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Fast-moving finite and infinite trains of solitons for nonlinear Schrödinger equations

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We study *infinite soliton trains* solutions of nonlinear Schrödinger equations, i.e. solutions behaving as the sum of infinitely many solitary waves at large time. Assuming the composing solitons have sufficiently large relative speeds, we prove the existence and uniqueness of such a soliton train. We also give a new construction of multi-solitons (i.e. finite trains) and prove uniqueness in an exponentially small neighbourhood, and we consider the case of solutions composed of several solitons and kinks (i.e. solutions with a non-zero background at infinity).

Keywords: soliton train; multi-soliton solution; multi-kink solution; nonlinear Schrödinger equation

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1. Introduction

We consider the following nonlinear Schrödinger (NLS) equation:

$$i\partial_t u + \Delta u = -g(|u|^2)u =: -f(u), \tag{1.1}$$

where $u = u(t, x)$ is a complex-valued function on $\mathbb{R} \times \mathbb{R}^d$, $d \geqslant 1$.

The aim of this paper is to construct special families of solutions to the energy-subcritical NLS equation (1.1). We shall look for *infinite soliton train*, *multi-soliton* and *multi-kink* solutions.

Recall that it is generally expected that global solutions to nonlinear dispersive equations such as the NLS equation eventually decompose at large time as a sum of solitons plus a scattering remainder (the *soliton resolution conjecture*). Except for the specific case of integrable equations, such results are usually out of reach (see nevertheless the recent breakthrough on the energy-critical wave equation in [10]). In the case of NLS equations, multi-solitons can be constructed via the inverse scattering transform in the integrable case ($d = 1$, $f(u) = |u|^2 u$). In non-integrable frameworks, multi-solitons have been known to exist since the pioneering work of Merle [21] (see § 1.2 for more details on the existing results of multi-solitons). The multi-solitons constructed up to now were made of a finite number of solitons and there was little evidence of the possibility of existence of infinite trains of solitons (note nevertheless the result [13] in the integrable case). The existence of such infinite soliton trains is, however, important, as they may provide examples or counter-examples of solutions with borderline behaviours (as is the case for the Korteweg–de Vries (KdV) equation; see [18]). In this paper, we show the existence of such infinite soliton trains for power nonlinearities. It turns out that our strategy is very flexible and allows us to prove many existence and uniqueness results of multi-soliton and multi-kink solutions for generic nonlinearities. In the remainder of this introduction, we state our main results on infinite trains (§ 1.1), multi-solitons (§ 1.2) and multi-kinks (§ 1.3) and give a summary of the strategy of the proofs (§ 1.4).

1.1. Infinite soliton trains

Our first main result is on the construction of a solution to (1.1) behaving like a sum of infinitely many solitons at large time. For this purpose we have to use scale invariance and work with the power nonlinearity $f_1(u) = |u|^\alpha u$, $0 < \alpha < \alpha_{\max}$, $\alpha_{\max} = +\infty$ for $d = 1, 2$ and $\alpha_{\max} = 4/(d-2)$ for $d \geqslant 3$. Let $\Phi_0 \in H^1(\mathbb{R}^d)$ be a fixed bound state that solves the elliptic equation

$$-\Delta\Phi_0 + \Phi_0 - |\Phi_0|^\alpha \Phi_0 = 0.$$

For $j \geqslant 1$, $\omega_j > 0$ (*frequency*), $\gamma_j \in \mathbb{R}$ (*phase*), $v_j \in \mathbb{R}^d$ (*velocity*), define a soliton $\tilde{R}_j$ by

$$\tilde{R}_j(t,x) := \exp(\mathrm{i}(\omega_j t - \tfrac{1}{4}|v_j|^2 + \tfrac{1}{2}v_j \cdot x + \gamma_j))\omega_j^{1/\alpha}\Phi_0(\sqrt{\omega_j}(x - v_j t)). \tag{1.2}$$

We consider the following soliton train:

$$R_\infty = \sum_{j=1}^{\infty} \tilde{R}_j. \tag{1.3}$$

Since (1.1) is a nonlinear problem, the function $R_\infty = R_\infty(t,x)$ is no longer a solution in general. Nevertheless, we shall show that in the vicinity of R_∞ one can still find a solution u to (1.1), which we refer to as an *infinite soliton train*. More precisely, the solution u to (1.1) is defined on $[T_0, +\infty)$ for some $T_0 \in \mathbb{R}$ and such that

$$\lim_{t\to+\infty} \|u - R_\infty\|_{X([t,\infty)\times\mathbb{R}^d)} = 0. \tag{1.4}$$

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Here $\|\cdot\|_{X([t,\infty)\times\mathbb{R}^d)}$ is some space-time norm measured on the slab $[t,\infty)\times\mathbb{R}^d$. A simple example is $X = L_t^\infty L_x^2$, in which case one can replace (1.4) by the equivalent condition

$$\lim_{t\to+\infty}\|u(t)-R(t)\|_{L^2}=0.$$

However, the definition (1.4) is more flexible, as it allows general Strichartz spaces (see (2.2)).

The main idea is that, in the energy-subcritical setting, all solitons have exponential tails (see (1.13)). When their relative speed is large, these travelling solitons are well separated and have very small overlaps that decay exponentially in time. At such high velocity and exponential separation, one does not need fine spectral details, and the whole argument can be carried out as a perturbation around the desired profile (e.g. the soliton series R_∞) in a well-chosen function space. As our proof is based on contraction estimates, the uniqueness follows immediately, albeit in a very restrictive function class.

We require that the parameters (ω_j, v_j) of the train satisfy the following assumption.

Assumption 1.1.

- *Integrability:* there exists $r_1 \geqslant 1$, $\frac{1}{2}d\alpha < r_1 < \alpha+2$, such that

$$A_\omega := \sum_{j=1}^{\infty}\omega_j^{1/\alpha-d/(2r_1)} < \infty. \tag{1.5}$$

- *High relative speeds:* the solitons travel sufficiently fast: there exists a constant $v_\star > 0$ such that

$$\sqrt{\min\{\omega_j,\omega_k\}}(|v_k-v_j|) \geqslant v_\star \quad \forall j\neq k. \tag{1.6}$$

Since R_∞ may be badly localized, we seek an infinite soliton train solution to (1.1) in the form $u = R_\infty + \eta$, where η satisfies the perturbation equation

$$\mathrm{i}\partial_t\eta + \Delta\eta = -f(R_\infty+\eta) + \sum_{j=1}^{\infty} f(\tilde{R}_j).$$

In Duhamel formulation, the perturbation equation for η reads

$$\eta(t) = -\mathrm{i}\int_t^\infty \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}\biggl(f(R_\infty+\eta) - \sum_{j=1}^{\infty} f(\tilde{R}_j)\biggr)\,\mathrm{d}\tau \quad \forall t\geqslant 0. \tag{1.7}$$

The following theorem gives the existence and uniqueness of the solution η to (1.7).

Theorem 1.2 (existence of an infinite soliton train solution). *Consider (1.1) with $f(u)=|u|^\alpha u$ satisfying $0<\alpha<\alpha_{\max}$. Let R_∞ be given as in (1.3), with parameters $\omega_j > 0$, $\gamma_j \in \mathbb{R}$ and $v_j\in\mathbb{R}^d$ for $j\in\mathbb{N}$, which satisfy assumption 1.1. There exist constants $C>0$, $c_1>0$ and $v_\sharp \gg 1$ such that (see (1.6)) if $v_\star > v_\sharp$, then there* *exists a unique solution* $\eta \in S([0,\infty))$ *(see (2.2) for the definition of Strichartz space) to (1.7) satisfying*

$$\|\eta\|_{S([t,\infty))} + \|\eta(t)\|_{L^{\alpha+2}} \leqslant C\mathrm{e}^{-c_1 v_\star t} \quad \forall t \geqslant 0. \tag{1.8}$$

Remark 1.3. By using theorem 1.2 and lemma 4.1, one can justify the existence of a solution $u = R_\infty + \eta$ satisfying (1.1) in the distributional sense. The uniqueness of such a solution is only proven for the perturbation η satisfying (1.7) and (1.8). In the mass-subcritical case $0 < \alpha < 4/d$, the soliton train R_∞ is in the Lebesgue space $C_t^0 L_x^2 \cap L_{tx}^\infty$, and one can show that the solution $u = R_\infty + \eta$ can be extended to all $\mathbb{R} \times \mathbb{R}^d$ and satisfies

$$u \in C_t^0 L_x^2(\mathbb{R} \times \mathbb{R}^d) \cap L_{t,\mathrm{loc}}^{2(d+2)/d} L_x^{2(d+2)/d}(\mathbb{R} \times \mathbb{R}^d)$$

(see (2.1)). Hence, it is a localized solution in the usual sense. In the mass-supercritical case $4/d \leqslant \alpha < \alpha_{\max}$, the soliton train $R_\infty = \sum_{j=1}^\infty \tilde{R}_j$ is no longer in L^2 since each composing piece $\tilde{R}_j$ has $O(1)$ L^2-norm. Nevertheless, we shall still build a regular solution to (1.7) since R_∞ has Lebesgue regularity $L_t^\infty L_x^{(d\alpha/2+)} \cap L_{tx}^\infty$, which is enough for the perturbation argument to work. We stress that in this case the solution η is only defined on $[0,\infty) \times \mathbb{R}^d$ and scatters forwards in time in L^2.

Remark 1.4. Typically, the parameters (ω_j, v_j) are chosen in the following order: first we take (ω_j) satisfying (1.5); then we inductively choose v_j such that the condition (1.6) is satisfied. For example, for $j \geqslant 1$, one can take $\omega_j = 2^{-j}$ and $v_j = 2^j \bar{v}$ for $\bar{v} \in \mathbb{R}^d$, $|\bar{v}| = v_\star$. Note that when $0 < \alpha < 4/d$ (mass-subcritical case) we can choose $r_1 \leqslant 2$. The soliton train is then in $L_t^\infty L_x^2$. We require $\frac{1}{2}d\alpha < r_1$ so that the exponent in (1.5) is positive. The condition $r_1 < \alpha + 2$ will be needed to show (4.2) in lemma 4.1.

Remark 1.5. Note that we did not introduce initial positions into the definition of $\tilde{R}_j$, so each soliton starts centred at 0. With some minor modifications, our construction can also work for the general case with the solitons starting centred at various x_j. For simplicity of presentation we shall not state the general case here.

Remark 1.6. Certainly theorem 1.2 can hold in more general situations. For example, instead of taking a fixed profile Φ_0 in (1.2), one can draw Φ_0 from a finite set of profiles $\mathcal{A} = \{\Phi_0^1, \dots, \Phi_0^K\}$, where each Φ_0^j is a bound state.

Remark 1.7. The rate of spatial decay of multi-solitons is still an open question in the NLS case (for KdV it is partly known: multi-solitons decay exponentially on the right). In theorem 1.2, the soliton train profile R_∞ around which we build our solution has only a polynomial spatial decay, which is non-uniform in time. Hence, we expect the solution $u = R_\infty + \eta$ to have the same decay.

1.2. Multi-solitons

From now on, we work with a generic nonlinearity and just assume that $f(u) = g(|u|^2)u$, where the function $g\colon [0,\infty) \to \mathbb{R}$ obeys some Hölder conditions mimicking the usual power-type nonlinearity. Precisely,

$$g \in C^0([0,\infty), \mathbb{R}) \cap C^2((0,\infty), \mathbb{R}), \quad g(0) = 0$$

and

$$|sg'(s)| + |s^2 g''(s)| \leqslant C(s^{\alpha_1} + s^{\alpha_2}) \quad \forall s > 0, \tag{1.9}$$

where $C > 0$, $0 < \alpha_1 \leqslant \alpha_2 < \frac{1}{2}\alpha_{\max}$.

A typical example is $g(s) = s^\alpha$ for some $0 < \alpha < \frac{1}{2}\alpha_{\max}$. A useful example to keep in mind is the combined nonlinearity $g(s) = s^{\alpha_1} - s^{\alpha_2}$ for some $0 < \alpha_1 < \alpha_2 < \frac{1}{2}\alpha_{\max}$. Other examples can easily be constructed. Henceforth we shall assume that $f(u) = g(|u|^2)u$ satisfies (1.9). In this case the corresponding nonlinearity $f(u)$ is usually called energy-subcritical, since there are lower bounds of the lifespans of the H^1 local solutions that depend only on the H^1-norm (not the profile) of initial data (cf. [5, 11]). The condition (1.9) is a natural generalization of the pure power nonlinearities. For much of our analysis it can be replaced by the weaker condition that $g(s)$ and $sg'(s)$ are Hölder continuous with suitable exponents. However, (1.9) is fairly easy to check and it suffices for most applications.

We now give a definition of a solitary wave that is slightly more general than (1.2). Given a set of parameters $\omega_0 > 0$ (*frequency*), $\gamma_0 \in \mathbb{R}$ (*phase*), $x_0, v_0 \in \mathbb{R}^d$ (*position* and *velocity*), a *solitary wave*, or a *soliton*, is a solution to (1.1) of the form

$$R_{\Phi_0,\omega_0,\gamma_0,x_0,v_0} := \Phi_0(x - v_0 t - x_0)\exp(\mathrm{i}(\tfrac{1}{2}v_0 \cdot x - \tfrac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)), \tag{1.10}$$

where $\Phi_0 \in H^1(\mathbb{R}^d)$ solves the elliptic equation

$$-\Delta\Phi_0 + \omega_0\Phi_0 - f(\Phi_0) = 0. \tag{1.11}$$

A non-trivial H^1 solution to (1.11) is usually called a *bound state*. Compared with (1.2), the main difference is that we do not use the parameter ω_j to rescale the solitons.

The existence of bound states is guaranteed (see [1]) if we assume, in addition to (1.1), that there exists $s_0 > 0$, such that

$$G(s_0) := \int_0^{s_0} g(\tilde{s})\,\mathrm{d}\tilde{s} > \omega_0 s_0. \tag{1.12}$$

Note that the condition (1.12) makes the nonlinearity focusing.

All bound states are exponentially decaying (see, for example, [3, § 3.3]), i.e.

$$\mathrm{e}^{\sqrt{\omega}|x|}(|\Phi_0| + |\nabla\Phi_0|) \in L^\infty(\mathbb{R}^d) \quad \text{for all } 0 < \omega < \omega_0. \tag{1.13}$$

A *ground state* is a bound state that minimizes among all bound states the *action*

$$S(\Phi_0) = \tfrac{1}{2}\|\nabla\Phi_0\|_2^2 + \tfrac{1}{2}\omega_0\|\Phi_0\|_2^2 - \frac{1}{2}\int_{\mathbb{R}^d} G(|\Phi_0|^2)\,\mathrm{d}x.$$

The ground state is usually unique modulo symmetries of the equation (see, for example, [17] for precise conditions on the nonlinearity ensuring uniqueness of the ground state). If $d \geqslant 2$, there exist infinitely many other solutions called *excited states* (see [1, 2] for more on ground states and excited states). The corresponding solitons are usually termed *ground-state solitons* (respectively, *excited-state solitons*).

A *multi-soliton* is a solution to (1.1) that, roughly speaking, looks like the sum of N solitons. To fix notation, let (see (1.10))

$$R(t,x) = \sum_{j=1}^{N} R_{\Phi_j,\omega_j,\gamma_j,x_j,v_j}(t,x) =: \sum_{j=1}^{N} R_j(t,x), \tag{1.14}$$

where each R_j is a soliton made from some parameters $(\omega_j,\gamma_j,x_j,v_j)$ and bound state Φ_j (we assume that (1.12) holds true for all ω_j).

If each Φ_j in (1.14) is a ground state, then the corresponding multi-soliton is called a *ground-state multi-soliton*. If at least one Φ_j is an excited state, we call it an *excited-state multi-soliton*.

We now review in more detail some known results on multi-solitons. Most results are on the pure power nonlinearity $f(u) = |u|^\alpha u$ with $0 < \alpha < \alpha_{\max}$ and ground states. If $\alpha = 4/d$ (respectively, $\alpha < 4/d$, $\alpha > 4/d$), then (1.1) is called (L^2) mass-critical (respectively, mass-subcritical, mass-supercritical). In the integrable case $d = 1$, $\alpha = 2$, Zakharov and Shabat [25] derived an explicit expression of multi-solitons by using the inverse scattering transform. For the mass-critical NLS equation, which is non-integrable in higher dimensions, Merle [21, corollary 3] constructed a solution blowing up at exactly N points at the same time, which gives a multi-soliton after a pseudo-conformal transformation. In the mass-subcritical case, the ground-state solitary waves are stable. Assuming the composing solitary waves R_j are ground states and have different velocities (i.e. $v_j \neq v_k$ if $j \neq k$ in (1.14)), Martel and Merle [19] proved the existence of an H^1 ground-state multi-soliton $u \in C([T_0,\infty), H^1)$ such that

$$\left\| u(t) - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} \leqslant C\mathrm{e}^{-\beta\sqrt{\omega_\star}v_\star t} \quad \forall t \geqslant T_0, \tag{1.15}$$

for some constant $\beta > 0$, where $T_0 \in \mathbb{R}$ is large enough, and the minimal relative velocity $v_\star$ and the minimal frequency $\omega_\star$ are defined by

$$v_\star := \min\{|v_j - v_k| \colon 1 \leqslant j \neq k \leqslant N\}, \tag{1.16}$$

$$\omega_\star = \min\{\omega_j,\ 1 \leqslant j \leqslant N\}. \tag{1.17}$$

Martel and Merle [19] also considered a general energy-subcritical nonlinearity $f(u) = g(|u|^2)u$ with $g \in C^1$, $g(0) = 0$ and satisfying $\|s^{-\alpha}g'(s)\|_{L^\infty_s(s\geqslant 1)} < \infty$ for some $0 < \alpha < \frac{1}{2}\alpha_{\max}$. Assuming a nonlinear stability condition around the ground state (see [19, eqn (16)]), they proved the existence of an H^1 ground-state multi-soliton satisfying the same estimate (1.15).

In [9], Côte *et al.* considered the mass-supercritical NLS equation ($f(u) = |u|^\alpha u$ with $4/d < \alpha < \alpha_{\max}$). Assuming the ground-state solitons R_j have different velocities, Côte *et al.* constructed an H^1 ground-state multi-soliton u satisfying (1.15). This result was sharpened in dimension 1 by Combet, who showed in [7] the existence of an N-parameter family of multi-solitons.

In [8], Côte and Le Coz considered the general energy-subcritical NLS equation with $f(u) = g(|u|^2)u$ satisfying assumptions similar to (1.9) and (1.12). Assuming the solitary waves R_j are excited states and have large relative velocities,

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i.e. assuming

$$v_\star \geqslant v_\sharp > 0$$

for $v_\sharp$ large enough, Côte and Le Coz constructed an excited-state multi-soliton $u \in C([T_0, \infty), H^1)$ for $T_0 \in \mathbb{R}$ large enough, which also satisfies (1.15).

The main strategy in [8,9,19,21] is the following: one takes a sequence of approximate solutions u_n solving (1.1) with final data $u_n(T_n) = R(T_n)$, $T_n \to \infty$; by using local conservation laws and coercivity of the Hessian (this has to be suitably modified in certain cases; see [8]), one derives uniform H^1 decay estimates of u_n on the time interval $[T_0, T_n]$, where T_0 is independent of n; the multi-soliton is then obtained after a compactness argument. We should point out that the uniqueness of multi-solitons is still left open by the above analysis (nevertheless, see [7,8] for the existence of one-parameter and N-parameter families of multi-solitons). Under restrictive assumptions on the nonlinearity (e.g. high regularity or a flatness assumption at the origin) and a high-relative-speed hypothesis, the stability of multi-solitons was obtained in [20,22–24], and instability in [8]. See also remark 1.12.

In this section we give new constructions of multi-solitons. We work in the context of the energy-subcritical problem (1.1) with $f(u)$ satisfying (1.9) and (1.12) We shall focus on *fast-moving* solitons, i.e. the minimum relative velocity $v_\star$ defined in (1.16) is sufficiently large. The composing solitons are in general bound states, which can be either ground states or excited states. In our next two results, we recover and improve the result from [8, theorem 1] in various settings. The improvements here are the lifespan and uniqueness. As for the infinite train, our new proof relies on a contraction argument around the desired profile. We begin with the pure power nonlinearity case.

Theorem 1.8 (existence, uniqueness of multi-solitons; power nonlinearity case). *Consider (1.1) with $f(u) = |u|^\alpha u$ satisfying $0 < \alpha < \alpha_{\max}$. Let R be the same as in (1.14) and define $v_\star$ as in (1.16). There exist constants $C > 0$, $c_1 > 0$ and $v_\sharp \gg 1$ such that if $v_\star > v_\sharp$, then there exists a unique solution $u \in C([0, \infty), H^1)$ to (1.1) satisfying*

$$\mathrm{e}^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + \mathrm{e}^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \leqslant C \quad \forall t \geqslant 0.$$

Here $c_2 = c_1 \min(1, \alpha) \leqslant c_1$. In particular $\|u(t) - R(t)\|_{H^1} \leqslant C\mathrm{e}^{-c_2 v_\star t}$.

Remark 1.9. As already mentioned, theorem 1.8 is a slight improvement of [8, theorem 1]. Here the multi-soliton is constructed on the time interval $[0, \infty)$, whereas in [8] this was done on $[T_0, \infty)$ for some $T_0 > 0$ large. In particular, we do not have to wait for the interactions between the solitons to be small to have the existence of our multi-soliton. However, we have no control over the constant C, so at small times our multi-soliton may be very far away from the sum of solitons. The uniqueness of solutions is a subtle issue; see remark 1.12.

The next result concerns the existence and uniqueness of multi-solitons in the general nonlinearity case, $f(u)$.

Theorem 1.10. *Consider (1.1) with $f(u) = g(|u|^2)u$ satisfying (1.9) and (1.12). Let R be the same as in (1.14) and define $v_\star$ as in (1.16). There exist constants*

$C > 0$, $c_1 > 0$, $c_2 > 0$, $T_0 \gg 1$ and $v_\sharp \gg 1$ such that if $v_\star > v_\sharp$, then there is a unique solution $u \in C([T_0,\infty), H^1)$ to (1.1) satisfying

$$\mathrm{e}^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + \mathrm{e}^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \leqslant C \quad \forall t \geqslant T_0.$$

Remark 1.11. Unlike theorem 1.8, the solution in theorem 1.10 exists only for $t \geqslant T_0$ with T_0 sufficiently large. To take $T_0 = 0$, our method requires extra conditions. For such results see § 6. We can extend theorem 1.17 similarly.

Remark 1.12. In theorems 1.8 and 1.10, the uniqueness of the multi-soliton solution holds in a very restrictive function class whose Strichartz-norm decays as $\mathrm{e}^{-c_1 v_\star t}$. A natural question is whether uniqueness holds in a wider setting. In general, this is a very subtle issue and in some cases one cannot get away with the exponential decay condition. Côte and Le Coz [8] considered the case when one of the composing solitons, say R_1, is unstable. Assuming $g \in C^\infty$ (see (1.1)) and the operator $L = -\mathrm{i}\Delta + \mathrm{i}\omega_1 - \mathrm{i}df(\Phi_1)$ has an eigenvalue $\lambda_1 \in \mathbb{C}$ with $\rho := \mathrm{Re}(\lambda_1) > 0$, they constructed a one-parameter family of multi-solitons $u_a(t)$ such that, for some $T_0 = T_0(a) > 0$,

$$\left\| u_a(t) - \sum_{j=1}^{N} R_j(t) - aY(t) \right\|_{H^1(\mathbb{R}^d)} \leqslant C\mathrm{e}^{-2\rho t} \quad \forall t \geqslant T_0.$$

Here $Y(t)$ is a non-trivial solution of the linearized flow around R_1, and $\mathrm{e}^{\rho t}\|Y(t)\|_{H^1}$ is periodic in t. This instability result shows that the exponential decay condition in the uniqueness statement cannot be removed in general for NLS equations with unstable solitary waves.

1.3. Multi-kinks

In this subsection, we push our approach further and attack the problem of the existence of multi-kinks, i.e. solutions built upon solitons and their non-localized counterparts, the kinks. Before stating our result, let us first mention some related works. When its solutions are considered with a non-zero background (i.e. $|u| \to \nu \neq 0$ at $\pm\infty$), the NLS equation (1.1) is often referred to as the Gross–Pitaevskii equation. For general nonlinearities, Chiron [6] investigated the existence of travelling-wave solutions with a non-zero background and showed that various types of nonlinearities can lead to a full zoology of profiles for the travelling waves. In the case of the 'classical' Gross–Pitaevskii equation, i.e. when $f(u) = (1 - |u|^2)u$ and solutions verify $|u| \to 1$ at infinity, the profiles of the travelling kink solutions $K(t,x) = \phi_c(x - ct)$ are explicitly known and given for $|c| < \sqrt{2}$ by the formula

$$\phi_c(x) = \sqrt{\frac{2 - c^2}{2}} \tanh\left(\frac{x\sqrt{2 - c^2}}{2}\right) + \mathrm{i}\frac{c}{\sqrt{2}}$$

with $\omega = 0$. (in particular, one can see that the limits at $-\infty$ and $+\infty$ are different, thus justifying the name 'kink'). In [4], Béthuel *et al.* proved the forward-in-time stability of a profile composed of several kinks travelling at different speeds. Note that, due to the non-zero background of the kinks, the profile cannot be simply taken as a sum of the kinks and one has to rely on another formulation of the Gross–Pitaevskii equation to define a multi-kink properly.

The main differences between our analysis and the works mentioned above are, first, that our kinks have a zero background on one side and a non-zero one on the other side, and second, that, due to the Galilean transform used to give a speed to the kink, our kinks have infinite energy (due to the non-zero background, the rotation in phase generated by the Galilean transform is no longer killed by the decay in the modulus). In particular, this prevents us from using energy methods, as was the case for multi-solitons in [8, 9, 19] or multi-kinks in [4].

We place ourselves in dimension $d = 1$. In such a context and under suitable assumptions on the nonlinearity f, (1.1) admits kink solutions. More precisely, given $\gamma, \omega, v, x_0 \in \mathbb{R}$, what we call a *kink* solution of (1.1) (or *half-kink*) is a function $K = K(t, x)$ defined similarly as a soliton by

$$K(t,x) := \exp(\mathrm{i}(\tfrac{1}{2}vx - \tfrac{1}{4}|v|^2 t + \omega t + \gamma))\phi(x - vt - x_0),$$

but where ϕ satisfies the profile equation on $\mathbb{R}$ *with a non-zero boundary condition* at one side of the real line, denoted by $\pm\infty$ and zero boundary condition on the other side (denoted by $\mp\infty$):

$$\left.\begin{aligned} -\phi'' + \omega\phi - f(\phi) = 0, \\ \lim_{x\to\mp\infty} \phi(x) = 0, \qquad \lim_{x\to\pm\infty} \phi(x) \neq 0. \end{aligned}\right\} \tag{1.18}$$

The existence of half-kinks is granted by the following proposition.

PROPOSITION 1.13. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be a C^1 function with $f(0) = 0$ and define $F(s) := \int_0^s f(t)\,\mathrm{d}t$. For $\omega \in \mathbb{R}$, let*

$$\zeta(\omega) := \inf\{\zeta > 0,\ F(\zeta) - \tfrac{1}{2}\omega\zeta^2 = 0\}$$

and assume that there exists $\omega_1 \in \mathbb{R}$ such that

$$\zeta(\omega_1) > 0, \quad f'(0) - \omega_1 < 0, \quad f(\zeta(\omega_1)) - \omega_1\zeta(\omega_1) = 0. \tag{1.19}$$

Then, for $\omega = \omega_1$, there exists a kink profile solution $\phi \in \mathcal{C}^2(\mathbb{R})$ of (1.18), i.e. ϕ is unique (up to translation), positive and satisfies $\phi > 0$, $\phi' > 0$ on $\mathbb{R}$ and the boundary conditions

$$\lim_{x\to-\infty} \phi(x) = 0, \qquad \lim_{x\to+\infty} \phi(x) = \zeta(\omega_1) > 0. \tag{1.20}$$

If, in addition,

$$f'(\zeta(\omega_1)) - \omega_1 < 0,$$

then for any $0 < \delta < \omega_1 - \max\{f'(0), f'(\zeta(\omega_1))\}$ there exists $C > 0$ such that

$$|\phi'(x)| + |\phi(x)\mathbf{1}_{\{x<0\}}| + |(\zeta_1(\omega_1) - \phi(x))\mathbf{1}_{\{x>0\}}| \leqslant C\mathrm{e}^{-\delta|x|}. \tag{1.21}$$

REMARK 1.14. By uniqueness we mean that when $\omega = \omega_1$ the only solutions connecting 0 to $\zeta(\omega_1)$ (i.e. satisfying (1.20)) are of the form $\phi(\cdot + c)$ for some $c \in \mathbb{R}$.

REMARK 1.15. Using the symmetry $x \to -x$ it is easy to see that proposition 1.13 also implies the existence and uniqueness of a kink solution ϕ satisfying

$$\lim_{x\to-\infty} \phi(x) = \zeta(\omega_1) > 0, \qquad \lim_{x\to+\infty} \phi(x) = 0.$$

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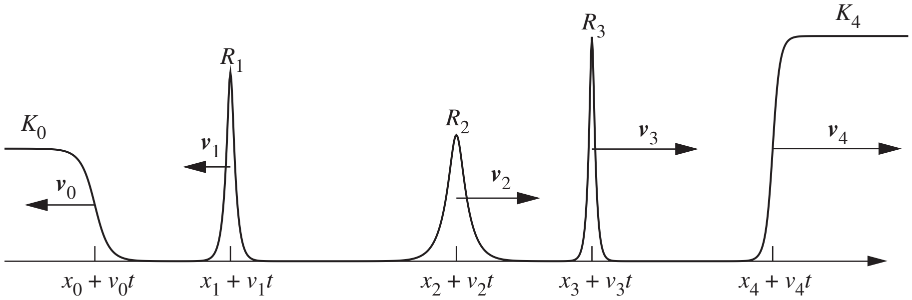

Figure 1. Schematic of the multi-kink profile KR in (1.22).

By replacing $<$ by $>$ in the assumptions of proposition 1.13 we immediately obtain the existence of a kink profile connecting 0 to $\zeta(\omega_1) < 0$.

REMARK 1.16. It is well known (see [1]) that if instead of (1.19) we assume that there exists $\omega_0 \in \mathbb{R}$ such that

$$\zeta(\omega_0) > 0, \quad f(\zeta(\omega_0)) - \omega\zeta(\omega_0) > 0,$$

then for $\omega = \omega_0$ there exists a soliton profile, i.e. a unique positive even solution $\phi \in \mathcal{C}^2(\mathbb{R})$ to (1.18) with boundary conditions

$$\lim_{x\to\pm\infty} \phi(x) = 0.$$

The profile on which we want to build a solution to (1.1) is the following. Take $N \in \mathbb{N}$, $(v_j, x_j, \omega_j, \gamma_j)_{j=0,\dots,N+1} \subset \mathbb{R}^4$ such that $v_0 < \cdots < v_{N+1}$. Assume that for ω_0 and ω_{N+1} there exist two kink profiles, ϕ_0 and ϕ_{N+1} (solutions of (1.18)), satisfying the boundary conditions

$$\lim_{x\to-\infty} \phi_0(x) \neq 0, \qquad \lim_{x\to+\infty} \phi_0(x) = 0,$$
$$\lim_{x\to-\infty} \phi_{N+1}(x) = 0, \qquad \lim_{x\to+\infty} \phi_{N+1}(x) \neq 0.$$

Denote by K_0 and K_{N+1} the corresponding kinks. For $j = 1, \dots, N$, assume as before that we are given localized soliton profiles $(\phi_j)_{j=1,\dots,N}$ and let R_j be the corresponding solitons. Consider the following approximate solution, composed of a kink on the left and on the right and solitons in the middle (see figure 1):

$$KR(t,x) := K_0(t,x) + \sum_{j=1}^{N} R_j(t,x) + K_{N+1}(t,x). \tag{1.22}$$

Our last result concerns solutions of that are composed of solitons and half-kinks.

THEOREM 1.17. *Consider (1.1) with $d = 1$ and $f(u) = g(|u|^2)u$ satisfying (1.9), and let KR be the profile defined in (1.22). Define $v_\star$ by*

$$v_\star := \inf\{|v_j - v_k|;\ j,k = 0,\dots,N+1,\ j \neq k\}.$$

Then there exist $v_\sharp > 0$ (independent of (v_j)) large enough, $T_0 \gg 1$ and constants $C, c_1, c_2 > 0$ such that if $v_\star > v_\sharp$, then there exists a (unique) multi-kink solution

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$u \in \mathcal{C}([T_0, +\infty), H^1_{\mathrm{loc}}(\mathbb{R}))$ to (1.1) satisfying

$$\mathrm{e}^{c_1 v_\star t}\|u - KR\|_{S([t,+\infty))} + \mathrm{e}^{c_2 v_\star t}\|\nabla(u - KR)\|_{S([t,+\infty))} \leqslant C \quad \forall t \geqslant T_0.$$

It will be clear from the proof that the theorem remains valid if we remove K_0 or K_{N+1} from the profile KR. It is also fine if $v_0 > 0$ or $v_{N+1} < 0$.

1.4. Strategy of the proofs

To simplify the presentation, we shall give a streamlined proof of theorems 1.2, 1.8, 1.10 and 1.17. The key tools are propositions 2.3 and 2.4, which reduce matters to the checking of a few conditions on the solitons. This is done in § 2. We stress that the situation here is a bit different from the usual stability theory in critical NLS problems (cf. [15,16]). There the approximate solutions often have finite space-time norms and the perturbation errors only need to be small in some dual Strichartz space. In our case the solitary waves carry infinite space-time norms on any non-compact time interval (unless one considers L^∞_t). For this we have to rework the stability theory a little around a solitary-wave-type solution. The price to pay is that the perturbation errors and source terms need to be exponentially small in time. This is the main place where the large relative velocity assumption is used. We give the proofs of theorems 1.8 and 1.10 in § 3, of theorem 1.2 in § 4 and, finally, of theorem 1.17 in § 5. In § 6, we conclude the paper by giving three results similar to theorem 1.10 with additional assumptions that allow us to take $T_0 = 0$.

2. The perturbation argument

We start this section by giving some preliminaries and notation.

2.1. Preliminaries and notation

For any two quantities A and B, we use $A \lesssim B$ (respectively, $A \gtrsim B$) to denote the inequality $A \leqslant CB$ (respectively, $A \geqslant CB$) for a generic positive constant C. The dependence of C on other parameters or constants is usually clear from the context and we will often suppress this dependence. Sometimes we will write $A \lesssim_k B$ if the implied constant C depends on the parameter k. We shall use the notation $C = C(X)$ if the constant C depends explicitly on some quantity X.

For any function $f \colon \mathbb{R}^d \to \mathbb{C}$, we use $\|f\|_{L^p}$ or $\|f\|_p$ to denote the Lebesgue L^p-norm of f for $1 \leqslant p \leqslant \infty$. We use $L^q_t L^r_x$ to define the space-time norm as

$$\|u\|_{L^q_t L^r_x(\mathbb{R}\times\mathbb{R}^d)} := \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}^d} |u(t,x)|^r \,\mathrm{d}x\right)^{q/r} \mathrm{d}t\right)^{1/q},$$

with the usual modifications when $q, r = \infty$, or when the domain $\mathbb{R}\times\mathbb{R}^d$ is replaced by a smaller region of space-time such as $I \times \mathbb{R}^d$. When $q = r$ we abbreviate $L^q_t L^q_x$ as $L^q_{t,x}$ or L^q_{tx}. We shall write $u \in L^q_{t,\mathrm{loc}} L^r_x(\mathbb{R}\times\mathbb{R}^d)$ if

$$\|u\|_{L^q_t L^r_x(K\times\mathbb{R}^d)} < \infty \quad \text{for any compact } K \subset \mathbb{R}. \tag{2.1}$$

We shall need the standard dispersive inequality: for any $2 \leqslant p \leqslant \infty$,

$$\|\mathrm{e}^{\mathrm{i}t\Delta} f\|_p \lesssim |t|^{-d(1/2-1/p)}\|f\|_{p/(p-1)} \quad \forall t \neq 0.$$

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The dispersive inequality can be used to deduce certain space-time estimates known as Strichartz inequalities. Recall that for dimension $d \geqslant 1$ we say a pair of exponents (q, r) is *(Schrödinger) admissible* if

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}, \quad 2 \leqslant q, r \leqslant \infty \text{ and } (d, q, r) \neq (2, 2, \infty).$$

For any fixed space-time slab $I \times \mathbb{R}^d$, we define the Strichartz norm

$$\|u\|_{S(I)} := \sup_{(q,r) \text{ admissible}} \|u\|_{L_t^q L_x^r(I \times \mathbb{R}^d)}. \tag{2.2}$$

For $d = 2$, we need to further impose $q > q_1$ in the above norm for some q_1 slightly larger than 2, so as to stay away from the forbidden endpoint. The choice of q_1 is usually simple. We use $S(I)$ to denote the closure of all test functions in $\mathbb{R} \times \mathbb{R}^d$ under this norm. We denote by $N(I)$ the dual space of $S(I)$.

We now state the standard Strichartz estimates. For the non-endpoint case, one can see, for example, [12]. For the end-point case, see [14].

Lemma 2.1. *If $u\colon I \times \mathbb{R}^d \to \mathbb{C}$ solves*

$$\mathrm{i}\partial_t u + \Delta u = F, \quad u(t_0) = u_0,$$

for some $t_0 \in I$, $u_0 \in L_x^2(\mathbb{R}^d)$, then

$$\|u\|_{S(I)} \lesssim_d \|u_0\|_2 + \|F\|_{N(I)}.$$

We need a few simple estimates on the nonlinearity. For any complex-valued function $F = F(z)$, recall the notation

$$F_z := \frac{1}{2}\left(\frac{\partial F}{\partial x} - \mathrm{i}\frac{\partial F}{\partial y}\right), \qquad F_{\bar z} := \frac{1}{2}\left(\frac{\partial F}{\partial x} + \mathrm{i}\frac{\partial F}{\partial y}\right).$$

If we write $F(z) = F^*(z, \bar z)$ with z and $\bar z$ treated as independent variables in F^*, then

$$F_z = \frac{\partial F^*}{\partial z} \quad \text{and} \quad F_{\bar z} = \frac{\partial F^*}{\partial \bar z}.$$

By the chain rule and fundamental theorem of calculus, it is easy to check that

$$\begin{aligned} \nabla(F(u(x))) &= F_z(u(x))\nabla u(x) + F_{\bar z}(u(x))\overline{\nabla u(x)}, \\ F(z_1) - F(z_2) &= (z_1 - z_2)\int_0^1 F_z(z_2 + \theta(z_1 - z_2))\,\mathrm{d}\theta \\ &\quad + (\overline{z_1 - z_2})\int_0^1 F_{\bar z}(z_2 + \theta(z_1 - z_2))\,\mathrm{d}\theta. \end{aligned} \tag{2.3}$$

These two identities will be used later.

Lemma 2.2 (Hölder continuity of f' and g). *Let $f(z) = g(|z|^2)z$ for $z \in \mathbb{C}$ and suppose g satisfies (1.9) and (1.12). Then for all $s_1, s_2 > 0$ we have*

$$\begin{aligned} &|g(s_1^2) - g(s_2^2)| + |s_1^2 g'(s_1^2) - s_2^2 g'(s_2^2)| \\ &\quad \lesssim |s_1 - s_2|^{\min\{2\alpha_1, 1\}}(s_1 + s_2)^{\max\{2\alpha_1 - 1, 0\}} \\ &\qquad + |s_1 - s_2|^{\min\{2\alpha_2, 1\}}(s_1 + s_2)^{\max\{2\alpha_2 - 1, 0\}}; \end{aligned} \tag{2.4}$$

and, for any $z_1, z_2 \in \mathbb{C}$,

$$\begin{aligned} |f_z(z_1) - f_z(z_2)| + |f_{\bar z}(z_1) - f_{\bar z}(z_2)| + |g(|z_1|^2) - g(|z_2|^2)| \\ \lesssim |z_1 - z_2|^{\min\{2\alpha_1, 1\}}(|z_1| + |z_2|)^{\max\{2\alpha_1 - 1, 0\}} \\ + |z_1 - z_2|^{\min\{2\alpha_2, 1\}}(|z_1| + |z_2|)^{\max\{2\alpha_2 - 1, 0\}}; \end{aligned} \tag{2.5}$$

$$|f(z_1) - f(z_2)| \lesssim |z_1 - z_2| \cdot ((|z_1| + |z_2|)^{2\alpha_1} + (|z_1| + |z_2|)^{2\alpha_2}). \tag{2.6}$$

Proof of lemma 2.2. By (1.9), we get, for any $s > 0$,

$$|(s^2 g'(s^2))'| \lesssim |sg'(s^2)| + |s^3 g''(s^2)| \lesssim s^{2\alpha_1 - 1} + s^{2\alpha_2 - 1}.$$

Clearly, for any $s_1, s_2 > 0$, using the above estimate we have

$$\begin{aligned} |s_1^2 g'(s_1^2) - s_2^2 g'(s_2^2)| &\lesssim |s_1^{2\alpha_1} - s_2^{2\alpha_1}| + |s_1^{2\alpha_2} - s_2^{2\alpha_2}| \\ &\lesssim \sum_{k=1}^{2} |s_1 - s_2|^{\min\{2\alpha_k, 1\}}(s_1 + s_2)^{\max\{2\alpha_k - 1, 0\}}. \end{aligned}$$

The estimate for $g(s^2)$ is similar. Therefore, (2.4) follows. Observe that

$$f_z(z) = g'(|z|^2)|z|^2 + g(|z|^2), \qquad f_{\bar z}(z) = g'(|z|^2) z^2.$$

Obviously, (2.5) holds for $g(|z|^2)$ and $f_z(z)$ using (2.4). For $f_{\bar z}(z)$, the estimate is similar: let $z_1 = \rho_1 \mathrm{e}^{\mathrm{i}\theta_1}$, $z_2 = \rho_2 \mathrm{e}^{\mathrm{i}\theta_2}$ with $|\theta_1 - \theta_2| \leqslant \pi$. We just need to note that

$$|f_{\bar z}(z_1) - f_{\bar z}(z_2)| = |g'(\rho_1^2)\rho_1^2 \mathrm{e}^{\mathrm{i}(\theta_1 - \theta_2)} - g'(\rho_2^2)\rho_2^2 \mathrm{e}^{\mathrm{i}(\theta_2 - \theta_1)}|$$

and

$$|z_1 - z_2| \sim |\rho_1 - \rho_2| \left|\cos\left(\frac{\theta_1 - \theta_2}{2}\right)\right| + (\rho_1 + \rho_2)\left|\sin\left(\frac{\theta_1 - \theta_2}{2}\right)\right|.$$

Estimating the real and imaginary parts separately gives the result. Finally, (2.6) follows from (2.3) and (2.5). □

With the preliminaries and notation out of the way, we now turn to the main matter of this section.

To prove our results, we shall state and prove a general proposition on the solvability of NLS equations around an approximate solution profile with exponentially decaying source terms. This proposition is very useful in that it reduces the construction of multi-soliton solutions to the verification of only a few conditions (see (2.7) and (2.11)). To simplify numeric notation we shall first deal with the pure power nonlinearity case.

Proposition 2.3. *Let* $0 < \alpha < \alpha_{\max}$. *Let* $H = H(t, x)\colon [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$, $W = W(t, x)\colon [0, \infty) \times \mathbb{R}^d \to \mathbb{C}$ *be given functions that, for some* $C_1 > 0$, $\lambda > 0$, *satisfy*

$$\|W(t)\|_{\alpha+2} + \mathrm{e}^{\lambda t}\|H(t)\|_{(\alpha+2)/(\alpha+1)} \leqslant C_1 \quad \forall t \geqslant 0. \tag{2.7}$$

Let $f_1(z) = |z|^\alpha z$ *and consider the equation*

$$\eta(t) = \mathrm{i}\int_t^\infty \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}(f_1(W + \eta) - f_1(W) + H)(\tau)\,\mathrm{d}\tau. \tag{2.8}$$

There exists a constant $\lambda_* = \lambda_*(\alpha, d, C_1) > 0$ sufficiently large such that if $\lambda \geqslant \lambda_*$, then the following hold:

- *there exists a unique solution η to (2.8) satisfying*

$$\|\eta(t)\|_{\alpha+2} \leqslant C_1 \mathrm{e}^{-\lambda t} \quad \forall t \geqslant 0; \tag{2.9}$$

- *all (L^2 level) Strichartz norms of η are finite and decay exponentially, i.e.*

$$\|\eta\|_{S([t,\infty))} \lesssim \mathrm{e}^{-\lambda t} \quad \forall t \geqslant 0; \tag{2.10}$$

- *if, in addition to (2.7), for some $C_2 > 0$, (H, W) also satisfies*

$$\|\nabla W(t)\|_{\alpha+2} + \mathrm{e}^{\lambda t}\|\nabla H(t)\|_{(\alpha+2)/(\alpha+1)} \leqslant C_2 \quad \forall t \geqslant 0, \tag{2.11}$$

  then $\eta \in L_t^\infty H_x^1$, and for some $C_3 = C_3(d, \alpha, C_1) > 0$,

$$\|\nabla\eta(t)\|_{\alpha+2} + \|\nabla\eta\|_{S([t,\infty))} \leqslant C_3 C_2 \mathrm{e}^{-\min\{\alpha,1\}\lambda t} \quad \forall t \geqslant 0, \tag{2.12}$$

  where both C_3 and λ_* are independent of C_2.

Proof of proposition 2.3. We write (2.8) as $\eta = V\eta$. We shall show that, for λ sufficiently large, V is a contraction in the ball

$$B = \{\eta \colon \|\eta\|_{\tilde{X}} := \|\mathrm{e}^{\lambda t}\|\eta(t)\|_{\alpha+2}\|_{L_t^\infty([0,\infty))} \leqslant C_1\}.$$

We first check that V maps B into B. Define

$$\theta := d\left(\frac{1}{2} - \frac{1}{\alpha+2}\right).$$

It is easy to check that $0 < \theta < 1$, since by assumption $0 < \alpha < \alpha_{\max}$. By the simple inequality

$$|f_1(z_1) - f_1(z_2)| \lesssim |z_1 - z_2| \cdot (|z_1|^\alpha + |z_2|^\alpha) \quad \forall z_1, z_2 \in \mathbb{C}, \tag{2.13}$$

we have

$$|f_1(W+\eta) - f_1(W)| \lesssim |\eta| \cdot (|W|^\alpha + |\eta|^\alpha). \tag{2.14}$$

By using the dispersive estimate, the assumptions on (W, H) and (2.14), we have

$$\begin{aligned}
\|\eta(t)\|_{\alpha+2} &\leqslant C\int_t^\infty |t-\tau|^{-\theta}(\||W(\tau)|^\alpha|\eta(\tau)|\|_{(\alpha+2)/(\alpha+1)} \\
&\qquad + \||\eta(\tau)|^{\alpha+1}\|_{(\alpha+2)/(\alpha+1)} + \|H(\tau)\|_{(\alpha+2)/(\alpha+1)})\,\mathrm{d}\tau \\
&\leqslant C\int_t^\infty |t-\tau|^{-\theta}(\|W(\tau)\|_{\alpha+2}^\alpha\|\eta(\tau)\|_{\alpha+2} + \|\eta(\tau)\|_{\alpha+2}^{\alpha+1} \\
&\qquad + \|H(\tau)\|_{(\alpha+2)/(\alpha+1)})\,\mathrm{d}\tau \\
&\leqslant C\int_t^\infty |t-\tau|^{-\theta}(C_1^\alpha C_1\mathrm{e}^{-\lambda\tau} + C_1^{\alpha+1}\mathrm{e}^{-\lambda(\alpha+1)\tau} + C_1\mathrm{e}^{-\lambda\tau})\,\mathrm{d}\tau \\
&\leqslant CC_1\mathrm{e}^{-\lambda t}I_1,
\end{aligned} \tag{2.15}$$

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where $C = C(d, \alpha)$ and, for $\tilde{\tau} = \tau - t$,

$$I_1 = C_1^\alpha \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda \tilde{\tau}} \, d\tilde{\tau} + C_1^\alpha \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda(\alpha+1)\tilde{\tau}} \, d\tilde{\tau} + \int_0^\infty (\tilde{\tau})^{-\theta} e^{-\lambda \tilde{\tau}} \, d\tilde{\tau}.$$

It is not difficult to check that, for λ sufficiently large,

$$CI_1 \leqslant \frac{C(C_1, d, \alpha)}{\lambda^{1-\theta}} \leqslant 1.$$

Hence, $\|\eta(t)\|_{\alpha+2} \leqslant C_1 e^{-\lambda t}$, and V maps B to B. By using (2.13) and a similar estimate as in (2.15), we can also show that, for any $\eta_1 \in B$, $\eta_2 \in B$,

$$\|(V\eta_1)(t) - (V\eta_2)(t)\|_{\tilde{X}} \leqslant \tfrac{1}{2} \|\eta_1 - \eta_2\|_{\tilde{X}}.$$

This completes the proof that V is a contraction on B.

Next (2.10) is a simple consequence of the Strichartz estimate. Denote by a the number such that

$$\frac{2}{a} + \frac{d}{\alpha+2} = \frac{d}{2}.$$

It is easy to check that $2 < a < \infty$ since $0 < \alpha < \alpha_{\max}$. By (2.13) and the Strichartz estimate, we have

$$\begin{aligned}
\|\eta\|_{S([t,\infty))} &\lesssim \|f_1(W+\eta) - f_1(W)\|_{L_\tau^{a/(a-1)} L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))} \\
&\quad + \|H\|_{L_\tau^{a/(a-1)} L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))} \\
&\lesssim \||\eta| \cdot (|W|^\alpha + |\eta|^\alpha)\|_{L_\tau^{a/(a-1)} L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))} \\
&\quad + \|H\|_{L_\tau^{a/(a-1)} L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))} \\
&\lesssim \|W\|^\alpha_{L_\tau^\infty L_x^{\alpha+2}([0,\infty))} \cdot \|\eta\|_{L_\tau^{a/(a-1)} L_x^{\alpha+2}([t,\infty))} \\
&\quad + \|\eta\|^{\alpha+1}_{L_\tau^{(\alpha+1)a/(a-1)} L_x^{\alpha+2}([t,\infty))} + \|H\|_{L_\tau^{a/(a-1)} L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))} \\
&\lesssim e^{-\lambda t} \quad \forall t \geqslant 0.
\end{aligned} \tag{2.16}$$

Finally, to show (2.12), we first prove that V maps B_1 into B_1, where

$$B_1 = B \cap \Big\{\eta \colon \sup_{t \geqslant 0} (e^{\min\{\alpha,1\}\lambda t} \|\nabla \eta(t)\|_{\alpha+2}) \leqslant C_2\Big\}.$$

We start with the identity

$$\begin{aligned}
&\nabla(f_1(W+\eta) - f_1(W)) \\
&\qquad = ((\partial_z f_1)(W+\eta) - (\partial_z f_1)(W))\nabla(W+\eta) + (\partial_z f_1)(W)\nabla\eta \\
&\qquad\quad + ((\partial_{\bar{z}} f_1)(W+\eta) - (\partial_{\bar{z}} f_1)(W))\overline{\nabla(W+\eta)} + (\partial_{\bar{z}} f_1)(W)\overline{\nabla\eta}.
\end{aligned} \tag{2.17}$$

Note that, for $0 < \alpha \leqslant 1$,

$$|(\partial_z f_1)(z_1) - (\partial_z f_1)(z_2)| \lesssim |z_1 - z_2|^\alpha \quad \forall z_1, z_2 \in \mathbb{C},$$

and for $\alpha > 1$,

$$|(\partial_z f_1)(z_1) - (\partial_z f_1)(z_2)| \lesssim (|z_1|^{\alpha-1} + |z_2|^{\alpha-1})|z_1 - z_2| \quad \forall z_1, z_2 \in \mathbb{C}.$$

Therefore,

$$
\begin{aligned}
&|\nabla(f_1(W+\eta)-f_1(W))| \\
&\quad\lesssim \begin{cases} |\eta|^\alpha|\nabla(W+\eta)|+|W|^\alpha|\nabla\eta| & \text{if } 0<\alpha\leqslant 1, \\ (|\eta|^{\alpha-1}+|W|^{\alpha-1})|\eta||\nabla(W+\eta)|+|W|^\alpha|\nabla\eta| & \text{if } \alpha>1. \end{cases}
\end{aligned} \tag{2.18}
$$

For simplicity we shall only discuss the case $0<\alpha\leqslant 1$. The argument for $\alpha>1$ is similar (even simpler) and will be omitted. By using (2.18), (2.9), (2.11) and the dispersive inequality, we have, for $t\geqslant 0$,

$$
\begin{aligned}
&\|\nabla\eta(t)\|_{\alpha+2} \\
&\quad\lesssim_{d,\alpha} \int_t^\infty |t-\tau|^{-\theta}(\||\eta|^\alpha|\nabla(W+\eta)|\|_{(\alpha+2)/(\alpha+1)} \\
&\qquad\qquad + \||W|^\alpha\nabla\eta\|_{(\alpha+2)/(\alpha+1)}+\|\nabla H\|_{(\alpha+2)/(\alpha+1)})\,\mathrm{d}\tau \\
&\quad\lesssim_{d,\alpha} \int_t^\infty |t-\tau|^{-\theta}(\|\eta\|_{\alpha+2}^\alpha(\|\nabla W\|_{\alpha+2}+\|\nabla\eta\|_{\alpha+2}) \\
&\qquad\qquad + \|W\|_{\alpha+2}^\alpha\|\nabla\eta\|_{\alpha+2}+\|\nabla H\|_{(\alpha+2)/(\alpha+1)})\,\mathrm{d}\tau \\
&\quad\lesssim_{d,\alpha,C_1} C_2\int_t^\infty |t-\tau|^{-\theta}\mathrm{e}^{-\lambda\alpha\tau}\,\mathrm{d}\tau + C_2\int_t^\infty |t-\tau|^{-\theta}\mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau \\
&\quad\lesssim_{d,\alpha,C_1} C_2\int_t^\infty |t-\tau|^{-\theta}\mathrm{e}^{-\lambda\alpha\tau}\,\mathrm{d}\tau \\
&\quad\leqslant C_2\mathrm{e}^{-\lambda\alpha t}\cdot C(d,\alpha,C_1)\int_0^\infty |\tilde\tau|^{-\theta}\mathrm{e}^{-\lambda\alpha\tilde\tau}\,\mathrm{d}\tilde\tau \\
&\quad= C_2\mathrm{e}^{-\lambda\alpha t}\cdot C(d,\alpha,C_1)\cdot(\lambda\alpha)^{-(1-\theta)}\int_0^\infty |\tilde\tau|^{-\theta}\mathrm{e}^{-\tilde\tau}\,\mathrm{d}\tilde\tau.
\end{aligned}
$$

Now, if we take $\lambda\geqslant\lambda_*$, and $\lambda_*=\lambda_*(d,\alpha,C_1)$ is independent of C_2 and sufficiently large such that

$$
C(d,\alpha,C_1)\cdot(\lambda_*\alpha)^{-(1-\theta)}\int_0^\infty |\tilde\tau|^{-\theta}\mathrm{e}^{-\tilde\tau}\,\mathrm{d}\tilde\tau \leqslant \tfrac{1}{2}, \tag{2.19}
$$

then clearly

$$
\|\nabla\eta(t)\|_{\alpha+2}\leqslant C_2\mathrm{e}^{-\lambda\alpha t}\quad \forall t\geqslant 0.
$$

By a similar argument, for the case $\alpha>1$ we also obtain

$$
\|\nabla\eta(t)\|_{\alpha+2}\leqslant C_2\mathrm{e}^{-\lambda t}\quad \forall t\geqslant 0.
$$

Hence, we have proved that V maps B_1 to B_1. Since V is a contraction on B and maps B_1 into B_1, it is obvious that we have constructed the solution satisfying

$$
\|\nabla\eta(t)\|_{\alpha+2}\leqslant C_2\mathrm{e}^{-\lambda\min\{\alpha,1\}t}\quad \forall t\geqslant 0. \tag{2.20}
$$

It remains for us to bound the Strichartz norm $\|\nabla\eta(t)\|_{S([t,\infty))}$. The argument is similar to that in (2.16). Let a be the same number such that

$$
\frac{2}{a}+\frac{d}{\alpha+2}=\frac{d}{2}.
$$

By (2.18) and Strichartz, we have

$$\begin{aligned}\|\nabla\eta\|_{S([t,\infty))} &\lesssim_d \||\eta|^\alpha|\nabla(W+\eta)|\|_{N([t,\infty))} + \||W|^\alpha|\nabla\eta|\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))}\\ &\lesssim_d \||\eta|^\alpha|\nabla W|\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))}\\ &\quad + \||\eta|^\alpha|\nabla\eta|\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))}\\ &\quad + \||W|^\alpha|\nabla\eta|\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))}\\ &\quad + \|\nabla H\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))}\\ &\lesssim_d \||\eta|^\alpha\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/\alpha}([t,\infty))}\||\nabla W|+|\nabla\eta|\|_{L_\tau^\infty L_x^{\alpha+2}([t,\infty))}\\ &\quad + \||W|^\alpha\|_{L_\tau^\infty L_x^{(\alpha+2)/\alpha}([t,\infty))}\|\nabla\eta\|_{L_\tau^{a/(a-1)}L_x^{\alpha+2}([t,\infty))}\\ &\quad + \|\nabla H\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/(\alpha+1)}([t,\infty))}. \end{aligned} \tag{2.21}$$

By (2.9), we have

$$\begin{aligned}\||\eta|^\alpha\|_{L_\tau^{a/(a-1)}L_x^{(\alpha+2)/\alpha}([t,\infty))} &\leqslant \|\|\eta\|_{\alpha+2}^\alpha\|_{L_\tau^{a/(a-1)}([t,\infty))}\\ &\leqslant C_1^\alpha\biggl(\int_t^\infty \mathrm{e}^{-\lambda\alpha a\tau/(a-1)}\,\mathrm{d}\tau\biggr)^{(a-1)/a}\\ &\leqslant C_1^\alpha\cdot\biggl(\lambda\alpha\frac{a}{a-1}\biggr)^{-(a-1)/a}\cdot\mathrm{e}^{-\lambda\alpha t}.\end{aligned}$$

Plugging the above estimates into (2.21) and using (2.11), (2.20), we obtain

$$\|\nabla\eta\|_{S([t,\infty))} \lesssim_{d,\alpha,C_1} C_2\mathrm{e}^{-\lambda\alpha t}.$$

This settles the estimate for $0<\alpha\leqslant 1$.

By a similar estimate, we also have, for $\alpha>1$,

$$\|\nabla\eta\|_{S([t,\infty))} \lesssim_{d,\alpha,C_1} C_2\mathrm{e}^{-\lambda t}.$$

This completes the proof of (2.12). □

The next proposition, unlike proposition 2.3, is based solely on Strichartz estimates. It will be used in the proof of theorems 1.10 and 1.17. Several assumptions and conditions have to be modified to take care of the general nonlinearity, $f(u)$.

Proposition 2.4. *Let f be the same as in (1.1), satisfying condition (1.9). Let $H=H(t,x)\colon[0,\infty)\times\mathbb{R}^d\to\mathbb{C}$, $W=W(t,x)\colon[0,\infty)\times\mathbb{R}^d\to\mathbb{C}$ be given functions that, for some $C_1>0$, $C_2>0$, $\lambda>0$, $T_0\geqslant 0$, satisfy*

$$\left.\begin{aligned}\|W(t)\|_\infty+\mathrm{e}^{\lambda t}\|H(t)\|_2\leqslant C_1\quad\forall t\geqslant T_0,\\ \|\nabla W(t)\|_2+\|\nabla W(t)\|_\infty+\mathrm{e}^{\lambda t}\|\nabla H(t)\|_2\leqslant C_2\quad\forall t\geqslant T_0.\end{aligned}\right\} \tag{2.22}$$

Consider the equation

$$\eta(t)=-\mathrm{i}\int_t^\infty \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}(f(W+\eta)-f(W)+H)(\tau)\,\mathrm{d}\tau,\quad t\geqslant T_0. \tag{2.23}$$

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There exist a constant $\lambda_* = \lambda_*(d, \alpha_1, \alpha_2, C_1) > 0$ (independent of C_2) and a time $T_* = T_*(d, \alpha_1, \alpha_2, C_1, C_2) > 0$ sufficiently large such that if $\lambda \geqslant \lambda_*$ and $T_0 \geqslant T^*$, then there exists a unique solution η to (2.23) on $[T_0, +\infty) \times \mathbb{R}^d$ satisfying

$$\mathrm{e}^{\lambda t}\|\eta\|_{S([t,\infty))} + \mathrm{e}^{\lambda c_1 t}\|\nabla\eta\|_{S([t,\infty))} \leqslant 1 \quad \forall t \geqslant T_0. \tag{2.24}$$

Here $c_1 > 0$ is a constant depending only on (α_1, d).

Remark 2.5. It is important to note that λ_* *does not depend* on C_2. This will be essential for the proof of theorems 1.10 and 1.17.

Proof of proposition 2.4. To simplify numeric notation we shall suppress all explicit dependence of constants on all parameters except the constant C_2.

We now sketch the main computations. Take $0 < \beta_1 \leqslant 2\alpha_1$ such that $\beta_1 < 1/100d$. Define

$$\beta_2 := \begin{cases} \dfrac{4}{d-2} & \text{if } d \geqslant 3, \\ m-1 & \text{if } d = 1, 2, \end{cases}$$
$$c_1 := \tfrac{1}{2}\beta_1.$$

Here, for $d = 1, 2$, m is an integer such that $m > 2\alpha_2 + 2$.

We shall omit the standard contraction argument, since it will be essentially a repetition and we check only the following property: if on $[T_0, +\infty)$ we have

$$\mathrm{e}^{\lambda t}\|\eta\|_{S([t,\infty))} + \mathrm{e}^{c_1\lambda t}\|\nabla\eta\|_{S([t,\infty))} \leqslant C,$$

then the following *a priori* estimate holds, provided λ and T_0 are chosen large enough:

$$\mathrm{e}^{\lambda t}\|\eta\|_{S([t,\infty))} + \mathrm{e}^{c_1\lambda t}\|\nabla\eta\|_{S([t,\infty))} \leqslant 1. \tag{2.25}$$

We start with $\|\eta\|_{S([t,\infty))}$. By lemma 2.2 and Strichartz, we have

$$\begin{aligned} \|\eta\|_{S([t,\infty))} &\lesssim \|f(W+\eta) - f(W)\|_{N([t,\infty))} + \|H\|_{N([t,\infty))} \\ &\lesssim \|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})\|_{N([t,\infty))} && (2.26\,a) \\ &\quad + \|H\|_{L^1_\tau L^2_x([t,\infty))}. && (2.26\,b) \end{aligned}$$

For the term (2.26 b), by using (2.22), we have

$$\|H\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty \mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau \leqslant \tfrac{1}{100}\mathrm{e}^{-\lambda t},$$

where the constant $\frac{1}{100}$ is obtained by taking λ large enough.

For (2.26 a), consider two cases. If $d \geqslant 3$, then by the boundedness of W, we have

$$|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})| \lesssim |\eta| + |\eta|^{1+4/(d-2)}. \tag{2.27}$$

Hence, for $d \geqslant 3$, using the fact that both $((2d+4)/d, (2d+4)/d)$ and (q^*, q) are admissible with

$$\frac{1}{q^*} = \frac{1}{q} - \frac{1}{d} = \frac{d-2}{2d+4}$$

yields

$$\begin{aligned}(2.26\,a) &\lesssim \|\eta\|_{L^1_\tau L^2_x([t,\infty))} + \|\eta|\eta|^{4/(d-2)}\|_{L^{2(d+2)/(d+4)}_{\tau,x}([t,\infty))} \\ &\lesssim \int_t^\infty \mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau + \|\eta\|_{L^{2(d+2)/d}_{\tau,x}([t,\infty))}\cdot\|\eta\|^{4/(d-2)}_{L^{2(d+2)/(d-2)}_{\tau,x}([t,\infty))} \\ &\lesssim \frac{1}{\lambda}\mathrm{e}^{-\lambda t} + \|\eta\|_{S([t,\infty))}\cdot\|\nabla\eta\|^{4/(d-2)}_{S([t,\infty))} \\ &\lesssim \frac{1}{\lambda}\mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda t}\cdot\exp\left(-\frac{4}{d-2}c_1\lambda t\right) \\ &\leqslant \tfrac{1}{100}\mathrm{e}^{-\lambda t},\end{aligned}$$

where we have used the fact that λ and $t \geqslant T_0$ are sufficiently large.

For $d = 1, 2$, we replace (2.27) by

$$|\eta(|W|^{\beta_1} + |W|^{\beta_2} + |\eta|^{\beta_1} + |\eta|^{\beta_2})| \lesssim |\eta| + |\eta|^m.$$

Then

$$\||\eta|^m\|_{N([t,\infty))} \lesssim \||\eta|^m\|_{L^1_\tau L^2_x([t,\infty))} \lesssim \int_t^\infty \|\eta(\tau)\|^m_{2m}\,\mathrm{d}\tau.$$

By (2.24) and interpolation (i.e. the Gagliardo–Nirenberg inequality), we have, for $\theta = d(1/2 - 1/2m)$,

$$\|\eta(\tau)\|_{2m} \lesssim \|\eta(\tau)\|_2^{1-\theta}\|\nabla\eta(\tau)\|_2^\theta \lesssim \mathrm{e}^{-((1-\theta)\lambda + c_1\lambda\theta)\tau}.$$

It is easy to check that $m(1-\theta) \geqslant 1$. Therefore,

$$\||\eta|^m\|_{N([t,\infty))} \lesssim \int_t^\infty \mathrm{e}^{-\lambda\tau}\,\mathrm{d}\tau \leqslant \tfrac{1}{100}\mathrm{e}^{-\lambda t}.$$

Hence, the estimate also holds for $d = 1, 2$. Consequently, for all $d \geqslant 1$, and $t \geqslant T_0$,

$$\|\eta\|_{S([t,\infty))} \leqslant \tfrac{1}{10}\mathrm{e}^{-\lambda t}.$$

Now we estimate $\|\nabla\eta\|_{S([t,\infty))}$. By the Strichartz estimate and (2.17),

$$\begin{aligned}\|\nabla\eta\|_{S([t,\infty))} &\lesssim \|\nabla(f(W+\eta) - f(W))\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))} \\ &\lesssim \||f_z(W+\eta) - f_z(W)|\cdot\nabla(W+\eta)\|_{N([t,\infty))} \\ &\quad + \||f_{\bar z}(W+\eta) - f_{\bar z}(W)|\cdot\overline{\nabla(W+\eta)}\|_{N([t,\infty))} \\ &\quad + \||f_z(W)|\nabla\eta\|_{N([t,\infty))} + \||f_{\bar z}(W)|\overline{\nabla\eta}\|_{N([t,\infty))} + \|\nabla H\|_{N([t,\infty))}.\end{aligned}$$

By lemma 2.2, we get

$$\begin{aligned}&\|\nabla\eta\|_{S([t,\infty))} \\ &\quad\lesssim \||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))} + \||\eta|^{\beta_1}|\nabla W|\|_{N([t,\infty))} &(2.28\,a)\\ &\qquad + \||\eta|^{\min\{\beta_2,1\}}(|W| + |\eta|)^{\max\{\beta_2-1,0\}}\cdot(|\nabla W| + |\nabla\eta|)\|_{N([t,\infty))} &(2.28\,b)\\ &\qquad + \|(|f_z(W)| + |f_{\bar z}(W)|)\nabla\eta\|_{L^1_\tau L^2_x([t,\infty))} + \|\nabla H\|_{L^1_\tau L^2_x([t,\infty))}. &(2.28\,c)\end{aligned}$$

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Consider (2.28 a). Let a be the number such that

$$\frac{2}{a}+\frac{d}{\beta_1+2}=\frac{d}{2}$$

and let $a'=a/(a-1)$. Then

$$\begin{aligned}\||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))} &\lesssim \||\eta|^{\beta_1}\nabla\eta\|_{L_\tau^{a'}L_x^{(\beta_1+2)/(\beta_1+1)}([t,\infty))}\\ &\lesssim \||\eta|^{\beta_1}\|_{L_\tau^{(1/a'-1/a)^{-1}}L_x^{(\beta_1+2)/\beta_1}([t,\infty))}\|\nabla\eta\|_{L_\tau^{a}L_x^{\beta_1+2}([t,\infty))}\\ &\lesssim \left(\int_t^\infty \|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot a/(a-2)}\,\mathrm{d}\tau\right)^{(a-2)/a}\cdot\|\nabla\eta\|_{S([t,\infty))}.\end{aligned}\tag{2.29}$$

It is not difficult to check that $\beta_1\cdot a/(a-2)<a$ (since $\beta_1<4/d$). By using the fact that $\|\eta\|_{L_\tau^aL_x^{\beta_1+2}([t,\infty))}\lesssim \mathrm{e}^{-\lambda t}$ and the Hölder inequality, for $t\geqslant T_0$ we have

$$\begin{aligned}\int_t^\infty \|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot a/(a-2)}\,\mathrm{d}\tau &\lesssim \sum_{k\geqslant t-1}\int_k^{k+1}\|\eta(\tau)\|_{\beta_1+2}^{\beta_1\cdot a/(a-2)}\,\mathrm{d}\tau\\ &\lesssim \sum_{k\geqslant t-1}\left(\int_k^{k+1}\|\eta(\tau)\|_{\beta_1+2}^{a}d\tau\right)^{1/a\cdot a\beta_1/(a-2)}\\ &\lesssim \sum_{k\geqslant t-1}\exp\left(-\lambda k\cdot\frac{a\beta_1}{a-2}\right)\\ &\lesssim \frac{1}{\lambda}\exp\left(-\lambda(t-1)\cdot\frac{a\beta_1}{a-2}\right).\end{aligned}$$

Plugging the above estimate into (2.29), we obtain

$$\||\eta|^{\beta_1}|\nabla\eta|\|_{N([t,\infty))}\lesssim\left(\frac{1}{\lambda}\right)^{(a-2)/a}\mathrm{e}^{-\lambda\beta_1(t-1)}\cdot\mathrm{e}^{-c_1\lambda t}\leqslant\tfrac{1}{100}\mathrm{e}^{-c_1\lambda t},\quad t\geqslant T_0,$$

for λ sufficiently large and $T_0\geqslant 1$.

Similarly, for $t\geqslant T_0$, using $\beta_1 a'=\beta_1 a/(a-1)<a$, we have

$$\begin{aligned}\||\eta|^{\beta_1}|\nabla W|\|_{N([t,\infty))} &\lesssim \||\eta|^{\beta_1}\|_{L_\tau^{a'}L_x^{(\beta_1+2)/\beta_1}([t,\infty))}\|\nabla W\|_{L_\tau^\infty L_x^{\beta_1+2}([t,\infty))}\\ &\lesssim \mathrm{e}^{-\lambda\beta_1(t-1)}C_2\\ &\lesssim \mathrm{e}^{-c_1\lambda t}\mathrm{e}^{-\lambda c_1(t-2)}C_2\leqslant\tfrac{1}{100}\mathrm{e}^{-c_1\lambda t}.\end{aligned}$$

Hence,

$$(2.28\,a)\leqslant\tfrac{1}{50}\mathrm{e}^{-c_1\lambda t}.$$

Next we deal with (2.28 b). Consider first the case $d\geqslant 6$. In this case $\beta_2\leqslant 1$. Therefore,

$$\begin{aligned}(2.28\,b) &\lesssim \||\eta|^{4/(d-2)}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\ &\lesssim \||\eta|^{4/(d-2)}\nabla\eta\|_{L_{\tau,x}^{2(d+2)/(d+4)}([t,\infty))}+\||\eta|^{4/(d-2)}|\nabla W|\|_{L_\tau^2L_x^{2d/(d+2)}([t,\infty))}\\ &\lesssim \|\nabla\eta\|_{S([t,\infty))}^{1+4/(d-2)}+\|\|\eta(\tau)\|_{L_x^2}^{4/(d-2)}\cdot\|\nabla W\|_{L_x^{((d+2)/2d-2/(d-2))^{-1}}}\|_{L_\tau^2([t,\infty))}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \exp\left(-c_1\lambda\left(1+\frac{4}{d-2}\right)t\right) + C_2\cdot\left(\int_t^\infty \|\eta(\tau)\|_2^{4/(d-2)\cdot 2}\,\mathrm{d}\tau\right)^{1/2}\\
&\lesssim \exp\left(-c_1\lambda\left(1+\frac{4}{d-2}\right)t\right) + C_2\cdot\left(\int_t^\infty \exp\left(-\frac{8}{d-2}\lambda\tau\right)\mathrm{d}\tau\right)^{1/2}\\
&\leqslant \tfrac{1}{200}\mathrm{e}^{-c_1\lambda t} + C_2\cdot \mathrm{e}^{-c_1\lambda T_0}\cdot \mathrm{e}^{-c_1\lambda t}\\
&\leqslant \tfrac{1}{100}\mathrm{e}^{-c_1\lambda t},
\end{aligned}
\tag{2.30}
$$

for λ and T_0 sufficiently large.

Consider next the case $3 \leqslant d \leqslant 5$. In this case $\beta_2 = 4/(d-2) > 1$. Therefore, using the boundedness of W, we have

$$
\begin{aligned}
(2.28\,b) &\lesssim \||\eta|\cdot(|W|+\eta)^{4/(d-2)-1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\eta|^{\beta_1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))} + \||\eta|^{4/(d-2)}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim |(2.28\,a)| + \||\eta|^{4/(d-2)}\nabla\eta\|_{L^{2(d+2)/(d+4)}_{\tau,x}([t,\infty))} + \||\eta|^{4/(d-2)}|\nabla W|\|_{N([t,\infty))}\\
&\leqslant \tfrac{1}{30}\mathrm{e}^{-c_1\lambda t} + \||\eta|^{4/(d-2)}|\nabla W|\|_{N([t,\infty))}.
\end{aligned}
$$

For $d = 5$, we can bound the term $\||\eta|^{4/(d-2)}|\nabla W|\|_{N([t,\infty))}$ in the same way as in (2.30) (it is easy to check that $2d/(d+2) < (d-2)/2$ for $d \geqslant 5$). For $d = 3, 4$, we have

$$
\begin{aligned}
\||\eta|^{4/(d-2)}|\nabla W|\|_{N([t,\infty))} &\lesssim \||\eta|^{4/(d-2)}|\nabla W|\|_{L^2_\tau L^{2d/(d+2)}_x([t,\infty))}\\
&\lesssim C_2\left(\int_t^\infty \|\eta(\tau)\|^{8/(d-2)}_{8d/(d^2-4)}\,\mathrm{d}\tau\right)^{1/2}.
\end{aligned}
\tag{2.31}
$$

Since $d = 3, 4$, it is easy to check that $2 < 8d/(d^2-4) < 2d/(d-2)$. By interpolation, for $\theta = \frac{1}{8}(d-2)^2$ we have

$$
\begin{aligned}
\|\eta(\tau)\|_{L_x^{8d/(d^2-4)}} &\lesssim \|\eta(\tau)\|_2^\theta \|\nabla\eta(\tau)\|_2^{1-\theta}\\
&\lesssim \mathrm{e}^{-\theta\lambda\tau}\mathrm{e}^{-(1-\theta)c_1\lambda\tau}\\
&\lesssim \mathrm{e}^{-\theta\lambda\tau}.
\end{aligned}
$$

Plugging this estimate into (2.31), for $d = 3, 4$ we obtain

$$
\begin{aligned}
\||\eta|^{4/(d-2)}|\nabla W|\|_{N([t,\infty))} &\lesssim C_2\left(\int_t^\infty \mathrm{e}^{-\lambda(d-2)\tau}\,\mathrm{d}\tau\right)^{1/2}\\
&\lesssim C_2\cdot\lambda^{-(d-2)/2}\mathrm{e}^{-(d-2)\lambda t/2}\\
&\leqslant \tfrac{1}{100}\mathrm{e}^{-c_1\lambda t},
\end{aligned}
$$

which is clearly enough for us.

It remains to bound $(2.28\,b)$ for $d = 1, 2$. Since in this case $\beta_2 = m - 1 > 1$, we have

$$
\begin{aligned}
(2.28\,b) &\lesssim \||\eta|(|W|+|\eta|)^{m-2}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\lesssim \||\eta|^{\beta_1}(|\nabla W|+|\nabla\eta|)\|_{N([t,\infty))}\\
&\qquad + \||\eta|^m|\nabla\eta|\|_{N([t,\infty))} + \||\eta|^m|\nabla W|\|_{N([t,\infty))}
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim |(2.28\,a)| + \||\eta|^m|\nabla W|\|_{L^1_\tau L^2_x([t,\infty))} + \||\eta|^m|\nabla\eta|\|_{L^{2(d+2)/(d+4)}_{\tau,x}([t,\infty))}\\
&\lesssim |(2.28\,a)| + C_2\|\eta\|^m_{L^m_\tau L^{2m}_x([t,\infty))} + \|\nabla\eta\|_{S([t,\infty))}\cdot\|\eta\|^m_{L^{m\cdot(d+2)/2}_{\tau,x}([t,\infty))}.
\end{aligned}
\tag{2.32}
$$

Now, by the Gagliardo–Nirenberg inequality,

$$
\begin{aligned}
\|\eta(\tau)\|^m_{2m} &\lesssim (\|\eta(\tau)\|_2^{1-d(1/2-1/2m)}\|\nabla\eta(\tau)\|_2^{d(1/2-1/2m)})^m\\
&\lesssim \|\eta(\tau)\|_2^{d/2}\\
&\lesssim \mathrm{e}^{-\lambda\tau/2}.
\end{aligned}
$$

Similarly,

$$
\|\eta(\tau)\|^m_{m(d+2)/2} \lesssim \|\eta(\tau)\|_2^{2d/(d+2)} \lesssim \mathrm{e}^{-\lambda\tau/2}.
$$

Plugging the above estimates into (2.32) and integrating in time, for $d=1,2$ we obtain

$$
(2.28\,b) \leqslant \tfrac{1}{100}\mathrm{e}^{-c_1\lambda t},
$$

which is acceptable for us. We have completed the estimate of $(2.28\,b)$ for all $d\geqslant 1$.

Finally, consider $(2.28\,c)$. Note $\||f_z(W)| + |f_{\bar z}(W)|\|_{L^\infty_{t,x}} \leqslant C$ by (2.5) and (2.22). Thus,

$$
\begin{aligned}
(2.28\,c) &\leqslant C\int_t^\infty (\|\nabla\eta\|_{L^\infty_t L^2_x([\tau,\infty))} + \|\nabla H(\tau)\|_{L^2_x})\,\mathrm{d}\tau\\
&\leqslant C\int_t^\infty (\mathrm{e}^{-c_1\lambda\tau} + C_2\mathrm{e}^{-\lambda\tau})\,\mathrm{d}\tau\\
&\leqslant \left(\frac{C}{c_1\lambda} + C_2\mathrm{e}^{-c_1\lambda t}\right)\mathrm{e}^{-c_1\lambda t} \leqslant \tfrac{1}{100}\mathrm{e}^{-c_1\lambda t}
\end{aligned}
$$

if we take λ and $t\geqslant T_0$ large enough.

We have finished the proof of the *a priori* estimate (2.25). Thus, the proposition is proved. □

Remark 2.6. Our proof does not work for the energy-critical case because the overlap of multi-solitons no longer decays exponentially, but is just power-like; our proof relies heavily on the exponential decay property.

3. The *N*-soliton case

In this section we give the proofs of theorems 1.8 and 1.10.

We first recall (1.14), the multi-soliton profile, and observe that the difference $\eta = u - R$ satisfies the equation

$$
\begin{aligned}
i\partial_t\eta + \Delta\eta &= -f(R+\eta) + \sum_{j=1}^N f(R_j)\\
&= -(f(R+\eta) - f(R)) - \left(f(R) - \sum_{j=1}^N f(R_j)\right).
\end{aligned}
\tag{3.1}
$$

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The following lemma gives the estimates on R and the source term

$$f(R) - \sum_{j=1}^{N} f(R_j).$$

LEMMA 3.1. *There exist constants* $\tilde{C}_1 > 0$ *depending on*

$$(N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N),$$

$\tilde{c}_1 > 0$ *depending only on* α_1, $\tilde{C}_2 > 0$ *depending on*

$$(N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (v_j)_{j=1}^N, (x_j)_{j=1}^N),$$

such that the following hold: for every $1 \leqslant r \leqslant \infty$ *and* $t \geqslant 0$,

$$\|R(t)\|_r + \sum_{j=1}^{N} \|R_j(t)\|_r \leqslant \tilde{C}_1, \tag{3.2}$$

$$\left\| f(R(t)) - \sum_{j=1}^{N} f(R_j(t)) \right\|_r \leqslant \tilde{C}_1 \mathrm{e}^{-\tilde{c}_1 \sqrt{\omega_\star} v_\star t}, \tag{3.3}$$

$$\|\nabla R(t)\|_r \leqslant \tilde{C}_2, \tag{3.4}$$

$$\left\| \nabla \left(f(R(t)) - \sum_{j=1}^{N} f(R_j(t)) \right) \right\|_r \leqslant \tilde{C}_2 \exp(-\tilde{c}_1 \sqrt{\omega_\star} v_\star t). \tag{3.5}$$

Here recall $\omega_\star = \min\{\omega_j,\ 1 \leqslant j \leqslant N\}$ *and* $v_\star = \min\{|v_k - v_j| \colon 1 \leqslant k \neq j \leqslant N\}$.

Proof of lemma 3.1. The estimates (3.2) and (3.4) follow directly from (1.10) and (1.13).

To simplify the notation, define

$$\Omega := (N, \alpha_1, \alpha_2, d, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N).$$

To prove (3.3), we start with the pointwise estimate. By (3.2) and lemma 2.2,

$$\begin{aligned}
&\left| f(R(t,x)) - \sum_{j=1}^{N} f(R_j(t,x)) \right| \\
&\quad = \left| \sum_{j=1}^{N} g(|R(t,x)|^2) R_j(t,x) - \sum_{j=1}^{N} g(|R_j(t,x)|^2) R_j(t,x) \right| \\
&\quad \leqslant \sum_{j=1}^{N} |g(|R(t,x)|^2) - g(|R_j(t,x)|^2)| \cdot |R_j(t,x)| \\
&\quad \lesssim_\Omega \sum_{j=1}^{N} (|R(t,x) - R_j(t,x)| + |R(t,x) - R_j(t,x)|^{2\alpha_1}) \cdot |R_j(t,x)| \\
&\quad \lesssim_\Omega \sup_{k \neq j} (|R_k(t,x)| \cdot |R_j(t,x)| + (|R_k(t,x)| \cdot |R_j(t,x)|)^{2\alpha_1}).
\end{aligned} \tag{3.6}$$

It suffices to treat the first term in the bracket of (3.6). The second term is estimated similarly.

By (1.13), for any $\delta < 1$,

$$|R_k(t,x)| \lesssim_{d,\delta} \exp(-\delta\sqrt{\omega_k}|x - v_k t - x_k|) \quad \forall k = 1,\dots,N.$$

Now fix some $\delta < 1$ for the rest of the proof.

Clearly, for any $k \neq j$,

$$|R_k(t,x)| \cdot |R_j(t,x)| \lesssim_{d,\delta} \exp(-\delta(\sqrt{\omega_k}|x - v_k t - x_k| + \sqrt{\omega_j}|x - v_j t - x_j|)). \tag{3.7}$$

By the triangle inequality, it is clear that, for all $j \neq k$, $x \in \mathbb{R}^d$, $t \geqslant 0$,

$$\begin{aligned}\sqrt{\omega_k}|x - v_k t - x_k| &+ \sqrt{\omega_j}|x - v_j t - x_j| \\ &\geqslant \min\{\sqrt{\omega_j},\sqrt{\omega_k}\}(|v_j - v_k|t - |x_k - x_j|) \\ &\geqslant \sqrt{\omega_\star}(v_\star t - |x_k - x_j|).\end{aligned} \tag{3.8}$$

Plugging (3.8) into (3.7), for any $k \neq j$, we obtain

$$\begin{aligned}|R_k(t,x)| &\cdot |R_j(t,x)| \\ &\lesssim_{\Omega} \exp(-\tfrac{1}{2}\delta\sqrt{\omega_\star}v_\star t) \\ &\qquad \times \exp(-\tfrac{1}{2}\delta(\sqrt{\omega_k}|x - v_k t - x_k| + \sqrt{\omega_j}|x - v_j t - x_j|)).\end{aligned} \tag{3.9}$$

Now (3.3) follows easily from (3.9) and (3.6).

Finally, to show (3.5) we only need to recall (2.3) and write

$$\nabla(f(R)) - \sum_{j=1}^{N}\nabla(f(R_j)) = \sum_{j=1}^{N}(f_z(R) - f_z(R_j))\nabla R_j + \sum_{j=1}^{N}(f_{\bar z}(R) - f_{\bar z}(R_j))\overline{\nabla R_j}.$$

Thanks to the above decomposition, the rest of the proof is essentially a repetition of that of (3.3). The only difference is that the constants will depend on the velocities v_j due to the terms ∇R_j. We omit further details. □

Now we are ready to complete the proof of theorem 1.8.

Proof of theorem 1.8. By (3.1), we need to solve the integral equation (2.8) for η on $[0,\infty) \times \mathbb{R}^d$, with $W = R$ and $H = f_1(R) - \sum_{j=1}^{N} f_1(R_j)$. By lemma 3.1, conditions (2.7) and (2.11) are satisfied. Thus, by proposition 2.3, there exists $\eta \in C([0,\infty), H^1)$ with $\|\langle\nabla\rangle\eta\|_{S([t,\infty))}$ decaying exponentially in t. Since the soliton piece $R \in C([0,\infty), H^1)$, so is $u(t)$. □

Proof of theorem 1.10. This is similar to the proof of theorem 1.8. We need to apply proposition 2.4 with $W = R$ and $H = f(R) - \sum_{j=1}^{N} f(R_j)$. By lemma 3.1, the condition (2.22) is satisfied. By proposition 2.4, there exists $\eta \in C([T_0,\infty), H^1)$ with $\|\langle\nabla\rangle\eta\|_{S([t,\infty))}$ (in particular $\|\eta(t)\|_{H^1}$) decaying exponentially in t. □

4. An infinite soliton train

In this section we construct an infinite soliton train solution to (1.1).

Thanks to proposition 2.3, the proof of theorem 1.2 is reduced to checking the regularity of the infinite soliton R_∞ and the tail estimates.

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LEMMA 4.1 (regularity of R_∞). *Let* R_∞ *be given as in (1.3) and recall* $f_1(z) = |z|^\alpha z$. *Then*

(1) *there is a constant* $\tilde{A}_1 > 0$ *depending only on* (A_ω, d, α), *such that*

$$\|R_\infty(t)\|_\infty + \|R_\infty(t)\|_{r_1} + \sum_{j=1}^{\infty}(\|\tilde{R}_j(t)\|_\infty + \|\tilde{R}_j(t)\|_{r_1}) \leqslant \tilde{A}_1 \quad \forall t \geqslant 0, \tag{4.1}$$

$$\|f_1(R_\infty(t))\|_{(\alpha+2-\epsilon_1)/(\alpha+1)} + \sum_{j=1}^{\infty}\|f_1(\tilde{R}_j(t))\|_{(\alpha+2-\epsilon_1)/(\alpha+1)} \leqslant \tilde{A}_1 \quad \forall t \geqslant 0, \tag{4.2}$$

where $0 < \epsilon_1 < 1$ *is a small constant depending on* (r_1, α);

(2) *there are constants* $\tilde{c}_1 > 0$, $\tilde{c}_2 > 0$ *depending only on* (α, d), $C_1 > 0$, $C_2 > 0$ *depending on* $(\tilde{A}_1, d, \alpha)$, *such that*

$$\left\|f_1(R_\infty(t)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t))\right\|_\infty \leqslant C_1 \mathrm{e}^{-\tilde{c}_1 v_\star t} \quad \forall t \geqslant 0, \tag{4.3}$$

$$\left\|f_1(R_\infty(t)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t))\right\|_{(\alpha+2)/(\alpha+1)} \leqslant C_2 \mathrm{e}^{-\tilde{c}_2 v_\star t} \quad \forall t \geqslant 0. \tag{4.4}$$

Proof of lemma 4.1. The inequalities (4.1), (4.2) are simple consequences of (1.5). The proof of the inequality (4.3) is similar to the proof of (3.3) and we sketch the modifications. By using (4.1) and (1.13) (fix $\eta < 1$), we have

$$\begin{aligned}
&\left|f_1(R_\infty(t,x)) - \sum_{j=1}^{\infty} f_1(\tilde{R}_j(t,x))\right| \\
&\quad \lesssim \sum_{j=1}^{\infty} ||R_\infty(t,x)|^\alpha - |\tilde{R}_j(t,x)|^\alpha| \cdot |\tilde{R}_j(t,x)| \\
&\quad \lesssim \sum_{j=1}^{\infty} |R_\infty(t,x) - \tilde{R}_j(t,x)|^{\min\{\alpha,1\}} |\tilde{R}_j(t,x)| \\
&\quad \lesssim \sum_{j=1}^{\infty} \left|\sum_{k\neq j} \omega_k^{1/\alpha} \exp(-\eta\sqrt{\omega_k}|x - v_k t|)\right|^{\min\{1,\alpha\}} \omega_j^{1/\alpha} \exp(-\eta\sqrt{\omega_j}|x - v_j t|) \\
&\quad \lesssim \sum_{j=1}^{\infty} \omega_j^{1/\alpha} \left|\sum_{k\neq j} \omega_k^{1/\alpha} \exp(-\eta(\sqrt{\omega_k}|x - v_k t| + \sqrt{\omega_j}|x - v_j t|))\right|^{\min\{1,\alpha\}}.
\end{aligned}$$

By (1.6), we have

$$\sqrt{\omega_k}|x - v_k t| + \sqrt{\omega_j}|x - v_j t| \geqslant v_\star t \quad \forall t \geqslant 0.$$

Hence, (4.3) follows from the above estimate and (1.5). Finally, (4.4) follows from interpolating the estimates (4.2), (4.3). □

We now complete the proof of theorem 1.2.

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Proof of theorem 1.2. We first rewrite (1.7) as

$$\eta(t) = \mathrm{i}\int_t^\infty \mathrm{e}^{\mathrm{i}(t-\tau)\Delta}\left(f_1(R_\infty+\eta) - f_1(R_\infty) + f_1(R_\infty) - \sum_{j=1}^\infty f_1(\tilde{R}_j)\right)\mathrm{d}\tau.$$

We then apply proposition 2.3 with $W = R_\infty$ and $H = f_1(R_\infty) - \sum_{j=1}^\infty f_1(\tilde{R}_j)$. By lemma 4.1, it is easy to check that the condition (2.7) is satisfied. The theorem follows easily. □

5. Half-kinks

We conclude this paper by giving the proofs of theorem 1.17 and proposition 1.13.

Proof of theorem 1.17. The proof is similar to that of theorem 1.10. The only difference is that, due to the non-zero background, the profile KR is no longer in $\mathcal{C}(\mathbb{R}, H^1)$ but only in $\mathcal{C}(\mathbb{R}, H^1_{\mathrm{loc}})$. □

Proof of proposition 1.13. Assume $\omega = \omega_1$ and define $\zeta_1 := \zeta(\omega_1)$. Take any $\phi_0 \in (0, \zeta_1)$ and let ϕ be the solution to (1.18) on the maximal interval of existence I and with initial data

$$\phi(0) = \phi_0, \qquad \phi'(0) = \sqrt{\omega_1\phi_0^2 - 2F(\phi_0)}.$$

We first prove that $\phi(x) \in (0, \zeta_1)$ for any $x \in I$. Indeed, assume on the contrary that there exists x_0 such that $\phi(x_0) = 0$ or $\phi(x_0) = \zeta_1$. From our choice of initial data for ϕ, it follows that, for any $x \in I$, ϕ satisfies the first integral identity

$$-\tfrac{1}{2}|\phi'(x)|^2 = F(\phi(x)) - \tfrac{1}{2}\omega_1|\phi(x)|^2. \tag{5.1}$$

In particular, (5.1) at $x = x_0$ implies

$$\phi'(x_0) = 0.$$

However, by the Cauchy–Lipschitz theorem it follows that $\phi \equiv 0$ or $\phi \equiv \zeta_1$ on I, which contradicts $\phi_0 \in (0, \zeta_1)$. Hence, for all $x \in I$ we have $\phi(x) \in (0, \zeta_1)$, which implies, in particular, that $I = \mathbb{R}$.

Since $\phi_0 \in (0, \zeta_1)$, we have $\phi'(0) > 0$ and, by continuity, $\phi'(x) > 0$ for x close to 0. We claim that in fact $\phi'(x) > 0$ on $\mathbb{R}$. Indeed, assume by contradiction that there exists x_0 such that $\phi'(x_0) = 0$. From the first integral (5.1), this implies that

$$F(\phi(x_0)) - \tfrac{1}{2}\omega_1|\phi(x_0)|^2 = 0.$$

Therefore, $\phi(x_0) = 0$ or $\phi(x_0) = \zeta_1$, but we have proved that to be impossible. Hence, $\phi' > 0$ on $\mathbb{R}$.

We consider now the limits of ϕ at $\pm\infty$. Define

$$l := \lim_{x\to-\infty}\phi(x), \qquad L := \lim_{x\to+\infty}\phi(x).$$

Let us show that $l = 0$ and $L = \zeta_1$. Indeed, by (5.1), we have $F(l) - \frac{1}{2}\omega_1 l^2 = 0$ (indeed otherwise it would imply $|\phi'| > \delta > 0$ for x large: a contradiction with the boundedness of ϕ). Since $\phi \in (0, \zeta_1)$ and ϕ is increasing, this implies $l = 0$ and $L = \zeta_1$.

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Let us now show that ϕ is unique up to translations. Assume by contradiction that there exists a solution $\tilde{\phi} \in \mathcal{C}^2(\mathbb{R})$ to (1.18) satisfying the connection property (1.20). Since we claim uniqueness only up to translation, we can assume that $\phi(0) \in (0, \zeta_1)$. In addition, since we have shown that ϕ varies continuously from 0 to ζ_1, we can also assume without loss of generality that $\phi(0) = \phi_0 = \tilde{\phi}(0)$. The first integral identity for $\tilde{\phi}$ is, for any $x \in \mathbb{R}$,

$$\tfrac{1}{2}|\tilde{\phi}'(x)|^2 - \tfrac{1}{2}\omega_1|\tilde{\phi}(x)|^2 + F(\tilde{\phi}(x)) = \tfrac{1}{2}|\tilde{\phi}'(0)|^2 - \tfrac{1}{2}\omega_1|\tilde{\phi}(0)|^2 + F(\tilde{\phi}(0)).$$

In particular, since $\lim_{x\to\pm\infty} \tilde{\phi}'(x) = 0$, and 0 and ζ_1 are zeros of $\zeta \to F(\zeta) - \frac{1}{2}\omega\zeta^2$, we have

$$\tfrac{1}{2}|\tilde{\phi}'(0)|^2 = \tfrac{1}{2}\omega_1|\tilde{\phi}(0)|^2 - F(\tilde{\phi}(0)).$$

As previously, it is not hard to see that ϕ' has a constant sign, which must be positive due to the limits of ϕ at $\pm\infty$. Therefore, $\tilde{\phi}'(0) = \phi'(0)$ and the uniqueness follows from the Cauchy–Lipschitz theorem. Differentiating the equation, we see that ϕ' verifies

$$-(\phi')'' + (\omega_1 - f'(\phi))\phi' = 0.$$

Since

$$\lim_{x\to-\infty}(\omega_1 - f'(\phi)) = \omega_1 - f'(0) > 0$$

and

$$\lim_{x\to+\infty}(\omega_1 - f'(\phi)) = \omega_1 - f'(\zeta(\omega_1)) > 0,$$

(1.21) follows from classical ordinary differential equation arguments. □

6. Multi-soliton up to time zero

In this section we add extra conditions to theorem 1.10 so that the solution exists in $[0, \infty)$.

THEOREM 6.1. *Consider (1.1) with $f(u) = g(|u|^2)u$ satisfying (1.9) and (1.12). Let R be the same as in (1.14) and define $v_\star$ as in (1.16). Suppose*

$$\bar{v} := \max_{k=1,\dots,N} |v_k| \leqslant M v_\star^M \quad \textit{for some } M \geqslant 1. \tag{6.1}$$

There exist constants $C > 0$, $c_1 > 0$, $c_2 > 0$ and $v_\sharp = v_\sharp(M) \gg 1$ such that if $v_\star > v_\sharp$, then there is a unique solution $u \in C([0,\infty), H^1)$ to (1.1) satisfying

$$\mathrm{e}^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} + \mathrm{e}^{c_2 v_\star t}\|\nabla(u - R)\|_{S([t,\infty))} \leqslant C \quad \forall t \geqslant 0.$$

REMARK 6.2. The extra condition (6.1) is satisfied, for example, if $v_j = \mu\tilde{v}_j$ for some fixed $\tilde{v}_j$ and μ is an increasing parameter.

Sketch of the proof. Following the proof of lemma 3.1, the assumption (2.22) of proposition 2.4 is satisfied with

$$T_0 = 1, \qquad \lambda = cv_\star, \qquad C_1 = C_0, \qquad C_2 = C_0\bar{v},$$

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where $c = C(\alpha_1)\sqrt{\min_{j=1,\ldots,N}\{\omega_j\}}$ and $C_0 = C_0(d, N, \alpha_1, \alpha_2, (\omega_j)_{j=1}^N, (x_j)_{j=1}^N)$ are independent of $(v_j)_{j=1}^N$. The smallness condition used in the proof of proposition 2.4 is of the form

$$\mathrm{e}^{-c\lambda_* t}(1 + C_2) \leqslant \varepsilon \tag{6.2}$$

for some small $\varepsilon > 0$ independent of C_2. It can be satisfied either by fixing $\lambda_* \gg 1$ independent of C_2 and then requiring $t \geqslant T_0$ with $T_0 = T_0(C_2)$ large (as in the proof of proposition 2.3), or by fixing $T_0 = 1$, using the assumption $C_2 = C_0\bar{v} \leqslant C_0 M v_\star^M$, and requiring $v_\star$ to be sufficiently large. In the latter case we get a solution $\eta(t)$ for $1 \leqslant t < \infty$. Since the soliton piece $R \in C([0,\infty), H^1)$ and $\|\eta(t = 1)\|_{H^1}$ can be chosen sufficiently small by enlarging λ_*, we can extend $\eta(t)$ up to time $t = 0$ with $O(1)$ estimates by local existence theory in H^1. □

The following result is L^2-theory for L^2-subcritical and critical nonlinearities.

THEOREM 6.3. *Consider (1.1) with $f(u) = g(|u|^2)u$ satisfying (1.9) and (1.12). Furthermore, assume $\alpha_2 \leqslant 2/d$. Let R be the same as in (1.14) and define $v_\star$ as in (1.16). There exist constants $C > 0$, $c_1 > 0$ and $v_\sharp \gg 1$, such that if $v_\star > v_\sharp$, then there is a unique solution $u \in C([0,\infty), L^2)$ to (1.1) satisfying*

$$\mathrm{e}^{c_1 v_\star t}\|u - R\|_{S([t,\infty))} \leqslant C \quad \forall t \geqslant 0.$$

Sketch of the proof. We shall modify the first part of the proof of proposition 2.4, which bounds $\eta = u - R$ in $S([t,\infty))$. In that part, estimates for $\nabla\eta$ are only used to bound the global nonlinear terms $\sum_{j=1,2}|\eta|^{2\alpha_j+1}$ in the dual Strichartz space $N([t,\infty))$. Suppose $\alpha_2 \leqslant 2/d$ and

$$\|\eta\|_{S([t,\infty))} \leqslant \mathrm{e}^{-\lambda t} \quad \forall t > 0.$$

For $m = 2\alpha_j + 1$, $r = m + 1$ and a such that $2/a + d/r = d/2$, we have

$$\||\eta|^m\|_{N([t,\infty))} \leqslant \||\eta|^m\|_{L^{a'}L^{r'}(t,\infty)} \leqslant \|\eta\|^m_{L^{a'm}L^{r'm}(t,\infty)},$$

where $r' = r/(r-1)$ and $a' = a/(a-1)$. Let q and b be such that

$$q = r'm, \qquad \frac{2}{b} + \frac{d}{q} = \frac{d}{2}.$$

We claim that $\alpha_j \leqslant 2/d$ is equivalent to

$$a'm \leqslant b. \tag{6.3}$$

Indeed, (6.3) amounts to

$$\frac{2}{a'} \geqslant \frac{2m}{b} = m\left(\frac{d}{2} - \frac{d}{q}\right) = m\frac{d}{2} - \frac{d}{r'},$$

i.e.

$$m\frac{d}{2} \leqslant \frac{d}{r'} + \frac{2}{a'} = d + 2 - \left(\frac{d}{r} + \frac{2}{a}\right) = \frac{d}{2} + 2,$$

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which is exactly $\alpha_j \leqslant 2/d$. Thus,

$$
\begin{aligned}
\| |\eta|^m \|_{N([t,\infty))} &\leqslant \left(\int_t^\infty \|\eta(s)\|_{L^q}^{a'm} \, \mathrm{d}s \right)^{1/a'} \\
&= \left(\sum_{k=0}^\infty \int_{t+k}^{t+k+1} \|\eta(s)\|_{L^q}^{a'm} \, \mathrm{d}s \right)^{1/a'} \\
&\leqslant \left(\sum_{k=0}^\infty \left(\int_{t+k}^{t+k+1} \|\eta(s)\|_{L^q}^{b} \, \mathrm{d}s \right)^{a'm/b} \right)^{1/a'} \\
&\leqslant \left(\sum_{k=0}^\infty (\mathrm{e}^{-b\lambda(t+k)})^{a'm/b} \right)^{1/a'} \\
&= C\mathrm{e}^{-m\lambda t}.
\end{aligned}
$$

We have used (6.3) in the second inequality. The rest of the proof is the same as the first part of the proof of proposition 2.4. □

The following result is valid for both L^2-subcritical and L^2-supercritical nonlinearities. Its proof extends that of proposition 2.3.

THEOREM 6.4. *Consider (1.1) with* $f(u) = g(|u|^2)u$ *satisfying (1.9) and (1.12). Let* $\beta_j = 2\alpha_j$, $j = 1, 2$, *with* $0 < \beta_1 \leqslant \beta_2 < \alpha_{\max}$. *Assume for* $d \geqslant 3$ *that*

$$\frac{\beta_2}{1+\beta_2} \leqslant \beta_1 \leqslant \beta_2 \quad \text{if } 0 < \beta_2 < \tfrac{1}{2}\alpha_{\max}, \tag{6.4}$$

$$\frac{\beta_2}{\alpha_{\max}+1-\beta_2} < \beta_1 \leqslant \beta_2 \quad \text{if } \tfrac{1}{2}\alpha_{\max} \leqslant \beta_2 < \alpha_{\max}, \tag{6.5}$$

and for $d = 1, 2$ *we assume (6.4) only. Then we can choose* r_1 *and* r_2 *such that*

$$0 \leqslant r_1 - 2 \leqslant \beta_1 \leqslant \beta_2 \leqslant r_2 - 2 < \alpha_{\max}, \tag{6.6}$$

$$r_1\beta_2 \leqslant r_1 r_2 - r_1 - r_2 \leqslant r_2\beta_1. \tag{6.7}$$

Let R *be the same as in (1.14) and define* $v_\star$ *as in (1.16). For any choice of* r_1, r_2 *satisfying (6.6), (6.7), there exist constants* $C > 0$, $c_1 > 0$ *and* $v_\sharp \gg 1$ *such that if* $v_\star > v_\sharp$, *then there is a unique solution* $u = R + \eta$ *to (1.1) on* $[0, +\infty)$ *satisfying*

$$\|\eta(t)\|_{L^{r_1} \cap L^{r_2}} \leqslant C\mathrm{e}^{-c_1 v_\star t} \quad \forall t \geqslant 0. \tag{6.8}$$

Moreover,

$$\|\eta\|_{S([t,\infty))} \leqslant C\mathrm{e}^{-c_1 v_\star t} \quad \forall t \geqslant 0.$$

Note the first strict inequality in (6.5) compared with (6.4). See figure 2 for the β_1–β_2 region when $d = 3$. Note also that (6.4) and (6.5) are equivalent (when $d \geqslant 3$) to

$$\beta_1 \leqslant \beta_2 \leqslant \frac{\beta_1}{1-\beta_1} \qquad \text{if } 0 < \beta_1 < \frac{\alpha_{\max}}{\alpha_{\max}+2}, \tag{6.9}$$

$$\beta_1 \leqslant \beta_2 < \frac{(\alpha_{\max}+1)\beta_1}{1+\beta_1} \quad \text{if } \frac{\alpha_{\max}}{\alpha_{\max}+2} \leqslant \beta_1 < \alpha_{\max}. \tag{6.10}$$

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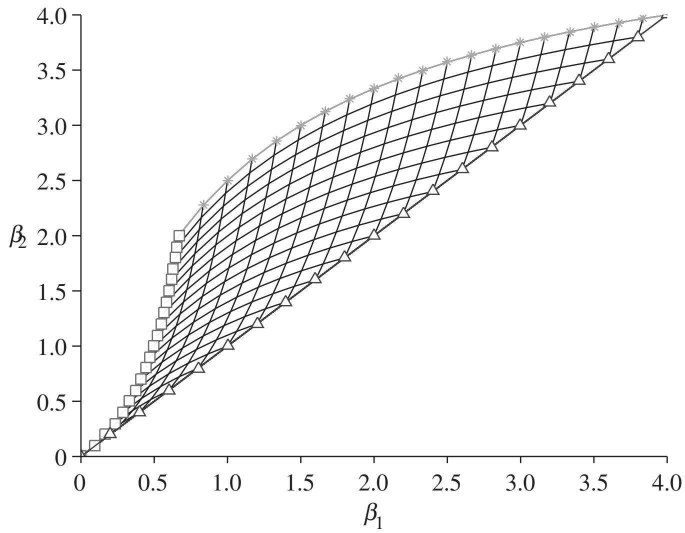

Figure 2. Region of admissible β_1, β_2 in theorem 6.4 for $d = 3$.

Sketch of the proof of theorem 6.4. For $j = 1, 2$ and $\theta_j = d(1/2 - 1/r_j) \in (0, 1)$, we have

$$\|\eta(t)\|_{L^{r_j}} \lesssim \int_t^\infty |t-\tau|^{-\theta_j} \sum_{k=1,2} \||\eta(\tau)|^{1+\beta_k}\|_{r_j'} \, d\tau + (\text{nice terms}),$$

where $r_j' = r_j/(r_j - 1)$. The nice terms can be estimated as in the proof of proposition 2.3. Note that

$$\||\eta|^{1+\beta_k}\|_{r_j'} = \|\eta\|_{(r_j')(1+\beta_k)}^{1+\beta_k}$$

can be estimated by using the Hölder inequality and (6.8) if

$$r_1 \leqslant \frac{r_j}{r_j - 1}(1 + \beta_k) \leqslant r_2 \quad \forall j, k. \tag{6.11}$$

For $j = 1$, the left inequality of (6.11) is always true. The right inequality is equivalent to $r_1(1 + \beta_2) \leqslant r_2(r_1 - 1)$, or

$$r_2 \leqslant r_1(r_2 - 1 - \beta_2). \tag{6.12}$$

For $j = 2$, the right inequality of (6.11) is always true. The left inequality is equivalent to $r_1(r_2 - 1) \leqslant r_2(1 + \beta_1)$, or

$$r_2(r_1 - 1 - \beta_1) \leqslant r_1. \tag{6.13}$$

Equations (6.12) and (6.13) are equivalent to (6.7). Furthermore, (6.6) and (6.7) can be combined into the following equivalent condition:

$$0 \leqslant r_1 - 2 \leqslant b_1(r_1, r_2) \leqslant \beta_1 \leqslant \beta_2 \leqslant b_2(r_1, r_2) \leqslant r_2 - 2 < \alpha_{\max}, \tag{6.14}$$

where

$$b_1(r_1, r_2) = r_1 - 1 - \frac{r_1}{r_2}, \qquad b_2(r_1, r_2) = r_2 - 1 - \frac{r_2}{r_1}.$$

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It turns out that when $2 \leqslant r_1 \leqslant r_2 < \alpha_{\max} + 2$ we always have

$$0 \leqslant r_1 - 2 \leqslant b_1(r_1, r_2) \leqslant b_2(r_1, r_2) \leqslant r_2 - 2 < \alpha_{\max}.$$

Thus, for any (β_1, β_2) in the right triangle with a vertex $(b_1(r_1, r_2), b_2(r_1, r_2))$ and hypotenuse on the line $\beta_1 = \beta_2$, the pair (r_1, r_2) satisfies (6.6) and (6.7).

Denote the curve $\Gamma(r_1)$ for fixed $2 \leqslant r_1 < 2 + \alpha_{\max}$ by

$$\Gamma(r_1) = \{(b_1(r_1, r_2), b_2(r_1, r_2)) \colon r_1 \leqslant r_2 \leqslant 2 + \alpha_{\max}\},$$

which satisfies

$$b_2 = \frac{b_1}{r_1 - 1 - b_1}, \qquad b_1 = \frac{(r_1 - 1)b_2}{1 + b_2},$$

and starts at $(r_1 - 2, r_1 - 2)$. It goes to infinity with asymptote $b_1 = r_1 - 1$ for $d = 1, 2$, and ends at $\Sigma(2 + \alpha_{\max})$ as defined below for $d \geqslant 3$. It moves to the right as r_1 increases.

Denote the curve $\Sigma(r_2)$ for fixed $2 < r_2 \leqslant 2 + \alpha_{\max}$ by

$$\Sigma(r_2) = \{(b_1(r_1, r_2), b_2(r_1, r_2)) \colon 2 \leqslant r_1 \leqslant r_2\},$$

which satisfies

$$b_2 = (r_2 - 1)\frac{b_1}{1 + b_1}.$$

It starts at $\Gamma(2)$ and ends at $(r_2 - 2, r_2 - 2)$. It moves upwards as r_2 increases.

For given $0 < \beta_1 < \beta_2 < \alpha_{\max}$, conditions (6.4), (6.5) imply that (β_1, β_2) is on the right of $\Gamma(2)$, and if $d \geqslant 3$, is below $\Sigma(\alpha_{\max})$. Thus, we can find $R_1 = R_1(\beta_1, \beta_2)$ and $R_2 = R_2(\beta_1, \beta_2)$ such that (β_1, β_2) is the intersection point of $\Gamma(R_1)$ and $\Sigma(R_2)$, and $R_1 \leqslant R_2$. To satisfy (6.14), we can choose either $(r_1, r_2) = (R_1, R_2)$ or any $2 \leqslant r_1 < R_1 \leqslant R_2 < r_2 < 2 + \alpha_{\max}$ as long as the intersection point $\Gamma(r_1) \cap \Sigma(r_2)$ is in an upper-left direction to (β_1, β_2).

The above shows we can estimate $|\eta|^{1+\beta_k}$ in $L^{r_j'}$ for $j, k = 1, 2$.

For the Strichartz estimate, since $(2/\theta_1, r_1)$ is admissible, with $a = (2/\theta_1)'$ we have

$$\begin{aligned}\|\eta\|_{S([t,\infty))} &\lesssim \|f(W + \eta) - f(W) + H\|_{L^a(t,\infty; L^{r_1'})} \\ &\lesssim \|\mathrm{e}^{-c_1 v_* \tau}\|_{L^a(t,\infty)} \\ &\lesssim v_*^{-1/a} \mathrm{e}^{-c_1 v_* t}.\end{aligned}$$

□

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