m(\chi, r)$ is not known for all $r \in [0, 1)$. But we can estimate the bounds for $m(\chi, r)$. In [15], Bombieri has obtained upper bounds and lower bounds for $m(\chi, r)$ for complex valued bounded analytic functions.

We now generalize the notion of Bohr radius for the class $z\chi = \{zf : f \in \chi\}$, i.e., for the class of functions $f \in \chi$ with $f(0) = 0$. The Bohr radius for the class $z\chi$ is the largest radius R_1 such that (a) $M_r(f) = \sum_{k=0}^\infty \|A_k\| \, r^{k+1} \leq 1$ for $r \in [0, R_1]$ and for all $f(z) = \sum_{k=0}^\infty A_k z^{k+1} \in z\chi$, (b) when $r \in (R_1, 1)$, there is a function $f \in z\chi$ such that $M_r(f) > 1$. In view of (1.3), it is easy to see that R_1 is the unique root of

$$(1.4) \qquad\qquad\qquad r\, m(\chi, r) = 1.$$

Since $m(\chi, r)$ is strictly increasing, the function $r \mapsto r\, m(\chi, r)$ is also strictly increasing in $r \in [0, 1)$, which shows that (1.4) has the unique root in $(0, 1)$. Bombieri [15] has proved that $R_1 = 1/\sqrt{2}$ for the complex valued bounded functions in $\mathbb{D}$. Later, Paulsen et al. [23] have extensively studied the radius R_1 for complex valued functions. In the present paper, we obtain R_1 for operator valued functions in $\chi = \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ in Lemma 3.3.

In 2019, Popescu [25] proved the following interesting result, which is an analogue of the classical theorem of Bohr for operator valued bounded analytic functions in $\mathbb{D}$.

Theorem A [25]   *Let $f \in H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ be an operator valued bounded holomorphic function with the expansion (1.2) such that $A_0 = a_0 I$, $a_0 \in \mathbb{C}$. Then*

$$(1.5) \qquad \sum_{n=0}^\infty \|A_n\| \, r^n \leq \|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))} \quad \text{for} \ \ |z| = r \leq \frac{1}{3}$$

and $1/3$ is the best possible constant. Moreover, the inequality is strict unless f is a constant.

The proof of Theorem A relies on the bound of the norm of coefficients $\|A_n\|$, which may be obtained as an application of the operator counterpart of the Schwarz-Pick inequality.

Lemma 1.6 [7] (counterpart of Schwarz-Pick inequality) *Let $B(z)$ be an analytic function with values in $\mathcal{B}(\mathcal{H})$ and satisfying $\|B(z)\| \leq 1$ on $\mathbb{D}$. Then*

$$(1-|a|)^{n-1}\left\|\frac{B^{(n)}(a)}{n!}\right\| \leq \frac{\|I - B(a)^*B(a)\|^{1/2}\,\|I - B(a)B(a)^*\|^{1/2}}{1-|a|^2}$$

for each $a \in \mathbb{D}$ and $n = 1, 2, \ldots$.

We note that for $f \in \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ of the form (1.2) with $A_0 = a_0 I, |a_0| < 1$. Without loss of generality, assume that $\|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))} \leq 1$. Then, by the virtue of Lemma 1.6, putting $a = 0$, we obtain

$$(1.7) \qquad \|A_n\| \leq \left\|I - |A_0|^2\right\| = 1 - |a_0|^2 \quad \text{for} \quad n \in \mathbb{N}.$$

In view of (1.7), the proof of Theorem A follows immediately. For the sharpness of the constant 1/3, we consider the following function

$$(1.8) \qquad \psi_a(z) = \left(\frac{a-z}{1-az}\right)I = A_0 + \sum_{k=1}^{\infty} A_k z^k \quad z \in \mathbb{D},$$

where $A_0 = aI, A_k = -(1-a^2)a^{k-1}I$ and for some $a \in [0,1)$. Then it is easy to see that $M_r(f) = \sum_{k=0}^{\infty} \|A_k\| r^k = (a + (1-2a^2)r)/(1-ar) > 1$ whenever $r > 1/(1+2a)$. Taking a is very close to 1, i.e., $a \to 1^-$, we obtain $M_r(f) > 1$ for $r > 1/3$, which shows that 1/3 is the best possible.

We say that $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ satisfies Bohr phenomenon if all the functions in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ satisfies the inequality (1.5) for $r \leq 1/3$. It is worth mentioning that the constant 1/3 does not dependent on the coefficients of functions. In the recent years, refined versions and improved versions of Bohr inequality in the case of bounded analytic functions have become central research interest in one and several complex variables. Several authors have established various refined and improved versions of Bohr's inequality (see [1, 5]). We now recall some of them. Let $f \in H^\infty(\mathbb{D}, \mathbb{C})$ be of the form

$$(1.9) \qquad f(z) = \sum_{k=0}^{\infty} a_k z^k \quad \text{for} \quad z \in \mathbb{D}.$$

Let S_r be the area of the image of the subdisk $\mathbb{D}_r = \{z : |z| < r\}$ under the map f given by (1.9). Then it is known that $S_r/\pi = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k}$ (see [20]).

Theorem B *Let $f \in \mathcal{B}(\mathbb{D}, \mathbb{C})$ of the form (1.9). Then*

(a) $\sum_{k=0}^{\infty} |a_k| r^k + \left(\frac{1}{1+|a_0|} + \frac{r}{1-r}\right) \sum_{k=1}^{\infty} |a_k|^2 r^{2k} \leq 1$ *for $|z| = r \leq 1/3$. The constant 1/3 is the best possible.*

(b) $\sum_{k=0}^{\infty} |a_k| r^k + \frac{8}{9}\left(\frac{S_r}{\pi}\right) \leq 1$ *for $r \leq 1/3$. The bound 8/9 and the constant 1/3 cannot be replaced by a larger quantity.*

(c) $|f(z)| + \sum_{k=N}^{\infty} |a_k| r^k \leq 1$ *for $|z| = r \leq R_{N,1}$, where $R_{N,1}$ is the unique root in $(0,1)$ of $2(1+r)r^N - (1-r)^2 = 0$. The constant $R_{N,1}$ is the best possible.*

It is important to note that the proof of the part (a) and part (b) can be obtained by putting $\gamma = 0$ in [20, Theorem 2] and [20, Theorem 1] respectively. The proof of the part (c) can be obtained by putting $\gamma = 0$ in [1, Theorem 2.7].

Like the quantity $S_r/\pi = \sum_{k=1}^{\infty} k|a_k|^2 r^{2k}$ for the functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$ in $H^\infty(\mathbb{D}, \mathbb{C})$, we define

$$(1.10) \qquad S(z) = \sum_{k=1}^{\infty} k\, \|A_k\|^2\, r^{2k}, \quad |z| = r < 1,$$

for the functions $f(z) = \sum_{k=0}^{\infty} A_k z^k \in H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$. Let $G(w)$ be a polynomial defined by

$$(1.11) \qquad G(w) = c_1 w + c_2 w^2 + \cdots + c_l w^l \text{ for } c_i \in \mathbb{R}^+, \quad i = 1, 2, \cdots, l.$$

1.2 Multidimensional analogues of operator valued Bohr's inequality

Let α be an n-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of nonnegative integers, $|\alpha|$ be the sum $\alpha_1 + \cdots + \alpha_n$ of its components, $\alpha!$ denotes the product $\alpha_1!\alpha_2!\ldots\alpha_n!$, z denotes an n-tuple $(z_1, z_2, \ldots, z_n)$ of complex numbers, and z^α denotes the product $z_1^{\alpha_1} z_2^{\alpha_2} \ldots z_n^{\alpha_n}$. Using the standard multi-index notation, we write an operator valued n-variable power series

$$(1.12) \qquad f(z) = \sum_{\alpha} A_\alpha z^\alpha, \quad A_\alpha \in \mathcal{B}(\mathcal{H}).$$

Let $\mathbb{D}^n = \{z \in \mathbb{C}^n : z = (z_1, \ldots, z_n), |z_j| < 1, j = 1, 2, \ldots, n\}$. Let $K_n(\mathcal{H})$ be the largest nonnegative number such that the power series (1.12) converges in $\mathbb{D}^n$ and $\|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))} \le 1$, then

$$(1.13) \qquad \sum_{\alpha} \|A_\alpha\|\, |z^\alpha| \le 1 \text{ for all } z \in K_n(\mathcal{H}) \cdot \mathbb{D}^n.$$

Definition 1.1 A domain $D \subset \mathbb{C}^n$ is said to be a Reinhardt domain centered at $0 \in D$ if for any $z = (z_1, \ldots, z_n) \in D$, and for each $\theta_k \in [0, 2\pi]$, $k = 1, 2, \ldots, n$, we have that $(z_1 e^{i\theta_1}, \ldots, z_n e^{i\theta_n}) \in D$. We say that $D \subset \mathbb{C}^n$ is a complete Reinhardt domain if for each $z = (z_1, \ldots, z_n) \in D$, and for each $|\xi_k| \le 1$, $k = 1, 2, \ldots, n$, we have that $\xi.z = (z_1\xi_1, \ldots, z_n\xi_n) \in D$. A domain $Q \subset \mathbb{C}^n$ is called a circular domain centered at $0 \in Q$ if for any $z = (z_1, \ldots, z_n) \in Q$, and for each $\theta \in [0, 2\pi]$, we have that $(z_1 e^{i\theta}, \ldots, z_n e^{i\theta}) \in Q$. We say that $Q \subset \mathbb{C}^n$ is a complete circular domain if for each $z = (z_1, \ldots, z_n) \in Q$, and for each $|\xi| \le 1$, we have that $\xi.z = (z_1\xi_1, \ldots, z_n\xi_n) \in Q$.

For a complete circular domain $Q \subseteq \mathbb{C}^n$ centered at $0 \in Q$, every analytic function in Q can be expressed into the following homogeneous polynomials

$$(1.14) \qquad f(z) = \sum_{k=0}^{\infty} P_k(z) \text{ for } z \in Q,$$

where $P_k(z) = \sum_{|\alpha|=k} A_\alpha z^\alpha$ is a homogeneous polynomial of degree k and $P_0(z) = f(0)$.

It is natural to raise the following question.

Question 1.15 Can we establish the analogue of Theorem B for the operator valued functions in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$? If so, then what is the multidimensional analogue of Theorem B for the functions in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$?

In Theorem B (b), it is worth mentioning that Bohr inequality is improved by adding one degree polynomial in S_r/π with the majorant series $\sum_{k=0}^{\infty} |a_k| r^k$. Therefore, we have the following question.

Question 1.16 Is it possible to establish an improved version of Theorem B (b) by adding the polynomial $G(S_r/\pi)$ with the majorant series $\sum_{k=0}^{\infty} \|A_k\| r^k$ for $f \in \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$? If so, then what is the multidimensional analogue of Theorem B for the functions in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$?

One of the main aims of this paper is to answer Questions 1.15 and 1.16 affirmatively.

2 Main results

Theorem 2.1 *If the series* (1.14) *converges in the domain Q such that the estimate* $\|f(z)\| < 1$ *holds in Q and* $f(0) = a_0 I$, $a_0 \in \mathbb{C}$, $|a_0| < 1$, *then*

$$\tag{2.1} \sum_{k=0}^{\infty} \|P_k(z)\| < 1$$

in the homothetic domain $(1/3)Q$. *Moreover, if Q is convex, the constant $1/3$ is the best possible.*

In particular, if f is complex valued bounded analytic functions in the domain Q, we can obtain the multidimensional analogues of Bohr's inequality for complex valued bounded analytic functions in the domain Q, which has been independently proved by Aizenberg [2]. We obtain the Bohr radius for the functions $f \in \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ with the initial coefficients $f(0) = 0$ in the following lemma. Recall that R_1 is the unique root of (1.4). In the following lemma, we assume that $\mathcal{B}(\mathcal{H})$ is complex Hilbert space.

Lemma 2.2 *Let $f \in \chi = \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$ with $f(0) = 0$. Then $R_1 = 1/\sqrt{2}$.*

If $f \in \mathcal{B}(\mathbb{D}, \mathbb{C})$, then the well known Bombieri [15] result follows from Lemma 2.2. In the following result, we obtain an multidimensional analogues of Lemma 2.2 by assuming $\mathcal{B}(\mathcal{H})$ is a complex Hilbert space.

Theorem 2.2 *If the series* (1.14) *converges in the domain Q with the estimate* $\|f(z)\| < 1$ *holds in Q and* $f(0) = 0$, *then*

$$\tag{2.3} \sum_{k=1}^{\infty} \|P_k(z)\| < 1$$

in the homothetic domain $(1/\sqrt{2})Q$. *Moreover, if Q is convex, then the constant $1/\sqrt{2}$ is the best possible.*

Corollary 2.4 *Suppose that the series* (1.12) *converges in the polydisk* $\mathbb{D}^n$ *such that* $f(0) = 0$ *and* $\|f(z)\| < 1$ *in* $\mathbb{D}^n$. *Then*

$$\sum_{k=1}^{\infty} \left\| \sum_{|\alpha|=k} A_\alpha z^\alpha \right\| \leq 1$$

in the polydisk $(1/\sqrt{2})\mathbb{D}^n$. *The constant* $1/\sqrt{2}$ *is the best possible.*

We establish the multidimensional analogues of Theorem B (c) for the operator valued analytic functions in the complete circular domain Q.

Theorem 2.3 *Suppose that Q is a complete circular domain centered at $0 \in Q \subset \mathbb{C}^n$. If the series* (1.14) *converges in Q such that $\|f(z)\| < 1$ for all $z \in Q$ and $f(0) = a_0 I$, $a_0 \in \mathbb{C}$, $|a_0| < 1$. Then for $p \in (0,1]$, we have*

$$(2.5) \qquad \qquad \|f(z)\|^p + \sum_{k=N}^{\infty} \|P_k(z)\| \leq 1$$

in the homothetic domain $(R_{N,p})Q$, where $R_{N,p}$ is the positive root in $(0,1)$ of the equation

$$(2.6) \qquad \qquad 2(1 + r)r^N - p(1 - r)^2 = 0.$$

Moreover, if Q is convex, then the constant $R_{N,p}$ is the best possible.

In the following result, we prove the multidimensional version of Theorem B (a) for the operator valued analytic functions in the complete circular domain Q.

Theorem 2.4 *Assume that the series* (1.14) *converges in the domain Q such that $\|f(z)\| < 1$ for all $z \in Q$ and $f(0) = a_0 I$, $a_0 \in \mathbb{C}$, $|a_0| < 1$. Then*

$$(2.7) \qquad \sum_{k=0}^{\infty} \|P_k(z)\| + \left(\frac{1}{1 + \|f(0)\|} + \frac{r}{1 - r} \right) \sum_{k=1}^{\infty} \|P_k(z)\|^2 \leq 1.$$

holds in the homothetic domain $(1/3)Q$. Moreover, if Q is convex, then $1/3$ is the best possible.

In the following, we obtain an multidimensional version of Theorem B (b) for the operator valued analytic functions in the complete circular domain Q.

Theorem 2.5 *If the series* (1.14) *and* (1.10) *converge in the domain Q such that $\|f(z)\| < 1$ for all $z \in Q$ and $f(0) = a_0 I$, $a_0 \in \mathbb{C}$, $|a_0| < 1$. If G is given by* (1.11), *then*

$$(2.8) \qquad \sum_{k=0}^{\infty} \|P_k(z)\| + G\left(\sum_{k=1}^{\infty} k \|P_k(z)\|^2 \right) \leq 1$$

holds in the homothetic domain $(1/3)Q$, *where the coefficients of G satisfy*

$$(2.9) \qquad 8c_1 \left(\frac{3}{8}\right)^2 + 24c_2 \left(\frac{3}{8}\right)^4 + \cdots + 8(2l-1)c_l \left(\frac{3}{8}\right)^{2l} \leq 1.$$

Moreover, if Q is convex, then $1/3$ *cannot be replaced by a larger quantity.*

3 Key lemmas and their proofs

We first obtain the upper and lower bounds of $m(\chi, r)$ for functions in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$.

Lemma 3.1 *Let* $\mathcal{B}(\mathcal{H})$ *be a complex Hilbert space and* $f \in \mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$. *Then*

(a) $m(\chi, r) \leq 1/\sqrt{1-r^2}$ *for* $r \in [0, 1)$ *and*
(b) $m(\chi, r) \geq (3 - \sqrt{8(1-r^2)})/(1-r)$ *for* $r \in [1/3, 1)$.

Proof Let $f(z) = \sum_{k=0}^{\infty} A_k z^k$, where $A_k \in \mathcal{B}(\mathcal{H})$ for $k \in \mathbb{N} \cup \{0\}$.

(a) In view of the Cauchy-Schwarz inequality, we have

$$(3.2) \qquad M_r(f) = \sum_{k=0}^{\infty} \|A_k\| r^k \leq \sqrt{\sum_{k=0}^{\infty} \|A_k\|^2} \sqrt{\sum_{k=0}^{\infty} r^{2k}} \quad \text{for } r \in [0, 1).$$

From the given assumption that f is in the unit ball of $H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))$. That is, $\|f\|_{H^\infty(\mathbb{D}, \mathcal{B}(\mathcal{H}))} \leq 1$. In particular, we have $\|f\|^2_{H^2(\mathbb{D}, \mathcal{B}(\mathcal{H}))} = \sum_{k=0}^{\infty} \|A_k\|^2 \leq 1$. Then the desired inequality follows from (3.2).

(b) To obtain the lower bound of $m(\chi, r)$, we consider the function ψ_a which is defined by (1.8). Now, by the well known argument used in proving [21, Lemma 8.1.13], we can easily obtain the lower bound of $m(\chi, r)$. Hence, we omit the details. ∎

In the following result, we establish an operator valued analogues of Theorem B (c) with the term $|f(z)|$ replaced by $\|f(z)\|^p$ for $p \in (0, 1]$ for functions f in $\mathcal{B}(\mathbb{D}, \mathcal{B}(\mathcal{H}))$.

Lemma 3.3 *Let* $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ *be a bounded analytic function of the form* (1.2) *such that* $A_0 = a_0 I$, $a_0 \in \mathbb{C}$, $|a_0| < 1$. *If* $\|f(z)\| \leq 1$ *in* $\mathbb{D}$, *then for* $p \in (0, 1]$, *we have*

$$(3.4) \qquad \|f(z)\|^p + \sum_{k=N}^{\infty} \|A_k\| r^k \leq 1 \quad \text{for } |z| = r \leq R_{N,p},$$

where $R_{N,p}$ *is the unique root in* $(0, 1)$ *of the equation* (2.6). *The constant* $R_{N,p}$ *is the best possible.*

Proof Let $\mathcal{B}(\mathcal{H})$ be the set of bounded linear operators on a complex Hilbert space $\mathcal{H}$. Then $\mathcal{B}(\mathcal{H})$ is a complex Banach space with respect to the norm

$$\|A\| = \sup_{h \in \mathcal{H} \setminus \{0\}} \frac{\|Ah\|}{\|h\|} = \sup_{h \in \mathcal{H}, \|h\|=1} \|Ah\|,$$

where $A \in \mathcal{B}(\mathcal{H})$. Let X^* be the dual space of $\mathcal{B}(\mathcal{H})$. For $x \in \mathcal{B}(\mathcal{H}) \smallsetminus \{0\}$, let

$$T(x) = \{l_x \in X^* : l_x(x) = \|x\| \text{ and } \|l_x\| = 1\}.$$

By the Hahn-Banach theorem, we have $T(x) \neq \phi$. Fix $z \in \mathbb{D} \smallsetminus \{0\}$ and let $v = z/|z| \in \partial\mathbb{D}$. Without loss of generality, we assume $f(z) \neq 0$. Let $\tau \in \mathcal{B}(\mathcal{H})$ be any fixed point such that $\|\tau\| = 1$. We define the following holomorphic function in $\mathbb{D}$ by

$$g(\rho) = l_\tau(f(\rho v)) \quad \text{for } \rho \in \mathbb{D},$$

where $l_\tau \in T(\tau)$. Clearly, $|g(\rho)| \leq 1$ in $\mathbb{D}$. By applying the Schwarz-Pick lemma (often referred to as Lindelöf's inequality) (see [5]) to the complex-valued function g, we obtain

$$(3.5) \qquad |g(\rho)| \leq \frac{|g(0)| + |\rho|}{1 + |g(0)||\rho|}, \quad \rho \in \mathbb{D}.$$

It is easy to see that

$$(3.6) \qquad |g(0)| = |l_\tau(f(0))| \leq \|l_\tau\| \, \|f(0)\| = \|f(0)\|.$$

Let $|g(0)| = t$ and $|\rho| = s$. Then $s \in [0,1)$ and hence, the right hand side term $(t + s)/(1 + ts)$ of (3.5) is an increasing function in the variable $t \in [0, \infty)$. Thus, (3.5) and (3.6) give

$$(3.7) \qquad |l_\tau(f(\rho v))| \leq \frac{\|f(0)\| + |\rho|}{1 + \|f(0)\| \, |\rho|}.$$

By choosing $\rho = |z|$ and $\tau = f(z)/\|f(z)\|$ in (3.7), we obtain

$$(3.8) \qquad \|f(z)\| \leq \frac{\|f(0)\| + |z|}{1 + \|f(0)\| \, |z|} \quad \text{for } z \in \mathbb{D}.$$

Let $f(z) = \sum_{k=0}^{\infty} A_k z^k$ be analytic in $\mathbb{D}$ such that $A_k \in \mathcal{B}(\mathcal{H})$ for all $k \in \mathbb{N} \cup \{0\}$ and $A_0 = a_0 I$, $a_0 \in \mathbb{C}$ with $|a_0| < 1$. Then, in view of (1.7) and (3.8), we have

$$(3.9) \quad \|f(z)\|^p + \sum_{k=N}^{\infty} \|A_k\| \, |z|^k \leq \left(\frac{|z| + |a_0|}{1 + |z||a_0|} \right)^p + (1 - |a_0|^2) \frac{|z|^N}{1 - |z|} = 1 + \Upsilon_{N,p}(|a_0|),$$

where

$$\Upsilon_{N,p}(|a_0|) = \left(\frac{r + |a_0|}{1 + r|a_0|} \right)^p - \frac{1 - r - (1 - |a_0|^2)r^N}{1 - r} \quad \text{for } |z| = r$$

and $|a_0| \in [0,1]$. Let $|a_0| = \alpha$. To prove the inequality (3.4), it is enough to show that $\Upsilon_{N,p}(\alpha) \leq 0$ for $\alpha \in [0,1]$. That is, we have to show that $\Phi_{N,p}(\alpha) \leq 0$, where

$$\Phi_{N,p}(\alpha) = (1 - r)(r + \alpha)^p - (1 + r\alpha)^p \left(1 - r - (1 - \alpha^2)r^N \right).$$

Clearly, $\Phi_{N,p}(1) = 0$. Therefore, if we show that $\Phi_{N,p}$ is an increasing function in α under the condition (2.6), then we are done. A simple computation shows that

$$\Phi'_{N,p}(\alpha) = p(1 - r)(r + \alpha)^{p-1} - pr \left(1 - r - (1 - \alpha^2)r^N \right)(1 + r\alpha)^{p-1} - 2\alpha r^N (1 + r\alpha)^p$$

and

$$\Phi''_{N,p}(\alpha) = p(p-1)(1-r)\left((r+\alpha)^{p-2} - r^2(1+r\alpha)^{p-2}\right)$$
$$+ p(p-1)r^2(1-\alpha^2)r^N(1+r\alpha)^{p-2} - 2\alpha p r^{N+1}(1+r\alpha)^{p-1} - 2r^N(1+r\alpha)^p.$$

Note that $(r+\alpha)^{p-2} - r^2(1+r\alpha)^{p-2}$ is positive for $\alpha \in [0,1]$ and $p \in (0,1]$, which shows that the first term in the expression of $\Phi''_{N,p}(\alpha)$ is negative for $p \in (0,1]$. Since the other terms in the expression of $\Phi''_{N,p}(\alpha)$ are also negative for $p \in (0,1]$, it follows that $\Phi''_{N,p}(\alpha) \leq 0$ for all $\alpha \in [0,1]$ and $p \in (0,1]$. Thus, $\Phi'_{N,p}$ is a monotonically decreasing function in $\alpha \in [0,1]$ for $p \in (0,1]$, which gives that

$$(3.10) \qquad \Phi'_{N,p}(\alpha) \geq \Phi'_{N,p}(1) = p(1-r)^2 - 2r^N(1+r) := \Psi(r).$$

We observe that $\Psi(r) \geq 0$ for $r \leq R_{N,p}$, where $R_{N,p}$ is the unique root of $\Psi(r) = 0$. Indeed, $\Psi'(r) = -2p(1-r) - 2\left(Nr^{N-1} + (N+1)r^N\right) \leq 0$ implies that Ψ is a decreasing function in $r \in [0,1]$. On the other hand, since $\Psi(0) = p > 0$ and $\Psi(1) = -4 < 0$, Ψ has the unique root in $(0,1)$ and let $R_{N,p}$ be that root. Since Ψ is a decreasing function in r, we have $\Psi(r) \geq \Psi(R_{N,p}) = 0$ for $r \leq R_{N,p}$. Therefore, $\Phi'_{N,p}(\alpha) \geq 0$ for $r \leq R_{N,p}$ and $p \in (0,1]$. Thus, $\Phi_{N,p}$ is an increasing function in α whenever $p \in (0,1]$. Therefore, $\Phi_{N,p}(\alpha) \leq \Phi_{N,p}(1) = 0$ for $r \leq R_{N,p}$ for $p \in (0,1]$. This shows that $\Upsilon_{N,p}(\alpha) \leq 0$ for $r \leq R_{N,p}$, $p \in (0,1]$ and hence, the inequality (3.4) follows from (3.9).

To prove the sharpness of the radius $R_{N,p}$, we consider the function ψ_a with $\psi_a(z) = \sum_{k=0}^{\infty} A_k z^k$ given by (1.8). A simple computation shows that

$$(3.11) \qquad \|\psi_a(-r)\|^p + \sum_{k=N}^{\infty} \|A_k\| r^k = \left(\frac{r+a}{1+ra}\right)^p + (1-a^2)\frac{a^{N-1}r^N}{1-ar}$$

$$= 1 - \frac{(1-a)B_{N,p}(a,r)}{(1+ar)^p(1-ar)},$$

where

$$B_{N,p}(a,r) = (1-ar)(1+ar)^p\left(\frac{1}{1-a}\left(1 - \left(\frac{r+a}{1+ra}\right)^p\right) - \left(\frac{1+a}{1-ar}\right)a^{N-1}r^N\right).$$

We note that the right hand expression of (3.11) is greater than or equals to 1 if, and only if, $B_{N,p}(a,r) \leq 0$. By letting $a \to 1^-$, we obtain

$$\lim_{a\to 1^-} B_{N,p}(a,r) = (1-r)(1+r)^p\left(p\left(\frac{1-r}{1+r}\right) - \frac{2r^N}{1-r}\right) < 0$$

for $r > R_{N,p}$. This proves the sharpness of the constant $R_{N,p}$. This completes the proof. ∎

In the next result, we prove an operator valued analogues of Theorem B (a).

Lemma 3.12　　Let $f : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ be a bounded holomorphic function of the form (1.2) such that $A_0 = a_0 I$, $a_0 \in \mathbb{C}$ with $|a_0| < 1$. If $\|f(z)\| \le 1$ in $\mathbb{D}$, then we have

$$(3.13) \qquad \sum_{k=0}^{\infty} \|A_k\|\, r^k + \left(\frac{1}{1 + \|f(0)\|} + \frac{r}{1-r} \right) \sum_{k=1}^{\infty} \|A_k\|^2\, r^{2k} \le 1 \quad for \quad |z| = r \le \frac{1}{3}.$$

The constant $1/3$ *is the best possible.*

Proof　　Let $f(z) = \sum_{k=0}^{\infty} A_k z^n$ in $\mathbb{D}$ with $\|f(z)\| \le 1$ in $\mathbb{D}$ such that $A_k \in \mathcal{B}(\mathcal{H})$ for all $k \in \mathbb{N} \cup \{0\}$ and $A_0 = a_0 I$, $a_0 \in \mathbb{C}$. Set $\|A_0\| = |a_0| = b \in [0,1]$. Then by (1.7), we obtain

$$\sum_{k=0}^{\infty} \|A_k\|\, r^k + \left(\frac{1}{1 + |a_0|} + \frac{r}{1-r} \right) \sum_{k=1}^{\infty} \|A_k\|^2\, r^{2k}$$

$$\le b + (1 - b^2)\left(\frac{r}{1-r} \right) + \left(\frac{1}{1+b} + \frac{r}{1-r} \right)(1-b^2)^2 \frac{r^2}{1-r^2} := \Psi_2(b).$$

Clearly, Ψ_2 can be expressed as $\Psi_2(b) = b + \alpha(1 - b^2) + \beta(1-b)(1-b^2) + \gamma(1-b^2)^2$ for $b \in [0,1]$, where

$$\alpha = \frac{r}{1-r}, \ \beta = \frac{r^2}{1-r^2} \ \text{and} \ \gamma = \frac{r^3}{(1-r)(1-r^2)}.$$

Clearly, α, β, and γ are non-negative. We note that

$$\Psi_2'(b) = 1 - 2\alpha b + \beta(3b^2 - 2b - 1) + 4\gamma(b^3 - b),$$
$$\Psi_2''(b) = -2\alpha + 2\beta(3b - 1) + 4\gamma(3b^2 - 1) \ \text{and} \ \Psi_2'''(b) = 6\beta b + 24b\gamma.$$

Since β and γ are non-negative, we have $\Psi_2'''(b) > 0$ for all $b \in [0,1]$. That is, Ψ_2'' is an increasing function of b, which implies that

$$\Psi_2''(b) \le \Psi_2''(1) = -2\alpha + 4\beta + 8\gamma = \frac{2r}{(1-r)(1-r^2)}\, \tau(r),$$

where $\tau(r) = 4r^2 + 2r(1-r) - (1-r^2) = (1+r)(3r-1)$. It is easy to see that $\tau(r) \le 0$ for $r \le 1/3$. Therefore, $\Psi_2''(b) \le 0$ for $b \in [0,1]$, which shows that Ψ_2' is a decreasing function in $b \in [0,1]$. Thus, for $r \le 1/3$, we obtain

$$\Psi_2'(b) \ge \Psi_2'(1) = 1 - 2\alpha = \frac{1 - 3r}{1 - r}.$$

Clearly, for $r \le 1/3$, we have $\Psi_2'(1) \ge 0$ and hence $\Psi_2'(b) \ge 0$ in $[0,1]$. This implies that Ψ_2 is an increasing function in $[0,1]$ and hence, we obtain $\Psi_2(b) \le \Psi_2(1) = 1$ for $r \le 1/3$. This proves the desired inequality (3.13). For the sharpness of the constant $1/3$, we consider the function ψ_a given by (1.8). A simple computation shows that

$$\|A_0\| + \sum_{k=1}^{\infty} \|A_k\|\, r^k + \left(\frac{1}{1 + |a_0|} + \frac{r}{1-r} \right) \sum_{k=1}^{\infty} \|A_k\|^2\, r^{2k}$$

$$= a + \frac{1-a^2}{a} \sum_{k=1}^{\infty} a^n r^n + \left(\frac{1}{1+a} + \frac{r}{1-r} \right) \frac{1-b^2}{b} \sum_{k=1}^{\infty} a^{2k} r^{2k} := 1 + (1-a)u(r) := v(r),$$

where

$$u(r) = \frac{(1+a)r}{1-ar} + \left(\frac{1}{1+a} + \frac{r}{1-r}\right)\frac{(1+a)(1-a^2)r^2}{1-a^2r^2} - 1.$$

We note that u is strictly increasing function in $(0,1)$. Hence, for $r > 1/3$, we have $u(r) > u(1/3)$. By letting a very close to 1, we obtain $\lim_{a\to 1^-} u(1/3) = 0$. Therefore, u is strictly positive function for $r > 1/3$, as a is very close to 1. Hence, $v(r) > 1$ for $r > 1/3$, which shows that the radius $1/3$ is the best possible. ∎

In the following result, we establish an operator valued analogues of Theorem B (b).

Lemma 3.14   *Let f be as in Lemma 3.12. If $\|f(z)\| \leq 1$ in $\mathbb{D}$, then we have*

$$(3.15) \qquad \sum_{k=0}^{\infty} \|A_k\| r^k + G(S(z)) \leq 1 \text{ for } r \leq \frac{1}{3},$$

where the coefficients of $G(w)$ are given by (1.11) satisfy (2.9). Furthermore, the constant $1/3$ cannot be replaced by a larger quantity. Here S is given by (1.10).

Proof   Let $\|A_0\| = |a_0| = b \in [0,1]$. Then using the inequality (1.7), we obtain

$$(3.16) \qquad S(z) \leq (1-b^2)^2 \sum_{k=1}^{\infty} kr^{2k} = (1-b^2)^2 \frac{r^2}{(1-r^2)^2}.$$

The inequality (3.16) along with (1.7) gives

$$\|A_0\| + \sum_{k=1}^{\infty} \|A_k\| r^k + U(S(z)) \leq b + (1-b^2)\frac{r}{1-r}$$

$$(3.17) \qquad + \sum_{m=1}^{l} c_m \left(\frac{(1-b^2)r}{1-r^2}\right)^{2m} = 1 + H(r),$$

where

$$(3.18) \qquad H(r) = (1-b^2)\frac{r}{1-r} + \sum_{m=1}^{l} c_m \left(\frac{(1-b^2)r}{1-r^2}\right)^{2m} - (1-b).$$

It is easy to see that $H(r)$ is an increasing function and hence $H(r) \leq H(1/3)$ for $r \leq 1/3$. A simple computation shows that

$$H(1/3) = \frac{1-b^2}{2}\left(1 + 2F_l(b) - \frac{2}{1+b}\right) := \frac{1-b^2}{2}J(b),$$

where

$$F_l(b) = \sum_{m=1}^{l} c_l(1-b^2)^{2m-1}\left(\frac{3}{8}\right)^{2m} \text{ and } J(b) = 1 + 2F_l(b) - \frac{2}{1+b}.$$

To show that $H(r) \leq 0$, it is enough to show that $J(b) \leq 0$ for $b \in [0,1]$. Since $b \in [0,1]$, a simple computation shows that

$$b(1+b)^2 (3/8)^2 \leq 4(3/8)^2, \ldots, b(1+b)^2(1-b^2)^{2m-2}(3/8)^{2l} \leq 4(3/8)^{2l}.$$

It is easy to see that

$$J'(b) = \frac{2}{(1+b)^2}\left(1 - 2c_1 b(1+b)^2\left(\frac{3}{8}\right)^2 - 6c_2 b(1+b)^2(1-b^2)^2\left(\frac{3}{8}\right)^4\right.$$

$$\left. - \cdots - 2(2l-1)c_l b(1+b)^2(1-b^2)^{2l-2}\left(\frac{3}{8}\right)^{2l}\right)$$

$$\geq \frac{2}{(1+b)^2}\left(1 - \left(8c_1\left(\frac{3}{8}\right)^2 + 24c_2\left(\frac{3}{8}\right)^4 + \cdots + 8(2l-1)c_l\left(\frac{3}{8}\right)^{2l}\right)\right)$$

$$\geq 0, \quad \text{if } 8c_1\left(\frac{3}{8}\right)^2 + 24c_2\left(\frac{3}{8}\right)^4 + \cdots + 8(2l-1)c_l\left(\frac{3}{8}\right)^{2l} \leq 1.$$

Therefore, $J(b)$ is an increasing function in $[0,1]$ if (2.9) holds. Hence, $J(b) \leq J(1) = 0$ for all $b \in [0,1]$, which gives the desired inequality (3.15). To show the sharpness of the constant $1/3$, we consider the function ψ_a given by (1.8). A simple computation using (1.7) shows that

$$\sum_{n=0}^{\infty} |a_n| r^n + U(S(z)) = a + (1-a^2)\frac{r}{1-ar} + \frac{c_1 r^2(1-a^2)^2}{(1-a^2 r^2)^2} + \cdots + \frac{c_m r^{2m}(1-a^2)^{2m}}{(1-a^2 r^2)^{2m}}$$

$$:= 1 - (1-a)\Phi_1(r),$$

where

$$\Phi_1(r) = -\frac{(1+a)r}{1-ar} - \frac{c_1 r^2(1-a)(1+a)^2}{(1-a^2 r^2)^2} - \cdots - \frac{c_m r^{2m}(1-a)^{2m-1}(1+a)^{2m}}{(1-a^2 r^2)^{2m}} + 1.$$

It is not difficult to show that $\Phi_1(r)$ is strictly decreasing function of r in $(0,1)$. Therefore, for $r > 1/3$, we have $\Phi_1(r) < \Phi_1(1/3)$. A simple computation shows that $\lim_{a\to 1}\Phi_1(1/3) = 0$. Therefore $\Phi_1(r) < 0$ for $r > 1/3$. Hence $1 - (1-a)\Phi_1(r) > 1$ for $r > 1/3$, which shows that $1/3$ is the best possible. This completes the proof. ∎

4 Proofs of the main results

Proof of Theorem 2.1   To obtain the inequality (2.1), we convert the multidimensional power series (1.14) into the power series of one complex variable and we want to make use of Lemma 3.3. Let $L = \{z = (z_1, \ldots, z_n) : z_j = a_j t, j = 1, 2, \ldots, n; t \in \mathbb{C}\}$ be a complex line. Then, in each section of the domain Q by the line L, the series (1.14) turns into the following power series of complex variable t:

$$(4.1) \qquad f(at) = \sum_{k=0}^{\infty} P_k(a) t^k = f(0) + \sum_{k=1}^{\infty} P_k(a) t^k.$$

Since $\|f(at)\| < 1$ for $t \in \mathbb{D}$ and $f(0) = a_0 I$, $|a_0| < 1$, by Lemma 3.3, we obtain

$$(4.2) \qquad \sum_{k=0}^{\infty} \|P_k(a) t^k\| < 1$$

for z in the section $L \cap \left(\frac{1}{3}Q\right)$. Since L is an arbitrary complex line passing through the origin, the inequality (4.2) is just (2.1). For the sharpness of the constant $1/3$, let

the domain Q be convex. Then Q is an intersection of half spaces

$$Q = \bigcap_{a \in J} \{ z = (z_1, \dots, z_n) : \text{Re}\, (a_1 z_1 + \cdots + a_n z_n) < 1 \}$$

for some J. Because Q is circular, we obtain

$$Q = \bigcap_{a \in J} \{ z = (z_1, \dots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1 \}.$$

Therefore, to show the constant $1/3$ is the best possible, it is enough to show that $1/3$ cannot be improved for each domain $G_a = \{ z = (z_1, \dots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1 \}$. In view of Theorem A, for some $b \in [0, 1)$, there exists a function $\psi_b : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ defined by (1.8) with $\| \psi_b(\xi) \| < 1$ for $\xi \in \mathbb{D}$, but for any $|\xi| = r > 1/3$, (2.1) fails to hold in the disk $\mathbb{D}_r = \{ \xi : |\xi| < r \}$. On the other hand, consider the function $\phi : G_a \to \mathbb{D}$ defined by $\phi(z) = a_1 z_1 + \cdots + a_n z_n$. Thus, the function $f(z) = (\psi_b \circ \phi)(z)$ gives the sharpness of the constant $1/3$ for each domain G_a. This completes the proof. ∎

Proof of Lemma 2.2   In view of Lemma 3.1 (a), with $r = 1/\sqrt{2}$, we have $m(\chi, 1/\sqrt{2}) \leq \sqrt{2}$ and by Lemma 3.1 (b) with $r = 1/\sqrt{2}$, we obtain $m(\chi, 1/\sqrt{2}) \geq \sqrt{2}$. These together give $(1/\sqrt{2})\, m(\chi, 1/\sqrt{2}) = 1$. Hence by (1.4), we conclude that $R_1 = 1/\sqrt{2}$. ∎

Proof of Theorem 2.2   In view of Lemma 2.2, by going the similar lines of argument as in Theorem 2.1, we can easily show that $\sum_{k=1}^{\infty} \| P_k(z) \| < 1$ in the homothetic domain $(1/\sqrt{2})Q$. To show the constant $1/\sqrt{2}$ is the best possible, let the domain Q be convex. Therefore, by the analogues proof of Theorem 2.1, it is enough to prove that the constant $1/\sqrt{2}$ cannot be improved in each domain $G_a = \{ z = (z_1, \dots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1 \}$. Since $1/\sqrt{2}$ is the best possible in Lemma 2.2, there exists an analytic function $h : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ defined by

$$h(\xi) = \xi \left(\frac{\frac{1}{\sqrt{2}} - \xi}{1 - \frac{\xi}{\sqrt{2}}} \right) I = \sum_{k=1}^{\infty} A_k \xi^k \quad \text{for} \quad \xi \in \mathbb{D},$$

where $A_1 = 1/\sqrt{2}$ and $A_k = -(1/2)(1/\sqrt{2})^{k-2}$ for $k \geq 2$ such that $\| h(\xi) \| < 1$ in $\mathbb{D}$ and $h(0) = 0$. But for any $|\xi| = r > 1/\sqrt{2}$,

$$\sum_{k=1}^{\infty} \| A_k \| r^k = \frac{r/\sqrt{2}}{1 - (r/\sqrt{2})} > 1,$$

which shows that (2.2) fails to hold in the disk $\mathbb{D}_r = \{ z : |z| < r \}$. Therefore, the function $f(z) = (h \circ \phi)(z)$ gives the sharpness of the constant $1/\sqrt{2}$ in each domain G_a, where $\phi : G_a \to \mathbb{D}$ defined by $\phi(z) = a_1 z_1 + \cdots + a_n z_n$. This completes the proof. ∎

Proof of Theorem 2.3   In view of Lemma 3.3 and the analogues proof of Theorem 2.1, as well as from (3.4), we can easily obtain the inequality (2.5) in the homothetic domain $(R_{N,p})Q$, where $R_{N,p}$ is the positive root in $(0, 1)$ of the equation (2.6). To prove the constant $R_{N,p}$ is the best possible whenever Q is convex, in view of the analogues proof of Theorem 2.1, it is enough to show that $R_{N,p}$ cannot be improved for each domain $G_a = \{ z = (z_1, \dots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1 \}$. Since $R_{N,p}$ is the best

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possible in Lemma 3.3, there exists an analytic function $h : \mathbb{D} \to \mathcal{B}(\mathcal{H})$ such that $\|h(\xi)\| < 1$ in $\mathbb{D}$, but (2.5) fails to hold in the disk $\mathbb{D}_r = \{z : |z| < r\}$ for each $|\xi| = r > R_{N,p}$. Thus, the function $f(z) = (h \circ \phi)(z)$ gives the sharpness of the constant $R_{N,p}$ in each domain G_a, where $\phi : G_a \to \mathbb{D}$ is defined by $\phi(z) = a_1 z_1 + \cdots + a_n z_n$. ∎

Proof of Theorem 2.4   In view of Lemma 3.12 and the analogues proof of Theorem 2.1, as well as from (3.13), we can obtain the inequality (2.7) in the homothetic domain $(1/3)Q$. To prove the constant $1/3$ is the best possible when Q is convex, in view of the analogues proof of Theorem 2.1, it is enough to show that $1/3$ cannot be improved for each domain $G_a = \{z = (z_1, \ldots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1\}$. Thus, the function $f(z) = (h \circ \phi)(z)$ gives the sharpness of the constant $1/3$ in each domain G_a, where $\phi : G_a \to \mathbb{D}$ is defined by $\phi(z) = a_1 z_1 + \cdots + a_n z_n$. ∎

Proof of Theorem 2.5   Using Lemma 3.14 and the analogues proof of Theorem 2.1, from (3.15), we can easily obtain the inequality (2.8) in the homothetic domain $(1/3)Q$, where the coefficients of G satisfy (2.9). When Q is convex, to prove the constant $1/3$ is the best possible, in view of the analogues proof of Theorem 2.1, it is enough to show that $1/3$ cannot be improved for each domain $G_a = \{z = (z_1, \ldots, z_n) : |a_1 z_1 + \cdots + a_n z_n| < 1\}$. Thus, the function $f(z) = (h \circ \phi)(z)$ gives the sharpness of the constant $1/3$ in each domain G_a, where $\phi : G_a \to \mathbb{D}$ defined by $\phi(z) = a_1 z_1 + \cdots + a_n z_n$. ∎

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School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Bhubaneswar 752050, Odisha, India
e-mail: avrao@iitbbs.ac.in   himadrihalder119@gmail.com