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GLOBALLY REALIZABLE COMPONENTS OF LOCAL DEFORMATION RINGS

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Abstract Let n be either 2 or an odd integer greater than 1, and fix a prime $p > 2(n+1)$. Under standard 'adequate image' assumptions, we show that the set of components of n-dimensional p-adic potentially semistable local Galois deformation rings that are seen by potentially automorphic compatible systems of polarizable Galois representations over some CM field is independent of the particular global situation. We also (under the same assumption on n) improve on the main potential automorphy result of Barnet-Lamb *et al.* [Potential automorphy and change of weight, *Ann. of Math. (2)* **179**(2) (2014), 501–609], replacing 'potentially diagonalizable' by 'potentially globally realizable'.

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Contents

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1. Introduction

1.1. Our results

The results of this paper are motivated by the questions of level raising/lowering for automorphic forms, and the weight part of Serre's conjecture, as well as a question about 'automorphic components' related to the Fontaine–Mazur conjecture and to global approaches to the p-adic local Langlands programme. However, our main results are stated purely in terms of Galois representations.

We need to introduce a few definitions before stating our main theorem. Let F be a CM field with maximal totally real subfield F^+, and write G_F for the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. (In this paper, all CM fields will be imaginary and so F/F^+ is a quadratic extension.)

We say that a representation $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is *reasonable* if it satisfies the hypotheses needed to apply the automorphy lifting theorems of [2]; that is, $p > 2(n+1)$, F does not contain a primitive pth root of unity, and $\overline{s}$ is polarizable, odd, and is irreducible when restricted to $G_{F(\zeta_p)}$. We say that a representation $s : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is reasonable if its reduction mod p is reasonable.

We say that a compatible system of l-adic representations of G_F is *weakly irreducible* if for a positive density set of primes l, its l-adic Galois representations are irreducible. Conjecturally, this is equivalent to all of the l-adic representations being irreducible, but this seems to be very hard to prove; weak irreducibility is a well-behaved substitute for that stronger condition. By the results of [2, 37], if a compatible system is odd, regular, and polarizable, then it is weakly irreducible if and only if it is potentially automorphic (in the sense that it potentially corresponds to a cuspidal automorphic representation).

The condition that the representation $\overline{s}$ is polarizable is best expressed in terms of the group $\mathcal{G}_n$ introduced in [16]; it is a reductive group with connected component $\mathrm{GL}_n \times \mathrm{GL}_1$ and component group of order 2, and as explained in [8, § 8.3], it is very closely related to the C-group of an n-dimensional unitary group over F^+ which splits over F. Then the polarizability of $\overline{s}$ is equivalent to the existence of a prolongation of $\overline{s}$ to a representation $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{F}}_p)$. In particular, this implies that $\overline{s}^c \cong \overline{s}^\vee \overline{\mu}|_{G_{F^+}}$ for some character $\overline{\mu}$ of G_{F^+}.

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Suppose that $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{F}}_p)$ is a prolongation of $\overline{s}$, and let v be a finite place of F^+. A *polarized component* for $\overline{\rho}|_{G_{F_v^+}}$ is, by definition, an irreducible component of a deformation ring for lifts of $\overline{\rho}|_{G_{F_v^+}}$ which are of some fixed inertial type, and of a fixed regular Hodge type if $v|p$. If $v|p$, then we say that such a component is *globally realizable* if it occurs globally, in the sense that there is some totally real field L^+ with a CM quadratic extension L, a place $w|p$ of L^+ for which $L_w^+ \cong F_v^+$, and a (polarizable, odd, reasonable) representation $r : G_L \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ which is part of a (polarizable, odd, weakly irreducible) compatible system, which prolongs to a representation $\rho' : G_{L^+} \to \mathcal{G}_n(\overline{\mathbf{Q}}_p)$ whose restriction $\rho'|_{G_{L_w^+}}$ gives rise to a point on this component (so that in particular, $\overline{\rho}'|_{G_{L_w^+}} \cong \overline{\rho}|_{G_{F_v^+}}$). If the place v splits in F as ww^c, then the deformation ring for $\overline{\rho}|_{G_{F_v^+}}$ can be identified with a deformation ring for $\overline{s}|_{G_{F_w}}$, as in [16]. Conjecturally, every component is expected to be globally realizable (and the analogous statement for places $v \nmid p$ is known), but proving this seems to be a very hard problem.

Our main theorem is the following (see Theorem 4.2.11 for a more precise statement and see § 1.4 for any unfamiliar terminology).

Theorem A. *Assume that either n is odd, or that $n = 2$. Let F be a CM field, and let $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be a reasonable representation, with prolongation $\overline{\rho}$.*

Let S be a finite set of finite places of F^+, such that S contains all of the places at which $\overline{\rho}$ is ramified and all of the places lying over p. For each place $v \in S$, let C_v be a component for $\overline{\rho}|_{G_{F_v^+}}$, which is globally realizable if $v|p$.

Then there exists an odd, regular, polarized, weakly irreducible compatible system $(\{s_\lambda\}, \{\mu_\lambda\})$ of G_F-representations with associated p-adic representation s, and a prolongation ρ of s, which satisfies the following:

(1) *ρ lifts $\overline{\rho}$, and for each place $v \in S$, the representation $\rho|_{G_{F_v^+}}$ lies on C_v.*

(2) *ρ is unramified outside S.*

Note that by the very definition of global realizability, the hypothesis that each C_v is globally realizable is a necessary condition for the conclusion of the theorem to hold.

The hypothesis that $\overline{s}$ is reasonable is needed in order to apply the theorems of [2], and some restriction on p, n and the size of the image of $\overline{s}$ is certainly necessary; for example, the results of [31] show that the analogous result fails for modular forms of weight 2 if $p = 2$ (in fact there are also dihedral counterexamples due to Serre with $p = 3$; see [14, § 4.4]). More generally, calculations in Galois cohomology suggest that if $p \leqslant n + 1$, and $\overline{s}|_{G_{F(\zeta_p)}}$ is reducible, then it unreasonable to hope for a global lifting result with control of the local representations at all places. Thus the only unnaturally restrictive hypothesis in Theorem A is the exclusion of even integers $n > 2$, which is a byproduct of our methods; this is because given a compatible systems of n-dimensional polarizable l-adic representations, we cannot deduce the oddness of all the representations in the compatible system from the oddness of a single representation.

An almost immediate corollary of our results is the following potential automorphy theorem, which may be of independent interest. Subject to the restriction that $n = 2$

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or n is odd, it improves on [2, Theorem 4.5.1] by replacing 'potentially diagonalizable' by 'globally realizable'. Note that if $v|p$ in F^+ splits in F, then the global realizability of $\rho|_{G_{F_v^+}}$ only depends on $s|_{G_{F_w}}$ for $w|v$ in F.

Theorem B (Corollary 4.2.12). *Assume that either n is odd or $n = 2$. Let F be a CM field, and let (s, μ) be a polarized representation, where*

$$s : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

is odd and ramified at only finitely many primes. Suppose that $\overline{s}$ is reasonable. Let ρ be the corresponding prolongation of s, and assume that $\rho|_{G_{F_v^+}}$ is globally realizable for each $v|p$. Then (s, μ) is potentially automorphic.

In fact (as noted in the abstract), we could weaken the hypothesis that $\rho|_{G_{F_v^+}}$ is globally realizable to requiring only that it is potentially globally realizable, because this is equivalent to global realizability by Corollary 4.2.13. Perhaps surprisingly, if $\overline{s}$ is automorphic, then we cannot deduce that s is also automorphic; this is because our methods make considerable use of potential automorphy results for other representations in a compatible system containing s. On the other hand, if we did know that weakly irreducible compatible systems are automorphic (rather than just potentially automorphic), then a version of the Breuil–Mézard conjecture for odd-dimensional globally realizable representations and a version of the weight part of Serre's conjecture for odd-dimensional unitary groups would both follow from combining Theorem A with the methods of [24].

1.2. History and motivation

We now give a somewhat leisurely overview of our motivations and of previous work on similar questions. Ultimately, the problems that we are working on are motivated by congruences between modular forms; more specifically, we are concerned with congruences between eigenforms. Such congruences can often best be understood in terms of the corresponding Galois representations, and in particular in terms of the restrictions of these (global) Galois representations to (local) decomposition groups. It is therefore natural to wonder whether there is a local to global principle for the existence of such congruences.

The first results in the literature that we know of that are explicitly formulated in this way are those of [18], which we now recall. Let $p > 3$ be prime, and let f be a newform of level prime to p and weight 2. We can (after choosing an embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$) associate a p-adic Galois representation $\rho_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ to f, and thus a mod p representation $\overline{\rho}_f : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$. We say that an irreducible representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ or $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$ is *modular of weight 2* if it is isomorphic to some $\overline{\rho}_f$ (respectively ρ_f).

Then the main result of [18] is as follows. Suppose we are given a modular representation $\overline{\rho} : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{F}}_p)$ of weight 2, and that for each prime $l \neq p$, we are given a lifting of $\overline{\rho}|_{G_{\mathbf{Q}_l}}$ to a p-adic representation $\rho_l : G_{\mathbf{Q}_l} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$. Suppose also that all but finitely many of the ρ_l are unramified. Then there is a lift $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\overline{\mathbf{Q}}_p)$

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of $\overline{\rho}$ which is modular of weight 2, with the property that for all $l \neq p$, we have an isomorphism of restrictions to inertia $\rho|_{I_{\mathbf{Q}_l}} \cong \rho_l|_{I_{\mathbf{Q}_l}}$.

This result is best possible, in the sense that any modular representation is necessarily only ramified at finitely many primes, and that (for example, because spaces of modular forms are finite-dimensional) it is unreasonable to pin down $\rho|_{G_{\mathbf{Q}_l}}$ more than specifying it on inertia. In particular, by local–global compatibility, the conductor of $\rho|_{I_{\mathbf{Q}_l}}$ determines the l-power part of the level of the modular form associated to ρ.

It is natural to wonder whether this result can be extended to cover modular forms of higher weight or with p dividing their level, and to allow some control of $\rho|_{G_{\mathbf{Q}_p}}$. Since the local behaviour at p of ρ_f is highly dependent on the weight of the newform f and the power of p dividing its level, these two questions are closely related, and the most general local to global result of this kind is best formulated in terms of components of local deformation rings. If we fix a weight $k \geqslant 2$, and a finite extension $K/\mathbf{Q}_p$, then we can consider the deformation ring R_K^k for liftings of $\overline{\rho}|_{G_{\mathbf{Q}_p}}$ which become semistable over K with Hodge–Tate weights $0, k-1$. If ρ is modular of weight k and some level, then for some sufficiently large K, $\rho|_{G_{\mathbf{Q}_p}}$ corresponds to a point of R_K^k. (For example, if the corresponding modular form has level prime to p, then we can take $K = \mathbf{Q}_p$.)

The spectrum of R_K^k has finitely many irreducible components, and given such a component, we can ask whether there exists a lift ρ with the property that $\rho|_{G_{\mathbf{Q}_p}}$ lies on this component. This is the correct analogue of what we are demanding at the places $l \neq p$; indeed, it turns out that specifying $\rho|_{I_{\mathbf{Q}_l}}$ is equivalent to demanding that $\rho|_{G_{\mathbf{Q}_l}}$ lies on a particular component of a deformation ring for $\overline{\rho}|_{G_{\mathbf{Q}_l}}$. Proving that the lifting problem still admits a solution when we specify a component at p is much harder than the case in which we only specify components away from p, but it follows from the results of [28] that such a lift exists under a mild ('Taylor–Wiles') condition on $\overline{\rho}$. In particular, this again gives a complete understanding of the possibilities for the weight and level of the corresponding modular form.

When we restrict to the case that $2 \leqslant k \leqslant p+1$ and $K = \mathbf{Q}_p$, the lifting problem is closely related to Serre's conjectures [40] on the weight and level of modular Galois representations. For example, while not formulated in this way, Serre's conjecture on the minimal weight and level is equivalent to asking that $\overline{\rho}$ admits a modular lift whose ramification away from p is as small as possible (as measured by the Swan conductor), and whose weight is as small as possible, compatible with the property of locally having a crystalline lift of the corresponding Hodge–Tate weights.

It is relatively straightforward to formulate conjectural generalizations of these results. For example, a detailed formulation of a generalization of Serre's conjectures to Hilbert modular forms was made in [7], and the weights are described in terms of the existence of local crystalline lifts in a similar fashion to that described above. One can make similar conjectures for automorphic representations on unitary groups over CM fields (or equivalently, for conjugate self-dual automorphic representations of GL_n over CM fields), and it is these generalizations that will concern us below. (However, we do not expect any straightforward generalization of these results to hold outside of settings which are discrete series at the infinite places.)

While some of the arguments of [18] and of the related papers on the weight and level parts of Serre's conjecture can be generalized to Hilbert modular forms, it seems hard to adapt them to prove the conjectures of [7] completely, and much harder still to study congruences between forms on GL_n in this way. However, in the mid 2000s, a new approach to these problems was discovered by Khare–Wintenberger [27] and the third author [22], which we will now describe in the setting of modular forms.

The approach is via the deformation theory of global Galois representations and automorphy lifting theorems. Suppose that as above we are given components of deformation rings for $\overline{\rho}|_{G_{\mathbf{Q}_l}}$ for each prime l, which are unramified for all but finitely many l. Then there is a corresponding deformation ring R^{univ} for the global representation $\overline{\rho}$, and the $\overline{\mathbf{Q}}_p$-points of its spectrum precisely correspond to the Galois representations that we are hoping to construct. If we can show that the set of $\overline{\mathbf{Q}}_p$-points is nonempty, then we can hope to show that the Galois representations are modular using modularity lifting theorems (the Taylor–Wiles method).

The tangent space to R^{univ} can be computed by Galois cohomology, and it turns out that R^{univ} always has dimension at least 1. (This computation relies on the weight k being at least 2, and more generally on us being in a discrete series context.) Heuristic arguments lead us to expect that R^{univ} is a finite $\mathbf{Z}_p$-algebra, and if this is the case, the lower bound on the dimension guarantees the existence of $\overline{\mathbf{Q}}_p$-points.

There is no known purely Galois-theoretic argument guaranteeing this finiteness in general (although it can sometimes be arranged at the cost of allowing additional ramification away from p by an argument of Ramakrishna [39]). However, modularity lifting theorems are proved by identifying deformation rings such as R^{univ} with Hecke algebras, which are finite over $\mathbf{Z}_p$ by definition, so in principle it is enough to prove an appropriate modularity lifting theorem (which can then be used in the final step of the argument to deduce that the Galois representations that we have constructed are actually modular).

Unfortunately, this argument is circular as written, because what the Taylor–Wiles method allows us to prove is that if some $\overline{\mathbf{Q}}_p$-point of R^{univ} is modular, then R^{univ} may be identified with a Hecke algebra; but it gives us no assistance with producing a $\overline{\mathbf{Q}}_p$-point in the first place. A key insight of Khare–Wintenberger is that this argument can be combined with base change and/or potential modularity to avoid the circularity. Suppose that F is a totally real finite extension of $\mathbf{Q}$, and that $\overline{\rho}|_{G_F}$ is irreducible. Then we may consider the deformation problem for $\overline{\rho}|_{G_F}$ given by the restrictions to places of F of the conditions we imposed over $\mathbf{Q}$, and the corresponding deformation ring R_F^{univ} is again of dimension at least one.

Now, by definition R^{univ} is an R_F^{univ}-algebra, and it is in fact a finite R_F^{univ}-algebra. It is therefore enough to prove a modularity lifting theorem for R_F^{univ}. This allows us to reprove many cases of the theorem of [18], in the following way. Suppose for simplicity that for each prime $l \neq p$, both the original modular representation $\rho_f|_{G_{\mathbf{Q}_l}}$ and the local representation ρ_l are finitely ramified (that is, they become unramified after restriction to a finite extension of $\mathbf{Q}_l$). Then we may choose a finite solvable totally real extension $F/\mathbf{Q}$ so that the restrictions to the finite places of F of these representations

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are actually unramified. In particular, the corresponding restrictions of $\rho_f|_{G_{\mathbf{Q}_l}}$ and the local representation ρ_l lie on the same component of the corresponding local deformation ring (indeed, the unramified local deformation ring is formally smooth).

By solvable base change, $\rho_f|_{G_F}$ is modular, and by the choice of F, it gives a point of R_F^{univ}. A modularity lifting theorem then shows that R_F^{univ} is a finite $\mathbf{Z}_p$-algebra. Thus the same is true of R^{univ}, and so R^{univ} has $\overline{\mathbf{Q}}_p$-points, which give the sought-after Galois representations; the modularity of these representations again follows from solvable base change. The more general case, in which the ramification can be potentially unipotent, can be handled in the same way when given some level-raising and level-lowering results over F; by choosing F appropriately, one reduces to a relatively straightforward case (see [41]).

A variant of this argument makes it possible to state and prove results about Galois representations that make no reference to automorphic forms (although the proofs make heavy use of automorphic techniques). To this end, rather than assuming that $\overline{\rho}$ is modular, assume only that it is irreducible and odd, in the sense that $\overline{\rho}(c)$ is non-scalar, where c is a complex conjugation. (Of course, since Serre's conjecture is a theorem, this implies that $\overline{\rho}$ is modular, but we can and will make an analogous assumption in more general contexts where the analogue of Serre's conjecture is open.) Then the same Galois cohomology calculations go through, and if we want to produce lifts of $\overline{\rho}$ with specified local properties, it is enough by the above arguments to find a finite (not necessarily solvable) extension of totally real fields $F/\mathbf{Q}$ for which $\overline{\rho}|_{G_F}$ is modular (that is, it comes from a Hilbert modular form).

An argument of Taylor [42, 43] can be used to prove such 'potential modularity' theorems. The idea is as follows: one can find a moduli space whose F-points correspond to abelian varieties, part of whose p-torsion is isomorphic to $\overline{\rho}|_{G_F}$, and the corresponding part of whose l-torsion, for some fixed prime $l \neq p$, is isomorphic to an induction of a character. Since inductions of characters are always modular, in favourable circumstances one can use modularity lifting theorems to prove that (part of) the l-adic Tate module of the corresponding abelian variety is modular, and thus that (part of) the p-adic Tate module is modular, and finally that $\overline{\rho}|_{G_F}$ is modular. That F-points exist for F sufficiently large follows from a theorem of Moret-Bailly, which also allows one to impose the kinds of local conditions that are needed in order to apply modularity lifting theorems.

As well as producing lifts of $\overline{\rho}$ with specified local properties, it turns out that potential modularity allows one to prove that each p-adic representation ρ that is constructed in this way is part of a compatible system of l-adic representations. Indeed, this property is automatic for Galois representations associated to automorphic forms, so that $\rho|_{G_F}$ is part of a compatible system. By solvable base change, the same is true for $\rho|_{G_{F'}}$ whenever F/F' is a solvable extension of totally real fields, and an argument with Brauer's theorem (see [19]) makes it possible to put these together to give the required compatible system.

We now digress briefly to discuss another aspect of compatible systems that will be of fundamental importance throughout this paper. In general, it seems to be hopeless to understand the components of local potentially semistable deformation rings of mod p representations in any concrete way, and this in turn places serious restrictions on automorphy lifting theorems. However, it is possible to understand them in the

Fontaine–Laffaille case, which by definition is the case that p is unramified in the base field, and the weight is small relative to p. It is also very hard to prove automorphy lifting theorems when the global mod p Galois representation has a small image (in particular, when the image is reducible). This can make it very difficult to prove the potential automorphy of a given Galois representation (or its automorphy, even if we know that it is residually automorphic). However, if instead we are given a weakly irreducible compatible system of l-adic Galois representations, then for all large l, the l-adic representation will be Fontaine–Laffaille, and its residual image will be suitably large. Accordingly, one can hope to prove the potential automorphy of the l-adic representation for some l, and immediately deduce the potential automorphy for all l. As we now explain, this has been carried out in considerable generality. In our arguments, this will allow us to use compatible systems of Galois representations as a kind of proxy for automorphic representations, without assuming the Fontaine–Mazur conjecture.

We briefly review the history of higher-dimensional potential automorphy theorems. Many of the arguments discussed above were generalized to polarizable (that is, essentially conjugate self-dual) n-dimensional Galois representations of CM fields in the papers [16, 26, 44] (the corresponding automorphic representations being those on general unitary groups). In particular, [44] uses automorphy lifting techniques to prove the kind of level-raising/lowering results that we applied after the base change above, and the third author's paper [23] deduced an n-dimensional version of the theorem of Diamond–Taylor in low weight.

Potential modularity, while powerful, has its limitations, the chief of which is that the method explained above only works for modular forms of weight 2 (that is, for Galois representations with Hodge–Tate weights $0, 1$), because for reasons of Griffiths transversality, the required moduli spaces only exist in this case. Allied to this is the difficulty mentioned above that the deformation rings for $\overline{\rho}_{|G_{\mathbf{Q}_p}}$ are much more complicated than those for $l \neq p$, and much less well understood (for example, it is certainly no longer the case that deformations are potentially unipotent on inertia). Indeed, when k is large compared to p, very little is known about the components of these deformation rings, and we do not know whether we can make base changes to make representations lie on the same component in any generality, which limits our ability to change components at p (recall that for $l \neq p$, in the discussion above we used that any finitely ramified representation can be made to lie on the unique unramified component after a suitable base change). (It is however worth noting that when $k = 2$ arguments of this kind are possible, even for Hilbert modular forms over totally real fields in which p is highly ramified, by the results of [29] and [21]. These results were a crucial part of the proof of the weight part of Serre's conjecture for Hilbert modular forms [24], and the lack of anything similar for higher-dimensional representations is one of the reasons that less is known about the weight part of Serre's conjecture in dimension greater than 2.)

In higher dimensions, the situation is worse; the potential automorphy theorems of [26] apply only in weight 0 (that is, the lowest discrete series; the corresponding Galois representations have Hodge–Tate weights $0, 1, \ldots, n-1$), and in fact only to ordinary Galois representations. While this was enough to prove the Sato–Tate conjecture (by proving the potential automorphy of the symmetric powers of the 2-dimensional Galois

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representations attached to elliptic curves, which are ordinary at most primes), it falls well short of proving the potential modularity of compatible systems of Galois representations in any generality. This shortcoming was resolved in [2], which also introduced a way of systematically changing the weight of the representations, or more generally of moving between components of deformation rings at p. The argument involves a refinement of a method of Harris [25] (the 'tensor product trick'), the basic idea of which is as follows: given a global Galois representation with regular Hodge–Tate weights, by taking the tensor product with another representation, one can produce a representation (of much higher dimension) of weight 0. It is then possible to apply potential modularity techniques to this representation.

Of course, to be useful, one has to have a way of 'undoing' the tensor product again (on both the Galois and automorphic sides). In general, this is a hard problem, but if the representation we tensor with is the induction of a character, it turns out to be relatively straightforward, essentially because the tensor product of an n-dimensional representation with the induction of a character is itself the induction of an n-dimensional representation.

We also obtain a way of moving between weights and between components of deformation rings at places dividing p, in the following way. Let $\overline{\rho}: G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be our original mod p representation, and let $r: G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be induced from a character. Fix a deformation ring R_1^{univ} for $\overline{\rho}$ corresponding to certain given local conditions. Given a deformation ρ of $\overline{\rho}$, we obtain a deformation $\rho \otimes r$ of $\overline{\rho} \otimes \overline{r}$, and we let R_2^{univ} be the corresponding deformation ring. This procedure makes R^{univ} into an R_2^{univ}-algebra, and an argument analogous to the one of Khare–Wintenberger that we mentioned above shows that it is even a finite R_2^{univ}-algebra.

It is not immediately obvious that this buys us anything, as to apply the techniques we used above we will need to prove an automorphy lifting theorem for R_2^{univ}. To this end, suppose that we also have another deformation ring R_3^{univ} for $\overline{\rho}$, and another lift r' of $\overline{r}$, and that the deformation problem corresponding to the tensor product of a deformation for R_3^{univ} with r' again corresponds to R_2^{univ}. If r and r' are both inductions of characters, and if we know that R_3^{univ} has automorphic points, then we can prove an automorphy lifting theorem for R_2^{univ}, deduce its finiteness over $\mathbf{Z}_p$, and then prove the existence of lifts (and automorphy lifting) for R_1^{univ}, the original problem of interest.

It may not be obvious that this is generally applicable, but in fact in combination with base change techniques it gives enough flexibility to prove the potential automorphy of compatible families, by moving between Fontaine–Laffaille and ordinary weight 0 deformation problems. A little thought shows that this argument allows us to move freely between components of local deformation rings at places dividing p, provided that the corresponding representations are *potentially diagonalizable*, in the sense that after some base change, the components contain a point which is a direct sum of crystalline characters. It turns out to be straightforward to show that Fontaine–Laffaille representations and ordinary representations are potentially diagonalizable, giving the claimed potential automorphy result. (The most general result about the existence of Galois representations proved in [2] is essentially Theorem A above, but with 'globally realizable' replaced by 'potentially diagonalizable'.)

While this gives a general potential automorphy result for compatible systems, it is unsatisfactory as an answer to our original lifting question, due to the restriction to potential diagonalizable representations. This restriction is problematic for two reasons: first, beyond the Fontaine–Laffaille and ordinary cases, nothing is known about the potential diagonalizability or otherwise of n-dimensional representations. (It is quite plausible that all potentially crystalline representations are potentially diagonalizable, but this seems to be very hard to establish.) In addition, potentially diagonalizable representations are by definition potentially crystalline, so they do not (unlike Theorem A) tell us anything about general potentially semistable representations.

1.3. A sketch of the proofs

We now briefly sketch the proof of Theorem A, omitting many of the trickier technical details; in particular, we completely ignore places away from p in our discussions, as the difficulties they present are similar to, but simpler than, the difficulties for places dividing p. We also suppress all mention of choices of polarization. In essence, our idea is to go beyond the potentially diagonalizable case, by allowing ourselves to tensor with representations that are not necessarily induced from characters. There are some obvious difficulties with this approach, chief among them that on both the automorphic and Galois sides, it is hard to 'undo' a tensor product. We overcome this by using compatible systems, rather than individual Galois representations.

Our first technique is a variant of the argument of [2] explained above, that of tensoring with auxiliary representations in order to move between different components. However, for our purposes, it is insufficient to tensor with a fixed global representation. Instead, we put ourselves in the following situation: suppose given representations $r, s : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ which belong to compatible systems, and suppose for simplicity that for each place $v|p$, we have $\overline{r}|_{G_{F_v}} \cong \overline{s}|_{G_{F_v}}$. Let A_v be the component corresponding to $r|_{G_{F_v}}$, and let B_v be the component corresponding to $s|_{G_{F_v}}$. For each v, let C_v be one of A_v and B_v, and let D_v be the other. Let R_C and R_D be the corresponding global deformation rings for $\overline{r}$ and $\overline{s}$, respectively. (We should really be considering prolongations of these representations to $\mathcal{G}_n$-valued representations of G_{F^+}, but we ignore this point for the purposes of this introduction.)

We would like to produce representations r', s' corresponding to points of R_C and R_D, which belong to compatible systems; in this way, we will be able to swap components between different residual representations. We initially accomplish this under very restrictive hypotheses, which we will later relax; note that we certainly need to assume at first that $\overline{r} \otimes \overline{s}$ is irreducible, and that $r \otimes s$ has regular Hodge–Tate weights. Now, the deformation ring R^{univ} corresponding to $r \otimes s$ is finite over $\mathbf{Z}_p$, because $r \otimes s$ is part of a compatible system, and thus potentially automorphic. Taking the tensor product of representations coming from R_C and R_D makes $R_C \widehat{\otimes} R_D$ into an R^{univ}-algebra, and we are able to show that it is in fact a finite R^{univ}-algebra, so that in the same way as before, we can see that R_C, R_D both have $\overline{\mathbf{Q}}_p$-points, which will correspond to the lifts r', s' that we want to produce.

However, we need to show that these lifts belong to compatible systems. The tensor product $r' \otimes s'$ does belong to a compatible system of representations $\{t_l\}$ (because it

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corresponds to a point of R^{univ}, and is thus potentially automorphic), so we would like to have a result ensuring that if one representation in a compatible system is a tensor product, they all are. This should be true quite generally, but it seems hard to prove, and we only establish it under rather restrictive hypotheses (which are ultimately sufficient for our needs).

In outline, we argue as follows. We assume that the Zariski closure of the image of s' contains $\mathrm{SL}_n(\overline{\mathbf{Q}}_p)$. Under a strong irreducibility hypothesis (which can be arranged at infinitely many primes l by imposing local conditions), we can show that the representations t_l decompose accordingly as tensor products $t_l = r'_l \otimes s'_l$, where the Zariski closure of the image of s'_l contains $\mathrm{SL}_n(\overline{\mathbf{Q}}_l)$. Here we make crucial use of a result of Larsen and Pink [30]. Then, if l is sufficiently large, the representations r'_l and s'_l belong to compatible families by the results of [2]; this is the only place that we need our assumption on n, as we have to know that r'_l and s'_l are odd in the sense of [3]. This oddness is automatic if n is odd, and can be proved if $n = 2$ by the methods of [9, 10], but it seems to be beyond the reach of current technology if $n > 2$ is even. Our assumptions then show that (after possibly twisting) the p-adic representations in these compatible systems are r' and s', as required.

With this component swapping result in hand, the basic outline of the proof of Theorem A is as follows. Suppose for simplicity that there is only a single place $v|p$ of F, let r be some lift of $\overline{r}$, and let C be the given globally realizable component. By definition, this means that there is another CM field L, a place $w|p$ and a representation s (which is part of a compatible system) such that $s|_{G_{L_w}}$ lies on C. Then if we apply our swapping result to $r|_{G_{FL}}$ and $s|_{G_{FL}}$, we can produce a lift of $\overline{r}|_{G_{FL}}$ which lies on the correct component at some place over v. If we could produce a lift with this property at all places above v, then we would be done by the usual Khare–Wintenberger method; in order to do this, we replace FL with its Galois closure, and s with its various Galois conjugates, and then inductively apply the swapping result to each of these conjugates in turn.

Unfortunately, the actual argument is much more complicated than this straightforward outline. The problem is that all of the results that we are applying have hypotheses that we have been ignoring; for example, we need $\overline{r} \otimes \overline{s}$ to be irreducible, we need $r \otimes s$ to have regular Hodge–Tate weights, and we need to satisfy the restrictive hypotheses of our main swapping result, which are a mixture of local and global assumptions.

We are able to handle the various local assumptions away from p by more base change tricks, but these cannot help with the global problems. Since the fields F, L and the representations $\overline{r}, \overline{s}$ are arbitrary, we cannot hope to arrange that their restrictions to the Galois closure of FL are irreducible. Instead, we make use of an idea introduced in [10], and use the theorem of Moret-Bailly mentioned above to construct auxiliary global representations with a large image, which locally admit potentially diagonalizable lifts of arbitrarily large weights.

These representations are constructed over extensions of F and L that we have little control over, and we have to go to some lengths to ensure that we can arrange all of the properties we need. Rather than swapping directly between $\overline{r}$ and $\overline{s}$, we instead make a long chain of swaps, going via many auxiliary representations, and making many base changes and descents by the Khare–Wintenberger argument.

1.4. Notation, conventions, and background material

All representations considered in this paper are assumed to be continuous with respect to the natural topologies, and we will never draw attention to this. If M/F is an extension of number fields, then we will write M^{gal} for the Galois closure of M over $\mathbf{Q}$, and $M^{F\text{-gal}}$ for the Galois closure of M over F. As usual, if K is a field of characteristic zero, then we write $G_K = \mathrm{Gal}(\overline{K}/K)$ for its absolute Galois group, and if K is furthermore a local field, we write I_K for the inertia subgroup of G_K.

1.4.1. Polarizable representations. We begin by recalling some definitions and results from [2, 16] concerning polarizable representations.

Recall from [16] that the reductive group $\mathcal{G}_n$ over $\mathbf{Z}$ is given by the semi-direct product of $\mathcal{G}_n^0 = \mathrm{GL}_n \times \mathrm{GL}_1$ by the group $\{1, \jmath\}$, where

$$\jmath(g, a)\jmath^{-1} = (a \cdot {}^t g^{-1}, a).$$

We let $\nu : \mathcal{G}_n \to \mathrm{GL}_1$ be the character which sends (g, a) to a and sends $\jmath$ to -1.

Let Γ be a group, with an index 2 subgroup Δ. Fix an element $\gamma_0 \in \Gamma \setminus \Delta$. Let R be a (commutative) ring. Then by [16, Lemma 2.1.1], there is a natural bijection between

- the set of homomorphisms $\rho : \Gamma \to \mathcal{G}_n(R)$ which induce isomorphisms $\Gamma/\Delta \xrightarrow{\sim} \mathcal{G}_n/\mathcal{G}_n^0$, and
- the set of triples $(r, \mu, \langle\,,\rangle)$ consisting of homomorphisms $r : \Delta \to \mathrm{GL}_n(R)$ and $\mu : \Gamma \to R^\times$, and a perfect R-linear pairing
$$\langle\,,\rangle : R^n \times R^n \to R,$$
which for all $x, y \in R^n$ and $\delta \in \Delta$ satisfies
  - $\langle x, r(\gamma_0^2)y\rangle = -\mu(\gamma_0)\langle y, x\rangle$, and
  - $\langle r(\delta)x, r(\gamma_0\delta\gamma_0^{-1})y\rangle = \mu(\delta)\langle x, y\rangle$.

This correspondence is given by taking $r =$ proj. onto the first factor of $\mathcal{G}_n^\circ$ of $\rho|_\Delta$, and $\mu = \nu \circ \rho$, and setting

$$\langle x, y\rangle = {}^t x A^{-1} y,$$

where $\rho(\gamma_0) = (A, -\mu(\gamma_0))\jmath$. We say that the pair (r, μ) is *polarized*[1] and that r is *polarizable*, and is μ-*polarized*. If we are given a polarized pair (r, μ), then we will sometimes refer to a corresponding homomorphism $\rho : \Gamma \to \mathcal{G}_n(R)$ (which depends on the choice of γ_0, as well as on a choice of pairing $\langle\,,\rangle$ witnessing the polarizability of (r, μ)) as a *prolongation* of the pair (r, μ).

Given a polarized pair (r, μ), we call μ the *multiplier character* of the pair (r, μ). Given two polarized representations (r_1, μ_1) and (r_2, μ_2), there is a polarized representation $(r_1 \otimes r_2, \delta_{\Gamma/\Delta}\mu_1\mu_2)$, where $\delta_{\Gamma/\Delta}$ denotes the unique non-trivial character of Γ/Δ (see [2, § 1.1] for the explicit description of this construction as an operation on $\mathcal{G}_n$-valued representations).

[1]This is established terminology, and so we use it here. Note though that in general the pair (r, μ) may not determine the pairing $\langle\,,\rangle$ uniquely, even up to a scalar multiple. However, if R is a field and r is absolutely irreducible, then, as we will observe below, the pairing $\langle\,,\rangle$ *is* uniquely determined up to a scalar.

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Suppose now that R is a complete local Noetherian ring, that $r : \Delta \to \mathrm{GL}_n(R)$ is such that $r \bmod \mathfrak{m}_R$ is absolutely irreducible, and that $\mu : \Gamma \to R^\times$ is a character such that

$$r^{\gamma_0} \cong r^\vee \otimes \mu|_\Delta. \tag{1.4.2}$$

Giving such an isomorphism is equivalent to giving a pairing

$$\langle\, , \rangle : R^n \times R^n \to R,$$

which for all $x, y \in R^n$ and $\delta \in \Delta$ satisfies $\langle r(\delta)x, r(\gamma_0\delta\gamma_0^{-1})y\rangle = \mu(\delta)\langle x, y\rangle$ for all $x, y \in R^n$ and $\delta \in \Delta$. Since isomorphism (1.4.2) is unique up to scaling by elements of $R^\times$ (because of our assumption that $r \bmod \mathfrak{m}_R$ is absolutely irreducible), we see that the corresponding pairing $\langle\, , \rangle$ is also unique up to scaling. In particular, if (r, μ) is polarized, then the pairing $\langle\, , \rangle$ that yields a prolongation of (r, μ) is unique up to scaling.

If ρ is one particular prolongation, corresponding to a pairing $\langle\, , \rangle$, then we see that conjugating ρ by the element $(1, \lambda^{-1}) \in \mathrm{GL}_n(R) \times \mathrm{GL}_1(R) = \mathcal{G}_n^\circ(R) \subset \mathcal{G}_n(R)$ scales $\langle\, , \rangle$ by λ; the relevant computation is that

$$\begin{aligned}(1, \lambda^{-1})(g, a)\jmath(1, \lambda) &= (1, \lambda^{-1})(g, a)\jmath(1, \lambda)\jmath^{-1}\jmath \\ &= (1, \lambda^{-1})(g, a)(\lambda, \lambda)\jmath \\ &= (\lambda g, a)\jmath.\end{aligned}$$

Thus we see that all possible prolongations of (r, μ) are obtained from the given prolongation ρ by such conjugations. We also see that the possible pairings arising from the choice of a prolongation are independent of the choice of the element $\gamma_0 \in \Gamma \setminus \Delta$ used to construct the bijection described above between (certain) homomorphisms $\rho : \Gamma \to \mathcal{G}_n(R)$ and (certain) triples $(r, \mu, \langle\, , \rangle)$.

We now consider the particular case that $\Gamma = G_{F^+}$, $\Delta = G_F$, where F is a (totally complex) CM field with maximal totally real subfield F^+. We say that the pair (r, μ) is *polarized and odd* if it is polarized, and for all complex conjugations $c \in G_{F^+}$, we have $\mu(c) = -1$. In particular, we have the following standard lemma.

Lemma 1.4.3. *Suppose that the characteristic of $R/\mathfrak{m}_R$ is not 2. If (r, μ) is polarized, n is odd, and $r \bmod \mathfrak{m}_R$ is absolutely irreducible, then (r, μ) is automatically odd.*

Proof. This follows from the fact that any odd-dimensional perfect pairing that is preserved up to scaling by a residually absolutely irreducible group action (in characteristics other than 2) is necessarily symmetric. Indeed, let c be any complex conjugation, take γ_0 equal to c, and let $\langle\, , \rangle$ denote the pairing arising from a choice of prolongation. Since $c^2 = 1$, we find (using the first of the properties satisfied by the pairing arising from a prolongation) that $\langle x, y\rangle = -\mu(c)\langle y, x\rangle$. On the other hand, as we already remarked, the pairing is necessarily symmetric. Thus we find that $\mu(c) = -1$, as required. □

We can restrict a global representation $\rho : G_{F^+} \to \mathcal{G}_n(R)$ to the decomposition group $G_{F_v^+}$ of any finite place v of F^+. Note that if v is inert or ramified in F, then $G_{F_v^+}$ is not contained in G_F, so we are in the situation above with $\Gamma = G_{F_v^+}$ and $\Delta = G_{F_v}$.

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If however v splits in F, then $G_{F_v^+}$ is contained in G_F, so that $\rho(G_{F_v^+}) \subset \mathcal{G}_n^0(R) = \mathrm{GL}_n(R) \times R^\times$, so that ($\mu$ being fixed) the data of the representation ρ is the same as the data of the corresponding representation $r : G_{F_v^+} \to \mathrm{GL}_n(R)$.

1.4.4. Compatible systems. Let F be a number field. We recall some definitions from [2, § 5]. Note that what we call a 'compatible system' is a 'weakly compatible system' in [2].

By a *compatible system* $\mathcal{R}$ *of* n-dimensional representations of G_F *defined over* M we shall mean a 5-tuple

$$(M, S, \{Q_v(X)\}, \{r_\lambda\}, \{H_\tau\}),$$

where

(1) M is a number field,
(2) S is a finite set of primes of F,
(3) for each prime $v \notin S$ of F, $Q_v(X)$ is a monic degree n polynomial in $M[X]$,
(4) for each prime λ of M (with residue characteristic l, say),

$$r_\lambda : G_F \to \mathrm{GL}_n(\overline{M}_\lambda)$$

is a semisimple representation such that

- if $v \notin S$ is a prime of F and $v \nmid l$, then r_λ is unramified at v and $r_\lambda(\mathrm{Frob}_v)$ has characteristic polynomial $Q_v(X)$,
- if $v|l$, then $r_\lambda|_{G_{F_v}}$ is de Rham and in the case $v \notin S$ crystalline,

(5) for $\tau : F \hookrightarrow \overline{M}$, H_τ is a multiset of n integers such that for any $\overline{M} \hookrightarrow \overline{M}_\lambda$ over M, the τ-labelled Hodge–Tate weights of r_λ are H_τ.

We will call $\mathcal{R}$ *regular* if for each $\tau : F \hookrightarrow \overline{M}$, every element of H_τ has multiplicity 1. We will refer to a rank 1 compatible system of representations as a compatible system of characters.

Remark 1.4.5. By abuse of terminology, we refer to a collection of Galois representations $\{r_\lambda\}$ as a compatible system if it extends to a 5-tuple $\mathcal{R}$ as above. In this case, we say that the compatible system $\{r_\lambda\}$ is unramified outside S if it extends to such a 5-tuple with the given finite set S. Note that if $\{r_\lambda\}$ is unramified outside S, then the individual representations r_λ are unramified outside $S \cup \{v|l\}$ where λ has residue characteristic l.

Remark 1.4.6. By a slight abuse of terminology, if F'/F is a finite extension of number fields, and S is a finite set of places of F, then we will sometimes say that a compatible system of representations of $G_{F'}$ is unramified outside S if it is unramified outside of the set of places of F' lying over S. Similarly, we will say that an extension F''/F' is unramified outside of S if it is unramified at all places of F' not lying over a place of S, and so on. In particular, we will frequently apply this convention to quadratic extensions F/F^+, where F is CM with maximal totally real field F^+.

Note that if M'/M is a finite extension, then a compatible system defined over M naturally determines a compatible system with M'-coefficients. We regard these two compatible systems as equivalent. Similarly one can enlarge S, and we also regard compatible systems associated in this way as equivalent. We may then consider the equivalence classes of the equivalence relation generated by these equivalences, and it follows easily from [37, Lemma 1.1] that to each equivalence class of compatible systems is associated a minimal choice of M, namely the field generated by the coefficients of the polynomials $Q_v(X)$.

For this reason, we generally suppress M in the below. Somewhat abusively, we shall often assume that M comes with a fixed embedding $M \hookrightarrow \overline{\mathbf{Q}}_p$ for each prime p, and hence talk of *the* p-adic representation

$$s : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

associated to $\{s_\lambda\}$.

We also introduce the following convenient shorthand terminology.

Definition 1.4.7. Let F be a number field, and let $r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$ be a representation. Then we say that r is Fontaine–Laffaille, or Fontaine–Laffaille at all primes dividing l (for emphasis), if l is unramified in F, and for all $\tau : F \hookrightarrow \overline{\mathbf{Q}}_l$, the τ-labelled Hodge–Tate weights of r are contained in an interval of length $(l-2)$ (the precise interval possibly depending on τ).

Note that in particular if $\{r_\lambda\}$ is a compatible system, then all but finitely many of the r_λ are Fontaine–Laffaille.

If F is CM (in this paper, all CM fields are imaginary), we denote its maximal totally real subfield by F^+. If F is CM, and if $\mathcal{M} = (M, S_{F^+}, \{X - \alpha_v\}, \{\mu_\lambda\}, \{w\})$ is a compatible system of characters of G_{F^+}, then we will call $(\mathcal{R}, \mathcal{M})$ a polarized (and odd) compatible system if for all primes λ of M the pair (r_λ, μ_λ) is polarized (and odd). (Here S_{F^+} denotes the set of places of F^+ lying below an element of S.) We will call $\mathcal{R}$ polarizable (and odd) if there exists an $\mathcal{M}$ such that $(\mathcal{R}, \mathcal{M})$ is a polarized (and odd) compatible system. Note that $\mu_\lambda(c_v)$ is independent of λ, so oddness of a polarized compatible system can be checked at a single λ.

Recall from [2, § 2.1] that a *polarized automorphic representation* of $\mathrm{GL}_n(\mathbf{A}_F)$ is a pair (π, χ) consisting of an automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$, and a character $\chi : \mathbf{A}_{F^+}^\times/(F^+)^\times \to \mathbf{C}^\times$ with $\chi_v(-1) = (-1)^n$ for all $v|\infty$, such that $\pi^c \cong \pi^\vee \otimes (\chi \circ \mathbf{N}_{F/F^+} \circ \det)$. We say that an automorphic representation π of $\mathrm{GL}_n(\mathbf{A}_F)$ is *polarizable* if there exists a χ such that (π, χ) is polarized.

If (π, χ) is a regular algebraic cuspidal polarized automorphic representation of $\mathrm{GL}_n(\mathbf{A}_F)$, then there is an associated polarized and odd compatible system $(\{r_\lambda(\pi)\}, \{\varepsilon^{1-n} r_\lambda(\chi)\})$, as explained in [2, § 5.1]. (See also [2, Theorem 2.1.1]; note that ε denotes the cyclotomic character, and $r_\lambda(\chi)$ is the compatible system associated to χ, regarded as an automorphic representation of $\mathrm{GL}_1(\mathbf{A}_{F^+})$. The assumption that $\chi_v(-1) = (-1)^n$ ensures the oddness of this compatible system.) We say that the pair of compatible systems $(\{r_\lambda(\pi)\}, \{\varepsilon^{1-n} r_\lambda(\chi)\})$ is automorphic. We write $r_p(\pi)$ for the associated p-adic

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representation, and we say that a representation $r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ is automorphic if it is isomorphic to $r_p(\pi)$ for some π; note that a compatible system is automorphic if and only if for some prime p, its associated p-adic representation is automorphic.

The following definition is one of several closely related definitions that one could make of what it means for a compatible system to be potentially automorphic; conjecturally, all of these definitions are equivalent, and are equivalent to automorphy, but this seems to be very difficult to prove.

Definition 1.4.8. If F is CM, then we say that a pair of compatible systems $(\{s_\lambda\}, \{\psi_\lambda\})$, with the s_λ being n-dimensional and the ψ_λ being characters, is *potentially automorphic* if for every finite Galois extension $F^{(\mathrm{avoid})}/F$, there is a finite Galois extension of CM fields L/F, which is linearly disjoint from $F^{(\mathrm{avoid})}/F$, and is such that $(\{s_\lambda|_{G_L}\}, \{\psi_\lambda|_{G_L}\})$ is automorphic.

Similarly, we say that a compatible system $\{s_\lambda\}$ is *potentially automorphic* if it may be extended to a potentially automorphic pair of compatible systems $(\{s_\lambda\}, \{\psi_\lambda\})$.

Definition 1.4.9. We will call $\mathcal{R}$ *pure* (of weight $w \in \mathbf{Z}$) if

- for each $v \notin S$, each root α of $Q_v(X)$ in $\overline{M}$ and each $\iota : \overline{M} \hookrightarrow \mathbf{C}$, we have
$$|\iota\alpha|^2 = q_v^w;$$
- and for each $\tau : F \hookrightarrow \overline{M}$ and each complex conjugation c in $\mathrm{Gal}(\overline{M}/\mathbf{Q})$, we have
$$H_{c\tau} = \{w - h : \ h \in H_\tau\}.$$

In the following definition, and throughout the body of the paper, 'density' means 'Dirichlet density'.

Definition 1.4.10. We say that a compatible system $\{s_\lambda\}$ is *weakly irreducible* if there is a positive density set of rational primes l so that for all primes $\lambda | l$, the representation s_λ is irreducible.

One expects that the irreducibility of a single Galois representation in a compatible system should imply the irreducibility of all representations, but this is unknown in general. On the other hand, the notion of weak irreducibility turns out to be easy to work with in light of the following results.

Lemma 1.4.11. *Let F be CM. Then a regular, odd, polarizable compatible system of representations of G_F is weakly irreducible if and only if it is potentially automorphic.*

Proof. Any automorphic compatible system is weakly irreducible by [37, Theorem 1.7]; it follows immediately that potentially automorphic compatible systems are also weakly irreducible. Conversely, a weakly irreducible compatible system is potentially automorphic by the results of [2]; see Theorem 2.1.16. □

Lemma 1.4.12. *Let F be CM, and let $\{r_\lambda\}$ be a weakly irreducible, regular, odd, polarizable compatible system of representations of G_F. Then $\{r_\lambda\}$ is pure in the sense of* [2, § 5.1].

Proof. Since $\{r_\lambda\}$ is potentially automorphic by Lemma 1.4.11, it is pure by [2, Corollary 5.4.3]. $\square$

Lemma 1.4.13. *Let F be CM, and let $\{r_\lambda\}$ be a regular, odd, polarizable compatible system of representations of G_F. Suppose furthermore that $\{r_\lambda\}$ is pure. Then we may write $\{r_\lambda\} = \oplus_{i=1}^{s}\{r_{i,\lambda}\}$, where each $\{r_{i,\lambda}\}$ is a weakly irreducible, regular, odd, polarizable compatible system of representations of G_F.*

Proof. This is immediate from [37, Theorem 2.1] and Lemma 1.4.11. $\square$

We will occasionally need to make use of compatibility at ramified places. While we have not built this into our definition of a compatible system, it follows from potential automorphy, as in the following result.

Proposition 1.4.14. *Let F be CM, and let $\{r_\lambda\}$ be a weakly irreducible, regular, odd, polarizable compatible system of representations of G_F with field of coefficients M.*

Let v be a finite place of F, and suppose that $v \nmid \mathbf{N}\lambda$ (respectively, that $v|\mathbf{N}\lambda$). Then, we have the following:

(1) *For each finite extension K/F_v, $r_\lambda|_{G_K}$ is unramified (respectively, crystalline) for some λ if and only if it is so for all λ.*
(2) *Suppose that (1) holds. Then there is a representation r_v of I_{K/F_v} over $\overline{M}$ such that for each λ, $r_\lambda|_{I_{K/F_v}} \cong r_v$ (respectively, $\mathrm{WD}(r_\lambda|_{G_{F_v}})|_{I_{K/F_v}} \cong r_v$).*

Proof. By Lemma 1.4.11, $\{r_\lambda\}$ is potentially automorphic, so it is strictly compatible by [2, Corollary 5.4.3]. Strict compatibility means by definition that the Weil–Deligne representation corresponding to $r_\lambda|_{G_{F_v}}$ is independent of v, so the consequences follow immediately. $\square$

We next establish some results describing how the property of weak irreducibility of a compatible system behaves under restriction. To begin with, suppose that F is a number field and that $\{r_\lambda\}$ is a compatible system of representations of G_F, and that the Zariski closure of the image of r_λ is G_λ. Let $G_\lambda^\circ \subset G_\lambda$ denote the connected component of the identity. The following is a theorem of Serre.

Theorem 1.4.15. *The pre-image of G_λ° in G_F is independent of λ.*

Proof. See [30, Proposition 6.14]. $\square$

As a corollary, we have the following result, which allows us to ensure that certain restrictions of weakly irreducible compatible systems remain weakly irreducible.

Lemma 1.4.16. *Let F be a number field, and let $\{r_\lambda\}$ be a compatible system of G_F-representations. Then there exists a finite extension $F^{(\mathrm{avoid})}/F$ with the following property: if L/F is a finite extension linearly disjoint from $F^{(\mathrm{avoid})}$, and if $r = r_\lambda$ is any representation in the compatible system which is irreducible, then $r|_{G_L}$ is irreducible. In particular, if $\{r_\lambda\}$ is weakly irreducible, then $\{r_\lambda|_{G_L}\}$ is weakly irreducible.*

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Proof. Suppose that $r = r_\lambda$ is irreducible. If L is any finite degree extension of F, then the Zariski closure of the image of $r|_{G_L}$ will also contain G_λ°, because G_λ° is connected and hence does not contain any finite index subgroups.

By Theorem 1.4.15, the pre-image of G_λ° in G_F is independent of λ. Let $F^{(\mathrm{avoid})}$ denote the corresponding fixed field. If L is disjoint from $F^{(\mathrm{avoid})}$, then the component group of $r|_{G_L}$ will be isomorphic to the component group of r, and hence the Zariski closures of r and $r|_{G_L}$ will coincide. In particular, $r|_{G_L}$ will be irreducible. □

We also have the following variant of the preceding result, which shows that *arbitrary* restrictions (to CM extensions) of weakly irreducible compatible systems (satisfying the appropriate hypotheses) remain weakly irreducible, provided that at least one member of the restricted compatible system is irreducible.

Lemma 1.4.17. *Let F be CM, and let $\{r_\lambda\}$ be a weakly irreducible, regular, odd, polarizable compatible system of representations of G_F. Let M be a CM extension of F. If some $r_\mu|_{G_M}$ is irreducible, then $\{r_\lambda|_{G_M}\}$ (which is* a priori *a regular, odd, polarizable compatible system of representations of G_M) is again weakly irreducible.*

Proof. By Lemma 1.4.12, the weakly compatible system $\{r_\lambda\}$ is odd, polarized, regular and pure. These properties are inherited by the system $\{r_\lambda|_{G_M}\}$, which is therefore a direct sum of weakly irreducible compatible systems by Lemma 1.4.13. Since $r_\mu|_{G_M}$ is irreducible, there can only be one compatible system in this direct sum, and $\{r_\lambda|_{G_M}\}$ is weakly irreducible, as required. □

We close the present discussion of weak irreducibility with the following result, which establishes the weak irreducibility of certain tensor products of compatible systems.

Lemma 1.4.18. *Let F be CM, and let $\{s_\lambda\}$, $\{t_\lambda\}$ be regular, odd, weakly irreducible polarizable compatible systems of representations of G_F. Assume that $\{s_\lambda \otimes t_\lambda\}$ is regular, and that at least one representation $s_\mu \otimes t_\mu$ is irreducible. Then $\{s_\lambda \otimes t_\lambda\}$ is weakly irreducible.*

Proof. By Lemma 1.4.12, each of $\{s_\lambda\}$, $\{t_\lambda\}$ is pure, so that $\{s_\lambda \otimes t_\lambda\}$ is regular, pure, odd, and polarizable. By Lemma 1.4.13, it is therefore a direct sum of weakly irreducible compatible systems; since $s_\mu \otimes t_\mu$ is irreducible, this direct sum can only consist of a single compatible system, as required. □

1.4.19. Deformation rings. When we consider deformation rings and automorphy lifting theorems, we can no longer use algebraically closed coefficient fields. To this end, we adopt the convention that $\mathcal{O}$ will denote the ring of integers in a finite extension $E/\mathbf{Q}_p$ with residue field $\mathbf{F}$, and that E will be chosen large enough such that all representations under consideration are defined over E (and all mod p representations are defined over $\mathbf{F}$); as always, the precise choice of E is unimportant.

As usual, let F be a CM field with maximal totally real subfield F^+. Following [4], we work in a slightly more general context than [2, 16], allowing ramification at primes of F^+ which are inert or ramified in F. This allows us to make cleaner statements, and is also necessary for some of our arguments with auxiliary primes.

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Fix a prime $p > 2$ and a polarized residual representation $(\overline{s}, \overline{\mu})$ of G_{F^+} with $\overline{s}$ absolutely irreducible and (as always in the body of this paper) $p > 2$. We choose once and for all an element $\gamma_0 \in G_{F_+} \setminus G_F$ (e.g., we could choose γ_0 to be the complex conjugation at one of the real places of F^+), and we let $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\mathbf{F})$ be a fixed prolongation of $(\overline{s}, \overline{\mu})$, following the procedure discussed in § 1.4.1.

Let $\mu : G_{F^+} \to \mathcal{O}^\times$ be a lift of $\overline{\mu}$. Let A be a complete local Noetherian $\mathcal{O}$-algebra. Then a *μ-polarized lifting* of $\overline{\rho}$ to A is a representation $\rho : G_{F^+} \to \mathcal{G}_n(A)$ with $\nu \circ \rho = \mu$ and $\rho \pmod{\mathfrak{m}_A} = \overline{\rho}$. A *$\mu$-polarized deformation* of $\overline{\rho}$ to A is a $1 + M_n(\mathfrak{m}_A)$-conjugacy class of liftings. As in [15, Lemma 1.5], μ-polarized liftings and deformations ρ in this sense are equivalent to the data of a lifting or deformation s of $\overline{s}$ which satisfies $s^c \cong \mu s^\vee$ (where the equivalence arises from taking $s = \rho|_{G_F}$).

We also need to consider the corresponding local deformation problems. We refer to [4, § 3] for the definitions of deformations of fixed inertial and Hodge types. Let v be a finite place of F, let $\overline{\rho}_v : G_{F_v^+} \to \mathcal{G}_n(\mathbf{F})$ be a representation with multiplier $\overline{\mu}$, and let μ be a lift of $\overline{\mu}$. Then a *μ-polarized lifting* of $\overline{\rho}_v$ to A is a representation $\rho_v : G_{F_v^+} \to \mathcal{G}_n(A)$ with $\nu \circ \rho_v = \mu$ and $\rho_v \pmod{\mathfrak{m}_A} = \overline{\rho}_v$. If we fix an inertial type $I_{F_v^+} \to \mathcal{G}_n(E)$, then in the case $l \neq p$, we may consider the universal framed deformation $\mathcal{O}$-algebra $R^{\square,\tau}$ of inertial type τ; this ring is non-zero for a finite and nonempty set of inertial types τ. We refer to an irreducible component of any $R^{\square,\tau}[1/p]$ as simply 'a μ-polarized component for $\overline{\rho}_v$'; such a component uniquely determines τ. By [4, Lemma 3.4.1], each irreducible component is invariant under conjugation, in the sense that conjugation by elements of $\ker(\mathcal{G}_n(R^{\square,\tau}) \to \mathcal{G}_n(\mathbf{F}))$ preserves each irreducible component. We will sometimes speak of polarized components, rather than μ-polarized components, when the choice of μ is clear from the context.

If $v|p$, then in the same way we let $R^{\square,\tau,\mathbf{v}}$ denote the universal framed deformation $\mathcal{O}$-algebra of $\overline{\rho}$ for μ-polarized potentially semistable lifts of inertial type τ and Hodge type $\mathbf{v}$. We again refer to an irreducible component of any $R^{\square,\tau,\mathbf{v}}[1/p]$ as simply 'a μ-polarized component for $\overline{\rho}_v$'. Note that such a component again uniquely determines τ and $\mathbf{v}$; we say that a component is *regular* if $\mathbf{v}$ is regular (that is, the labelled Hodge–Tate weights are distinct); we will always assume this in our main results.

Return now to the global situation of a polarized residual representation $(\overline{s}, \overline{\mu})$ with prolongation $\overline{\rho}$. Suppose that v splits in F as ww^c. A choice of embedding $\overline{F^+} \to \overline{F_v^+}$ gives rise to a choice of $w|v$ in F. With respect to this choice, the representation $\overline{\rho}|_{G_{F_v^+}}$ has an image in $\mathcal{G}_n^\circ(\mathbf{F}) = \mathrm{GL}_n(\mathbf{F}) \times \mathrm{GL}_1(\mathbf{F})$, and the projection to the first factor is the representation $\overline{s}|_{G_{F_w}}$. (A different choice of embedding $\overline{F^+} \to \overline{F_v^+}$ corresponding to $w^c|v$ in F would have the effect of replacing $\overline{s}|_{G_{F_w}}$ by $\mu \otimes \overline{s}^\vee|_{G_{F_w}} \simeq \overline{s}|_{G_{F_{w^c}}}$.) If $\rho_v : G_{F_v^+} \to \mathcal{G}_n(A)$ is a μ-polarized lifting of $\overline{\rho}|_{G_{F_v^+}}$, then the projection to $\mathrm{GL}_n(A)$ gives rise to a lift $s_w : G_{F_w} = G_{F_v^+} \to \mathrm{GL}_n(A)$ of $\overline{s}|_{G_{F_w}}$.

Lemma 1.4.20. *If v splits in F, then the assignment $\rho_v \mapsto s_w$ is an equivalence of categories between the μ-polarized liftings of $\overline{\rho}|_{G_{F_v^+}}$ and the liftings of $\overline{s}|_{G_{F_w}}$.*

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Proof. Under this identification, the representation ρ_v is simply the representation $\rho_v = (s_w, \mu) : G_{F_v^+} \to \mathcal{G}_n^\circ(A) \subset \mathcal{G}_n(A)$, from which the result is clear (cf. the discussion in [16, § 2.3]). □

By Lemma 1.4.20, if v splits in F, then we can identify μ-polarized components with components of the corresponding lifting rings for $\overline{s}|_{G_{F_w}}$ after choosing a prime $w|v$ (or after choosing an embedding $\overline{F^+} \to \overline{F_v^+}$ which gives a canonical choice of $w|v$). We will sometimes do this without comment later in the paper.

Convention 1.4.21. *Let F/F^+ be a CM extension. Given a prime v in F^+, we choose an embedding $\overline{F^+} \to \overline{F_v^+}$ which in turn determines a choice of prime in F above v which we denote by w.*

Convention 1.4.22. *If $\overline{s} : G_F \to \mathrm{GL}_n(\mathbf{F})$ is as above, with prolongation $\overline{\rho} : G_{F^+} \to \mathcal{G}_n(\mathbf{F})$, and if (following Convention 1.4.21) w denotes a prime of F lying over the prime v of F^+, then we will often write 'μ-polarized component for $\overline{s}|_{G_{F_w}}$' rather than 'μ-polarized component for $\overline{\rho}|_{G_{F_v^+}}$'.*

Given another representation $\overline{\rho}' : G_{F_v^+} \to \mathcal{G}_m(\mathbf{F})$, and polarized components C, D for $\overline{\rho}$, $\overline{\rho}'$, respectively, then there is a well-defined component $C \otimes D$ for $\overline{\rho} \otimes \overline{\rho}'$. Similarly, if L/F_v^+ is a finite extension, there is a well-defined $\mu|_{G_L}$-polarized component $C|_L$ for $\overline{\rho}|_{G_L}$. (In the case that v is a split prime, this is [4, Lemma 3.5.1], and the general case is proved in exactly the same way.)

We will frequently make use of the following lemma without further comment.

Lemma 1.4.23. *Let F be a CM field, and let $\{r_\lambda\}$ be a regular, odd, polarizable, weakly irreducible compatible system of representations of G_F. Then for each λ and each finite place w of F, the representation $r_\lambda|_{G_{F_w}}$ lies on a unique component of the corresponding deformation ring.*

Proof. It suffices to prove that $r_\lambda|_{G_{F_w}}$ defines a smooth point of the corresponding deformation ring. By [4, Corollary 3.3.5], it is enough to prove that there is a finite extension L/F_w such that $r_\lambda|_{G_L}$ is pure in the sense of [46, § 1]. Since $\{r_\lambda\}$ is potentially automorphic by Lemma 1.4.11, this follows from the main theorems of [12, 13] (which show that automorphic Galois representations are pure). □

We now return to global deformation problems.

Proposition 1.4.24. *Let F be a CM field, and let $p > 2$ be prime. Let $(\overline{r}, \overline{\mu})$ be an absolutely irreducible polarized representation of G_F. Suppose that $\overline{r}$ is odd, and that $\overline{r}|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and let μ be a de Rham lift of $\overline{\mu}$. Let S be a finite set of finite places of F^+ such that $\overline{r}$ and μ are unramified outside S. For each place $v \in S$, let C_v be a μ-polarized component for $\overline{r}|_{G_{F_v}}$, which is regular if $v|p$.*

Let R^{univ} be the universal deformation ring for μ-polarized deformations of $\overline{r}$ which are unramified outside S, and lie on the component C_v for each $v \in S$. Then R^{univ} has Krull dimension at least one.

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Proof. This is [4, Corollary 5.1.1] (note that the condition there of being 'discrete series and odd' is by definition the same as being odd in the sense of this paper). □

1.4.25. Automorphy lifting and adequate representations. We end this section by recalling some results concerning automorphy lifting theorems. Let F be a CM field with maximal totally real subfield F^+. We say that a finite place v of F is *split* if $v|_{F^+}$ splits in F. In order to apply the (potential) automorphy results of [2], we need to assume that all of the places $v|p$, and all of the places at which our Galois representations are ramified, are split places; we will avoid making such assumptions in our main results by making base changes.

We have the following theorem; the notion of adequacy is recalled in Definition 1.4.29.

Theorem 1.4.26. *Let F be a CM field, and let $p > 2$ be prime. Suppose that $p \nmid n$ and that $\zeta_p \notin F$. Let (r, μ) be a polarized automorphic Galois representation, where $r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$, and assume that $\overline{r}(G_{F(\zeta_p)})$ is adequate.*

Let S be a finite set of places of F^+ which includes all places at which (r, μ) is ramified, and all places dividing p, and for each $v \in S$ let C_v denote the local component at v on which $\rho|_{G_{F_v}}$ lies, where $\rho : G_{F^+} \to \mathcal{G}_n(\overline{\mathbf{Q}}_p)$ is the prolongation of (r, μ). Assume that every place in S is a split place.

Let R_C denote the global deformation $\mathcal{O}$-algebra for $\overline{r}$ which parameterizes deformations of $\overline{\rho}$ that are μ-polarized, that are unramified outside S, and that for each $v \in S$, lie on the corresponding component C_v. Then R_C is a finite $\mathcal{O}$-algebra, and any representation corresponding to a $\overline{\mathbf{Q}}_p$-point of R_C is automorphic.

Proof. The automorphy statement is essentially [10, Theorem 7.1], and the finiteness statement follows easily from the proof of *loc. cit.* (cf. [47, Theorem 10.1]). We only need to justify the slightly weaker hypotheses that we are making here, in comparison to assumptions 4(c) and 4(d) in the statement of [10, Theorem 7.1]. Assumption 4(d) was only made because local–global compatibility at places dividing p was unknown at the time that [10] was written, but it is now available in the required generality thanks to [1]. Assumption 4(c) is satisfied by Lemma 1.4.23. □

As a consequence, we have the following useful finiteness result.

Lemma 1.4.27. *Let p be an odd prime, and let F be a CM field with $\zeta_p \notin F$. Let $\{(s_\lambda, \mu_\lambda)\}$ be a weakly irreducible, odd, regular, polarized compatible system of n-dimensional representations of G_F. Assume that $p \nmid n$, let (s, μ) be the p-adic representation coming from the given compatible system, with corresponding prolongation ρ, and assume that $\overline{s}(G_{F(\zeta_p)})$ is adequate.*

Let S be a finite set of finite places of F^+ which contains all of the places dividing p, and is such that ρ is unramified outside S. For each $v \in S$, let C_v denote the local component at v on which $\rho|_{G_{F_v}}$ lies. Let R_C denote the global deformation $\mathcal{O}$-algebra which parameterizes deformations of $\overline{\rho}$ that are μ-polarized, that are unramified outside S, and that, for each $v \in S$, lie on the corresponding component C_v. Then R_C is a finite $\mathcal{O}$-algebra.

Proof. By Lemma 1.4.11, $\{(s_\lambda, \mu_\lambda)\}$ is potentially automorphic. By [2, Lemma 1.2.3], we can reduce to the case that $\{(s_\lambda, \mu_\lambda)\}$ is in fact automorphic, and all the places in S are split places, in which case it follows from Theorem 1.4.26 that R_C is a finite $\mathcal{O}$-algebra. □

1.4.28. Adequate representations. Let k be a field of characteristic p. We always assume it is sufficiently large to contain all the eigenvalues of any representation under consideration. Let V be a vector space over k and let $G \subset \mathrm{GL}(V)$ be a group which acts absolutely irreducibly. We first recall from [47] what it means for G to be adequate.

Definition 1.4.29. G is *adequate* if the following conditions hold.

(1) $H^0(G, \mathrm{ad}^0(V)) = 0$.
(2) $H^1(G, k) = 0$.
(3) $H^1(G, \mathrm{ad}^0(V)) = 0$.
(4) For every irreducible G-submodule $W \subset \mathrm{ad}^0 V$, there is an element $g \in G$ with an eigenvalue α such that $\mathrm{tr}\, e_{g,\alpha} W \neq 0$ (where $e_{g,\alpha}$ is the projection to the generalized α-eigenspace of g).

Lemma 1.4.30. *Suppose that G acts absolutely irreducibly on V. Then the following are equivalent.*

(1) *Condition (4) of Definition 1.4.29.*
(2) *The set of semisimple elements of G spans $\mathrm{ad}(V) \otimes_k \overline{k}$ as a $\overline{k}$-vector space.*

Proof. This follows from Lemma A.1 of the appendix to [47], namely the equivalence between (i) and (iii). □

Lemma 1.4.31. *Suppose that V and V' are absolutely irreducible representations of a group Γ. Suppose that the projective images of Γ on V and V' are disjoint, that is, the group Γ surjects onto the product of the projective images of Γ on V and V', and denote the projective images by $\mathbf{P}G$ and $\mathbf{P}G'$. Then the images of Γ on $\mathrm{ad}(V)$ and $\mathrm{ad}(V')$ are $\mathbf{P}G$ and $\mathbf{P}G'$, respectively, and the image of Γ on $\mathrm{ad}(V \otimes V')$ is $\mathbf{P}G \times \mathbf{P}G'$.*

Proof. The fact that Γ acts on $\mathrm{ad}(V)$ as $\mathbf{P}G$ is completely formal. Hence under the assumption that the projective images $\mathbf{P}G$ and $\mathbf{P}G'$ are disjoint, Γ acts on $\mathrm{ad}(V) \oplus \mathrm{ad}(V')$ via $\mathbf{P}G \times \mathbf{P}G'$. The kernel of the map

$$\mathrm{GL}(\mathrm{ad}(V)) \times \mathrm{GL}(\mathrm{ad}(V')) \to \mathrm{GL}(\mathrm{ad}(V \otimes V'))$$

consists precisely of pairs of scalar matrices (λ, λ^{-1}). But it is not possible for any $g \in G$ (or $\mathbf{P}G$) to act on $\mathrm{ad}(V)$ as a non-trivial scalar – this would force the action of g on V itself to be scalar and then to be trivial on $\mathrm{ad}(V)$. □

Remark 1.4.32. The proof of Lemma 1.4.31 is just another way of saying that the map

$$\mathrm{PGL}(V) \times \mathrm{PGL}(V') \to \mathrm{GL}(\mathrm{ad}(V \otimes V'))$$

is injective.

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The following lemma is similar to Lemma A.2 of the appendix to [47].

Lemma 1.4.33. *Suppose that V and V' are absolutely irreducible representations of a group Γ. Suppose that the projective images of Γ on V and V' are disjoint, and that the images of Γ on V and V' satisfy condition (4) of Definition 1.4.29. Then the image of Γ on $V \otimes V'$ satisfies condition (4) of Definition 1.4.29.*

Proof. Let G and G' denote the images of Γ in V and V', respectively. By Lemma 1.4.30, the semisimple elements g of G and g' of G' span $\mathrm{ad}(V)$ and $\mathrm{ad}(V')$, respectively. In particular, the elements $g \otimes g'$, which are also semisimple, span $\mathrm{ad}(V) \otimes \mathrm{ad}(V') = \mathrm{ad}(V \otimes V')$.

Let g and g' be any pair of semisimple elements in G and G', respectively. By Lemma 1.4.31, there is a $\gamma \in \Gamma$ which acts projectively on V and V' by g and g', respectively. Hence it acts on V and V' by λg and $\lambda' g'$, respectively, for scalars λ and λ'. Hence it acts on $V \otimes V'$ by a scalar multiple of $g \otimes g'$. In particular, it spans the same line in $\mathrm{ad}(V \otimes V')$ as $g \otimes g'$. Hence these elements span $\mathrm{ad}(V \otimes V')$, as required. □

Lemma 1.4.34. *Suppose that V and V' are absolutely irreducible representations of a group Γ of dimensions $n, n' > 2(p+1)$, respectively, whose projective images are disjoint. Then the image of Γ acting on $V \otimes V'$ is adequate.*

Proof. By Theorem A.9 of the appendix to [47], the images of Γ acting on V and V' are both adequate. That condition (4) of Definition 1.4.29 holds for the image H of Γ acting on $V \otimes V'$ then follows from Lemma 1.4.33.

The adjoint representation of H has image $\mathbf{P}G \times \mathbf{P}G'$ by Lemma 1.4.31, so there is a surjective map $H \to \mathbf{P}G \times \mathbf{P}G'$ whose kernel Z is central in H (and acts by scalars on $V \otimes V'$). Certainly, Z injects into $\overline{k}^{\times}$ and so has order prime to p. Let M and M' be $\mathbf{P}G$- and $\mathbf{P}G'$-modules, respectively. Since Z has order prime to p, inflation–restriction gives

$$H^1(H, M \otimes M') = H^1(\mathbf{P}G \times \mathbf{P}G', M \otimes M').$$

Another application of inflation–restriction gives an exact sequence

$$H^1(\mathbf{P}G, M) \otimes (M')^{\mathbf{P}G'} \to H^1(\mathbf{P}G \times \mathbf{P}G', M \otimes M') \to M^{\mathbf{P}G} \otimes H^1(\mathbf{P}G', M').$$

Letting $M = M' = k$ or $M = \mathrm{ad}(V)$ and $M' = \mathrm{ad}(V')$, we see the two exterior groups vanish by the adequacy of the images of Γ on V and V', and hence so does the middle group. Absolute irreducibility is easy, and the lemma follows. □

Lemma 1.4.35. *Let $G \subset \mathrm{GL}(V)$, and let $H \subset G$ be a normal subgroup with G/H of order prime to p, such that H is adequate. Then G is adequate.*

Proof. If condition (4) of Definition 1.4.29 holds for H, it obviously holds for G. So it suffices to check the cohomological conditions. We have (since G/H has order prime to p) that

$$H^1(G, M) = H^1(H, M)^{G/H}.$$

Hence if the right-hand side vanishes, then so does the left-hand side. Similarly, if a representation of G is absolutely irreducible after restriction to H, it is absolutely irreducible. □

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Lemma 1.4.36. *Let $A \subset \mathrm{GL}_n(k)$ be absolutely irreducible with $n > 2(p+1)$. If $B \subset \mathrm{GL}_n(\overline{k})$ has a projective image containing $\mathrm{PSL}_n(l)$ for some sufficiently large extension l/k (depending on A), then the image of $A \otimes B$ is adequate.*

Proof. By using Lemma 1.4.35, we may assume that B has a projective image exactly $\mathrm{PSL}_n(l)$ for some l. This is a simple group. By taking l large enough so that the projective image of A has order less than that of $\mathrm{PSL}_n(l)$, we deduce that the projective images of A and B have no non-trivial common quotients. It follows by Goursat's lemma that the image of $A \otimes B$ surjects onto the product of the projective images of A and B. We now finish by invoking Lemma 1.4.34. □

The following is an immediate consequence of Lemma 1.4.36.

Lemma 1.4.37. *Suppose that $p > 2(n+1)$, that L is a number field, and that $\overline{a} : G_L \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is an irreducible representation. Then if q is a sufficiently large power of p (depending on $\overline{a}$), and $\overline{b} : G_L \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ has a projective image containing $\mathrm{PSL}_n(\mathbf{F}_q)$, then $(\overline{a} \otimes \overline{b})(G_L)$ is adequate.*

2. Globally realizable representations

2.1. Global realizability

Let $E/\mathbf{Q}_p$ be a finite extension with ring of integers $\mathcal{O}$ and residue field $\mathbf{F}$. Let $K/\mathbf{Q}_p$ be a finite extension, and let

$$\overline{r} : G_K \to \mathrm{GL}_n(\mathbf{F})$$

be a representation.

Definition 2.1.1. Say the representation $\overline{r}$ admits *many diagonalizable lifts* if the following holds: for any $C \geqslant 0$, $\overline{r}$ admits a potentially diagonalizable lift with the property that, for each embedding $\sigma : K \hookrightarrow \overline{\mathbf{Q}}_p$, the σ-labelled Hodge–Tate weights of the lift all differ by at least C.

Remark 2.1.2. We expect that *every* representation $\overline{r}$ admits many diagonalizable lifts. In this paper, we will use a base change trick (based on Lemmas 2.1.3 and 4.1.8) to avoid needing to know this.

Lemma 2.1.3. *For any representation $\overline{r} : G_K \to \mathrm{GL}_n(\mathbf{F})$, there is a finite extension L/K such that $\overline{r}|_{G_L}$ admits many diagonalizable lifts. Moreover, any L/K such that $\overline{r}$ and the mod p cyclotomic character $\overline{\varepsilon}$ become trivial over G_L has this property.*

Proof. Choose L so that each $\overline{r}|_{G_L}$ is trivial, and the mod p cyclotomic character of G_L is also trivial. For each integer $C > 0$, $1 \oplus \varepsilon^C \oplus \cdots \oplus \varepsilon^{(n-1)C}$ is a potentially diagonalizable (in fact, diagonal) crystalline lift of $\overline{r}|_{G_L}$, all of whose σ-labelled Hodge–Tate weights differ by at least C. □

Convention 2.1.4. *We will frequently consider potentially diagonalizable lifts of an $\overline{r}$ which admits many diagonalizable lifts. Whenever we do so, we will always choose the lifts to* *have Hodge–Tate weights that are sufficiently spread out (in the sense that the condition of Definition 2.1.1 holds for some sufficiently large C) that all representations formed in the arguments that we make (which will involve tensoring representations together) have regular Hodge–Tate weights. In order to streamline the paper, we will not make this explicit in any of our arguments.*

Definition 2.1.5 (Polarized Local Isomorphisms). Let F be a CM field. Suppose that $\overline{a} : G_F \to \mathrm{GL}_n(\mathbf{F})$ and $\overline{b} : G_F \to \mathrm{GL}_n(\mathbf{F})$ are absolutely irreducible polarizable representations with respect to a character $\overline{\mu}$, so (in particular) they both prolong to representations

$$\rho(\overline{a}), \rho(\overline{b}) : G_{F^+} \to \mathcal{G}_n(\mathbf{F}),$$

each of which is uniquely determined up to conjugation by an element in $\mathcal{G}_n^0(\mathbf{F})$. Let v be a prime in F^+, and let w be a prime above v in F. We define a polarized isomorphism $\overline{a}|_{G_{F_w}} \simeq \overline{b}|_{G_{F_w}}$ to be an isomorphism of representations which extends to an isomorphism of polarized representations:

$$\rho(\overline{a})|_{G_{F_v^+}} \simeq \rho(\overline{b})|_{G_{F_v^+}}.$$

If $v \in F^+$ splits in F, then any isomorphism between $\overline{a}|_{G_{F_w}}$ and $\overline{b}|_{G_{F_w}}$ extends to such an isomorphism, because $G_{F_v^+} = G_{F_w} \subset G_F$, and

$$\rho(\overline{a})|_{G_F} = \overline{a} \times \overline{\mu}|_{G_F} : G_F \to \mathrm{GL}_n(\mathbf{F}) \times \mathrm{GL}_1(\mathbf{F}) = \mathcal{G}_n^\circ(\mathbf{F}) \subset \mathcal{G}_n(\mathbf{F})$$

(and similarly for $\rho(\overline{b})|_{G_F}$), so that $\rho(\overline{a})|_{G_{F_v^+}} = \rho(\overline{a})|_{G_{F_w}} = \overline{a}|_{G_{F_w}} \times \overline{\mu}|_{G_{F_w}}$ (resp. $\rho(\overline{b})|_{G_{F_v^+}} = \rho(\overline{b})|_{G_{F_w}} = \overline{b}|_{G_{F_w}} \times \overline{\mu}|_{G_{F_w}}$). On the other hand, if v is inert or ramified in F/F^+ and $\overline{a}|_{G_{F_w}} = \overline{b}|_{G_{F_w}}$ is reducible, then this restriction may admit more than one polarization, and so the requirement that the representations $\rho(\overline{a})|_{G_{F_v^+}}$ and $\rho(\overline{b})|_{G_{F_v^+}}$ be isomorphic may be a non-trivial condition.

Definition 2.1.6. Let F be a CM field. We say that a representation $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is *reasonable* if

- $\zeta_p \notin F$, and $\overline{s}|_{G_{F(\zeta_p)}}$ is irreducible;
- $\overline{s}$ is polarizable and odd;
- $p > 2(n+1)$.

Definition 2.1.7. Let F be a CM field. We say that a representation $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is *pleasant* if

- $\zeta_p \notin F$, and $\overline{s}|_{G_{F(\zeta_p)}}$ is irreducible;
- $\overline{s}$ is polarizable and odd;
- $p > 2(n+1)$;
- all the primes $v|p$ in F^+ split in F;
- for each place $w|p$ of F, $\overline{s}|_{G_{F_w}}$ admits many diagonalizable lifts.

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Lemma 2.1.8. *Let $\overline{s}$ be a reasonable representation of G_F. Let $F^{(\mathrm{avoid})}/F$ be a finite extension. Then there is a finite extension L/F of CM fields, which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F, such that $\overline{s}|_{G_L}$ is pleasant.*

Proof. Replace $F^{(\mathrm{avoid})}$ with $F^{(\mathrm{avoid})} \cdot \overline{F}^{\ker \overline{r}}(\zeta_p)$. Let $E = E^+F$, where E^+/F^+ is any totally real extension linearly disjoint from $F^{(\mathrm{avoid})}/F^+$ with the property that for each place $w_E|p$ of E, both $\overline{s}|_{G_{E_{w_E}}}$ and $\overline{\varepsilon}|_{G_{E_{w_E}}}$ are trivial, where $\overline{\varepsilon}$ is the mod-p cyclotomic character. It follows from Lemma 2.1.3 that $\overline{s}|_{G_{E_{w_E}}}$ then admits many diagonalizable lifts, and moreover, the same is true if one replaces E^+ by any further finite extension. It now suffices to ensure the primes $v|p$ in E^+ split in E. To achieve this, we cross with a quadratic extension. Namely, let $L^+ = M^+E^+$, where M^+/F^+ is a quadratic extension with the property that $M^+_v \simeq F_v$ for $v|p$ in F^+, and such that $L = L^+F$ is linearly disjoint from $F^{(\mathrm{avoid})}$ over F. □

Definition 2.1.9. Let $K/\mathbf{Q}_p$ be a finite extension, and let $\overline{\rho} : G_K \to \mathcal{G}_n(\mathbf{F})$ be a representation with multiplier $\overline{\mu}_K$. Let μ_K be a de Rham lift of $\overline{\mu}_K$. A μ_K-polarized component C for $\overline{\rho}$ is *globally realizable* if there exist a CM number field F and an odd, regular, polarized, weakly irreducible compatible system $(\{s_\lambda\}, \{\mu_\lambda\})$ over F, with corresponding p-adic representation (s, μ), with the following properties:

(1) The residual representation $\overline{s}$ is reasonable.

(2) There exists a prolongation $\xi : G_{F^+} \to \mathcal{G}_n(\mathbf{F})$ of (s, μ), and a place v of F^+, such that $F^+_v \simeq K$, $\mu|_{G_{F^+_v}} = \mu_K$, $\overline{\xi}|_{G_{F^+_v}} \cong \overline{\rho}$, and the representation $\xi|_{G_{F^+_v}}$ lies on C.

We say that a de Rham lift $\rho : G_K \to \mathcal{G}_n(\mathcal{O})$ of $\overline{\rho}$ is globally realizable if it lies on a globally realizable component.

Remark 2.1.10. In Definition 2.1.9, if the condition holds for one prolongation ξ, then it holds for any prolongation. Indeed, we saw in § 1.4 that any two prolongations are conjugate by some element of $1 \times \mathrm{GL}_1 \subset \mathcal{G}^\circ_n$, and the components of the local deformation ring are invariant under conjugation by [4, Lemma 3.4.1].

Remark 2.1.11. If v splits in F, then by Lemma 1.4.20, we can identify the μ-polarized deformation ring for $\overline{\rho}$ with the lifting ring for $\overline{s}|_{G_{F_w}} : G_{F_w} \to \mathrm{GL}_n(K)$ (which is independent of μ_K). Then, in the setting of Definition 2.1.9, it follows from [16, Lemma 4.1.6] that the condition of a component being globally realizable is independent of the choice of μ_K (as we can twist compatible systems by algebraic characters).

Remark 2.1.12. Note that by definition, if a component is globally realizable, then it is regular.

Remark 2.1.13. While it is not obvious from the definition, as a consequence of our main results, we can show that if $n = 2$ or n is odd, then any potentially globally realizable component is globally realizable. More precisely, a component C for $\rho : G_K \to \mathcal{G}_n(\mathbf{F})$ is

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globally realizable if and only if there exists a finite extension L/K such that $C|_L$ is globally realizable. See Corollary 4.2.13.

Remark 2.1.14. Any (regular) potentially diagonalizable representation is globally realizable; this is easily proved using the methods of [20, Appendix A], and in particular if $n = 2$ or n is odd, it is a simple consequence of Corollary 4.2.13 (which shows that it is enough to prove this after an arbitrary base change), together with [2, Lemma A.2.5] (which shows how to globalize local representations which are induced from characters).

The notion of being globally realizable can also be formulated in automorphic terms.

Lemma 2.1.15. *A μ_K-polarized component C for $\overline{\rho}$ is globally realizable if and only if there exist a CM number field F, a regular algebraic cuspidal polarized automorphic representation (π, χ) of GL_n/F, and a prolongation $\rho_p(\pi)$ of $r_p(\pi)$ such that we have the following:*

(1) *There is a prime v in F^+ such that $F_v^+ \simeq K$, $\overline{\rho}_p(\pi)|_{G_{F_v}} \cong \overline{\rho}$, $(\varepsilon^{1-n} r_p(\chi))|_{G_K} = \mu_K$, and the representation $\rho_p(\pi)|_{G_{F_v^+}}$ lies on C.*

(2) *The residual representation $\overline{r}_p(\pi)$ is reasonable.*

Proof. For the 'if' direction, note that if these conditions are satisfied, then we may take $(\{s_\lambda\}, \{\mu_\lambda\})$ in the definition of global realizability to be $(\{r_\lambda(\pi)\}, \{\varepsilon^{1-n} r_\lambda(\chi)\})$. Conversely, if C is globally realizable, then we apply Theorem 2.1.16, taking $F^{(\mathrm{avoid})}$ to be $\overline{F}^{\ker \overline{s}}(\zeta_p)$, and S to be the set of places of F which lie over p. Then the conditions in the lemma are satisfied by the automorphic representation corresponding to the compatible system $(\{s_\lambda|_{G_L}\}, \{\mu_\lambda|_{G_{L^+}}\})$. □

Theorem 2.1.16. *Let $(\{s_\lambda^{(i)}\}, \{\mu_\lambda^{(i)}\})$, $i = 1, \ldots, r$ be compatible systems of odd, regular, weakly irreducible polarized Galois representations over a CM field F. Let S be a finite set of finite places of F^+, and let $F^{(\mathrm{avoid})}/F$ be a finite extension. Then there is a finite Galois extension L/F of CM fields with the following properties:*

- *L is linearly disjoint from $F^{(\mathrm{avoid})}$ over F.*
- *Every place in S splits completely in L^+.*
- *Each $(\{s_\lambda^{(i)}|_{G_L}\}, \{\mu_\lambda^{(i)}|_{G_{L^+}}\})$ is automorphic.*

Proof. By [2, Theorem 5.4.1], each compatible system $(\{s_\lambda^{(i)}\}, \{\mu_\lambda^{(i)}\})$ is potentially automorphic over some finite extension L_i/F, which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F. (Strictly speaking, that result assumes that all of the $s_\lambda^{(i)}$ are irreducible, but as explained in the introduction to [37], all that is actually needed is that there is a positive density set of rational primes l such that for each $\lambda | l$, $s_\lambda^{(i)}$ is irreducible.)

It suffices to show that L/F can be chosen simultaneously for all i, in such a way that all places of F above S split completely. This can be arranged by a slight refinement of the arguments of [2]; we explain the main idea here, referring the interested reader to the proof of [20, Proposition A.6] for a more detailed treatment of a similar result.

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As a first step, note that since each $\{s_\lambda^{(i)}\}$ is potentially automorphic, it follows from [37, Lemma 1.5, Theorem 1.7] that there is a positive density set of rational primes l such that if $\lambda | l$, then each $\{s_\lambda^{(i)}\}$ is irreducible. Therefore by [2, Proposition 5.3.2], we can choose l and $\lambda | l$ such that each $s_\lambda^{(i)}$ is Fontaine–Laffaille at all primes dividing l, each $s_\lambda^{(i)}$ is irreducible, and indeed $\overline{s}_\lambda^{(i)}|_{G_{F(\zeta_l)}}$ is irreducible. We also assume that $l > 2(\max \dim s_\lambda^i + 1)$.

In the main argument of [2], the field L/F is constructed by a (finite) number of applications of the theorem of Moret-Bailly, applied to a particular moduli space T over F^+ (see the proof of [2, Theorem 3.1.2]). By the version of the theorem of Moret-Bailly given in [2, Proposition 3.1.1], we can arrange that the places in S all split completely in L^+ provided that T has F_v^+-rational points for all places $v \in S$. This need not be the case, but we can in any case choose a finite solvable extension of totally real fields M^+/F^+ so that T has M_w^+-rational points for each place w of M^+ lying over a place in S. We then replace T by the Weil restriction $\mathrm{Res}_{M^+/F^+} T$, and running the arguments of [2], we find a finite Galois extension L^+/F^+, linearly disjoint from $F^{(\mathrm{avoid})}$ over F^+, in which all places in S split completely, with the property that if we set $L = L^+F$, then the restrictions $\{s_\lambda^{(i)}|_{G_{LM^+}}\}$ are automorphic. Since the extension LM^+/L is solvable, it follows that each $\{s_\lambda^{(i)}|_{G_L}\}$ is automorphic, as required. □

3. Compatible systems

Our aim in this section is to prove results showing that if one representation in a compatible system is a tensor product, then the compatible system is a tensor product. We do this under somewhat restrictive hypotheses (see Theorem 3.4.3), which we will suffice for the results of the following section due to some base change tricks and arguments with auxiliary places. We also prove a number of other results about compatible systems that we will use in § 4. Our results are mostly Lie-theoretic, and in particular we make crucial use of the results of [30].

3.1. Component groups

Recall that by Theorem 1.4.15, any compatible system has a well-defined component group. We have the following technical lemma.

Lemma 3.1.1. *Let F be a number field, let $\overline{r} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be irreducible, and suppose that $p > \max(n, 3)$. Let L/F denote the field $F(\ker(\overline{r}))$. Let $\{r_\lambda\}$ be any compatible system of Galois representations such that $\overline{r}_p = \overline{r}$ and such that $\det(r_p)$ has an infinite image. Let F'/F be a finite Galois extension which is linearly disjoint from L/F. Then the component group of $\{r_\lambda|_{G_{F'}}\}$ is independent of F'/F.*

Proof. Let G denote the Zariski closure of $\mathrm{im}(r_p)$, and let G^0 denote the connected component of G. Let $\mathrm{im}^\circ(r_p)$ denote the intersection of $\mathrm{im}(r_p)$ with G^0. Let F° denote the fixed field of $\mathrm{im}^\circ(r_p)$. By Theorem 1.4.15, F° is independent of p, and $G/G^0 \simeq \mathrm{Gal}(F^\circ/F)$. It suffices to show that F° is completely contained in L. The image $\mathrm{im}(r_p)$ of r_p inside $\mathrm{GL}_n(\overline{\mathbf{Z}}_p)$ naturally admits a surjection onto $\mathrm{Gal}(F^\circ/F)$. If $\mathrm{Gal}(F^\circ/F)$ has

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order prime to p, then this surjection factors through $\mathrm{im}(\overline{r}_p)$, since the kernel of $\mathrm{im}(r_p) \to \mathrm{im}(\overline{r}_p)$ is a pro-p group. But this implies that F° is contained in L. Thus we may assume that $\mathrm{Gal}(F^\circ/F)$ has order divisible by p, and hence that the component group G/G° has order divisible by p.

Note that G acts irreducibly because $\overline{r}$ is irreducible. By assumption, an element of order p in G/G° induces an outer automorphism of G° of order p. If this automorphism is trivial, then, by Schur's lemma, this automorphism may be modified by an inner automorphism to be a scalar which does not lie in G°. The assumption that the determinant has an infinite image, however, implies that (since G is reductive and irreducible) the centre Z is infinite, and hence that $Z^\circ = Z$. This is a contradiction, and hence this order p element induces a non-trivial outer automorphism of G°, and hence also of its Lie algebra $\mathfrak{g}$. The automorphism group of any simple Lie algebra has order at most $3 < p$. Thus this order p automorphism must act by permuting the simple factors. Yet G acts on a space of dimension n, and hence there are at most n simple factors. Hence we obtain a non-trivial element of order p in S_n, which is impossible for $p > n$. □

3.2. Representation theory

In this section, we begin by proving some basic representation-theoretic lemmas for reductive groups G. All the representations we consider below are assumed to be finite-dimensional.

3.2.1. Reductive linear algebraic groups. Let k be an algebraically closed field of characteristic zero. If G is a connected reductive linear algebraic group over k, then we let G^{der} denote the derived subgroup of G – it is a connected semisimple linear algebraic group – and let Z denote the centre of G. The natural morphism of connected reductive linear algebraic groups

$$G^{\mathrm{der}} \times Z^\circ \to G$$

(where as usual Z° denotes the connected component of the identity in Z) is surjective, and its kernel is contained in (the anti-diagonally embedded copy of) the intersection $G^{\mathrm{der}} \cap Z^\circ$, and thus is contained in the centre of G^{der}; in particular, it is finite.

We let $\widetilde{G}^{\mathrm{der}}$ denote the simply connected cover of G^{der}; it is again a connected semisimple linear algebraic group, and the kernel of the natural surjection $\widetilde{G}^{\mathrm{der}} \to G^{\mathrm{der}}$ is finite and central. We write $\widetilde{G} := \widetilde{G}^{\mathrm{der}} \times Z^\circ$ (and note that the possible ambiguity in our use of the notation $\widetilde{G}^{\mathrm{der}}$ is ameliorated by the fact that $\widetilde{G}^{\mathrm{der}}$ is naturally identified with the derived subgroup of $\widetilde{G}$). The composite morphism

$$\widetilde{G} = \widetilde{G}^{\mathrm{der}} \times Z^\circ \to G^{\mathrm{der}} \times Z^\circ \to G$$

is again surjective, and its kernel is finite and central.

Since $\widetilde{G}^{\mathrm{der}}$ is semisimple and simply connected, it may be written as a direct product of almost simple linear algebraic groups. Thus $\widetilde{G}$ is a direct product of such groups and a torus.

If H and J are linear algebraic groups, then any irreducible representation W of the product $H \times J$ may be factored (uniquely, up to isomorphism) as a tensor product

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$W \cong U \otimes_k V$, where U (resp. V) is an irreducible representation of H (resp. J). Applying this remark in the context of the preceding discussion (thinking of a representation of G as a representation of $G^{\mathrm{der}} \times Z^\circ$ via inflation), we find that the irreducible representations of G are obtained from the irreducible representations V of G^{der} by choosing a character of Z° which coincides with the given action of $G^{\mathrm{der}} \cap Z^\circ$ on V (Schur's lemma ensures that this action is indeed given by a character) and extending the G^{der}-action on V to an action of G (thought of as a quotient of $G^{\mathrm{der}} \times Z^\circ$) via having Z° act through this choice of character.

If $\mathfrak{g}$ denotes the Lie algebra of G (or equivalently of $\widetilde{G}$), so that $\mathfrak{g}^{\mathrm{der}}$ is the Lie algebra of G^{der} (or equivalently of $\widetilde{G}^{\mathrm{der}}$), then passing to the induced $\mathfrak{g}^{\mathrm{der}}$-action induces an equivalence of categories between the category of finite-dimensional $\widetilde{G}^{\mathrm{der}}$-representations over k and the category of finite-dimensional $\mathfrak{g}^{\mathrm{der}}$-representations over k. In particular, this equivalence induces a bijection between the isomorphism classes of irreducible representations of $\widetilde{G}^{\mathrm{der}}$ and the isomorphism classes of irreducible representations of $\mathfrak{g}^{\mathrm{der}}$.

The following lemma (and its proof) is a very special case of a theorem of Rajan [38].

Lemma 3.2.2. *Let U and V be two non-trivial representations of a simple Lie algebra $\mathfrak{g}$ over an algebraically closed field of characteristic zero. Then $U \otimes V$ is reducible.*

Proof. Without loss of generality, we may assume that U and V are irreducible. Let the highest weights of U and V be λ and μ, respectively. Then, if $U \otimes V$ is irreducible, it must be the irreducible representation of highest weight $\lambda + \mu$. By the Weyl character formula, this implies that

$$1 = \frac{\dim(U)\dim(V)}{\dim(U \otimes V)} = \prod_{\Phi^+} \frac{\langle \rho + \lambda, \alpha\rangle \langle \rho + \mu, \alpha\rangle}{\langle \rho, \alpha\rangle \langle \rho + \lambda + \mu, \alpha\rangle}.$$

Each individual factor has the form

$$\frac{\langle \rho, \alpha\rangle^2 + \langle \rho, \alpha\rangle(\langle \lambda, \alpha\rangle + \langle \mu, \alpha\rangle) + \langle \lambda, \alpha\rangle\langle \mu, \alpha\rangle}{\langle \rho, \alpha\rangle^2 + \langle \rho, \alpha\rangle(\langle \lambda, \alpha\rangle + \langle \mu, \alpha\rangle)} \geqslant 1.$$

Since the pairing is non-negative, we obtain a contradiction unless $\langle \lambda, \alpha\rangle\langle \mu, \alpha\rangle = 0$ for each root $\alpha \in \Phi^+$. Because $\mathfrak{g}$ is simple, there exists a maximal root $\beta \in \Phi^+$ such that, for any dominant weight ν, one has $\langle \nu, \beta\rangle \geqslant \langle \nu, \alpha\rangle$ for any $\alpha \in \Phi^+$. In particular, assuming without loss of generality that $\langle \lambda, \beta\rangle = 0$, we deduce that $\langle \lambda, \alpha\rangle = 0$ for all roots in Φ^+, which implies that $\lambda = 0$ and the corresponding representation is trivial. □

We say that an irreducible representation W of the connected reductive linear algebraic group G is tensor indecomposable if, for any isomorphism $W \simeq U \otimes V$ of G-representations, either U or V is a character.

Lemma 3.2.3. *Let $G = G^\circ$ be a connected reductive Lie group over an algebraically closed field of characteristic zero, and let $\widetilde{G}$ denote the finite cover of G constructed in the preceding discussion. If W is an irreducible representation of G of dimension > 1, thought of a representation of $\widetilde{G}$ via inflation, then W has a factorization*

$$W \simeq \bigotimes V_i$$

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as a tensor product of tensor indecomposable representations of $\widetilde{G}$, where the V_i are unique up to re-ordering and twisting and have dimension > 1.

Proof. As noted above, the algebraic group $\widetilde{G}$ is a direct product of almost simple linear algebraic groups G_i and a torus T. Any irreducible representation of G is then a tensor product of irreducible representations of the G_i up to twist by a character of T. It then suffices to show that the tensor indecomposable representations are precisely the representations of a single simple factor G_i up to twist. This follows immediately from Lemma 3.2.2 (which shows that an irreducible representation of any G_i is automatically tensor indecomposable). □

We next establish some purely representation-theoretic results.

Lemma 3.2.4. *Let V and W be finite-dimensional linear representations of a group G over an algebraically closed field. Suppose that $V \otimes W$ decomposes as a direct sum of characters. Suppose, in addition, there are at most three isomorphism classes of characters which occur in direct sum. Then V and W also admit such a decomposition.*

Proof. First consider the case when both V and W are irreducible. Note that for any character χ,

$$\dim \mathrm{Hom}_G(V \otimes W, \chi) = \dim \mathrm{Hom}_G(V, W^{\vee}\chi) \leqslant 1,$$

where the latter inequality follows from the irreducibility of V and W together with Schur's lemma. Since the number of distinct characters is at most three, it follows that $\dim(V \otimes W) \leqslant 3$, and thus at least one of V or W is a character, and the result follows.

In the general case, when V and W are not necessarily irreducible, choose irreducible subrepresentations $V' \subset V$ and $W' \subset W$. Then $V' \otimes W' \subset V \otimes W$, and so from what we have already proved, we find that each of V' and W' is necessarily a character. We then see that

$$V = V \otimes W' \otimes (W')^{-1} \subset (V \otimes W) \otimes (W')^{-1}$$

is a direct sum of at most three characters (possibly with multiplicities), and similarly for

$$W = (V')^{-1} \otimes V' \otimes W \subset (V')^{-1} \otimes V \otimes W. \qquad \square$$

Remark 3.2.5. The preceding result is false when there are four distinct characters. Indeed, one can take $V = W$ to be the irreducible 2-dimensional representation (over $\overline{\mathbf{Q}}$, say) of the quaternion group Q of order 8.

We recall the definition of a strongly irreducible representation.

Definition 3.2.6. A representation of a group G (either a Lie group or a Galois group) is *strongly irreducible* if it remains irreducible after restriction to any finite index subgroup $H \subset G$.

Remark 3.2.7. A representation of a reductive Lie group G is strongly irreducible if and only if it remains irreducible after restriction to the connected component $G^{\circ} \subset G$.

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If a topological group G acts continuously on W and a subgroup $H \subset G$ fixes a closed subspace $0 \subsetneq V \subsetneq W$, then the closure of H in G also fixes V. Hence continuous representations of a Galois group G are strongly irreducible if and only if they remain irreducible after restriction to any *closed* finite index subgroup H.

Lemma 3.2.8. *Let V and W be irreducible representations of a group G over an algebraically closed field k of characteristic zero, and suppose that the action of G on V is strongly irreducible. Then, up to scaling, there is at most one non-trivial bilinear G-equivariant pairing:*

$$W \times V \to k(\mu),$$

where $k(\mu)$ is the twist of the trivial representation by a character μ of G, and μ is allowed to range over all characters of G.

Proof. Any such pairing gives rise to an isomorphism $V \simeq W^\vee(\mu)$. If there existed two such pairings with the same μ which were not the same up to scalar, then there would be two corresponding isomorphisms in $\mathrm{Hom}_G(V, W^\vee(\mu))$. Using either of the identifications of V with $W^\vee(\mu)$, we deduce that

$$2 \leqslant \dim \mathrm{Hom}_G(V, W^\vee(\mu)) = \dim \mathrm{Hom}_G(V, V),$$

which contradicts the irreducibility of V by Schur's lemma. If there were two such pairings with *different* μ, then denoting one character by μ and the other by $\mu \otimes \chi$ for a non-trivial character χ, we deduce that $V \simeq W^\vee(\mu)$ and $V \simeq W^\vee(\mu \otimes \chi)$ and thus $V \simeq V(\chi)$. Taking determinants of both sides, it follows that $\chi^n = 1$. Hence χ defines a map: $G \to \mu_n \subset k^\times$, and in particular the image of χ is finite. Let H denote the kernel of this map, which is of finite index. Then we deduce that $\dim \mathrm{Hom}_H(V, V) \geqslant 2$. By Schur's lemma, this implies that $V|_H$ is reducible, contradicting our assumptions. □

3.2.9. Galois representations. Throughout this subsection, we let F be a (not necessarily CM) number field.

Lemma 3.2.10. *Let $r : G_F \to \mathrm{GL}_m(\overline{\mathbf{Q}}_p)$ be a representation with image $\Gamma := r(G_F)$, and suppose that the (semisimple) residual representation $\overline{r}$ has image $\overline{r}(G_F) = \overline{\Gamma} \subset \mathrm{GL}_m(\overline{\mathbf{F}}_p)$ containing $\mathrm{SL}_m(\mathbf{F}_q)$ for q a sufficiently large power of p (we can take $q = p$ if $p > 5$, and $q = 25$ if $p = 5$). Then the Zariski closure G of Γ contains $\mathrm{SL}_m(\overline{\mathbf{Q}}_p)$.*

Proof. After conjugation, the image is a closed subgroup of $\mathrm{GL}_m(\mathcal{O})$ for $\mathcal{O}$ the ring of integers in some finite extension of $\mathbf{Q}_p$. By the main result of [33], the image therefore contains a conjugate of $\mathrm{SL}_m(W(\mathbf{F}_q))$, from which the result follows immediately (since SL_m is unirational, by [5, Theorem 18.2]). □

Lemma 3.2.11. *Let $\{r_\lambda\}$ be a weakly compatible system of G_F-representations of dimension $m = n^2$. Suppose that $r = r_p = a \otimes b$, where a and b are n-dimensional representations which have images whose Zariski closure contains SL_n, and the corresponding Lie algebra of r_p contains $\mathfrak{sl}_n \times \mathfrak{sl}_n$. Then the component group of the compatible system $\{\mathrm{ad}(r_\lambda)\}$ is trivial.*

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Proof. By Theorem 1.4.15, the pre-image of G_λ° in G_F is independent of λ, and so the component group is independent of λ, and thus we may choose $\lambda = p$. We have $\mathrm{ad}(r) = \mathrm{ad}(a) \otimes \mathrm{ad}(b)$. By assumption, the Zariski closures of the images of $\mathrm{ad}(a)$ and $\mathrm{ad}(b)$ are both PGL_n, and hence the Zariski closure of $\mathrm{ad}(r)$ is $G = \mathrm{PGL}_n \times \mathrm{PGL}_n$. Since $G = G^\circ$, the component group is trivial. □

The following result relates irreducibility and strong irreducibility of Galois representations.

Lemma 3.2.12. *Let $r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$ be an irreducible Galois representation which is Hodge–Tate with regular Hodge–Tate weights. If r is not strongly irreducible, then r is induced from a strongly irreducible representation over some finite extension M/F.*

Proof. This is proved in the course of the proof of [11, Corollary 4.4]; we recall the argument. For any finite extension E/F, either $r|_{G_E}$ is irreducible or it decomposes into a sum of *distinct* irreducible representations. This follows immediately from the fact that $r|_{G_E}$ has distinct Hodge–Tate weights at any prime $w|l$. (Note that $r|_{G_E}$ is necessarily semisimple: if V denotes any irreducible subrepresentation, then the various translates of V by elements of G_F are stable under the corresponding conjugates of G_E, and so we see that $r|_{G_E}$ becomes completely decomposable under restriction to a finite index subgroup, so must already have been semisimple.) Suppose then that $r|_{G_E}$ is reducible for some finite extension E/F. Replacing E by its normal closure over F, we may assume that the extension E/F is Galois, and the claim that r is induced is immediate from [11, Lemma 4.3]. If $r = \mathrm{Ind}_{G_M}^{G_F} s$, then s is also Hodge–Tate with regular Hodge–Tate weights, so by induction on n we may assume that r is induced from a strongly irreducible representation. □

The following lemma will prove useful for lifting Galois representations along central extensions.

Lemma 3.2.13. *Suppose that $r : G_F \to \mathrm{GL}_{mn}(\overline{\mathbf{Q}}_p)$ is a Galois representation whose image has Zariski closure inside the image of the map $\mathrm{GL}_n \times \mathrm{GL}_m \to \mathrm{GL}_{nm}$. Then there exist Galois representations r_A and r_B of dimensions n and m, respectively, such that $r \simeq r_A \otimes r_B$.*

Proof. There is a central extension

$$0 \to Z \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p) \times \mathrm{GL}_m(\overline{\mathbf{Q}}_p) \to \Gamma(\overline{\mathbf{Q}}_p) \to 0,$$

where Γ denotes the image of $\mathrm{GL}_n \times \mathrm{GL}_m$ in GL_{nm}, and Z is the $\overline{\mathbf{Q}}_p$-points of a torus embedded anti-diagonally. The result then follows directly from [17, Proposition 5.3]. □

Lemma 3.2.14. *Fix $n \geqslant m$, and let $r : G_F \to \mathrm{GL}_{mn}(\overline{\mathbf{Q}}_p)$ be a strongly irreducible representation such that the Lie algebra of the Zariski closure of the image of r is isomorphic to $\mathfrak{t} \times \mathfrak{h} \times \mathfrak{sl}_n$, where*

- *$\mathfrak{t}$ is a torus of rank at most 1,*
- *$\mathfrak{h}$ is semisimple, and*

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- *the corresponding mn-dimensional representation of $\mathfrak{h} \times \mathfrak{sl}_n$ is the tensor product of an m-dimensional representation of $\mathfrak{h}$ with the standard representation of $\mathfrak{sl}_n$.*

Then either

(1) *r decomposes as a tensor product $a \otimes b$, where a is of dimension m and b is of dimension n, and furthermore the Zariski closure of the image of b contains SL_n; or*

(2) *$m = n$, and r is the tensor induction of an n-dimensional representation of G_L for some quadratic extension L of F.*

Moreover, in case (1), the representations a and b are unique up to twisting by a character, or possibly permuting a and b if $m = n$ and $\mathfrak{h} = \mathfrak{sl}_n$.

Proof. Let G denote the Zariski closure of the image of r, and let G° denote the connected component of G. For now, let us regard r as a (faithful) representation of G. We shall exhibit a factorization of r as a tensor product of representations of some cover of G (possibly after passing to a quadratic extension) and then promote this to an actual factorization of Galois representations by Lemma 3.2.13.

By our assumptions, together with the general discussion of (3.2.1), we may find a finite cover of G° by a group of the form $T \times H \times \mathrm{SL}_n$, where H is a product of almost simple groups, and T is a torus. If we let r° denote the restriction of r to G°, regarded as a representation of $T \times H \times \mathrm{SL}_n$ by inflation, then r° is irreducible (as r is strongly irreducible by assumption), and so we may write $r^\circ = a^\circ \otimes b^\circ \otimes c^\circ$, where a° is an irreducible representation of H, b° is the standard n-dimensional representation of SL_n, and c° is an irreducible (and hence 1-dimensional) representation of T. (We do not assert that any of a°, b°, or c° are representations of G°.)

After twisting by a character, we may assume (for example, by [36, Lemma 2.3.15]) that the determinant of r has a finite image, and hence that the determinant of r° is trivial. Thus we may in fact assume that c°, and hence $\mathfrak{t}$ and T, is trivial, and we do so from now on. Thus we assume that G° admits a finite cover by $H \times \mathrm{SL}_n$, and that r° admits a corresponding tensor factorization $a^\circ \otimes b^\circ$ with b° being the standard representation; we now attempt to extend this tensor factorization to a corresponding tensor factorization of r.

Since G° is normal in G, we obtain a conjugation action of G on G°, and hence on its universal cover $H \times \mathrm{SL}_n$. (Recall that the formation of universal covers is functorial in pointed spaces, and note that the conjugation action of G on G° acts via automorphisms of the pointed space $(G^\circ, 1)$.) Suppose first that $\mathfrak{h} \neq \mathfrak{sl}_n$; then there are no non-trivial morphisms $\mathfrak{sl}_n \to \mathfrak{h}$ (since $\mathfrak{h}$ has a faithful representation of dimension $m \leqslant n$), so that any automorphism of $\mathfrak{h} \times \mathfrak{sl}_n$ must fix the $\mathfrak{sl}_n$ factor, and correspondingly any automorphism of $H \times \mathrm{SL}_n$ must fix the SL_n-factor.

The component group G/G° is then endowed with a homomorphism

$$G/G^\circ \to \mathrm{Out}(\mathrm{SL}_n) \tag{3.2.15}$$

to the group of outer automorphisms of SL_n, which we claim is trivial. To see this, note first that if $n = 2$, then $\mathrm{Out}(\mathrm{SL}_2)$ is trivial, and so we are done. If $n \geqslant 3$, then the outer automorphism group of SL_n is cyclic of order 2. If G/G° surjects onto this outer

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automorphism group, then the restriction of r° to the SL_n-factor, which is a direct sum of m copies of b°, is isomorphic to its outer twist, which is a direct sum of m copies of $(b^\circ)^\vee$. Since $n \geqslant 3$, the standard representation of SL_n is not self-dual, and hence this is not possible. Thus (3.2.15) is trivial, as claimed.

The action of G on the SL_n-factor by conjugation is thus an inner action, and so induces a homomorphism $G \to \mathrm{PGL}_n$, compatible with the given map $\mathrm{SL}_n \to G^\circ$. Let $K \subseteq G$ denote the kernel of this homomorphism; the compatibility just remarked upon shows that K contains the image of H in G°. Then we obtain a surjection $K \times \mathrm{SL}_n \to G$ compatible with the given surjection $H \times \mathrm{SL}_n \to G^\circ$. Correspondingly, we obtain a tensor factorization $r = a \otimes b$ of r inflated to the cover $K \times \mathrm{SL}_n$ of G compatible with the factorization $r^\circ = a^\circ \otimes b^\circ$ of G°. This now induces a factorization of Galois representations by Lemma 3.2.13. Note that b° is the standard representation of SL_n, so the Zariski closure of the image of b contains SL_n, as claimed.

We now prove that this factorization is unique. Suppose that $r \simeq a' \otimes b' \simeq a \otimes b$. We already have uniqueness of these representations over G° by Lemma 3.2.3. Hence it follows that $\mathrm{Hom}(a, a')$ as a G-representation has a summand which becomes trivial when restricted to G°, and hence has a summand on which G acts through the finite quotient G/G°. If this factor is 1-dimensional, then a and a' are isomorphic up to twist. If this factor has dimension > 1, then, over G°, we see that $\mathrm{Hom}(a, a')|_{G^\circ} = \mathrm{Hom}(a, a)|_{G^\circ}$ has at least two trivial factors, which implies that a is reducible over G°, contradicting the strong irreducibility of r. The same logic applies to b, as required.

Suppose finally that $\mathfrak{h} = \mathfrak{sl}_n$. The argument proceeds as above, except now we have to allow the possibility that G/G° also swaps the factors. Assuming we are in this case, replacing G by G' where G' is the kernel of the map $G \to G/G^\circ \to S_2$, we obtain a tensor factorization of Galois representations as above over some quadratic extension. But then the image of r must coincide (up to twist) with the tensor induction of the corresponding n-dimensional representation (of a or b) from this quadratic extension. □

Lemma 3.2.16. *Consider p-adic representations a and b of G_F of dimensions m and n with $m \leqslant n$. Suppose that we have the following:*

(1) *The representation $a \otimes b$ is irreducible.*

(2) *The residual representation $\overline{b}$ has an image containing $\mathrm{SL}_n(\mathbf{F}_q)$ for q a sufficiently large power of p (in the sense of* Lemma 3.2.10*).*

Let A and B denote the Zariski closures of the images of a and b, respectively. Let A^{der} and B^{der} denote the corresponding derived subgroups, and $A^{\mathrm{der},\circ}$ and $B^{\mathrm{der},\circ}$ the connected components of these groups. Let G denote the Zariski closure of the image of $a \otimes b$. Then the corresponding representation of $G^{\mathrm{der},\circ}$ is the natural representation of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ}$ corresponding to the tensor product of the two natural representations. This identifies $G^{\mathrm{der},\circ}$ with the image of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ}$ in the automorphism group of the exterior tensor product of the two natural representations.

Proof. By Lemma 3.2.10, we have $B^{\mathrm{der},\circ} = \mathrm{SL}_n$. Certainly, the connected subgroup $G^{\mathrm{der},\circ} \subset \mathrm{SL}_{n^2}$ lies *inside* the image Γ of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ}$ under the exterior tensor product. Since Γ is connected (since it is the image of a connected group under an isogeny), it

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contains no proper finite index subgroups. Thus to prove $G^{\mathrm{der},\circ} \hookrightarrow \Gamma$ is an isomorphism, it suffices to prove it is an isogeny, which we can do on the level of Lie algebras.

Denote the Lie algebras of A and B by $\mathfrak{a} \oplus \mathfrak{t}_A$ and $\mathfrak{b} \oplus \mathfrak{t}_B$, where $\mathfrak{a}$ and $\mathfrak{b}$ are semisimple, and $\mathfrak{t}_A$ and $\mathfrak{t}_B$ are tori of rank at most one. Note that $\mathfrak{b} = \mathfrak{sl}_n$. After twisting, we may assume that $\mathfrak{t}_B$ is trivial, and that the Lie algebra of G is $\mathfrak{g} \oplus \mathfrak{t}_G$, where $\mathfrak{t}_G \simeq \mathfrak{t}_A$. Let $\mathfrak{d} \oplus \mathfrak{t}_D$ denote the Lie algebra (decomposed as a semisimple part $\mathfrak{d}$ and a torus $\mathfrak{t}_D$) of the Zariski closure of the image of $a \oplus b$. There is an inclusion $\mathfrak{d} \oplus \mathfrak{t}_D \subset \mathfrak{a} \oplus \mathfrak{b} \oplus \mathfrak{t}_A$, which induces an isomorphism $\mathfrak{t}_D \simeq \mathfrak{t}_A$. Hence $\mathfrak{d} \hookrightarrow \mathfrak{a} \oplus \mathfrak{b}$. On the other hand, since $(a \oplus b)^{\otimes 2}$ contains $a \otimes b$, there is a surjection $\mathfrak{d} \twoheadrightarrow \mathfrak{g}$. There is an isomorphism

$$\mathrm{ad}^0(a \otimes b) \simeq \mathrm{ad}^0(a) \oplus \mathrm{ad}^0(b) \oplus \big(\mathrm{ad}^0(a) \otimes \mathrm{ad}^0(b)\big).$$

The corresponding Lie algebras of the images of $\mathrm{ad}^0(a)$ and $\mathrm{ad}^0(b)$ are $\mathfrak{a}$ and $\mathfrak{b}$, respectively, and the Lie algebra of the image of $\mathrm{ad}^0(a \otimes b)$ is $\mathfrak{g}$. Hence there are maps

$$\mathfrak{a} \oplus \mathfrak{b} \supseteq \mathfrak{d} \twoheadrightarrow \mathfrak{g} \rightarrow \mathfrak{a} \oplus \mathfrak{b}.$$

The corresponding maps $\mathfrak{d} \rightarrow \mathfrak{a}$ and $\mathfrak{d} \rightarrow \mathfrak{b}$ (either coming from the inclusion into $\mathfrak{a} \oplus \mathfrak{b}$ or via the map to $\mathfrak{g}$) may be identified with each other, because the semisimple part of the Lie algebra of a is canonically identified with the Lie algebra of $\mathrm{ad}^0(a)$. Moreover, these maps are both surjective, and thus $\mathfrak{g}$ is identified with $\mathfrak{d}$. By Goursat's lemma, the inclusion $\mathfrak{g} \subseteq \mathfrak{a} \oplus \mathfrak{b}$ is the pullback to $\mathfrak{a} \oplus \mathfrak{b}$ of the graph of an isomorphism

$$\mathfrak{a}/\mathfrak{n}_A \simeq \mathfrak{b}/\mathfrak{n}_B,$$

for some ideals $\mathfrak{n}_A$ or $\mathfrak{n}_B$ which may be identified with the kernels of the projections from $\mathfrak{g} \rightarrow \mathfrak{b}$ and $\mathfrak{g} \rightarrow \mathfrak{a}$, respectively. Since $\mathfrak{b} = \mathfrak{sl}_n$ is simple, either both sides are trivial, in which case $\mathfrak{d} \simeq \mathfrak{g} \simeq \mathfrak{a} \oplus \mathfrak{b}$ (and we are done), or there is a surjection $\mathfrak{a} \twoheadrightarrow \mathfrak{b}$. In this latter case, by rank considerations (since A acts faithfully on a space of dimension $m \leqslant n$), we deduce that $\mathfrak{a} \simeq \mathfrak{sl}_n$ and that the map above induces an isomorphism of Lie algebras $\mathfrak{a} \simeq \mathfrak{b}$, and thus $\mathfrak{g} \simeq \mathfrak{a} \simeq \mathfrak{b}$ is diagonally embedded in $\mathfrak{a} \oplus \mathfrak{b}$. This implies that, still on the Lie algebra level, the representation $a \otimes b$ must come from the tensor product of an n-dimensional representation of $\mathfrak{sl}_n$ with a second n-dimensional representation of the same $\mathfrak{sl}_n$. In either case (standard tensor standard or standard tensor dual), the corresponding representation would be reducible (as also follows from a special case of Lemma 3.2.2). This implies (returning to the Lie group level) that, over some finite extension, $a \otimes b$ is either isomorphic to the direct sum of a 1-dimensional representation and an irreducible $n^2 - 1$-dimensional representation or an irreducible $\binom{n}{2}$-dimensional representation and an irreducible $\binom{n+1}{2}$-dimensional representation (depending on whether the representations of $\mathfrak{sl}_n$ are the same or dual to each other). But since $a \otimes b$ itself is irreducible by hypothesis, it can only decompose over a finite extension into irreducible representations of the same dimension. The claim follows. □

Theorem 3.2.17. *Let $\{r_\lambda\}$ be a compatible system of G_F-representations of dimension mn. Suppose there exists a prime p with $r = r_p$ satisfying the following:*

(1) *There exist p-adic representations a and b of G_F of dimensions m and n with $m \leqslant n$ such that $r \simeq a \otimes b$.*

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(2) *The representation $a \otimes b$ is irreducible.*

(3) *The residual representation $\overline{b}$ has an image containing $\mathrm{SL}_n(\mathbf{F}_q)$ for q a sufficiently large power of p (in the sense of* Lemma 3.2.10*).*

Suppose first that $(m, n) \neq (2, 2)$. Let λ be a place with residue characteristic l such that r_λ is strongly irreducible. Then there exist representations a_λ and b_λ, of dimensions m and n, respectively, such that $r_\lambda = a_\lambda \otimes b_\lambda$. Moreover, the image of b_λ has Zariski closure containing $\mathrm{SL}_n(\overline{\mathbf{Q}}_l)$, and a_λ and b_λ are unique up to twist by a character and up to permutation – the latter being possible only when $n = m$.

Suppose now that $m = n = 2$. In this case, assume also that $\{r_\lambda\}$ is odd, regular, polarizable, and weakly irreducible. Then there exists a set of primes l of density one such that for each $\lambda | l$, r_λ is strongly irreducible and admits a decomposition $r_\lambda = a_\lambda \otimes b_\lambda$ satisfying the conditions in the previous paragraph.

Proof. Let A and B denote the Zariski closures of the images of a and b, respectively. By Lemma 3.2.16, if G denotes the Zariski closure of the image of $a \otimes b$, then we may identify the corresponding representation of $G^{\mathrm{der},\circ}$ with the natural representation of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ}$ corresponding to the tensor product of the two natural representations.

Let λ denote a prime for which r_λ is strongly irreducible. Let G_λ denote the Zariski closure of the image of r_λ, and let G_λ^{der} denote the corresponding derived subgroup, and $G_\lambda^{\mathrm{der},\circ}$ the connected component of this group. By the strong irreducibility assumption, the corresponding representation of $G_\lambda^{\mathrm{der},\circ}$ is irreducible. By a theorem of Serre [30, Proposition 6.12], the formal character of a compatible system of Galois representations is independent of λ. In particular, the formal characters of the corresponding representations of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ} = A^{\mathrm{der},\circ} \times \mathrm{SL}_n$ and $G_\lambda^{\mathrm{der},\circ}$ coincide. It is possible for the formal characters of irreducible representations of connected groups to coincide even when the groups differ (for example, there exist 27-dimensional irreducible representations of G_2 and SL_3 with the same formal character). However, what *is* true [30, Theorem 5.6, Proposition 5.7] is that every such equality arises from taking the tensor product of a list of (explicitly given) basic similarity relations, described explicitly in §§ 5.3.1–5.3.4 of [30]. In particular, note that the standard representation of SL_n for any $n > 2$ does not admit *any* basic similarity relations (which one can also deduce by a consideration of ranks), while in the case $n = 2$, the only similarity relation which intervenes in our situation is that given by the coincidence of the formal character of the standard representation of Sp_4 with that of the external tensor product of two copies of the standard representation of SL_2. Applied to our situation, it follows that if $n > 2$ (which we assume for the time being, returning to the case $n = 2$ at the end of the proof), there exists a connected semisimple group $H_\lambda^{\mathrm{der},\circ}$ such that the representation of $G_\lambda^{\mathrm{der},\circ}$ is the tensor product of an irreducible n-dimensional representation of $H_\lambda^{\mathrm{der},\circ}$ with the standard representation of SL_n. Note that the Lie algebra $\mathfrak{h}$ of $H_\lambda^{\mathrm{der},\circ}$ need not *a priori* be equal to the Lie algebra $\mathfrak{a}$ of $A^{\mathrm{der},\circ}$, but this does not concern us. The result then follows from Lemma 3.2.14, once we show that the representation is not a tensor induction from a quadratic extension.

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We now prove that this case cannot occur. If r_λ is a tensor induction with Lie algebra containing $\mathfrak{h} \times \mathfrak{sl}_n$, then we must have $\mathfrak{h} = \mathfrak{sl}_n$ acting via the standard representation. Hence, once more by [30, Theorem 5.6, Proposition 5.7] (and the fact that the standard representation of SL_n does not admit any basic similarity relations), we deduce that the Lie algebra of $a \otimes b$ also contains $\mathfrak{sl}_n \times \mathfrak{sl}_n$, and thus that the Zariski closure of a contains SL_n. (To see that the Zariski closure of the image of a contains SL_n rather than a quotient of SL_n by some finite group, we use the tautological fact that a has a faithful representation in dimension n.) By Lemma 3.2.11, we deduce that the component group of $\mathrm{ad}(r_\lambda)$ is trivial. Yet suppose that G is the Zariski closure of the image of r_λ, and let H be the index two subgroup from which r_λ is tensor induced. Then

$$\mathrm{ad}(\mathrm{Tensor\ Ind}_H^G V) \simeq \mathbb{1} \oplus \mathrm{Ind}_H^G \mathrm{ad}^0(V) \oplus \mathrm{Tensor\ Ind}_H^G \mathrm{ad}^0(V).$$

In particular, we see (looking at the second factor) that the image of G acting on $\mathrm{ad}(r_\lambda)$ surjects onto G/H, and so the component group is non-trivial, a contradiction.

We now return to the case $n = 2$, where we have the additional assumption that the 4-dimensional compatible system $\{r_\lambda\}$ is odd, regular, polarizable, and weakly irreducible, and thus potentially automorphic by Lemma 1.4.11. By [48, Theorem 2], for a density one set of l, r_λ is irreducible for all $\lambda | l$. We claim that for any λ for which r_λ is irreducible, r_λ is also strongly irreducible. If this fails to be the case, then since r_λ is regular, it follows (as in the proof of [11, Corollary 4.4]) that r_λ is induced from a quadratic extension of F, and hence $r_\lambda \simeq r_\lambda \otimes \chi$ for some non-trivial quadratic character χ. Since χ lives in a compatible system, we see that every r_λ is induced from a common quadratic extension, and in particular no r_λ is strongly irreducible, contradicting our assumptions. In particular, since r is assumed irreducible, it is strongly irreducible.

Since r_λ is strongly irreducible for a set of primes l of density 1, it suffices to show (given the argument above in the case of general m, n) that the set of primes l for which there exists $\lambda | l$ with $G_\lambda^{\mathrm{der},\circ}$ having Lie algebra $\mathfrak{sp}_4$ is a set of density zero. Suppose not; then by Lemma 3.3.1, we deduce that the compatible system $\{\wedge^2 r_\lambda\}$ decomposes as a direct sum of a 1-dimensional and a 5-dimensional compatible system. Since $\wedge^2(a \otimes b) = \det(b) \otimes \mathrm{Sym}^2(a) \oplus \det(a) \otimes \mathrm{Sym}^2(b)$, it follows that at least one of $\mathrm{Sym}^2(a)$ and $\mathrm{Sym}^2(b)$ must have a 1-dimensional factor. But then either a or b is induced from a quadratic extension, so r is induced from a quadratic extension, contradicting the strong irreducibility of r which we proved in the previous paragraph. □

3.3. A lemma on 4-dimensional polarizable automorphic compatible systems

The following lemma was used in the proof of Theorem 3.2.17. Recall that for a Galois representation r_λ, we denote the Zariski closure of the image of r_λ by G_λ.

Lemma 3.3.1. *Let F be a CM field, and let $\{r_\lambda\}$ be a 4-dimensional compatible system of representations which is odd, regular, polarizable, and weakly irreducible. Suppose that there exists a set of primes l of positive upper density with the property that for some $\lambda | l$, we have $G_\lambda^{\mathrm{der},\circ} = \mathrm{Sp}_4$. Then the compatible system of 6-dimensional representations $\{\wedge^2 r_\lambda\}$ decomposes as a direct sum of two compatible systems of dimensions 5 and 1, respectively.*

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Proof. Let T be a set of primes l as in the statement of the lemma. If $l \in T$, then we let λ denote the choice of a particular $\lambda | l$ with the property that $G_\lambda^{\mathrm{der},\circ} = \mathrm{Sp}_4$. Then the Zariski closure of the image of r_λ is a subgroup of $\mathrm{GSp}_4(\overline{M}_\lambda)$ containing $\mathrm{Sp}_4(\overline{M}_\lambda)$. By [2, Proposition 5.3.2], after possibly replacing T with a smaller set of primes (still of positive upper density), we also may assume that, for any fixed finite extension H/F,

$$\overline{r}_\lambda|_{G_{H(\zeta_l)}} \to \mathrm{GSp}_4(\overline{\mathbf{F}}_l)$$

is irreducible for $l \in T$. We apply this with H equal to the compositum of all the quadratic extensions of F unramified outside the set of primes of bad reduction for the compatible system.

Consider the (semisimple) Galois representations:

$$\wedge^2 \overline{r}_\lambda|_{G_{F(\zeta_l)}} \to \mathrm{GL}_6(\overline{\mathbf{F}}_l).$$

For each $l \in T$, the representation $\wedge^2 \overline{r}_\lambda$ admits a 1-dimensional summand. We claim that for all but finitely many $l \in T$, the complementary 5-dimensional summand is also irreducible. To see this, consider the various possible images of $\overline{r}_\lambda : G_F \to \mathrm{GSp}_4(k)$ with $k = \overline{\mathbf{F}}_l$ under the additional assumption that they act irreducibly. The classification of such maximal subgroups (as first computed in [34]) shows that, for $l > 2$, we have one of the following:

(1) The image contains $\mathrm{Sp}_4(\mathbf{F}_l)$.

(2) The image stabilizes a decomposition $k^4 = k^2 \oplus k^2$, and $\overline{r}_\lambda$ is thus induced from a quadratic extension of F.

(3) The projective image is contained in the group $\mathrm{PGL}_2(k)$ acting via the symmetric cube representation.

(4) The projective image has absolutely bounded order.

For a more modern reference, one could also consult [6], in particular, tables 8.12 and 8.13, which list the maximal subgroups of $\mathrm{Sp}_4(q) := \mathrm{Sp}_4(\mathbf{F}_q)$ and from which one can read off the maximal subgroups of the almost simple extension $\mathrm{PGSp}_4(\mathbf{F}_q)$ of $\mathrm{Sp}_4(\mathbf{F}_q)/Z(\mathrm{Sp}_4(\mathbf{F}_q))$ by $\langle \delta \rangle = \mathbf{Z}/2\mathbf{Z}$ using the rightmost column. For the convenience of the reader, we note that groups listed there of type $\mathscr{C}_1$ correspond to reducible representations, those of type $\mathscr{C}_2$, $\mathscr{C}_3$, and $\mathscr{C}_5$ correspond to groups of type (2), and the remaining groups of type $\mathscr{C}_6$ or of class $\mathscr{S}$ have absolutely bounded image (type (4)) with the exception of $\mathrm{SL}_2(\mathbf{F}_q)$ which corresponds to (3).

In case (1), the representation $\wedge^2 \overline{r}_\lambda$ decomposes as a direct sum of an irreducible 5-dimensional representation and a 1-dimensional representation. Moreover, because $\mathrm{Sp}_4(\mathbf{F}_l)$ is quasisimple (a perfect central extension of a simple group), the projective image of the representation does not change after restriction to the solvable extension $F(\zeta_l)$.

We claim that cases (2) and (4) can only hold for finitely many l. In case (2), it follows, similarly to the proof of [11, Lemma 2.6], that for l sufficiently large, the representation is induced from a quadratic extension unramified at l. But the quadratic extension must also be unramified outside the fixed finite set S of primes of bad reduction for the compatible

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system, and so must be contained in H, contradicting our assumption on the irreducibility of $\overline{r}_\lambda$ restricted to $H(\zeta_l)$. For case (4), note that since r_λ is regular, the order of the projective image of $\overline{r}_\lambda$ (even after restriction to inertia at primes above λ) tends to infinity with l, as follows from Fontaine–Laffaille theory.

Finally, in case (3), $\wedge^2$ of the symmetric cube representation of a subgroup of $\mathrm{GL}_2(k)$ is the direct sum of a character plus (a twist of) the symmetric fifth power. If l is large, the only subgroups of $\mathrm{GL}_2(k)$ for which the 4-dimensional representation is irreducible but the 5-dimensional representation is not are those whose projective image is S_4; since the image of the symmetric cube representation of such a subgroup then has bounded projective order, arguing again by Fontaine–Laffaille theory, as in case (4), we see that this case may also be ruled out if l is sufficiently large. In conclusion, we see that if l is sufficiently large, then the 5-dimensional summand is irreducible, as required.

Shrinking T further, we see that we may assume that for all $l \in T$, the representation $\wedge^2 r_\lambda$ decomposes as the sum of a character and a 5-dimensional representation s_λ with the properties that s_λ is Fontaine–Laffaille, and $\overline{s}_\lambda|_{G_{F(\zeta_l)}}$ is irreducible. The representation s_λ is regular and essentially conjugate self-dual (since r_λ is), and is also odd (by Lemma 1.4.3). We can certainly assume that $l \geqslant 11$, so it follows that s_λ is potentially automorphic by [2, Theorem C], and hence extends to the desired compatible system. □

3.4. Factorization of compatible systems

Our main result in this section is Theorem 3.4.3. We begin with two preparatory lemmas. The following is a variant of the main results of [9, 10].

Lemma 3.4.1. *Let F be a CM field, and let $r : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$ be an irreducible regular polarizable representation of G_F. Suppose either that n is odd, or that $n = 2$. If $n = 2$, suppose further that $l \geqslant 11$, that $\mathrm{Sym}^2 r$ is irreducible and is Fontaine–Laffaille at all places dividing l, and that $\mathrm{Sym}^2 \overline{r}|_{G_{F(\zeta_l)}}$ is irreducible. Then r is odd.*

Proof. In the case that n is odd, this is immediate from Lemma 1.4.3. Suppose now that $n = 2$. Let $\mathbf{P}r$ denote the projective representation $\mathbf{P}r : G_F \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$ associated to r. The polarizability of r implies that $\mathbf{P}r^c \simeq \mathbf{P}r^\vee \simeq \mathbf{P}r$, and hence that $\mathbf{P}r$ extends to a representation $\mathbf{P}r : G_{F^+} \to \mathrm{PGL}_2(\overline{\mathbf{Q}}_p)$. Thus $\mathrm{Ad}^0(r) = \mathrm{Ad}^0(\mathbf{P}r)$ also extends to a representation $\mathrm{Ad}^0(r) : G_{F^+} \to \mathrm{GL}_3(\overline{\mathbf{Q}}_p)$. It follows from [2, Corollary 4.5.2] (and the case $n = 3$ of the result being proved) that $\mathrm{Ad}^0(r)$ is potentially automorphic, and thus (by [45, Proposition A]) the image of any complex conjugation is non-scalar, and thus the image of any complex conjugation under $\mathbf{P}r$ is non-scalar.

We show that this implies that r is odd. Because we are in dimension 2, there is certainly a non-degenerate pairing on $\overline{\mathbf{Q}}_p^2$ and a character μ of G_{F^+} which satisfies

$$\langle r(\sigma)x, r^c(\sigma)y\rangle = \mu(\sigma)\langle x, y\rangle$$

for any complex conjugation c. Since $\mathbf{P}r$ is not dihedral, the pairing is unique, and to prove oddness, it suffices to show that $\langle x, y\rangle = \langle y, x\rangle$. If not, then the pairing is symplectic,

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and $\langle x, x\rangle = 0$ for all x. This implies that

$$\langle r(\sigma)x, r^c(\sigma)x\rangle = 0$$

for all σ. Because the dimension is 2, we have $\langle x, y\rangle = 0$ for $x \neq 0$ only when y is a multiple of x. Since r is irreducible, it follows that $\mathbf{P}r(\sigma) = \mathbf{P}r^c(\sigma)$ for all σ, which, by Schur's lemma, implies that $\mathbf{P}r(c)$ is scalar, contradicting the result above. □

The following lemma gives a local condition for a representation not to be of the form $\rho \otimes (\rho^c)^\vee$ up to twist for some representation ρ.

Lemma 3.4.2. *Let F be a CM field. Let v be a prime in F^+ which is inert in F, and denote by w the corresponding prime in F. Denote by c the non-trivial element of* $\mathrm{Gal}(F/F^+) = \mathrm{Gal}(F_w/F_v^+)$. *Suppose that l is a prime distinct from the characteristic of v and w. Let*

$$\psi : G_{F_w} \to \overline{\mathbf{Q}}_l^\times$$

be a non-trivial ramified character such that $\psi^c|_{I_{F_w}} = \psi^{-1}|_{I_{F_w}}$. Suppose that

$$s_w : G_{F_w} \to \mathrm{GL}_{n^2}(\overline{\mathbf{Q}}_l)$$

is a representation such that

$$s_w|_{I_{F_w}} \cong \psi|_{I_{F_w}}^{\oplus n} \oplus \mathbb{1}^{\oplus(n^2-n)}.$$

Then, if $n > 2$, we cannot write s_w in the form

$$s_w \simeq \theta \otimes (\rho \otimes (\rho^c)^\vee),$$

where $\rho : G_{F_w} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$ and $\theta : G_{F_w} \to \overline{\mathbf{Q}}_l^\times$ is a character. If additionally $\psi|_{I_{F_w}}$ has order > 2, then there is no such decomposition when $n = 2$ either.

Proof. Suppose for the sake of contradiction that we can write $s_w \simeq \theta \otimes (\rho \otimes (\rho^c)^\vee)$. Consider the representations ρ and $(\rho^c)^\vee$ restricted to I_{F_w}. We are assuming for the sake of contradiction that

$$\rho \otimes (\rho^c)^\vee|_{I_{F_w}} \simeq \theta|_{I_{F_w}}^{-1} \otimes (\psi|_{I_{F_w}}^{\oplus n} \oplus \mathbb{1}^{\oplus(n^2-n)}).$$

By Lemma 3.2.4, we deduce that ρ and ρ^c restricted to I_{F_w} are both given by direct sums of characters.

Suppose that, after restriction to I_{F_w}, the representation ρ decomposes as the direct sum of n copies of a single character ϕ. The inertia group I_{F_w} is normal in G_{F_w} and in $G_{F_v^+}$. For a representation ϱ of I_{F_w} and for $\sigma \in G_{F_v^+}$, it thus makes sense to define a representation ϱ^σ of I_{F_w} by $\varrho^\sigma(g) = \varrho(\sigma g \sigma^{-1})$. Since ρ is a representation of G_{F_w}, we see that $\rho^\sigma \simeq \rho$ for any $\sigma \in G_{F_w}$. Hence if $\sigma \in G_{F_w}$, then we have

$$\phi^{\oplus n} \simeq \rho|_{I_{F_w}} \simeq \rho^\sigma|_{I_{F_w}} \simeq (\rho|_{I_{F_w}})^\sigma \simeq (\phi^\sigma)^{\oplus n},$$

and thus $\phi = \phi^\sigma$. It follows that if $c \in G_{F_v^+}$ is any lift of the non-trivial element of $G_{F_v^+}/G_{F_w} = \mathrm{Gal}(F_w/F_v^+)$, then ϕ^c does not depend on this lift. In particular, still assuming that $\rho|_{I_{F_w}}$ decomposes as a direct sum of n copies of ϕ, we have

$$\rho^c|_{I_{F_w}} \simeq (\phi^c)^{\oplus n}.$$

This would imply that $\rho \otimes (\rho^c)^\vee$ is a direct sum of n^2 equal characters of I_{F_w}, contradicting the fact that ψ is ramified, and thus non-trivial on inertia. In particular, ρ (and by symmetry, ρ^c) contains at least 2 distinct characters.

Let φ be a character of I_{F_w} occurring in ρ^c, so that φ^{-1} occurs in $(\rho^c)^\vee$. Then

$$\rho|_{I_{F_w}} = (\rho|_{I_{F_w}} \otimes \varphi^{-1}) \otimes \varphi \subset (\rho \otimes (\rho^c)^\vee)|_{I_{F_w}} \otimes \varphi = \theta^{-1}\varphi \otimes s_w|_{I_{F_w}},$$

so $\rho|_{I_{F_w}} \subset \theta^{-1}\varphi \otimes (\psi^{\oplus n} \oplus \mathbb{1}^{\oplus(n^2-n)})$ and it follows that ρ restricted to I_{F_w} is a direct sum of at most two distinct characters with some multiplicity. From the previous discussion, precisely two such characters occur. The same argument gives the same result for ρ^c. Thus we may write

$$\rho|_{I_{F_w}} = \phi_1^r \oplus \phi_2^{n-r}, \quad (\rho^c)^\vee|_{I_{F_w}} = \varphi_1^s \oplus \varphi_2^{n-s},$$

for some $0 < r < n$ and $0 < s < n$. Since each pair of characters is distinct, we must have $\phi_i\varphi_j \neq \phi_i\varphi_k$ and $\phi_j\varphi_i \neq \phi_k\varphi_i$ when $j \neq k$. In order for the tensor product to contain precisely two distinct characters, this forces the equalities $\phi_i\varphi_j = \phi_j\varphi_i$ and $\phi_i\varphi_i = \phi_j\varphi_j$. Since one character occurs in $\rho \otimes (\rho^c)^\vee$ exactly n times (and the other $n^2 - n$ times), we deduce, after some appropriate re-ordering, that

$$rs + (n-r)(n-s) = n.$$

This has no solutions in integers $0 < r < n$ and $0 < s < n$ when $n > 2$, a contradiction. (Indeed, we have $2n = rs + rs + (n-r)(n-s) + (n-r)(n-s) \geqslant r + s + (n-r) + (n-s) = 2n$ with equality if and only if $r = s = (n-r) = (n-s) = 1$.)

Now let us consider the case $n = 2$. In this case, the required character identities imply that $\phi_i/\phi_j = \varphi_i/\varphi_j$ is a character of order dividing 2. On the other hand, the ratio of the two distinct characters occurring in $\rho \otimes (\rho^c)^\vee|_{I_{F_w}}$ may be identified both with ψ and with ϕ_i/ϕ_j with $i \neq j$, which implies that ψ has order 2, a contradiction. □

Combining the two previous lemmas (as well as lemmas from previous sections), we are now able to prove a theorem which allows us to factor a compatible system into the tensor product of two other compatible systems given such a factorization at one prime p.

Theorem 3.4.3. *Let F be a CM field and let $p > 2$ be prime. Suppose that n is odd or that $n = 2$. Let $(\{s_\lambda\}, \{\mu_\lambda\})$ be an odd, polarized, regular, weakly irreducible compatible system of representations with associated p-adic representation (s, μ). Suppose that we can write*

$$(s, \mu) = (a, \mu_1) \otimes (b, \mu_2),$$

where a and b are n-dimensional representations which are de Rham at all places $v|p$. Suppose also that we have the following:

- *There is a positive density set of places l such that for each $\lambda|l$, the representation s_λ is strongly irreducible.*
- *s is strongly irreducible.*
- *The residual representation $\overline{b}$ has an image containing $\mathrm{SL}_n(\mathbf{F}_q)$ for q a sufficiently large power of p in the sense of* Lemma 3.2.10.

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- *There is a finite place $x \nmid p$ of F^+ which is inert in F, and a character $\psi : G_{F_x} \to \overline{\mathbf{Q}}_p^\times$, such that*
  - $\psi^c|_{I_{F_x}} = \psi^{-1}|_{I_{F_x}}$,
  - $\psi|_{I_{F_x}}$ *has (finite) order greater than* 2,
  - $a|_{G_{F_x}}$ *is unramified, and*
  - $b|_{I_{F_x}} \cong \psi|_{I_{F_x}} \oplus \mathbb{1}^{\oplus(n-1)}$.

Then there are odd, polarized, regular, weakly irreducible compatible systems $(\{a_\lambda\}, \{\mu_{1,\lambda}\})$, $(\{b_\lambda\}, \{\mu_{2,\lambda}\})$ *whose associated p-adic representations are respectively* (a, μ_1) *and* (b, μ_2)*; so in particular* $\{s_\lambda\} = \{a_\lambda \otimes b_\lambda\}$.

Proof. Under our first assumption, we deduce from [2, Proposition 5.3.2] that there exists a set of primes l of density 1 with $l > 2(n+1)$ and $x \nmid l$, such that for each $\lambda | l$, s_λ is Fontaine–Laffaille at all places dividing l, s_λ is strongly irreducible, and $\overline{s}_\lambda|_{G_{F(\zeta_l)}}$ is irreducible. If $n = 2$, we furthermore can and do assume that $\mathrm{Sym}^2 s_\lambda$ is irreducible and Fontaine–Laffaille, and $l \geqslant 11$. (The irreducibility of $\mathrm{Sym}^2 s_\lambda$ is an easy consequence of the assumption that s_λ is strongly irreducible – for example, one can deduce it from Lemma 3.2.8.)

By Theorem 3.2.17, we may choose l such that we can write $s_\lambda = a_l \otimes b_l$, where a_l and b_l are both n-dimensional representations of G_F, the Zariski closure of b_l contains $\mathrm{SL}_n(\overline{\mathbf{Q}}_l)$, and the unordered pair $\{a_l, b_l\}$ is unique up to twist. It follows from [36, Theorem 3.2.10] that we can choose a_l and b_l to be unramified at all but finitely many places, and de Rham at all places dividing l. (We apply the result to the surjection from $\mathrm{GL}_n \times \mathrm{GL}_n$ to its image in GL_{n^2}; [36, Hypothesis 3.2.4] is satisfied because F is CM and s_λ is polarizable.) It then follows from [32, Proposition 3.3.4] that we can furthermore ensure that a_l and b_l are in fact crystalline at all places dividing l. Moreover, the regularity of s_λ immediately implies the regularity of a and b.

Since s_λ is Fontaine–Laffaille at all places dividing l and $\overline{s}_\lambda|_{G_{F(\zeta_l)}}$ is irreducible, we see that each of a_l and b_l is Fontaine–Laffaille at all places dividing l, and that $\overline{a}_l|_{G_{F(\zeta_l)}}$ and $\overline{b}_l|_{G_{F(\zeta_l)}}$ are irreducible. Since s_λ is strongly irreducible, a_l and b_l are strongly irreducible.

We now show that a_l and b_l are both polarizable. Since $s_\lambda^c \cong \mu_\lambda s_\lambda^\vee$, it follows that

$$a_l \otimes b_l \cong \mu_\lambda (a_l^c)^\vee \otimes (b_l^c)^\vee,$$

and by the uniqueness of a_l and b_l up to twist, we see that there are characters $\psi_l, \varphi_l : G_F \to \overline{\mathbf{Q}}_l^\times$ such that either $a_l \cong \psi_l (b_l^c)^\vee$, $b_l \cong \varphi_l (a_l^c)^\vee$ or that $a_l \cong \psi_l (a_l^c)^\vee$, $b_l \cong \varphi_l (b_l^c)^\vee$. In either case, we have $s_\lambda^c \cong \psi_l \varphi_l s_\lambda^\vee$, and it follows from Lemma 3.2.8 that $\psi_l \varphi_l = \mu_\lambda^c = \mu_\lambda$.

In the first case, we have $s_\lambda \cong a_l \otimes b_l \cong \varphi_l a_l \otimes (a_l^c)^\vee$. This contradicts Lemma 3.4.2 (the hypotheses of which are satisfied by our assumptions on $a|_{G_{F_x}}$ and $b|_{G_{F_x}}$ and by Proposition 1.4.14).

We are therefore in the second case, and we need to show that ψ_l, φ_l both extend to characters of G_{F^+}. Taking the conjugate dual of the isomorphism $b_l \cong \varphi_l (b_l^c)^\vee$, we see that also $b_l \cong \varphi_l^c (b_l^c)^\vee$, so that $b_l \cong (\varphi_l / \varphi_l^c) b_l$. Since b_l is strongly irreducible, we have $\varphi_l = \varphi_l^c$, and φ_l extends to G_{F^+}. Similarly, ψ_l also extends, as required.

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Since a_l, b_l are polarizable, it follows from Lemma 3.4.1 that they are both odd. We can now apply [2, Theorem 5.5.1] to the polarized representations (a_l, ψ_l) and (b_l, φ_l), obtaining compatible systems $(\{a_\lambda\}, \{\psi_\lambda\})$ and $(\{b_\lambda\}, \{\varphi_\lambda\})$ whose corresponding l-adic representations are $(a_l, \psi_l), (b_l, \varphi_l)$, respectively. Since $(s_\lambda, \mu_\lambda) = (a_l, \psi_l) \otimes (b_l, \varphi_l)$, we have $(s, \mu) = (a_p, \psi_p) \otimes (b_p, \phi_p)$, where (a_p, ψ_p) and (b_p, φ_p) are the p-adic representations corresponding to $(\{a_\lambda\}, \{\psi_\lambda\})$ and $(\{b_\lambda\}, \{\varphi_\lambda\})$, respectively.

Since we also have $(s, \mu) = (a, \mu_1) \otimes (b, \mu_2)$, it follows from Lemma 3.2.14 (since we are assuming that s is strongly irreducible) that the pairs $\{a_p, b_p\}$ and $\{a, b\}$ agree up to twist. After possibly exchanging $\{a_\lambda\}$ and $\{b_\lambda\}$, we may suppose that b_p is a twist of b. Since b is strongly irreducible, and both b and b_p are both polarizable and de Rham, the twist must be by an algebraic character of G_F. Replacing $\{b_\lambda\}$ by the twist by the corresponding compatible system of characters, and $\{a_\lambda\}$ by the inverse twist, we have constructed the sought-after compatible systems (note that we must have $\mu_2 = \varphi_p$ by another application of Lemma 3.2.8, so that also $\mu_1 = \psi_p$). □

3.5. A technical lemma

We end this section with a technical lemma that will be used in § 4.1. We begin with some equally technical preliminaries.

Lemma 3.5.1. *Let H be a reductive linear algebraic group with a faithful irreducible linear representation V, such that the restriction of V to H° remains irreducible. Let Z_H denote the centre of H. Then there is an injective map*

$$H/Z_H H^\circ \to \mathrm{Out}(H^{\circ,\mathrm{der}}).$$

In particular, if Z_H is connected, then the component group of H injects into $\mathrm{Out}(H^{\circ,\mathrm{der}})$.

Proof. If Z_{H° denotes the centre of H°, then the natural morphism $H^{\circ,\mathrm{der}} \times Z_{H^\circ} \to H^\circ$ is surjective (since H° is a connected reductive linear algebraic group). Thus we see that the irreducible representation V of H° remains irreducible when restricted to $H^{\circ,\mathrm{der}}$; and we also see that the conjugation action of H° on $H^{\circ,\mathrm{der}}$ is via inner automorphisms, so that there is indeed a well-defined morphism $H/Z_H H^\circ \to \mathrm{Out}(H^{\circ,\mathrm{der}})$.

Suppose now that some element $h \in H$ acts on $H^{\circ,\mathrm{der}}$ via an inner automorphism. We must show that $h \in Z_H H^\circ$. Multiplying h by an element of $H^{\circ,\mathrm{der}}$, we may assume that h actually centralizes $H^{\circ,\mathrm{der}}$. Since V is an irreducible representation of $H^{\circ,\mathrm{der}}$, we see that h acts on V by scalars, and thus commutes with the action of H. Since H acts faithfully on V, we see that $h \in Z_H$, as required. □

Lemma 3.5.2. *Let H be a connected semisimple algebraic group with a faithful irreducible representation of dimension $\leqslant 2^d$. If $p > \max(d, 3)$, then p does not divide the order of $\mathrm{Out}(H^{\circ,\mathrm{der}})$.*

Proof. Suppose that we have an element of $\mathrm{Out}(H^{\circ,\mathrm{der}})$ of order divisible by p. Any such outer automorphism induces a non-trivial outer automorphism of the corresponding Lie algebra. Since the simple Lie algebras only have automorphisms of order at most 3, any such automorphism must consist of a permutation of the simple factors. But if there is

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a faithful representation of dimension at most 2^d, then there can be at most d simple factors, and S_d has no elements of order p if $p > d$. □

The following lemma is the main result of this subsection. Recall that if M/F is a finite extension, we let $M^{F\text{-gal}}$ denote the Galois closure of M over F.

Lemma 3.5.3. *Let F be a number field, and let*

$$\overline{a}, \overline{b} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$$

be such that $\overline{a} \otimes \overline{b}$ is irreducible. Let $\{r_\lambda\}$ denote a weakly compatible system of n^2-dimensional regular representations of G_F such that $r_p = a \otimes b$, where a is a deformation of $\overline{a}$, b is a deformation of $\overline{b}$, and $p > \max(n, 3)$. Assume that the image of $\overline{b}$ contains $\mathrm{SL}_n(\mathbf{F}_q)$ for q sufficiently large, in the sense of Lemma 3.2.10*. If for some λ, r_λ is induced from an extension M/F, then $[M^{F\text{-gal}} : F]$ has order prime to p.*

Proof. We may (and so) assume, after tensoring $\{r_\lambda\}$ by a compatible system of characters if necessary, that the image of $\det(r_\lambda)$ is infinite. (Note that any twist of r_λ is also induced from M/F.) Let $G_\lambda^\circ \subset G_\lambda$ denote the connected component of the Zariski closure G_λ of r_λ. Let N/F denote the fixed field of the inverse image of G_λ°. Since the connected component G_λ° is a normal subgroup of G_λ, it follows that N/F is Galois. On the other hand, if r_λ is induced from M, then certainly $M \subset N$, and hence $M^{F\text{-gal}} \subset N$. Hence it suffices to show that the component group of G_λ has order prime to p. For this, we work at the prime p (since the order of the component group is independent of λ by Theorem 1.4.15).

To this end, we now let G denote the Zariski closure of r_p and G° the connected component of the identity in G. By Lemma 3.2.16, we may assume that $G^{\mathrm{der},\circ}$ is the natural representation of $A^{\mathrm{der},\circ} \times B^{\mathrm{der},\circ}$, where A and B are the Zariski closures of the images of a and b, respectively, and $B^{\mathrm{der},\circ} = \mathrm{SL}_n(\overline{\mathbf{Q}}_p)$. Let Z denote the centre of G. Because $\det(G)$ is infinite, it follows that the Lie algebra $\mathfrak{g}$ contains a non-trivial torus $\mathfrak{t}$, and hence $\exp(\mathfrak{t}) \subset Z$ is infinite. Because G acts irreducibly, by Schur's lemma, it follows that Z consists of scalars, and thus $Z = \overline{\mathbf{Q}}_p^\times = Z^\circ$ is connected.

Because r_p is regular, it follows from Lemma 3.2.12 that r_p is induced from a strongly irreducible representation s_p over some extension M_p/F. By comparison with the decomposition of $G^{\mathrm{der},\circ}$ above, this strongly irreducible representation has dimension mn for some $m|n$ with $m[M_p : F] = n$. In particular, the Galois closure $M_p^{F\text{-gal}}/F$ of M_p over F has the property that $\mathrm{Gal}(M_p^{F\text{-gal}}/F)$ is a subgroup of S_n, and thus has order prime to p.

Let H denote the Zariski closure of the image of $s_p|_{G_{M_p^{F\text{-gal}}}}$. Since $[M_p^{F\text{-gal}} : F]$ has order prime to p, it is enough to prove that the component group of H has order prime to p. There is an inclusion $G^\circ \subset H \subset G$, and $Z_H = Z = Z^\circ$ is connected. Moreover, since s_p is strongly irreducible, it follows that H has a faithful irreducible representation of dimension $mn \leqslant n^2 \leqslant 2^n$ such that the restriction to H° is also irreducible. The result now follows from Lemmas 3.5.1 and 3.5.2. □

4. Building Lifts

4.1. Deformations of $\overline{a}\otimes\overline{b}$ and strong irreducibility

In this section, we show that strong irreducibility of compatible systems can be ensured by imposing ramification conditions at finite places. These ramification conditions become trivial after a finite base change, and we will use the Khare–Wintenberger method to remove them from our final results.

We begin with the following preparatory lemma.

Lemma 4.1.1. *Let $K/\mathbf{Q}_h$ be a finite extension of degree prime to p for some prime $h \equiv 1 \bmod p$. Suppose that $l \neq h$, and let*

$$\rho : G_K \to \mathrm{GL}_m(\overline{\mathbf{Q}}_l)$$

be a representation such that $\rho|_{I_K}$ has finite p-power order. If $p \nmid m$, then ρ admits a 1-dimensional subquotient which is the restriction of a character of $G_{\mathbf{Q}_h}$.

Note that the lemma applies to both $l = p$ and $l \neq p$.

Proof. Since not every irreducible constituent of ρ can be of degree a multiple of p, we may reduce to the case that ρ is irreducible. The image of inertia has p-power order and $h \neq p$. Hence the representation is tamely ramified. Any such irreducible representation is of the form $\rho = \mathrm{Ind}^{G_K}_{G_L} \psi$ for a character ψ of G_L, where L/K is unramified (although we do not use this fact) and the degree $[L : K] = m$. It follows that

$$n := [L : \mathbf{Q}_h] = [L : K][K : \mathbf{Q}_h]$$

is prime to p. Also, since $\rho|_{I_K}$ has p-power order, we see that $\psi|_{I_L}$ has p-power order.

It is elementary to see that, since $p \nmid n$ and $p|(h-1)$, the largest power of p dividing $|k_L^\times|$ is the same power of p which divides $|\mathbf{F}_h^\times| = h-1$. (Indeed, $(h-1)$ divides $|k_L^\times|$, which in turn divides $h^n - 1$, and we have $(h^n-1)/(h-1) \equiv n \pmod p$.) By local class field theory, it follows that the character ψ of G_L (whose restriction to inertia is, as we have observed, of p-power order) is the restriction of a character of $G_{\mathbf{Q}_h}$, which we also denote by ψ. Yet then

$$\rho = \mathrm{Ind}^{G_K}_{G_L} \psi = \psi \otimes \mathrm{Ind}^{G_K}_{G_L} 1.$$

The representation $\mathrm{Ind}^{G_K}_{G_L} 1$ contains a copy of the trivial representation. Since ρ is irreducible, it follows that $L = K$, and $\rho = \psi$, as claimed. □

Let F be a CM field. Consider a pair of irreducible representations:

$$\overline{a}, \overline{b} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$$

such that $\overline{a}\otimes\overline{b}$ is irreducible and $\overline{a}$ and $\overline{b}$ are polarizable. We now build on Lemma 3.5.3, showing how we can control the extensions M/F from which representations inside a compatible system containing a lift of $\overline{a}\otimes\overline{b}$ could possibly be induced.

We will shortly prove Lemma 4.1.3, which enables us to show that, for certain deformation problems, any deformation of $\overline{a}\otimes\overline{b}$ which is induced must be induced from one of a finite number of possible fields, independent of certain classes of auxiliary ramification sets Σ. We start with the following preparatory lemma.

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Lemma 4.1.2. *Let $[F_v : \mathbf{Q}_v]$ be a finite extension, and let*

$$r : G_{F_v} \to \mathrm{GL}_n(\overline{\mathbf{Q}}_l)$$

be either potentially unramified if $l \neq v$ or potentially crystalline if $l = v$. Assume that the restriction of the Weil–Deligne representation $\mathrm{WD}(r)$ to the inertia group I_{F_v} factors through a group of p-power order. Suppose that r is induced from a finite extension M_v/F_v whose Galois closure $M_v^{F_v\text{-gal}}$ over F_v has order prime to p. Then M_v/F_v is unramified.

Proof. The formation of Weil–Deligne representations is compatible with inductions. Hence, if $r = \mathrm{Ind}_{G_{M_v}}^{G_{F_v}} s$, then $\mathrm{WD}(r) = \mathrm{Ind}_{W_{M_v}}^{W_{F_v}} \mathrm{WD}(s)$. It follows that the kernel of W_{F_v} acting on $\mathrm{WD}(r)$ is contained in W_{M_v} and thus also contained in $W_{M_v^{F_v\text{-gal}}}$. But the image of $\mathrm{WD}(r)$ restricted to I_{F_v} has p-power order by assumption, and thus the inertia subgroup of $\mathrm{Gal}(M_v^{F_v\text{-gal}}/F_v)$ has p-power order. But $\mathrm{Gal}(M_v^{F_v\text{-gal}}/F_v)$ has order prime to p by assumption, and thus M_v/F_v is unramified. □

Lemma 4.1.3. *Fix a finite set S of places of F^+ containing all primes above p and all the primes at which $\overline{a}$ or $\overline{b}$ is ramified. Let $\{r_\lambda\}$ denote a weakly irreducible compatible system of odd, polarizable, regular n^2-dimensional representations of G_F such that $r_p = c \otimes d$, where c is a deformation of $\overline{a}$, and d is a deformation of $\overline{b}$, and $p > \max(n, 3)$. Assume that the image of $\overline{b}$ contains $\mathrm{SL}_n(\mathbf{F}_q)$ for q some sufficiently large power of p, in the sense of* Lemma 3.2.10.

Suppose that $\{r_\lambda\}$ is unramified outside a finite set $S \cup \Sigma$ of places of F^+ (in the sense of Remark 1.4.5*), and that for primes $v \in \Sigma$, $r_p(I_{F_v})$ is finite. Suppose that λ is a prime such that the representation r_λ is induced from M/F. Then*

(1) *$[M^{F\text{-gal}} : F]$ has order prime to p,*

(2) *$[M : F] \leqslant n^2$, and*

(3) *M/F is unramified outside S.*

In particular, if S is fixed, there are only finitely many such M independently of the choice of λ, the compatible system $\{r_\lambda\}$, and the set Σ.

Proof. By shrinking Σ if necessary, we may (and do) assume that Σ is disjoint from S; in particular, the representation $\overline{a} \otimes \overline{b}$ is unramified at primes of Σ. In particular, since the primes above p lie in S, we may assume that Σ does not contain any such prime.

The fact that $[M^{F\text{-gal}} : F]$ has order prime to p is an immediate consequence of Lemma 3.5.3. Let us consider the possible M from which r_λ may be induced. The degree $[M : F]$ is certainly bounded by n^2. If we prove that M/F is unramified at primes not dividing S, then there are only finitely many such M/F as an immediate consequence of a theorem of Hermite (following Minkowski). Consider the representation r_p locally at primes $v \in \Sigma$. By assumption, the image $r_p(I_{F_v})$ of the inertia subgroup I_{F_v} for $v \in \Sigma$ factors through a finite quotient. On the other hand, the representation $\overline{a} \otimes \overline{b}$ is unramified at v. Hence the image of inertia factors through a finite quotient of p-power order. By Proposition 1.4.14, the image of I_{F_v} factors through a finite quotient of p-power order for all representations $\mathrm{WD}(r_\lambda|_{G_{F_v}})$.

Suppose that r_λ is induced from M. Let v be a finite place of F. Then $r_\lambda|_{G_{F_v}}$ is induced from M_v. Moreover, the Galois closure $M_v^{F_v\text{-gal}}$ of M_v over F_v is contained in the localization of the Galois closure $M^{F\text{-gal}}$ of F at v, and thus $\mathrm{Gal}(M_v^{F_v\text{-gal}}/F_v)$ also has order prime to p. If $v \notin S$, then either $v \in \Sigma$ and the image of I_{F_v} in $\mathrm{WD}(r_\lambda|_{G_{F_v}})$ factors through a finite quotient of p-power order or $v \notin \Sigma$ and the image of I_{F_v} in $\mathrm{WD}(r_\lambda|_{G_{F_v}})$ is trivial. In either case, the claim that M_v/F_v is unramified follows immediately from Lemma 4.1.2. □

Given an integer n and a finite set S of places of F^+, we now introduce certain auxiliary primes v for fields M/F of the kind which arose in Lemma 4.1.3.

Lemma 4.1.4. *For each non-trivial M/F unramified outside S with $\mathrm{Gal}(M^{F\text{-gal}}/F)$ of order prime to p, either $M \subset F(\zeta_p)$ or there exist infinitely many finite places v of F^+ and $w|v$ in F with the following properties:*

1. *$N_{F/\mathbf{Q}}(w) = h$ is prime, and $h \equiv 1 \pmod{p^m}$ for some $p^m > n^2$.*
2. *h is unramified both in M and in the fixed fields of $\overline{a}$ and $\overline{b}$ over F. In particular, w is unramified in $M^{F\text{-gal}}/F$.*
3. *The images of $\overline{a}(\mathrm{Frob}_w)$ and $\overline{b}(\mathrm{Frob}_w)$ have orders prime to p.*
4. *The decomposition group of $M^{F\text{-gal}}/F$ at w is non-trivial.*
5. *v (resp. w) lies outside any given finite set of places of F^+ (resp. F).*

Proof. A choice of w in F determines a unique prime v of F^+ with $w|v$. Conditions (2) and (5) exclude only finitely many places. (By definition, any prime which ramifies in $M/\mathbf{Q}$ is either ramified in $F/\mathbf{Q}$, which is fixed, or is ramified in M/F, which, by definition, is unramified outside the fixed set S.) The remaining three conditions are a Chebotarev condition relative to the image of G_F in

$$\mathrm{Gal}(F(\zeta_{p^m})/F) \times \mathrm{Gal}(M^{F\text{-gal}}/F) \times \mathrm{GL}_n(\overline{\mathbf{F}}_p) \times \mathrm{GL}_n(\overline{\mathbf{F}}_p),$$

where the maps to the two copies of $\mathrm{GL}_n(\overline{\mathbf{F}}_p)$ factor via $\overline{a}$ and $\overline{b}$, respectively. It suffices to show that the image of G_F contains an element whose projection is trivial in the first group (for (1)), non-trivial in the second group (for (4)), and semisimple in the last two groups (for (3)). The first two conditions can be satisfied simultaneously unless $M^{F\text{-gal}} \subset F(\zeta_{p^m})$. Since $[M^{F\text{-gal}} : F]$ is prime to p, this is equivalent to $M \subset M^{F\text{-gal}} \subset F(\zeta_p)$. Thus if $M \not\subset F(\zeta_p)$, then there exists a $\sigma \in G_F$ which is trivial in $\mathrm{Gal}(F(\zeta_{p^m})/F)$ and non-trivial in $\mathrm{Gal}(M^{F\text{-gal}}/F)$. It follows that σ^{p^k} also has this property for any k, because $\mathrm{Gal}(M^{F\text{-gal}}/F)$ has order prime to p. On the other hand, the image of any sufficiently large p-power of any element of $\mathrm{GL}_n(\overline{\mathbf{F}}_p)$ has order prime to p. Thus the image of σ^{p^k} for sufficiently large k in the product above has the desired shape. □

Definition 4.1.5 (A suitable choice). Continuing on with our running assumptions on $\overline{a}$ and $\overline{b}$, suppose in addition that the assumptions of Lemma 4.1.3 are in effect. In particular, both $\overline{a}$ and $\overline{b}$ are n-dimensional residual representations of G_F which are polarizable, the representation $\overline{a} \otimes \overline{b}$ is absolutely irreducible, and there exists a weakly irreducible compatible system $\{r_\lambda\}$ such that $r_p = c \otimes d$ is a deformation of $\overline{a} \otimes \overline{b}$. Let S be a set containing all primes above p and all primes where $\overline{a}$ and $\overline{b}$ are ramified. Fix characters μ_1, μ_2 unramified outside S such that $\overline{a}$ and $\overline{b}$ are $\overline{\mu}_1$- and $\overline{\mu}_2$-polarizable, respectively. For $v \in S$, consider any collection of local μ_1-polarized components C_v and μ_2-polarized components D_v for $\overline{a}$ and $\overline{b}$, respectively. Suppose, furthermore, that $C_v \otimes D_v$ is regular for all $v|p$. We now choose an auxiliary set Σ disjoint from S and components C_v, D_v for $v \in \Sigma$ as follows. (Ultimately, we shall apply Lemma 4.1.3 with precisely this set Σ.) For each of the finitely many fields M/F not contained in $F(\zeta_p)$ and satisfying conditions (1), (2), and (3) in the conclusion of Lemma 4.1.3, we choose *two* primes v in F^+ with $w|v$ in F according to Lemma 4.1.4, where the auxiliary set of places given by (5) is the set of primes which ramify in any of the finitely many fields M/F. Moreover, for each M/F, we choose this pair of places v so that they have distinct residue characteristics. We now let Σ be the union of these pairs of places v of F^+ for all M/F. Our choice guarantees that, for any prime l, and for any given M/F, there exists a $v \in \Sigma$ corresponding to M/F with residue characteristic prime to l. We now choose C_v and D_v as follows. Recall that $F_v^+ \simeq F_w \simeq \mathbf{Q}_h$. Let $\psi : G_{\mathbf{Q}_h} \to \overline{\mathbf{Q}}_p^\times$ be a character such that $\psi|_{I_{F_w}}$ has order p^m, which exists because $h - 1 \equiv 0 \bmod p^m$. We consider deformations c and d at v such that

$$c|_{I_{F_w}} = \bigoplus_{i=0}^{n-1} \psi^i|_{I_{F_w}}, \quad d|_{I_{F_w}} = \bigoplus_{i=0}^{n-1} \psi^{ni}|_{I_{F_w}}.$$

Such deformations exist (locally) because $\overline{a}(\mathrm{Frob}_w)$ and $\overline{b}(\mathrm{Frob}_w)$ are semisimple. Since the characters $\psi^i|_{I_{F_w}}$ (and similarly the characters $\psi^{ni}|_{I_{F_w}}$) are pairwise distinct, this defines unions of components of the corresponding local deformation rings. We choose C_v and D_v to be any of the resulting components. The corresponding inertial type of the n^2-dimensional compatible system (which is well defined across the entire compatible system, by Proposition 1.4.14) at each $v \in \Sigma$ is

$$\bigoplus_{i=0}^{n^2-1} \psi^i|_{I_{F_w}} = \bigoplus_{i=0}^{n^2-1} \psi^i|_{I_{F_v^+}}.$$

(Note that $I_{F_w} = I_{F_v^+}$.) In particular, it is a direct sum of distinct characters of p-power order – here we use the assumption that $p^m > n^2$. As usual, the corresponding polarized deformation rings of $\overline{a}$ and $\overline{b}$ are denoted by $R_{C,\Sigma}$ and $R_{D,\Sigma}$, respectively. We refer to the choice of auxiliary set Σ and corresponding components C_v and D_v as a *suitable choice*.

Equipped with the notion of a suitable choice, we now prove the following.

Lemma 4.1.6. *Let $\{r_\lambda\}$ denote a weakly irreducible, polarizable, regular, odd compatible system of n^2-dimensional representations of G_F such that $r_p = c \otimes d$, where c and d are deformations of $\overline{a}$ and $\overline{b}$ of types $R_{C,\Sigma}$ and $R_{D,\Sigma}$, respectively for a suitable choice of Σ, C_v and D_v as in* Definition 4.1.5. *Assume that $p > \max(n, 3)$, and that the image of $\overline{b}$ contains $\mathrm{SL}_n(\mathbf{F}_q)$ for q some sufficiently large power of p, in the sense of* Lemma 3.2.10.
Suppose that some r_λ is induced from M/F. Then $M \subset F(\zeta_p)$.

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Proof. Assume that r_λ is induced from M/F. The choice of C_v and D_v for $v \in \Sigma$ ensures that, for primes $v \in \Sigma$, $r_p(I_{F_v})$ is finite. Hence, by Lemma 4.1.3, M/F is one of finitely many fields which is unramified outside S, has degree bounded by n^2, and $M^{F\text{-gal}}/F$ has order prime to p. Suppose that $M \not\subset F(\zeta_p)$. By the suitable choice of Σ as in Definition 4.1.5, there exists a $v \in \Sigma$ corresponding to M which satisfies the conditions of Lemma 4.1.4 *and* has residue characteristic not dividing $N(\lambda)$. Let us write

$$r_\lambda = \mathrm{Ind}_{G_M}^{G_F} s : G_F \to \mathrm{GL}_{n^2}(\overline{\mathbf{Q}}_l).$$

Since the inertial type of r_λ at v consists of distinct characters of $I_{\mathbf{Q}_h}$, it suffices to show that this is incompatible with being an induction.

Let x be a place of $M^{F\text{-gal}}$ lying over w (note that $F_w = \mathbf{Q}_h$). Since $\mathrm{Gal}(M^{F\text{-gal}}/F)$ has order prime to p, it follows that $[M_x^{F\text{-gal}} : F_w] = [M_x^{F\text{-gal}} : \mathbf{Q}_h]$ has order prime to p. We have a representation

$$s|_{G_{M_x^{F\text{-gal}}}} : G_{M_x^{F\text{-gal}}} \to \mathrm{GL}_m(\overline{\mathbf{Q}}_l),$$

where $m[M : F] = n^2$, and so $p \nmid m$ (since $p \nmid n$). It follows from Lemma 4.1.1 that $s|_{M_x^{F\text{-gal}}}$ contains at least one subquotient ω which is the restriction of a character of $G_{\mathbf{Q}_h}$ (note that $s(I_{M_x^{F\text{-gal}}})$ has finite p-power order because $r_\lambda(I_{F_w})$ does by the definition of a suitable choice of Σ). By the definition of an induction, there is an identification

$$r_\lambda|_{G_{M^{F\text{-gal}}}} = \bigoplus_{\sigma \in \mathrm{Gal}(M^{F\text{-gal}}/F)/\,\mathrm{Gal}(M^{F\text{-gal}}/M)} s^\sigma|_{G_{M^{F\text{-gal}}}},$$

where $s^\sigma(g) = s(\sigma g \sigma^{-1})$.

The decomposition group of w in $\mathrm{Gal}(M^{F\text{-gal}}/F)$ is non-trivial by the choice of v and w (more precisely, by condition (4) of Lemma 4.1.4). Moreover, because $M^{F\text{-gal}}$ is the Galois closure of M, the intersection of the conjugates of $\mathrm{Gal}(M^{F\text{-gal}}/M)$ inside $\mathrm{Gal}(M^{F\text{-gal}}/F)$ is trivial. Hence, for a suitable choice of $x|w$ in $M^{F\text{-gal}}$, we may ensure that there exists an element σ in the decomposition group of x above w in $\mathrm{Gal}(M^{F\text{-gal}}/F)$ that does not lie in $\mathrm{Gal}(M^{F\text{-gal}}/M)$. It follows that $s|_{G_{M^{F\text{-gal}}}} \oplus s^\sigma|_{G_{M^{F\text{-gal}}}}$ is a summand of $r_\lambda|_{G_{M^{F\text{-gal}}}}$. But σ lies inside the decomposition group of x, and hence $\sigma x = x$, and $s^\sigma|_{G_{M_x^{F\text{-gal}}}}$ is the conjugate by σ of $s|_{G_{M_x^{F\text{-gal}}}}$. Since ω occurs as a subquotient as $s|_{G_{M_x^{F\text{-gal}}}}$, it follows that ω^σ is a subquotient of $s^\sigma|_{G_{M_x^{F\text{-gal}}}}$, and hence $\omega \oplus \omega^\sigma$ is a subquotient of $r_\lambda|_{G_{M_x^{F\text{-gal}}}}$. But ω is the restriction of a character of $G_{\mathbf{Q}_h}$, and thus $\omega^\sigma = \omega$, and $\omega \oplus \omega$ is a subquotient of $r_\lambda|_{G_{M_x^{F\text{-gal}}}}$. By assumption, the restriction $r_\lambda|_{I_{F_w}}$ is a direct sum of distinct characters. Since w is unramified in $M^{F\text{-gal}}/F$ by the construction of Σ (cf. Lemma 4.1.4 (2)), the restriction $r_\lambda|_{G_{M_x^{F\text{-gal}}}}$ must also be a direct sum of distinct characters, and hence we have a contradiction. □

Lemma 4.1.7. *In the context of* Lemma 4.1.6, *assume also that* $\overline{r}_p|_{G_{F(\zeta_p)}}$ *is absolutely irreducible. Then* r_p *is strongly irreducible, and for a positive density of primes* l, *the representations* r_λ *are strongly irreducible for all* $\lambda|l$ *(that is, the compatible system* $\{r_\lambda\}$ *is 'weakly strongly irreducible').*

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Proof. By Lemma 3.2.12, the representations r_λ which are absolutely irreducible are strongly irreducible unless they are induced. Since $\{r_\lambda\}$ is weakly irreducible by assumption, for a positive density of primes l, the representations r_λ are irreducible for all $\lambda|l$, and certainly r_p is irreducible. Hence it suffices to show that the representations r_λ and r_p are not induced from some finite extension M/F. By the previous lemma (Lemma 4.1.6), this can happen only if $M \subset F(\zeta_p)$. Thus we will be done if we can show that $r_p|_{G_{F(\zeta_p)}}$ is absolutely irreducible, and that $\{r_\lambda|_{G_{F(\zeta_p)}}\}$ is weakly irreducible.

Since $\overline{r}_p|_{G_{F(\zeta_p)}}$ is absolutely irreducible by assumption, $r_p|_{G_{F(\zeta_p)}}$ is absolutely irreducible. Since $F(\zeta_p)/F$ is CM, it follows from Lemma 1.4.17 that $\{r_\lambda|_{G_{F(\zeta_p)}}\}$ is weakly irreducible, as required. □

We end this subsection with some results (versions of the Khare–Wintenberger argument) that will allow us to remove the auxiliary conditions discussed above from our final results. We remind the reader of the conventions introduced in Definition 2.1.5 and Conventions 1.4.21 and 1.4.22, which will be in force throughout the rest of this section; namely, we write w (possibly decorated by subscripts and superscripts) for a place of a CM field lying over the place v of its totally real subfield, and we do not explicitly mention prolongations.

Lemma 4.1.8 (Descending the existence of a compatible system). *Let S be a finite set of finite places of F^+ which contains all the places dividing p, and let $(\overline{a}, \overline{\mu})$ be a polarized representation which is unramified outside of S. Let $\mu : G_{F^+} \to \overline{\mathbf{Z}}_p^\times$ be a de Rham lift of $\overline{\mu}$ which is unramified outside of S. For each $v \in S$, let A_v be a μ-polarized component for $\overline{a}|_{G_{F_w}}$. Assume that $\overline{a} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is reasonable.*

Suppose that there is a finite Galois extension of CM fields L/F, linearly disjoint from $\overline{F}^{\ker \overline{a}}(\zeta_p)$ over F, and an odd, polarized, regular, weakly irreducible compatible system of representations $(\{s_\lambda\}, \{\mu'_\lambda\})$ of G_L, with associated p-adic representation (s, μ'), with the following properties:

1. *$\overline{a}|_{G_L}$ is reasonable.*
2. *$\overline{s} \cong \overline{a}|_{G_L}$.*
3. *$\mu' = \mu|_{G_L}$.*
4. *s is unramified outside of the places lying over S.*
5. *For each place v_L of L^+ lying over a place $v \in S$, $s|_{G_{L_{w_L}}}$ lies on $A_v|_{L^+_{v_L}}$.*

Then there is an odd, polarized, regular, weakly irreducible compatible system of representations $(\{a_\lambda\}, \{\mu_\lambda\})$ of G_F, with associated p-adic representation (a, μ), with the following properties:

1. *a lifts $\overline{a}$.*
2. *a is unramified outside of S.*
3. *For each place $v \in S$, $a|_{G_{F_w}}$ lies on A_v.*

Proof. By Theorem 2.1.16, after replacing L by a finite extension, we can and do assume that $(\{s_\lambda\}, \{\mu'_\lambda\})$ is automorphic. Let R_A be the universal $\mathcal{O}$-deformation algebra for $\overline{a}$,

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for the deformations which are

- μ-polarized and odd,
- unramified outside of S,
- if $v \in S$, then the corresponding lift lies on A_v.

Let $R_{A'}$ be the universal $\mathcal{O}$-deformation algebra for $\overline{s}$, for the deformations which are

- μ'-polarized and odd,
- unramified outside of the primes lying over S,
- if $v \in S$, and $v_L|v$ is a place of L^+, then the corresponding lift lies on $A_v|_{L^+_{v_L}}$.

By Proposition 1.4.24, R_A has positive dimension, and by [2, Lemma 1.2.3], it is finite over $R_{A'}$. Since $R_{A'}$ is finite over $\mathcal{O}$ by Lemma 1.4.27, we deduce that R_A is finite over $\mathcal{O}$. It follows that R_A has $\overline{\mathbf{Q}}_p$-points, and we let a be the representation corresponding to such a point.

It remains to check that (a, μ) is part of a weakly irreducible compatible system. Since (by construction) (s, μ') is automorphic, it follows from Theorem 1.4.26 that $(a|_{G_L}, \mu')$ is automorphic. Then (a, μ) is part of a compatible system by the usual argument with Brauer's theorem; to be precise, it follows from the proof of [2, Theorem 5.5.1], as the appeal to [2, Theorem 4.5.1] in that proof is only in order to prove potential automorphy, which we have already established. (Note that the hypothesis on the component groups of the Galois representations made in the proof of [2, Theorem 5.5.1] is guaranteed by Lemma 3.1.1; we are free to twist our representations by an algebraic character in order to guarantee that the determinant has infinite order.) This compatible system is weakly irreducible by Lemma 1.4.11. □

Corollary 4.1.9 (Level lowering for a compatible system). *Let S be a finite set of finite places of F^+, containing all of the places dividing p, and let $(\overline{a}, \overline{\mu})$ be a polarized representation which is unramified outside of S. Let $\mu : G_{F^+} \to \overline{\mathbf{Z}}_p^\times$ be a de Rham lift of $\overline{\mu}$ which is unramified outside of S. Assume that $\overline{a} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is reasonable. For each $v \in S$, let A_v be a μ-polarized component for $\overline{a}|_{G_{F_w}}$. Let Σ be a finite set of finite places of F^+, which is disjoint from S, and for each $v \in \Sigma$, let A_v be a μ-polarized component for $\overline{a}|_{G_{F_w}}$ which is potentially unramified.*

Suppose that there is an odd, polarized, regular, weakly irreducible system of representations $(\{s_\lambda\}, \{\mu_\lambda\})$ of G_F, with associated p-adic representation (s, μ), with the following properties:

(1) *$\overline{s} \cong \overline{a}$.*

(2) *s is unramified outside of $S \cup \Sigma$.*

(3) *For each place $v \in S \cup \Sigma$, $s|_{G_{F_w}}$ lies on A_v.*

Then there is an odd, polarized, regular, weakly irreducible system of representations $(\{a_\lambda\}, \{\mu_\lambda\})$ of G_F, with associated p-adic representation (a, μ), with the following properties:

(1) *a lifts $\overline{a}$.*

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(2) *a is unramified outside of S.*

(3) *For each place $v \in S$, $a|_{G_{F_w}}$ lies on A_v.*

Proof. We may choose a finite Galois extension L/F, linearly disjoint from $\overline{F}^{\ker \overline{a}}(\zeta_p)$ over F, with the properties that $\overline{a}|_{G_L}$ remains reasonable, and for each $v \in \Sigma$, and each place $v_L|v$ of L, $s|_{G_{L_{w_L}}}$ is unramified. The result then follows from Lemma 4.1.8, applied to $(\{s_\lambda|_{G_L}\}, \{\mu_\lambda|_{G_{L^+}}\})$. □

4.2. Local swapping

In this subsection, we prove our main theorem (Theorem 4.2.11), building on a series of lemmas. We begin with the following lemma, which will be used to move components between different mod p representations. Much of the rest of this section is devoted to relaxing the rather restrictive hypotheses made in this lemma, culminating in Lemma 4.2.9.

We again remind the reader that we are using the conventions introduced in Definition 2.1.5 and Conventions 1.4.21 and 1.4.22. We extend Convention 1.4.21 in the obvious way to subscripts and superscripts, so that w_1 is a place of F over v_1 in F^+, w_L is a place of L over v_L in L^+, and so on.

Lemma 4.2.1 (Local swapping I). *Suppose that either $n > 1$ is odd or $n = 2$. Let F be a CM field, and let $(\{a_\lambda\}, \{\mu_\lambda\})$ and $(\{b_\lambda\}, \{\nu_\lambda\})$ be two weakly irreducible, odd, regular, polarized compatible systems of n-dimensional representations with corresponding p-adic representations (a, μ) and (b, ν). Let S be a finite set of finite places of F^+ containing all of the places lying over p, such that each of a, b, μ, or ν is unramified outside of S. For each $v \in S$, write A_v for the μ-polarized component determined by $a|_{G_{F_w}}$, and B_v for the ν-polarized component determined by $b|_{G_{F_w}}$. Assume the following:*

(1) *For each place $v \in S$ with $v|p$, the component $A_v \otimes B_v$ is regular.*

(2) *The representations $\overline{a}$ and $\overline{b}$ are reasonable, and $(\overline{a} \otimes \overline{b})(G_{F(\zeta_p)})$ is adequate.*

(3) *The image $\overline{b}(G_F)$ contains $\mathrm{SL}_n(\mathbf{F}_p)$.*

(4) *There is a finite place $x \nmid p$ with $x \in S$ of F^+ which is inert in F, and a character $\psi\colon G_{F_x} \to \overline{\mathbf{Q}}_p^\times$ such that*
   - *$\psi^c|_{I_{F_x}} = \psi^{-1}|_{I_{F_x}}$,*
   - *$\psi|_{I_{F_x}}$ has (finite) order greater than 2,*
   - *$a|_{G_{F_x}}$ is unramified, and*
   - *$b|_{I_{F_x}} \cong \psi|_{I_{F_x}} \oplus \mathbb{1}^{\oplus(n-1)}$.*

We let $T \subset S \setminus \{x\}$ be a set of places with the property that if $v \in T$, then there are an equality $\mu|_{G_{F_v^+}} = \nu|_{G_{F_v^+}}$ and a polarized isomorphism $\overline{a}|_{G_{F_w}} \cong \overline{b}|_{G_{F_w}}$. Furthermore, if $v \in T$, then we set $C_v = B_v$ and $D_v = A_v$, while if $v \in S \setminus T$, then we set $C_v = A_v$ and $D_v = B_v$.

Then there exist odd, regular, polarized, weakly irreducible compatible systems $(\{c_\lambda\}, \{\mu_\lambda\})$ and $(\{d_\lambda\}, \{\nu_\lambda\})$ with corresponding p-adic representations (c, μ) and (d, ν), having

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the following properties:

- $\overline{c} \cong \overline{a}$ *and* $\overline{d} \cong \overline{b}$.
- *For each place* $v \in S$, $c|_{G_{F_w}}$ *lies on* C_v *and* $d|_{G_{F_w}}$ *lies on* D_v.
- c *and* d *are unramified outside of* S.

Proof. Note first that since $\overline{b}$ is reasonable, we have $p > 2(n+1) \geqslant 6$, so in particular $\overline{b}(G_F)$ is large enough that Lemma 3.2.10 applies. Note also that since $\overline{a}|_{G_{F_x}}$ is unramified but $\overline{b}|_{G_{F_x}}$ is ramified, we have $x \notin T$. By Corollary 4.1.9, it suffices to construct the desired compatible systems after increasing the set S, provided that the additional components A_v that we choose are potentially unramified. Note that the assumption that $(\overline{a} \otimes \overline{b})(G_{F(\zeta_p)})$ is adequate includes the assumption that $\overline{a} \otimes \overline{b}$ is absolutely irreducible; accordingly, we allow ramification in $S \cup \Sigma$, where Σ, and the components C_v and D_v for $v \in \Sigma$, are a suitable choice as in Definition 4.1.5. We set $A_v = C_v$ and $B_v = D_v$ for $v \in \Sigma$.

We consider the following four global deformation $\mathcal{O}$-algebras $R_{A,\Sigma}$, $R_{B,\Sigma}$, $R_{C,\Sigma}$, and $R_{D,\Sigma}$, defined as follows (from this point onwards, we drop the Σ from the notation):

(1) R_A and R_C are deformation rings for $\overline{a}$; R_B and R_D are deformation rings for $\overline{b}$.

(2) The deformations are polarized and odd; the multiplier characters of R_A, R_B, R_C, R_D are, respectively, μ, ν, μ, ν.

(3) If $v \in S \cup \Sigma$, then the restriction to G_{F_w} of the universal deformation corresponding to R_A, R_B, R_C, or R_D lies on the component A_v, B_v, C_v, or D_v, respectively.

(4) The representations are unramified outside of $S \cup \Sigma$.

We also consider a fifth deformation $\mathcal{O}$-algebra $R_{A\otimes B}$ for $\overline{a} \otimes \overline{b}$, which is defined to have the following properties:

(1) The deformations are polarized and odd, with multiplier $\mu\nu\delta_{F/F^+}$.

(2) If $v \in S \cup \Sigma$, then the corresponding lift lies on the component $A_v \otimes B_v = C_v \otimes D_v$.

(3) The representations are unramified outside of $S \cup \Sigma$.

Note that $\{a_\lambda \otimes b_\lambda\}$ is weakly irreducible by Lemma 1.4.18. It follows from Lemma 1.4.27 (with S replaced by $S \cup \Sigma$) that $R_{A\otimes B}$ is a finite $\mathcal{O}$-algebra. We claim that R_C and R_D are also both finite over $\mathcal{O}$.

To prove this, we first note that any deformation coming from R_C tensored with one from R_D gives, functorially, a deformation of type $R_{A\otimes B}$. The representation $a \otimes b$ also gives such a point. By Yoneda's lemma, there exist corresponding morphisms:

$$R_{A\otimes B} \to R_A \widehat{\otimes}_{\mathcal{O}} R_B, \quad R_{A\otimes B} \to R_C \widehat{\otimes}_{\mathcal{O}} R_D.$$

Since $R_{A\otimes B}$ is finite over $\mathcal{O}$, it then suffices to show that the morphism $R_{A\otimes B} \to R_C \widehat{\otimes}_{\mathcal{O}} R_D$ is finite.

By Nakayama's lemma, we are reduced to showing that if $\mathcal{A}$ is an Artinian k-algebra such that $\overline{a}$ and $\overline{b}$ admit deformations $\widetilde{a}$ and $\widetilde{b}$ to $\mathcal{A}$ (of types C and D, respectively) so that $\widetilde{a} \otimes \widetilde{b}$ is the trivial deformation of $\overline{a} \otimes \overline{b}$, then the corresponding map $R_C \widehat{\otimes}_{\mathcal{O}} R_D \to \mathcal{A}$ factors through some subalgebra $\mathcal{A}'$ of $\mathcal{A}$ of uniformly bounded length. To show this, let M

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be the fixed field of the kernel of $\overline{a} \oplus \overline{b}$. Then we find that

$$\widetilde{a}|_{G_M} \otimes \widetilde{b}|_{G_M} = (\overline{a} \otimes_k \mathcal{A})|_{G_M} \otimes (\overline{b} \otimes_k \mathcal{A})|_{G_M}$$

is a free $\mathcal{A}$-module with a trivial action of G_M. Now, we claim that if V and W are free $\mathcal{A}$-modules with an action of G_M such that the diagonal action on $V \otimes_{\mathcal{A}} W$ is trivial, then G_M acts on V and W by scalars. Assume for the sake of contradiction that G_M does not act on V via a character. Then there exists an element $v \in V$ and $g \in G_M$ such that gv is not a multiple of v. Yet then (choosing w to be any element of $W \setminus \mathfrak{m}_{\mathcal{A}} W$) $g(v \otimes w)$ cannot possibly be a multiple of $v \otimes w$, a contradiction. Hence the action of G_M on V (and W) is by scalars.

Let $R_{\det(C)}$, $R_{\det(D)}$, etc. denote the corresponding deformation rings for the determinants of our representations. We have the following diagram:

$$\begin{array}{ccc} R_{\det(A\otimes B)} & \longrightarrow & R_{\det(C)} \widehat{\otimes} R_{\det(D)} \\ \downarrow & & \downarrow \\ R_{A\otimes B} & \longrightarrow & R_C \widehat{\otimes} R_D \end{array}$$

We begin by showing that $R_{\det(C)}$ and $R_{\det(D)}$ are finite over $\mathcal{O}$. The Hodge-theoretic conditions imply that any two characters of type $\det(C)$ (or $\det(D)$) differ by a finite order character unramified outside $S \cup \Sigma$, and with ramification at $v|p \in S \cup \Sigma$ bounded purely by the corresponding type. The finiteness of $R_{\det(C)}$ and $R_{\det(D)}$ over $\mathcal{O}$ is now an immediate consequence of class field theory. (Fixing one such pair of characters τ_C and τ_D, the deformation rings over k are identified with group rings $k[\Gamma]$ for some finite ray class group Γ.)

Hence we may additionally assume that the determinants of $\widetilde{a}$ and $\widetilde{b}$ are fixed. From the argument above, we have also shown that the action of G_M on the corresponding $\mathcal{A}$-modules V and W is via a scalar. Since there are only finitely many characters of G_M of order n which are unramified outside any fixed finite set of primes, we deduce that there exists a finite Galois extension N/F such that the action of G_N on V and W is trivial. But this implies that the corresponding maps from R_C and R_D to $\mathcal{A}$ factor through the quotient of the universal deformation ring over k of $\overline{a}$ and $\overline{b}$ as representations of the finite group $\mathrm{Gal}(N/F)$. But the finiteness of these deformation rings follows exactly as in the proof of Lemma 1.2.3 of [2].

Since R_C and R_D are finite over $\mathcal{O}$, and have dimension at least one by Proposition 1.4.24, this shows that R_C and R_D both have non-trivial $\overline{\mathbf{Q}}_p$-valued points. Make a choice of such points, and let c, d be the corresponding p-adic representations. Recall that $\{a_\lambda \otimes b_\lambda\}$ is weakly irreducible, and thus potentially automorphic by Lemma 1.4.11. In particular, given any finite Galois extension $F^{(\mathrm{avoid})}/F$, we can find a CM Galois extension L/F which is linearly disjoint from $F^{(\mathrm{avoid})}/F$ and is such that $\{(a_\lambda \otimes b_\lambda)|_{G_L}\}$ is automorphic. Furthermore, by replacing $F^{(\mathrm{avoid})}$ by $F^{(\mathrm{avoid})}\overline{F}^{\ker \overline{a}\otimes\overline{b}}(\zeta_p)$, we may assume that $(\overline{a} \otimes \overline{b})(G_{L(\zeta_p)})$ is adequate, and $\zeta_p \notin L$. Making a further quadratic base change if necessary, we can also assume that all places at which $a \otimes b$ is ramified, and all places lying over p, are split places. Then $(c \otimes d)|_{G_L}$ is automorphic by Theorem 1.4.26. As in the proof of Lemma 4.1.8, it follows from

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the proof of [2, Proposition 5.5.1] that $c \otimes d$ is part of a regular, odd, polarizable weakly irreducible compatible system $\{t_\lambda\}$ (again, using Lemma 3.1.1 to guarantee the hypothesis on the component groups made in [2, Proposition 5.5.1]).

We will now apply Theorem 3.4.3 to the compatible system $\{t_\lambda\}$, and so deduce that c, d are the p-adic representations corresponding to compatible systems $\{c_\lambda\}$, $\{d_\lambda\}$, which (by construction) satisfy all the requirements of the lemma. To complete the proof, it suffices to verify that the hypotheses of Theorem 3.4.3 are satisfied by $\{t_\lambda\}$. The strong irreducibility assumptions are satisfied by Lemma 4.1.7 and the conditions at the places in Σ, which implies that the hypotheses of Lemma 4.1.6 hold. The hypothesis on the image of $\overline{b}$ is satisfied by assumption. Finally, the place x required in the hypotheses of Theorem 3.4.3 can be taken to be the place x appearing in hypothesis (4) of the present lemma: indeed, the hypotheses on the behaviour of a and b at x involve just the restriction of these representations to the inertia group at x, and thus they are constant along the components A_x and B_x. As noted above, $x \notin T$, and thus $C_x = A_x$ and $D_x = B_x$, so that these same hypotheses are satisfied by c and d. This completes the verification of the hypotheses of Theorem 3.4.3, and so also completes the proof of the lemma. □

Remark 4.2.2. The polarizability requirement in the previous lemma is essential, even for $n = 1$. In particular, there are genuine global obstructions (arising from units) to producing characters with prescribed ramification properties at all primes. However, the polarizable condition implies that (in our setting) the corresponding characters are unitary and come from the -1-part of the corresponding ray class groups; in particular, there are no unit obstructions provided that $p \neq 2$ (and in this paper, p is never 2).

We now introduce potentially diagonalizable lifts into the picture. We remind the reader of Convention 2.1.4 (that we will assume without explicit mention that the gaps between the Hodge–Tate weights of our potentially diagonalizable lifts are sufficiently large); we will also sometimes assume without further comment that in base change arguments the weights and types of the potentially diagonalizable representations have been chosen compatibly.

Lemma 4.2.3 (Existence of potentially diagonalizable lifts). *Let $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ be a pleasant representation in the sense of* Definition 2.1.7*, namely:*

- *$\zeta_p \notin F$, and $\overline{s}|_{G_{F(\zeta_p)}}$ is irreducible.*
- *$\overline{s}$ is polarizable and odd.*
- *$p > 2(n+1)$.*
- *All the primes $v|p$ in F^+ split completely in F.*
- *For each place $w|p$ of F, $\overline{s}|_{G_{F_w}}$ admits many diagonalizable lifts.*

Let $\overline{\mu}$ be a character such that $(\overline{s}, \overline{\mu})$ is polarized. Let μ be a de Rham lift of $\overline{\mu}$. Let S be a finite set of finite places of F^+, including all places at which $\overline{s}$ or μ is ramified, and all places lying over p.

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Then there is an odd, regular, polarized weakly irreducible compatible system $(\{s_\lambda\}, \{\mu_\lambda\})$ *of* G_F*-representations whose associated* p*-adic representation is* (s, μ)*, where:*

- s *lifts* $\overline{s}$,
- s *is unramified outside of* S*, and*
- s *is potentially diagonalizable at all places* $v|p$.

Furthermore, if we fix a μ*-polarized component* C_v *of* $\overline{s}$ *for each place* $v \nmid p$ *in* S*, then we may assume that* $s|_{G_{F_w}}$ *lies on* C_v *for each* v.

Proof. The existence of s follows immediately from [4, Theorem 5.2.1] (the hypotheses of which hold by [2, Theorems 4.3.1, 4.5.1]). That s is part of a compatible system follows from [2, Theorem 5.5.1]. □

We will also make use of the following variant of the previous result, where we no longer require that $\overline{s}|_{G_{F_w}}$ admit potentially diagonalizable lifts for $w|p$ in F, but we allow a finite base change.

Lemma 4.2.4 (Existence of potentially diagonalizable lifts II). *Let* $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ *be a reasonable representation in the sense of Definition 2.1.6, that is:*

- $\zeta_p \notin F$*, and* $\overline{s}|_{G_{F(\zeta_p)}}$ *is irreducible.*
- $\overline{s}$ *is polarizable and odd.*
- $p > 2(n+1)$.

Let $\overline{\mu}$ *be a character such that* $(\overline{s}, \overline{\mu})$ *is polarized. Let* μ *be a de Rham lift of* $\overline{\mu}$*. Let* S *be a finite set of finite places of* F^+*, including all places at which* $\overline{s}$ *or* μ *is ramified, and all places lying over* p*. Let* $F^{(\mathrm{avoid})}/F$ *be a finite extension.*

Then there is a finite Galois extension of CM fields L/F*, linearly disjoint from the field* $F^{(\mathrm{avoid})}(\zeta_p)$ *over* F*, and an odd, regular, polarized weakly irreducible compatible system* $(\{s_\lambda\}, \{\mu_\lambda\})$ *of* G_L*-representations whose associated* p*-adic representation is* $(s, \mu|_{G_L})$*, where:*

- $\overline{s}|_{G_L}$ *is pleasant,*
- s *lifts* $\overline{s}|_{G_L}$,
- s *is unramified outside of the places lying over* S_L*, and*
- s *is potentially diagonalizable at all places* $v|p$.

Furthermore, if we fix a μ*-polarized component* C_v *of* $\overline{s}$ *for each place* $v \nmid p$ *in* S*, then we may assume that* $s|_{G_{F_{w_L}}}$ *lies on* $C_v|_{L^+_{v_L}}$ *for each* $v_L|v$.

Proof. By Lemma 2.1.8, we may find a finite extension L/F linear disjoint from $F^{(\mathrm{avoid})}(\zeta_p)/F$ such that $\overline{s}|_{G_L}$ is pleasant. The result then follows immediately from Lemma 4.2.3. □

Remark 4.2.5 (Arguments both reasonable and pleasant). The notions of reasonable and pleasant (Definitions 2.1.6 and 2.1.7, which were also just recalled in the statements of Lemmas 4.2.4 and 4.2.3, respectively) are very closely related. Pleasant representations are automatically reasonable, and reasonable representations are pleasant after a finite extension (Lemma 2.1.8). Since, in the ultimate application, we only assume our residual representations are reasonable, we have endeavoured to structure the arguments below to only require reasonable hypotheses on the relevant residual representations. However, at a number of points (including in the *conclusions* of Lemmas 4.2.6 and 4.2.7 as well as during the proofs of Lemma 4.2.9 and Theorem 4.2.11), we deduce that certain residual representations satisfy the stronger condition of pleasantness, even though, almost all of the time, we only ever use the fact that these resulting representations are reasonable. (One notable exception is the proof of Lemma 4.2.6, where, to invoke Lemma 4.2.3, we require that $\overline{r}$ is reasonable.) Thus the close reader should bear in mind that when it is stated that a certain residual representation is pleasant, the main implication to take away is that it is reasonable.

The following lemma and its corollaries will be used in our later arguments to replace a given representation with one which behaves well under base change.

Lemma 4.2.6 (Auxiliary representations). *Let (a, μ) be polarized and odd, where $\overline{a} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is reasonable, and let S be a finite set of finite places of F^+, containing all the places lying over p, and all places at which (a, μ) is ramified. Let $F^{(\mathrm{avoid})}/F$ be a finite extension, and let q be a power of p.*

Then there is a finite Galois extension L/F with $\zeta_p \notin L$, which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F, and a weakly irreducible, odd, polarized, regular compatible system $(\{r_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $(r, \mu|_{G_{L^+}})$, which satisfies the following:

1. *$\overline{a}|_{G_L}$ is pleasant.*
2. *$\overline{r}$ and $\overline{r}|_{G_{L^{\mathrm{gal}}}}$ are pleasant, and $\overline{r}(G_{L^{\mathrm{gal}}}) \supset \mathrm{SL}_n(\mathbf{F}_q)$; here L^{gal} denotes the Galois closure of L over $\mathbf{Q}$.*
3. *$(\overline{r} \otimes \overline{a}|_{G_L})(G_{L(\zeta_p)})$ is adequate.*
4. *Each place in S splits completely in L^+. For each place v_L of L^+ lying over a place $v \in S$, we have polarized isomorphisms $\overline{r}|_{G_{L_{w_L}}} \cong \overline{a}|_{G_{L_{w_L}}} = \overline{a}|_{G_{F_w}}$.*
5. *There is a finite place x of L^+ which is inert in L and does not lie over any place in S, and a character $\psi : G_{L_x} \to \overline{\mathbf{Q}}_p^\times$ with $\psi^c|_{I_{L_x}} = \psi^{-1}|_{I_{L_x}}$, such that $\mu|_{G_{L_x^+}}$ is unramified, $\psi|_{I_{L_x}}$ has finite order greater than 2, and $\overline{r}|_{I_{L_x}} \cong \overline{\psi}|_{I_{L_x}} \oplus \mathbb{1}^{\oplus(n-1)}$.*
6. *There is an isomorphism $r|_{I_{L_x}} \cong \psi|_{I_{L_x}} \oplus \mathbb{1}^{\oplus(n-1)}$.*
7. *For any place v of L^+ not lying over S at which $r|_{G_{L_w}}$ is ramified, $r(I_{L_w})$ is finite.*
8. *$r|_{G_{L_w}}$ is potentially diagonalizable for all places $w|p$ of F.*

Proof. By Lemma 4.2.3, it is enough to construct the odd, polarized pair $(\overline{r}, \overline{\mu}|_{G_{L^+}})$ satisfying properties (1)–(5), since we may choose local components which force conditions (6)–(8). Replacing $F^{(\mathrm{avoid})}$ with $F^{(\mathrm{avoid})}\overline{F}^{\ker \overline{a}}(\zeta_p)$, and applying Lemma 2.1.8, we see that we can ignore the requirements that $\overline{a}|_{G_L}$ is pleasant, and that $\zeta_p \notin L$.

Choose a finite place $y \notin S$ of F^+ which is inert in F and is such that $\mu|_{G_{F_y^+}}$ is unramified, and choose a character $\psi : G_{F_y} \to \overline{\mathbf{Q}}_p^\times$ with $\psi^c|_{I_{F_y}} = \psi^{-1}|_{I_{F_y}}$ such that $\psi|_{I_{F_y}}$ has finite order greater than 2. (Note that we can do this for every place y which is inert in F.)

Replacing q with a power of q if necessary, we may assume that $\overline{\psi}$, $\overline{\mu}$, and all of the representations $\overline{a}|_{G_{F_w}}$ with $v \in S$ are defined over $\mathbf{F}_q$. We may also assume that $\mathrm{PSL}_n(\mathbf{F}_q)$ is simple, and is not isomorphic to any Jordan–Hölder factor of $\mathrm{Gal}(F^{\mathrm{gal}}/\mathbf{Q})$, and that q is large enough so that condition (3) follows automatically from condition (2) by Lemma 1.4.37. (Note that if $\overline{r}(G_{L^{\mathrm{gal}}}) \supset \mathrm{SL}_n(\mathbf{F}_{q^k})$ for some k, then in particular, $\overline{r}(G_{L^{\mathrm{gal}}}) \supset \mathrm{SL}_n(\mathbf{F}_q)$.)

Arguing exactly as in the proof of [20, Proposition A2], by [10, Proposition 3.2] (see also [35, Theorem 1.2]) we can find a finite totally real extension L^+/F^+, linearly disjoint from $F^{(\mathrm{avoid})}$ over F^+, in which all places above $S \cup \{y\}$ split completely, and a surjective odd reasonable $\overline{\mu}|_{G_{L^+}}$-polarized representation $\overline{r} : G_L \to \mathrm{GL}_n(\mathbf{F}_q)$ (where $L = L^+F$) with the property that for each place v_L of L lying over a place $v \in S$, we have a polarized isomorphism $\overline{r}|_{G_{L_{w_L}}} \cong \overline{a}|_{G_{L_{w_L}}}$. In addition, for each place x of L^+ lying over y, we can ensure that $\overline{r}|_{I_{L_x}} \cong \psi|_{I_{L_x}} \oplus \mathbb{1}^{\oplus(n-1)}$. Furthermore, by the same restriction of scalars trick that we used in the proof of Theorem 2.1.16, we can assume that L^+ is of the form M^+F^+, where $M^+/\mathbf{Q}$ is Galois.

It remains to check property (2). Since $M^+/\mathbf{Q}$ is Galois, we have $L^{\mathrm{gal}} = M^+F^{\mathrm{gal}} = LF^{\mathrm{gal}}$, and since we are assuming that $\mathrm{PGL}_n(\mathbf{F}_q)$ is simple and is not isomorphic to any Jordan–Hölder factor of $\mathrm{Gal}(F^{\mathrm{gal}}/\mathbf{Q})$, it follows that the projective image of $\overline{r}|_{L^{\mathrm{gal}}(\zeta_l)}$ contains $\mathrm{PGL}_n(\mathbf{F}_q)$, from which the required property follows at once. □

Corollary 4.2.7 (Auxiliary representations with specified components). *Suppose that either n is odd or $n = 2$. Let F be a CM field, and let $(\{a_\lambda\}, \{\mu_\lambda\})$ be a weakly irreducible, odd, regular, polarized compatible system of n-dimensional representations with corresponding p-adic representation (a, μ). Assume that $\overline{a}$ is reasonable. Let S be a finite set of finite places of F^+ containing all of the places lying over p, and chosen so that (a, μ) is unramified outside of S.*

For each $v \in S$, write A_v for the μ-polarized component determined by $a|_{G_{F_w}}$. Let $F^{(\mathrm{avoid})}/F$ be a finite extension, and let q be a power of p.

Then there is a finite Galois extension L/F with $\zeta_p \notin L$, which is linearly disjoint from $F^{(\mathrm{avoid})}$ over F, such that for each set T of places of L^+ which divide p, there is a weakly irreducible, odd, polarized, regular compatible system $(\{s_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $(s, \mu|_{G_{L^+}})$, which satisfies the following:

(1) *$\overline{a}|_{G_L}$ is pleasant.*

(2) *$\overline{s}$ is pleasant, and is independent of the choice of T (up to isomorphism).*

(3) *$\overline{s}|_{G_{L^{\mathrm{gal}}}}$ is pleasant, and $\overline{s}(G_{L^{\mathrm{gal}}}) \supset \mathrm{SL}_n(\mathbf{F}_q)$.*

(4) *$(\overline{s} \otimes \overline{a}|_{G_L})(G_{L(\zeta_p)})$ is adequate.*

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(5) *For each place* v_L *of* L^+ *lying over a place* $v \in S$*, we have a polarized isomorphism* $\overline{s}|_{G_{L_{w_L}}} \cong \overline{a}|_{G_{L_{w_L}}}$.

(6) *For each place* $v_L \notin T$ *of* L^+ *lying over a place* $v \in S$*,* $s|_{G_{L_{w_L}}}$ *lies on* $A_v|_{G_{L^+_{v_L}}}$.

(7) $s|_{G_{L_{w_L}}}$ *is potentially diagonalizable for all places* w_L *lying over places in* T.

(8) *There is a finite place* x *of* L^+ *which is inert in* L *and does not lie over a place in* S*, and a character* $\psi : G_{L_x} \to \overline{\mathbf{Q}}_p^\times$ *with* $\psi^c|_{I_{L_x}} = \psi^{-1}|_{I_{L_x}}$*, such that* $\mu|_{G_{L^+_x}}$ *is unramified,* $\psi|_{I_{L_x}}$ *has finite order greater than* 2*, and* $s|_{I_{L_x}} \cong \psi|_{I_{L_x}} \oplus \mathbb{1}^{\oplus(n-1)}$.

(9) *For any place* v *of* L^+ *not lying over* S *at which* $s|_{G_{L_w}}$ *is ramified,* $s(I_{L_w})$ *is finite.*

(10) *Both* L/F*,* x*, and* ψ *can be chosen independently of the choice of* T.

Proof. By Lemma 4.2.6, we may construct a compatible system $(\{r_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ which satisfies all the required conditions of the theorem except perhaps (6), and is potentially diagonalizable at all places above p. We now apply Lemma 4.2.1 to the compatible systems $(\{a_\lambda|_{G_L}\}, \{\mu_\lambda|_{G_{L^+}}\})$ and $(\{r_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ with respect to the set T. The result is a compatible system $(\{s_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ with $\overline{s} = \overline{r}$ independent of T and such that $(s, \mu|_{G_{L^+}})$ lies on the component associated to a for all $v_L \notin T$ and the component associated to r (which is a potentially diagonalizable component) for all $v_L \in T$, as required. Note that L is the field on whose absolute Galois group $\overline{s}$ is defined and so can be chosen independently of T, and similarly x and ψ can be chosen independently of T (as an examination of the relevant arguments shows). □

Corollary 4.2.8. *Maintaining the notation and assumptions of* Corollary 4.2.7*, we can instead produce* L/F *and* $\{s_\lambda\}$ *satisfying conclusions* (1)–(7) *of* Corollary 4.2.7*, such that in addition* s *is unramified outside of* S.

Proof. This follows from Corollary 4.2.7 by making a further base change to remove the ramification at places (including x) not lying over S. □

We can now establish the following improvement on Lemma 4.2.1, where we no longer need to make assumptions (1)–(4) there; we even allow $\overline{a} = \overline{b}$.

Lemma 4.2.9 (Local swapping II). *Suppose that either* n *is odd or* $n = 2$*. Let* F *be a CM field, and let* $(\{a_\lambda\}, \{\mu_\lambda\})$ *and* $(\{b_\lambda\}, \{\nu_\lambda\})$ *be two weakly irreducible, odd, regular, polarized compatible systems of* n*-dimensional representations of* G_F *with corresponding* p*-adic representations* (a, μ) *and* (b, ν)*. Let* S *be a finite set of finite places of* F^+*, containing all of the places lying over* p*, and chosen such that* (a, μ) *and* (b, ν) *are unramified outside of* S*. For each* $v \in S$*, write* A_v *for the* μ*-polarized component determined by* $a|_{G_{F_v}}$*, and* B_v *for the* ν*-polarized component determined by* $b|_{G_{F_v}}$*. Assume that the representations* $\overline{a}$ *and* $\overline{b}$ *are reasonable.*

Let $T \subset S$ *be a set of places with the property that if* $v \in T$*, then* $\mu|_{G_{F_v^+}} = \nu|_{G_{F_v^+}}$*, and there is a polarized isomorphism* $\overline{a}|_{G_{F_w}} \cong \overline{b}|_{G_{F_w}}$*. If* $v \in T$*, then we set* $C_v = B_v$ *and* $D_v = A_v$*, and if* $v \in S \setminus T$*, then we set* $C_v = A_v$ *and* $D_v = B_v$.

Then there exist odd, regular, polarized, weakly irreducible compatible systems $(\{c_\lambda\}, \{\mu_\lambda\})$ *and* $(\{d_\lambda\}, \{\nu_\lambda\})$ *with corresponding* p*-adic representations* (c, μ) *and* (d, ν)*,*

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with the following properties:

- $\overline{c} \cong \overline{a}$ *and* $\overline{d} \cong \overline{b}$.
- *For each place* $v \in S$, $c|_{G_{F_w}}$ *lies on* C_v *and* $d|_{G_{F_w}}$ *lies on* D_v.
- c *and* d *are unramified outside of* S.

Proof. Since the statement is symmetric in $(\{a_\lambda\}, \{\mu_\lambda\})$ and $(\{b_\lambda\}, \{\nu_\lambda\})$, it is enough to prove that $(\{c_\lambda\}, \{\lambda_\mu\})$ exists. We apply Corollary 4.2.7 *twice* to the compatible system $(\{b_\lambda\}, \{\nu_\lambda\})$. The point of this construction is to provide auxiliary compatible systems to which we can apply Lemma 4.2.1. In the first application, we take T in Corollary 4.2.7 to be the set of places of F^+ lying over p, and in the second application, we take T to be the set of places in our $S \setminus T$ which lie over p. The extension L/F, the place x, and the character ψ can be chosen independently of T by Corollary 4.2.7 part (10), and hence we may make the same choice on both cases. We deduce the existence of a finite Galois extension of CM fields L/F, and odd, regular, polarizable, weakly irreducible compatible systems $(\{s_\lambda\}, \{\nu_\lambda|_{G_{L^+}}\})$ and $(\{t_\lambda\}, \{\nu_\lambda|_{G_{L^+}}\})$ of G_L-representations, with associated p-adic representations $(s, \nu|_{G_{L^+}})$ and $(t, \nu|_{G_{L^+}})$, respectively, such that we have the following:

- $\overline{s} \cong \overline{t}$.
- $\overline{a}|_{G_L}$, $\overline{b}|_{G_L}$, and $\overline{s} = \overline{t}$ are pleasant.
- $\overline{s}(G_L)$ contains $\mathrm{SL}_n(\mathbf{F}_p)$.
- $(\overline{s} \otimes \overline{a}|_{G_L})(G_{L(\zeta_p)})$ is adequate.
- For each place v_L of L^+ lying over a place $v \in S$, we have polarized isomorphisms $\overline{s}|_{G_{L_{w_L}}} = \overline{t}|_{G_{L_{w_L}}} \cong \overline{a}|_{G_{L_{w_L}}}$.
- For each place v_L of L^+ not dividing p and lying over a place $v \in T$, $s|_{G_{L_{w_L}}}$ and $t|_{G_{L_{w_L}}}$ lie on $B_v|_{L^+_{v_L}}$.
- s is potentially diagonalizable at every place dividing p.
- t is potentially diagonalizable at every place dividing p and lying over a place in $S \setminus T$.
- At each place v_L of L lying over a place $v|p$ with $v \in T$, $t|_{G_{L_{w_L}}}$ lies on $B_v|_{L^+_{v_L}}$.
- There is a finite place x of L^+ which is inert in L and does not lie over a place in S, and a character $\psi : G_{L^+_x} \to \overline{\mathbf{Q}}_p^\times$ with $\psi^c|_{I_{L_x}} = \psi^{-1}|_{I_{L_x}}$, such that $\mu|_{G_{L^+_x}}$ is unramified, $\psi|_{I_{L_x}}$ has finite order greater than 2, and
$$t|_{I_{L_x}} \cong s|_{I_{L_x}} \cong \psi|_{I_{L_x}} \oplus \mathbb{1}^{\oplus(n-1)}.$$

Applying Lemma 4.2.1 to $(\{a_\lambda|_{G_L}\}, \{\mu_\lambda|_{G_{L^+}}\})$ and $(\{s_\lambda\}, \{\nu_\lambda|_{G_{L^+}}\})$, we deduce the existence of an odd, regular, polarizable, weakly irreducible compatible system $(\{u_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $(u, \mu|_{G_{L^+}})$, with the following properties:

- $\overline{u} \cong \overline{a}|_{G_L}$.
- For each place v_L of L^+ lying over a place $v \in S$ not dividing p, $u|_{G_{L_{w_L}}}$ lies on $A_v|_{G_{L^+_{v_L}}}$.
- For every place $v_L|p$ of L^+ lying over a place $v \in S \setminus T$, $u|_{G_{L_{w_L}}}$ lies on $A_v|_{L^+_{v_L}}$.

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- u is potentially diagonalizable at every place dividing p and lying over a place in T.
- u is unramified outside of S.

Applying Lemma 4.2.1 to $(\{u_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ and $(\{t_\lambda\}, \{\nu_\lambda|_{G_{L^+}}\})$, we then obtain an odd, regular, polarized, weakly irreducible compatible system $(\{e_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $(e, \mu|_{G_{L^+}})$, with the following properties:

- $\overline{e} \cong \overline{a}|_{G_L}$.
- For each place v_L lying over a place $v \in S \setminus T$, $e|_{G_{L_{w_L}}}$ lies on $A_v|_{L^+_{v_L}}$.
- For each place v_L lying over a place $v \in T$, $e|_{G_{L_{w_L}}}$ lies on $B_v|_{L^+_{v_L}}$.
- e is unramified outside of S.

The existence of $(\{c_\lambda\}, \{\mu_\lambda\})$ now follows from Lemma 4.1.8, applied to $(\{e_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$. □

Corollary 4.2.10 (Merging components). *Suppose that either n is odd or $n = 2$. Let F be a CM field, and let S be a finite set of finite places of F^+, containing all of the places lying over p. Let T be a set of places of F^+ which divide p. Let $(\{a_\lambda\}, \{\mu_\lambda\})$ and $\left\{(\{r_\lambda^v\}, \{\nu_\lambda^v\})\right\}_{v\in T}$ be weakly irreducible, odd, regular, polarized compatible systems of n-dimensional representations of G_F with corresponding p-adic representations (a, μ) and (r^v, ν^v).*

For each $v \in S$, write A_v for the μ_v-polarized component determined by $a|_{G_{F_w}}$. Assume that all of the representations $\overline{a}$ and $\{\overline{r}^v\}_{v\in T}$ are reasonable, and that the representations (a, μ) and $\{(r^v, \nu^v)\}_{v\in T}$ are unramified outside of S. Assume also, for each $v \in T$, that $\mu|_{G_{F_v^+}} = \nu^v|_{G_{F_v^+}}$, and that there is a polarized isomorphism $\overline{a}|_{G_{F_w}} \cong \overline{r}^v|_{G_{F_w}}$.

Then there exists an odd, regular, polarized, weakly irreducible compatible system $(\{c_\lambda\}, \{\mu_\lambda\})$ with corresponding p-adic representation (c, μ), with the following properties:

- *$\overline{c} \cong \overline{a}$.*
- *For each place $v \in S \setminus T$, $c|_{G_{F_w}}$ lies on A_v.*
- *For each place $v \in T$, $c|_{G_{F_w}}$ lies on the component determined by $r^v|_{G_{F_w}}$.*
- *c is unramified outside of S.*

Proof. Let $v_1, \ldots, v_m$ be the places in T. We claim that for each $0 \leqslant i \leqslant m$, we can find an odd, regular, polarized, weakly irreducible compatible system $(\{c_\lambda^i\}, \{\mu_\lambda\})$ with corresponding p-adic representation (c^i, μ), with the following properties:

- $\overline{c}^i \cong \overline{a}$.
- For each place $v \in S \setminus T$, $c^i|_{G_{F_w}}$ lies on A_v.
- For each $j \leqslant i$, $c^i|_{G_{F_{w_j}}}$ lies on the component determined by $r^{v_j}|_{G_{F_{w_j}}}$.
- For each $j > i$, $c^i|_{G_{F_{w_j}}}$ lies on the component determined by $a|_{G_{F_{w_j}}}$.
- c^i is unramified outside of S.

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Assuming we can do this, we can take $\{c_\lambda\} := \{c_\lambda^m\}$. We prove the existence of the $\{c_\lambda^i\}$ by induction on i, taking $\{c_\lambda^0\} := \{a_\lambda\}$. Then the existence of $\{c_\lambda^{i+1}\}$ is immediate from Lemma 4.2.9 applied to $(\{c_\lambda^i\}, \{\mu_\lambda\})$ and $(\{r_\lambda^{v_{i+1}}\}, \{\nu_\lambda^{v_{i+1}}\})$ (taking the set T there to be $\{v_{i+1}\}$). □

Our main theorem is the following (recall that the notion of a globally realizable component was defined in Definition 2.1.9).

Theorem 4.2.11. *Assume that either n is odd or $n = 2$. Let F be a CM field, and let $(\overline{s}, \overline{\mu})$ be a polarized representation, where $\overline{s} : G_F \to \mathrm{GL}_n(\overline{\mathbf{F}}_p)$ is reasonable. Let S be a finite set of finite places of F^+, such that S contains all of the places at which $(\overline{s}, \overline{\mu})$ is ramified and all of the places lying over p. Let μ be a de Rham lift of $\overline{\mu}$ which is unramified outside of S. For each place $v \in S$, let C_v be a μ-polarized component for $\overline{s}|_{G_{F_w}}$.*

Assume that C_v is globally realizable for each $v \in S$ which divides p. Then there exists an odd, regular, polarized, weakly irreducible compatible system $(\{s_\lambda\}, \{\mu_\lambda\})$ of G_F-representations with associated p-adic representation (s, μ), which satisfies

(1) *s lifts $\overline{s}$, and for each place $v \in S$, the representation $s|_{G_{F_w}}$ lies on C_v,*
(2) *s is unramified outside S.*

Proof. Fix a place $v_1 \in S$ which lies over p. By definition, the hypothesis that C_{v_1} is globally realizable means that we can find a CM field E, a place $v_{E,2}|p$ of E^+ such that $E^+_{v_{E,2}} \cong F^+_{v_1}$, a finite set S_E of places of E^+ containing all of the places lying over p, and an odd, regular, polarized, weakly irreducible compatible system $(\{r_\lambda\}, \{\mu_{E,\lambda}\})$ of G_E-representations with associated p-adic representation (r, μ_E), such that $\overline{r}$ is reasonable, r is unramified outside of S_E, and $r|_{G_{E_{w_{E,2}}}}$ lies on C_{v_1} (which as usual implies in particular that $\mu_E|_{G_{E^+_{v_{E,2}}}} \simeq \mu|_{G_{F^+_{v_1}}}$ and that there is a polarized isomorphism $\overline{r}|_{G_{E_{w_{E,2}}}} \cong \overline{s}|_{G_{F_{w_1}}}$). The reason we use the distinct numbers 1 and 2 in the subscripts is that we shall ultimately construct a field M containing both E and F, and there will be no *a priori* relation between the primes of M^+ over v_1 in F^+ and $v_{E,2}$ in E^+.

Applying Lemma 4.2.4 to $\overline{s}$, we may find a finite Galois extension L/F of CM fields and an odd, regular, polarizable, weakly irreducible compatible system $(\{\widetilde{s}_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $\widetilde{s}$, such that $\overline{s}|_{G_L}$ is pleasant, $\widetilde{s}$ lifts $\overline{s}|_{G_L}$, and $\widetilde{s}$ is potentially diagonalizable at all places over p. (Here $\{\mu_\lambda\}$ is the compatible system containing the character μ.) This compatible system is our initial seed; we now begin constructing auxiliary compatible systems in order to apply our component swapping to ultimately construct the desired representation.

Applying Corollaries 4.2.7 and 4.2.8 to $(\{\widetilde{s}_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$, we can, after possibly further extending L, find an odd, regular, polarizable, weakly irreducible compatible system $(\{t_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation t, such that we have the following:

- $\overline{s}|_{G_L}$, $\overline{t}$ and $\overline{t}|_{G_{(LE)^{\mathrm{gal}}}}$ are pleasant. (The pleasantness of $\overline{s}|_{G_L}$ and $\overline{t}$, and also of $\overline{t}|_{L^{\mathrm{gal}}}$, is ensured by the very statements of Corollaries 4.2.7 and 4.2.8. The pleasantness of $G_{(LE)^{\mathrm{gal}}}$ can then be achieved by taking q in Corollaries 4.2.7 and 4.2.8 to be

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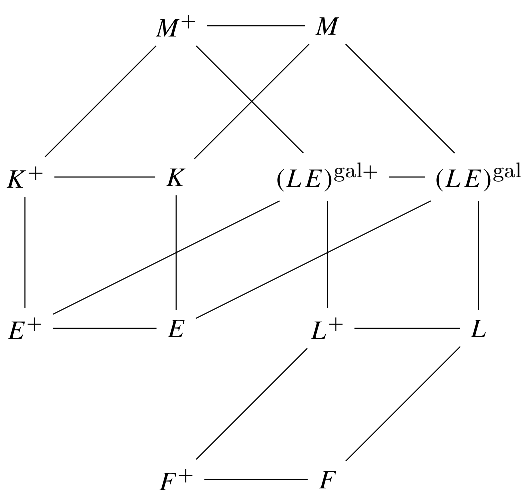
sufficiently large, and in particular large enough that $\mathrm{PGL}_n(\mathbf{F}_q)$ is simple and not isomorphic to any Jordan–Hölder factor of $\mathrm{Gal}(E^{\mathrm{gal}}/\mathbf{Q})$.)

- For each place v_L of L^+ lying over a place $v \in S$, we have a polarized isomorphism $\bar{t}|_{G_{L_{w_L}}} \cong \bar{s}|_{G_{L_{w_L}}}$.
- At each place v_L of L^+ lying over a place $v \in S$ which does not divide p, $t|_{G_{L_{w_L}}}$ lies on $C_v|_{L^+_{v_L}}$.
- $t|_{G_{L_{w_L}}}$ is potentially diagonalizable for all places $v_L|p$ of L^+.
- t is unramified outside of the places lying over S.

Now applying Corollaries 4.2.7 and 4.2.8 to $(\{r_\lambda\}, \{\mu_{E,\lambda}\})$, we may find a finite Galois extension K/E of CM fields and an odd, regular, polarizable, weakly irreducible compatible system $(\{u_\lambda\}, \{\nu_\lambda|_{G_{K^+}}\})$ of G_K-representations with associated p-adic representation $(u, \nu|_{G_{K^+}})$, such that we have the following:

- $\bar{u}|_{G_{(LK)^{\mathrm{gal}}}}$ and $\bar{t}|_{G_{(LK)^{\mathrm{gal}}}}$ are pleasant. (This can be achieved by noting that we may choose K/E to be linearly disjoint not only from $(LE)^{\mathrm{gal}}$ but also from the Galois closure of the splitting field of $\bar{t}$.)
- For each place v_K of K lying over a place in S_E, we have a polarized isomorphism $\bar{u}|_{G_{K_{v_K}}} \cong \bar{r}|_{G_{K_{v_K}}}$.
- $u|_{G_{K_{w_K}}}$ is potentially diagonalizable for all places $v_K|p$ of K^+ lying over places in S_E, except for the places $v_{K,2}$ lying over $v_{E,2}$, for which $u|_{G_{K_{w_{K,2}}}}$ lies on $C_{v_1}|_{K^+_{v_{K,2}}}$.
- $\nu|_{G_{K^+_{v_{K,2}}}} \simeq \mu_E|_{G_{K^+_{v_{K,2}}}}$ and $\mu_E|_{G_{E^+_{v_{E,2}}}} \simeq \mu|_{G_{F^+_{v_1}}}$. The first condition follows from the corollary we are invoking and the second condition was already assumed to be true.
- u is unramified outside of the places lying over S_E.

For ease of notation, we now write $M = (LK)^{\mathrm{gal}}$; note then that $M^{\mathrm{gal}} = M$. We also draw the following picture which represents the inclusions between various fields which occur in this argument:



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Choose a place $v_{M,2}$ above $v_{K,2}$ and $v_{E,2}$ in M^+. We now conjugate $\{u_\lambda\}$ by elements of $\mathrm{Gal}(M/\mathbf{Q})$. Because $\mathrm{Gal}(M/\mathbf{Q})$ acts transitively on the primes above p in M^+, there exists in particular a $\sigma \in \mathrm{Gal}(M/\mathbf{Q})$ with $\sigma v_{M,2} = v_{M,1}$ for any choice of prime $v_{M,1}$ above our fixed place v_1 of F^+. Note that, by construction, there will be a polarized isomorphism

$$\overline{u}^\sigma|G_{M_{w_{M,1}}} \simeq \overline{u}|G_{M_{w_{M,2}}} \simeq \overline{r}|G_{M_{w_{M,2}}} \simeq \overline{s}|G_{M_{w_{M,1}}} \simeq \overline{t}|G_{M_{w_{M,1}}},$$

where the second isomorphism comes from the construction of $\{u_\lambda\}$, and the third isomorphism was one of the defining properties of $\{r_\lambda\}$. Similarly, there is an identification of characters

$$\nu^\sigma|G_{M^+_{v_{M,1}}} \simeq \nu|G_{M^+_{v_{M,2}}} \simeq \mu|G_{M^+_{w_{M,1}}}.$$

We now apply Lemma 4.2.9 to $(\{t_\lambda|_{G_M}\}, \{\mu_\lambda|_{G_{M^+}}\})$ and $(\{u^\sigma_\lambda|_{G_M}\}, \{\nu_\lambda|_{G_{M^+}}\})$ relative to the set $T = \{v_{M,1}\}$. We deduce the existence of an odd, regular, polarized, weakly irreducible compatible system $(\{a_\lambda\}, \{\mu_\lambda|_{G_{M^+}}\})$ of G_M-representations with associated p-adic Galois representation $(a, \mu|_{G_{M^+}})$, with the following properties:

- $\overline{a} = \overline{t}|_{G_M}$ is pleasant.
- At the place $v_{M,1}$ of M^+ lying over v_1 in F^+, the representation $a|_{G_{M_{w_{M,1}}}}$ lies on $C_{v_1}|_{M^+_{v_{M,1}}}$.
- a is potentially diagonalizable at all places dividing p other than the places over $v_{M,1}$.
- At each place v_M of M^+ lying over a place $v \in S$ which does not divide p, $a|_{G_{M_{w_M}}}$ lies on $C_v|_{M^+_{v_M}}$.
- a is unramified outside of the places lying over S.

As above, we now conjugate $\{a_\lambda\}$ by elements of $\mathrm{Gal}(M/F) = \mathrm{Gal}(M^+/F^+)$. Because $\mathrm{Gal}(M^+/F^+)$ acts transitively on the primes above v_1 in M^+, for any such prime $\widetilde{v}_{M,1}|p$, there exists a $\sigma \in \mathrm{Gal}(M/F)$ with $\sigma\widetilde{v}_{M,1} = v_{M,1}$, and thus we deduce that there exists an odd, regular polarized, weakly irreducible compatible system $(\{a^\sigma_\lambda\}, \{\mu^\sigma_\lambda\})$ of G_M-representations with associated p-adic Galois representation (a^σ, μ^σ), with the following properties:

- $\widetilde{v}_{M,1}$ is a prime in M^+ above v_1 in F^+.
- $\overline{a}^\sigma$ is pleasant.
- $\mu^\sigma|_{G_{M^+_{\widetilde{v}_{M,1}}}} = \mu|_{G_{M^+_{\widetilde{v}_{M,1}}}}$, and there is a polarized isomorphism

  $$\overline{a}^\sigma|_{G_{M_{\widetilde{w}_{M,1}}}} = \overline{t}^\sigma|_{G_{M_{\widetilde{w}_{M,1}}}} = \overline{t}|_{G_{M_{\sigma\widetilde{w}_{M,1}}}} = \overline{t}|_{G_{M_{w_{M,1}}}} \cong \overline{t}|_{G_{M_{\widetilde{w}_{M,1}}}}$$

  for $\widetilde{w}_{M,1}|\widetilde{v}_{M,1}$ in M. This is because the representations $\overline{t}|_{G_{M_{\widetilde{w}_{M,1}}}}$ are all isomorphic to the restrictions of $\overline{s}|_{G_{F_{w_1}}}$ to $G_{M_{w_1}}$, and so do not depend on the choice of $\widetilde{v}_{M,1}$ above v_1 in M.
- $a^\sigma|_{G_{M_{\widetilde{w}_{M,1}}}}$ lies on $C_{v_1}|_{M^+_{\widetilde{v}_{M,1}}}$.
- a^σ is potentially diagonalizable at all places of M dividing p other than those lying over $\widetilde{v}_{M,1}$.

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We now apply Corollary 4.2.10 to the compatible systems $(\{t_\lambda|_{G_M}\}, \{\mu_\lambda|_{G_{M^+}}\})$ and $(\{a_\lambda^\sigma\}, \{\mu_\lambda^\sigma\})$, where we let σ range over a set of elements of $\mathrm{Gal}(M/F) = \mathrm{Gal}(M^+/F^+)$ such that $\sigma\widetilde{v}_{M,1} = v_{M,1}$, where $\widetilde{v}_{M,1}$ ranges exactly over the primes of M^+ above v_1 in F^+. We deduce the existence of an odd, regular, polarized, weakly irreducible compatible system $(\{b_\lambda\}, \{\mu_\lambda|_{G_{M^+}}\})$ of G_M-representations with associated p-adic Galois representation $(b, \mu|_{G_{M^+}})$, with the following properties:

- $\overline{b} = \overline{t}|_{G_M}$ is reasonable.
- At *every* place $\widetilde{v}_{M,1}|p$ of M^+ which lies over v_1, $b|_{G_{M_{\widetilde{w}_{M,1}}}}$ lies on $C_{v_1}|_{M^+_{\widetilde{v}_{M,1}}}$.
- b is potentially diagonalizable at all places of M dividing p and not lying over v_1.
- At each place v_M of M^+ lying over a place $v \in S$ which does not divide p, $b|_{G_{M_{w_M}}}$ lies on $C_v|_{M^+_{v_M}}$.
- b is unramified outside of the places lying over S.

We now use this deformation of $\overline{t}|_{G_M}$ to descend to a deformation of $\overline{t}$ over G_L. Namely, applying Lemma 4.1.8, we deduce the existence of an odd, regular, polarizable, weakly irreducible compatible system $(\{c_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic representation $(c, \mu|_{G_{L^+}})$, such that we have the following:

- $\overline{c} \cong \overline{t}$.
- At each place v_L of L^+ lying over a place $v \in S$ which does not divide p, $c|_{G_{L_{w_L}}}$ lies on $C_v|_{L^+_{v_L}}$.
- At each place $v_{L,1}$ of L^+ lying over v_1, $c|_{G_{L_{w_{L,1}}}}$ lies on $C_{v_1}|_{L^+_{v_{L,1}}}$.
- $c|_{G_{L_{w_L}}}$ is potentially diagonalizable for all places $v_L|p$ not lying over v_1.
- c is unramified outside of the places lying over S.

Hence we have succeeded in finding deformations of a representation $\overline{t}$ (which looks locally like $\overline{s}|_{G_L}$ at each $v|p$ but is globally different) which lies on the desired component at the places lying over v_1 and is potentially diagonalizable at all other primes.

We now use local swapping to find a corresponding deformation of $\overline{s}|_{G_L}$ which has the correct behaviour at the places lying over v_1 and is potentially diagonalizable at all other primes. That is, by applying Lemma 4.2.9 to $(\{\widetilde{s}_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ and $(\{c_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ (taking the subset T of that lemma to be the set of all places lying over p), we deduce the existence of an odd, regular, polarized, weakly irreducible compatible system $(\{d_\lambda\}, \{\mu_\lambda|_{G_{L^+}}\})$ of G_L-representations with associated p-adic Galois representation $(d, \mu|_{G_{L^+}})$, such that we have the following:

- $\overline{d} \cong \overline{s}|_{G_L}$.
- At each place v_L of L^+ lying over a place $v \in S$ which does not divide p, $d|_{G_{L_{w_L}}}$ lies on $C_v|_{L^+_{v_L}}$.
- At each place $v_{L,1}$ of L^+ lying over v_1, $d|_{G_{L_{w_{L,1}}}}$ lies on $C_{v_1}|_{L^+_{v_{L,1}}}$.
- $d|_{G_{L_{w_L}}}$ is potentially diagonalizable for all places $v_L|p$ not lying over v_1.
- d is unramified outside of the places lying over S.

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We now descend from L to F: Applying Lemma 4.1.8, we construct an odd, regular, polarized, weakly irreducible compatible system $(\{e_\lambda\}, \{\mu_\lambda\})$ of G_F-representations with associated p-adic representation (e, μ), such that we have the following:

- $\overline{e} \cong \overline{s}$.
- At each place $v \in S$ which does not divide p, $e|_{G_{F_w}}$ lies on C_v.
- $e|_{G_{F_{w_1}}}$ lies on C_{v_1}.
- $e|_{G_{F_w}}$ is potentially diagonalizable for all places $v|p$ of F^+ other than v_1.
- e is unramified outside of the places lying over S.

Since v_1 was arbitrary, there exists a compatible system with these properties for each choice of v_1. Applying Corollary 4.2.10 to these compatible systems, we obtain the required compatible system $(\{s_\lambda\}, \{\mu_\lambda\})$. □

As an immediate consequence, we have the following potential automorphy theorem.

Corollary 4.2.12. *Assume that either n is odd or $n = 2$. Let F be a CM field, and let (s, μ) be a polarized representation, where*

$$s : G_F \to \mathrm{GL}_n(\overline{\mathbf{Q}}_p)$$

is odd and ramified at only finitely many primes. Suppose that $\overline{s}$ is reasonable. Let ρ be the corresponding prolongation of s, and assume that $\rho|_{G_{F_v^+}}$ is globally realizable for each $v|p$. Then (s, μ) is potentially automorphic.

Proof. Let $F^{(\mathrm{avoid})}/F$ be a finite Galois extension. By Theorems 4.2.11 and 2.1.16, together with Lemma 2.1.8, there is a finite Galois extension of CM fields L/F, linearly disjoint from $F^{(\mathrm{avoid})}/F$ and a polarizable regular algebraic cuspidal automorphic representation of $\mathrm{GL}_n(\mathbf{A}_L)$ such that $\overline{r}_p(\pi) \cong \overline{s}|_{G_L}$ is pleasant, $s|_{G_L}$ is only ramified at split primes, and for each place v of L^+, $r_p(\pi)|_{G_{L_v}}$ lies on the component determined by $s|_{G_{L_v}}$. Then $s|_{G_L}$ is automorphic by Theorem 1.4.26. □

Finally, as promised in Remark 2.1.13, we show that 'potentially globally realizable' representations are globally realizable.

Corollary 4.2.13. *Assume that either n is odd or $n = 2$. A component C for $\overline{\rho} : G_K \to \mathcal{G}_n(\mathbf{F})$ is globally realizable if and only if there exists a finite extension L/K such that $C|_L$ is globally realizable.*

Proof. If C is globally realizable, then it is easy to see from the definitions that $C|_L$ is globally realizable, by choosing an appropriate extension of CM fields E/F. Conversely, suppose that $C|_L$ is globally realizable. Let $\mu_{\overline{\rho}}$ be the multiplier character for C, so that $\mu_{\overline{\rho}}|_{G_L}$ is the multiplier character for $C|_L$. By the definition of global realizability, $\mu_{\overline{\rho}}|_{G_L}$ is the restriction of a de Rham character of a totally real field, so it is a power of the cyclotomic character times a finite order character; so the same is true of $\mu_{\overline{\rho}}$.

Exactly as in the proof of [20, Proposition A2], by [10, Proposition 3.2] (see also [35, Theorem 1.2]) we can find a CM field F with maximal totally real subfield F^+, such that we have $F_v^+ \cong K$ for all $v|p$, and a representation $\overline{\varrho} : G_{F^+} \to \mathcal{G}_n(\mathbf{F})$ with multiplier $\overline{\mu}$, such

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that for each $v|p$, $\overline{\varrho}|_{G_{F_v^+}} \cong \overline{\rho}$, and $\overline{\varrho}|_{G_F}$ is reasonable. In particular, we have $\overline{\mu}|_{G_{F_v^+}} = \overline{\mu}_{\overline{\rho}}$ for all $v|p$.

We claim that we can find a (necessarily de Rham) lift μ of $\overline{\mu}$ which satisfies $\mu|_{G_{F_v^+}} = \mu_{\overline{\rho}}$ for all $v|p$. Indeed, as explained above, we can write $\mu_{\overline{\rho}} = \varepsilon^r \chi_{\overline{\rho}}$ for some integer r and some finite order character $\chi_{\overline{\rho}}$. By [16, Lemma 4.1.1], there exists a finite order global character χ such that $\chi|_{G_{F_v^+}} = \chi_{\overline{\rho}}$ for each $v|p$. By construction, it follows that $\varepsilon^r \chi|_{G_{F_v^+}} = \mu_{\overline{\rho}}$ for all $v|p$, and thus

$$\overline{\varepsilon^r \chi} = \overline{\mu} \cdot \psi$$

for some finite order residual character ψ which is trivial for all $v|p$. But then $\mu = \varepsilon^r \chi \widetilde{\psi}^{-1}$ has the required property, where $\widetilde{\psi}$ is the Teichmüller lift of ψ.

Let E/F be a finite Galois extension of CM fields, linearly disjoint from $\overline{F}^{\ker \overline{\varrho}}(\zeta_p)$ such that for each place $v_E|p$ of E^+, we have $E^+_{v_E} \cong L$. The result follows from Theorem 4.2.11 (applied with F equal to our E, and $\overline{s}$ our $\overline{\varrho}|_{G_E}$), together with Lemma 4.1.8 (applied with $\overline{a}$ there being our $\overline{\varrho}|_{G_F}$, and L our E). □

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