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Accessibility and centralizers for partially hyperbolic flows

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To the memory of Anatole Katok

Abstract. Stable accessibility of partially hyperbolic systems is central to their stable ergodicity, and we establish its C^1-density among partially hyperbolic flows, as well as in the categories of volume-preserving, symplectic, and contact partially hyperbolic flows. As applications, we obtain on one hand in each of these four categories of flows the C^1-density of the C^1-stable topological transitivity and triviality of the centralizer, and on the other hand the C^1-density of the C^1-stable K-property of the natural volume in the latter three categories.

Key words: accessibility, centralizer, dynamical systems, partially hyperbolic systems and dominated splittings
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1. *Introduction*

The renaissance of partial hyperbolicity that began in the 1990s centered on the quest for stably ergodic dynamical systems [**30**]. The Hopf argument as the central technical device brought the notion of accessibility to the fore, and this motivated results to the effect that stable accessibility of partially hyperbolic diffeomorphisms (Definition 2.1) is C^1-dense ([**18**, Main Theorem], [**3**], [**30**, Theorem 8.5]). Our first aim is to show that this also holds for flows. Among the applications is our secondary aim: a C^1-open dense set of partially hyperbolic flows that commute with no other flow.

1.1. *Statement of results.*

THEOREM 1.1. (Generic accessibility) *For any smooth compact manifold M and $r \geq 1$, C^1-stable accessibility (Definition 2.4) is C^1-dense among C^r flows in each of the following:*

- *partially hyperbolic flows;*
- *volume-preserving partially hyperbolic flows;*
- *symplectic partially hyperbolic flow; and*
- *contact partially hyperbolic flows.*

Our motivation for establishing genericity of accessibility was to adapt arguments of Burslem [14] in order to establish generic triviality of the flow centralizer (Definition 1.7), in particular a relative paucity of faithful $\mathbb{R}^k$ Anosov actions.

THEOREM 1.2. (No centralizer) *On any smooth compact manifold M and for any $r \geq 1$, C^r-flows which C^1-stably have trivial flow centralizer are C^1-dense among C^r flows in each of the following:*

- *partially hyperbolic flows;*
- *volume-preserving partially hyperbolic flows;*
- *symplectic partially hyperbolic flow; and*
- *contact partially hyperbolic flows.*

There are more direct applications of accessibility.

COROLLARY 1.3. (Generic transitivity) *On any smooth compact manifold M and for any $r \geq 1$, a C^1 open and dense set of C^r flows is topologically transitive in each of the following:*

- *volume-preserving partially hyperbolic flows;*
- *symplectic partially hyperbolic flow; and*
- *contact partially hyperbolic flows.*

Proof. Theorem 1.1 provides an open dense set of accessible such flows; their time-1 maps are accessible volume-preserving partially hyperbolic diffeomorphisms for which almost every point has a dense orbit ([10], [30, Theorem 8.3]); those points then have dense flow-orbits. □

Indeed, strong ergodic properties, such as the K-property [19, Definition 3.4.2], are similarly common if one adds the assumption of center-bunching [13].

COROLLARY 1.4. (K-property) *On any smooth compact manifold M and for any $r \geq 2$, there is a C^1-open and dense set of C^r flows for which the natural volume has the K-property (for all time-t maps) in each of the following:*

- *volume-preserving partially hyperbolic flows;*
- *symplectic partially hyperbolic flow; and*
- *contact partially hyperbolic flows.*

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Proof. Theorem 1.1 provides an open dense set of accessible such flows; for any $t \neq 0$ their time-t maps are accessible volume-preserving center-bunched partially hyperbolic diffeomorphisms and hence have the K-property [**13**, Theorem 0.1]. □

Remark 1.5. The conclusion of Corollary 1.4 implies that of Corollary 1.3, but the latter does not need center-bunching.

Remark 1.6. Being a finite-time condition, the center-bunching property appears to be open, so one could restate Corollary 1.4 as follows. On any smooth compact manifold M and for any $r \geq 2$, the set of C^r-flows for which the natural volume C^1-stably has the K-property (for all time-t maps) is C^1 dense in each of the following:

- volume-preserving partially hyperbolic flows;
- symplectic partially hyperbolic flow; and
- contact partially hyperbolic flows.

An alternative restatement would be as follows. On a smooth compact manifold M and for any $r \geq 2$, consider the set PH of C^r flows on M in each of the following:

- volume-preserving partially hyperbolic flows;
- symplectic partially hyperbolic flow; and
- contact partially hyperbolic flows.

Then the C^1-closure in PH of the set of flows for which the invariant volume *stably* has the K-property contains all center-bunched flows in PH.

As Corollary 1.4 indicates, accessibility is central to the ergodic theory of partially hyperbolic dynamical systems (Definition 2.1), the theory of which was revived two decades after its founding [**11**, **12**] when Pugh and Shub sought non-hyperbolic examples of volume-preserving dynamical systems that are *stably* ergodic [**24**, **31**–**33**]. Stable ergodicity was established by adapting the Hopf argument [**19**, §7.1.2], and accessibility is the key ingredient [**13**].

In this context, flows (as opposed to diffeomorphisms) have received little attention. Accessibility of a partially hyperbolic flow and of its time-1 map are equivalent, so a theory for diffeomorphisms suffices for establishing ergodicity for partially hyperbolic flows. Indeed, the initial examples of stably ergodic partially hyperbolic diffeomorphisms were time-1 maps of hyperbolic flows. Moreover, the *stability* of ergodicity and accessibility of the time-1 map implies their stability for the flow.

Considering flows becomes salient, however, when investigating the *prevalence* of (stable) accessibility and ergodicity. Diffeomorphisms are rarely the time-1 map of a flow, so density or genericity results for diffeomorphisms do not automatically imply like results for flows. The issue is that one needs to argue that the flows themselves rather than just their time-1 maps can be perturbed in a desired fashion.

1.2. *Centralizers.* Our interest in accessibility (that is, in Theorem 1.2) arose from a desire to understand the centralizer of flows beyond hyperbolic ones [**4**, **19**]. Partially hyperbolic flows are a natural next step, and this led us to wanting to adapt the pertinent result [**14**, Theorem 1.2] to flows—and its proof uses accessibility in an essential way.

The centralizer of a dynamical system reflects the symmetries of that system, and this leads to the expectation that the centralizer of a (sufficiently complex; see [19, Example 1.8.8]) dynamical system is often small. (It also reflects non-uniqueness of conjugacies [19, p. 97].) Since the notion is relative to an ambient group [19, Remark 1.8.9], we make the needed terminology explicit.

Definition 1.7. The *flow centralizer* of a C^r flow consists of the C^r flows that commute with it, and there are different types of triviality [27]. We say that the centralizer is *trivial* if it consists of constant scalings of the flow, that is, the generating vector fields of commuting flows are constant multiples of the given vector field; in this case we also say that the flow is *self-centered* or that it commutes with no other flow.

A vector field has *quasi-trivial* flow centralizer if vector fields that commute with it are its multiple by a smooth scalar factor.

Remark 1.8. (Hyperbolicity implies small centralizer [19, §9.1]) An Anosov flow has trivial flow centralizer [19, Corollary 9.1.4], and this extends to kinematic-expansive flows on a connected space with at most countably many chain components, all of which are topologically transitive [19, Theorem 9.1.3].

Quasi-triviality of the centralizer holds for (Bowen–Walters) expansive flows [29] and indeed C^r-generic flows [27] (including volume-preserving ones [9]).

An open and dense subset of C^∞ Axiom-A flows with a strong transversality condition has (properly) trivial flow centralizer [34], as do transitive Komuro expansive flows [7] (this includes the Lorenz attractor), and C^1-generic sectional Axiom-A flows [6, 8].

Indeed, hyperbolic flows usually have small centralizers [19, Theorem 9.1.3], and we extend this and the requisite accessibility result to partially hyperbolic flows.

One can also look for diffeomorphisms that commute with a flow; the set of these is the *diffeomorphism centralizer* of the flow. Even Anosov flows can have non-trivial diffeomorphism centralizers [28, §5].

While there has been interesting work beyond the hyperbolic context (the centralizer is quasi-trivial for a C^1-generic flow with at most finitely many sinks or sources [28], trivial if the flow moreover has at most countably many chain-recurrence classes), our results produce *open* dense sets with the desired properties, whereas elsewhere, often only 'residual' is known.

If one thinks of the centralizer question as the possibility of embedding a flow in a faithful $\mathbb{R}^2$-action (or a diffeomorphism into a faithful $\mathbb{Z}^2$-action), then a deeper probe could focus on the classification (or rigidity) of $\mathbb{R}^k$-actions for $k \geq 2$ (e.g., aiming to show that they are necessarily algebraic if the dynamics is hyperbolic). Great efforts have already been devoted to this aim [17, 26], and quite recently, these have been pushed into the partially hyperbolic realm for discrete time (for smooth, ergodic perturbations of certain algebraic systems, the smooth centralizer is either virtually $\mathbb{Z}^l$ or contains a smooth flow [5, 16]). Also, the centralizer of a partially hyperbolic $\mathbb{T}^3$-diffeomorphism homotopic to an Anosov automorphism is virtually trivial unless the diffeomorphism is smoothly conjugate to its linear part [22].

2. *Background*

In this section we review needed definitions and previous results. We provide much of the basic background elsewhere [19], but we define two main notions here.

Definition 2.1. (Partial hyperbolicity) An *embedding* f is said to be *(strongly) partially hyperbolic* on a compact f-invariant set Λ if there exist numbers $C > 0$,

$$0 < \lambda_1 \leq \mu_1 < \lambda_2 \leq \mu_2 < \lambda_3 \leq \mu_3, \quad \text{with } \mu_1 < 1 < \lambda_3,$$

and an invariant splitting into non-trivial stable, central and unstable subbundles

$$T_x M = E^s(x) \oplus E^c(x) \oplus E^u(x), \quad d_x f E^\tau(x) = E^\tau(f(x)),\ \tau = s, c, u,$$

such that if $n \in \mathbb{N}$, then (with $⦀A⦀ := \|A^{-1}\|^{-1}$)

$$C^{-1}\lambda_1^n \leq ⦀d_x f^n \upharpoonright E^s(x)⦀ \leq \|d_x f^n \upharpoonright E^s(x)\| \leq C\mu_1^n,$$
$$C^{-1}\lambda_2^n \leq ⦀d_x f^n \upharpoonright E^c(x)⦀ \leq \|d_x f^n \upharpoonright E^c(x)\| \leq C\mu_2^n,$$
$$C^{-1}\lambda_3^n \leq ⦀d_x f^n \upharpoonright E^u(x)⦀ \leq \|d_x f^n \upharpoonright E^u(x)\| \leq C\mu_3^n.$$

In this case we set $E^{cs} := E^c \oplus E^s$ and $E^{cu} := E^c \oplus E^u$.

A *flow* is said to be *partially hyperbolic* on a compact flow-invariant set if its time-1 map is partially hyperbolic on it, and *uniformly hyperbolic* if the center direction of the time-1 map consists only of the flow direction. In either case we say that a dynamical system is partially hyperbolic on an invariant set.

A *partially hyperbolic contact flow* is a partially hyperbolic flow generated by the *Reeb* vector field $R = R_\alpha$ of some contact form α; this is the vector field defined by $d\alpha(R, \cdot) \equiv 0$ and $\alpha(R) \equiv 1$ (the first condition constrains R_α to a one-dimensional subspace in each tangent space, and then the second condition selects a unique (non-zero!) element of that subspace [23, Lemma/Definition 1.1.9]).

Here, a *contact form* is a smooth 1-form α such that $\alpha \wedge (d\alpha)^n$ is a volume; this implies that the underlying manifold is $(2n+1)$-dimensional.

Remark 2.2. (Contact flows versus Reeb flows) The usual definition of a contact flow is that the flow preserves a contact form (or a contact structure; these are also called *infinitesimal automorphisms* [23, Definition 1.5.7]). A (cooriented) contact structure is a hyperplane field $\xi = \ker \alpha$ for some (locally defined) contact form α. The classical examples of contact flows (such as geodesic flows and surgered variants of them [20]) are in fact the *Reeb* flow of the invariant contact form α, that is, the flow generated by the Reeb vector field R of α.

Making this our definition removes ambiguity, but it is illuminating to clarify the relation between Reeb flows and flows that preserve a contact structure or form†. This is done to some extent in [21, §2.2]: if the flow of a nowhere-vanishing vector field X preserves a contact structure $\ker \alpha$ *transverse* to X, then it is the Reeb flow of the contact form $\alpha/\alpha(X)$; this transversality holds for topologically transitive flows without fixed

† If there are no fixed points, then the latter two are equivalent [23, p. 180].

points and for Anosov 3-flows. (The set of non-transversality of a vector field that preserves a contact structure is a smooth codimension-1 submanifold of Euler characteristic 0 that separates the ambient manifold invariantly into two open sets; in dimension 3 its connected components are 2-tori on which the flow is not expansive.) Flows with trivial flow centralizer that preserve a contact *form* are its Reeb flows (up to constant scaling).

For a partially hyperbolic set Λ and $x \in \Lambda$ there exist local stable and unstable manifolds that define global stable and unstable manifolds denoted by $W^s(x)$ and $W^u(x)$, respectively.

Remark 2.3. (Persistence of partial hyperbolicity) For a flow Φ and a partially hyperbolic set Λ for Φ with splitting $T_\Lambda M = E^u \oplus E^c \oplus E^s$ and continuous invariant cone fields $\mathcal{C}^u, \mathcal{C}^s, \mathcal{C}^{cu}$, and $\mathcal{C}^{cs}$, containing E^u, E^s, E^{cu}, and E^{cs}, respectively, there exist neighborhoods U_0 of Λ and $\mathcal{U}_0$ of Φ and cone fields $\mathcal{C}_0^u$, $\mathcal{C}_0^s$, $\mathcal{C}_0^{cu}$, and $\mathcal{C}_0^{cs}$ over U_0 such that if $\Psi \in \mathcal{U}_0$ and $\Lambda' \subset U_0$ is a compact Ψ-invariant set, then Λ' is partially hyperbolic with a splitting $T_\Delta M = E^s_\Psi \oplus E^c_\Psi \oplus E^u_\Psi$ such that E^u_Ψ, E^s_Ψ, E^{cu}_Ψ, and E^{cs}_Ψ are contained in $\mathcal{C}_0^u$, $\mathcal{C}_0^s$, $\mathcal{C}_0^{cu}$, and $\mathcal{C}_0^{cs}$, respectively.

To avoid confusion we will sometimes refer to these neighborhoods as $U_0(\Phi, \Lambda)$ and $\mathcal{U}_0(\Phi, \Lambda)$ when we consider different flows and different partially hyperbolic sets.

Definition 2.4. (Accessibility) Two points p, q in a partially hyperbolic set $\Lambda \subset M$ are *accessible* if there are points $z_i \in M$ with $z_0 = p$, $z_\ell = q$, such that $z_i \in V^\alpha(z_{i-1})$ for $i = 1, \dots, \ell$ and $\alpha = s$ or u. The collection of points $z_0, z_1, \dots, z_\ell$ is called the *us-path* connecting p and q and is denoted variously by $[p, q]_f = [p, q] = [z_0, z_1, \dots, z_\ell]$. (Note that there is an actual path from p to q that consists of pieces of smooth curves on local stable or unstable manifolds with the z_i as endpoints.)

Accessibility is an equivalence relation and the collection of points accessible from a given point p is called the *accessibility class* of p.

A partially hyperbolic set Λ is *bisaturated* if $W^u(x) \subset \Lambda$ and $W^s(x) \subset \Lambda$ for all $x \in \Lambda$, and a bisaturated partially hyperbolic set is said to be *accessible* if the accessibility class of any point is the entire set, or, in other words, if any two points are accessible.

If the entire manifold is partially hyperbolic for a flow, then it is bisaturated. In this case, the flow is *accessible* if the entire manifold is an accessibility class.

A pair (Φ, Λ) of a dynamical system and a partially hyperbolic set of it is *accessible* on $X \subset M$ if for every $p \in X \cap \Lambda$ and $q \in X$ there is an su-path from p to q. If Λ is bisaturated, this implies that either $X \cap \Lambda = \varnothing$ or $X \subset \Lambda$. Furthermore, a pair (Φ, Λ) of a dynamical system and a partially hyperbolic set of it is *stably accessible* on $X \subset M$ if there exist neighborhoods U of Λ and $\mathcal{U}$ of Φ such that if $\widetilde{\Phi} \in \mathcal{U}$ and $\widetilde{\Lambda} \subset U$ is a $\widetilde{\Phi}$-invariant bisaturated compact set, then $(\widetilde{\Phi}, \widetilde{\Lambda})$ is accessible on X.

Brin quadrilaterals illuminate what is needed for accessibility and are important in our proof (Definition 3.13, Figure 1). The difficulty addressed by accessibility is the ability to achieve displacements in the center direction even by motion solely along stable and unstable arcs. Brin quadrilaterals consist of pairs of stable and unstable arcs arranged so as to return to the starting point or its center leaf; when the stable and unstable foliations are

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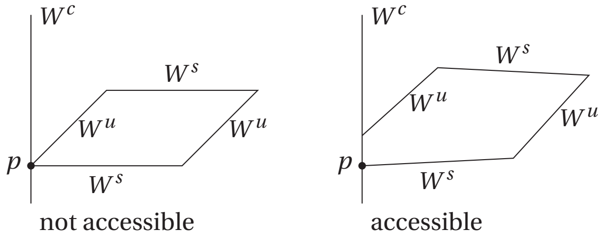

FIGURE 1. A 'bad' and 'good' Brin quadrilateral.

jointly integrable, such quadrilaterals close up and there is no displacement in the center direction. When such a quadrilateral does not close up, displacement in the center direction is possible, and when the center is one-dimensional, this implies accessibility. In general, and in our situation, one needs to arrange for such quadrilaterals to produce displacement in any of the center directions (Figure 3).

Although we are interested in flows that are partially hyperbolic over the entire manifold, our general result on accessibility (Theorem 3.1) holds for bisaturated partially hyperbolic sets.

We obtain accessibility (Theorem 1.1) by adapting from [3, 18] the proof of the following result.

THEOREM 2.5. (Avila, Crovisier, Dolgopyat and Wilkinson [18, Main Theorem], [3, footnote p. 13]) *If M is a smooth compact manifold and $r \geq 1$, then stable accessibility is C^1-dense among*
- *all,*
- *volume-preserving, and*
- *symplectic*

partially hyperbolic C^r diffeomorphisms of M.

From Theorem 1.1, we obtain Theorem 1.2 by adapting ingredients from the proof of the following result.

THEOREM 2.6. (Burslem [14, Theorem 1.2]) *In the set of C^r partially hyperbolic diffeomorphisms of a compact manifold M ($r \geq 1$), there is a C^1-open and C^1-dense subset V whose elements all have discrete diffeomorphism centralizer.*

3. *Accessibility*

In [18] the authors prove that accessibility is C^1 open and dense in the space of C^r partially hyperbolic diffeomorphisms for $r \geq 1$. Furthermore, the result holds in the symplectic and conservative settings. In [3] the result is extended to bisaturated partially hyperbolic sets, and a gap in [18] is fixed. We build on this work.

3.1. *Proof outline and the C^1-restriction.*

The fundamental difficulty in producing accessibility is that one needs to perturb the invariant subbundles (or foliations) in specific

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ways, but that such perturbations can only be achieved indirectly by perturbing the dynamics. This, in turn, affects the invariant subbundles in locations beyond those where one seeks to produce the desired effects—which might undo what one has previously achieved. What makes this approach possible is that the resulting perturbations of the invariant subbundles are quickly attenuated further along an orbit. This needs to be controlled well enough to ensure that the minor perturbations far away in time cause no problems, and one ingredient is to undertake the desired perturbations along an orbit with a large first-return time, so these undesired effects have dampened enough by the time the orbit returns. It is helpful here that the kind of perturbations we need are to bring the invariant subbundles/manifolds into 'general position' rather than the opposite. Nonetheless, these features seem to necessitate using the C^1 topology. To emphasize this point: undertaking C^r-perturbations for $r \geq 2$ necessarily requires a $1/\epsilon^{r-1}$-neighborhood, and this makes it impossible to arrange for large first-return times, which in turn necessitates controlling the perturbative effects across multiple returns. This is not known to be impossible but constitutes a formidable difficulty. Interestingly, this appears to be the sole place where difficulties with C^r-perturbations arise, so should anyone be able to address this point, a number of C^r-genericity theorems would follow directly.

To achieve the global property of accessibility one mixes the global and the local in a two-step approach: accessibility modulo disks and accessibility on disks. The first step produces a (careful) choice of disks 'in the center direction' such that the partially hyperbolic dynamical system is accessible modulo these disks: one can get from any one point to any other via a finite path consisting of segments that are variously stable, unstable, or in one of these disks.

Our proof techniques for this are essentially the same as theirs. The first step is to find a collection of disks, called a c-admissible disk family, such that each disk is sufficiently small, intersects the bisaturated set, and for which points take a long time to return under the flow. In the dynamically coherent setting these can be chosen as small disks in the center foliation, but in the not dynamically coherent setting we choose disks that are sufficiently 'close' to the center direction. These disks are chosen in such a way that the family of disks is sufficiently dense in the manifold, but still pairwise disjoint. Furthermore, for any point p in the bisaturated set and any disk D there is a us-path that will start at p and end at a point in D, and this is a robust property. This is accessibility modulo disks.

The perturbation construction then locally modifies the flow in such a way that the disk segments can be replaced by stable-unstable paths, that is, such that each disk is in an accessibility class. It is this perturbation we need to implement for flows rather than maps. And this perturbation does not affect the previously obtained accessibility modulo these disks, because that is robust. Thus, the second step is to prove that a small perturbation of the flow can ensure that the system is accessible on each c-admissible disk, and this is also a robust property. Then we can build a us-path not only to points on the disk, but also to points close to the disk. So, given two points x and y in the bisaturated set, the first step ensures that there is a us-path from x to a point close to y in a c-admissible disk, and the second ensure that there is a us-path from the point in the disk to y.

Thus, the central modification is the introduction of these local perturbation arguments for flows rather than diffeomorphisms. The remaining parts of the prior arguments are

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to the effect that the perturbation does indeed have the desired properties, and those arguments apply via time-1 maps.

As we work through these arguments, we concentrate on these local perturbations, describing what needs to be achieved, with some indications of proofs, and then producing those needed perturbations. We conclude this preview with a brief description of what issues we needed to address when finding suitable perturbations for flows.

For diffeomorphisms a natural way to achieve a desired local perturbation is to postcompose the diffeomorphism with a map that is close to the identity and equal to it outside the neighborhood in question. This map is picked just such as to move points from where the diffeomorphism mapped them to where one wants the perturbation to map them. This is the approach taken by Avila, Crovisier, Dolgopyat, and Wilkinson. Some care is taken that the postcomposed map is in the right category (volume-preserving or symplectic, respectively). Then so is the composition. This approach has no counterpart for flows because postcomposing a flow is meaningful only for a time-t map, and postcomposing a time-t map with anything is not going to give anything that corresponds in a clear way to a flow. (It may be amusing or intriguing that this postcomposed map is constructed as the time-1 map of a flow, but this does not help make the approach work for perturbing flows.)

What we do instead is to introduce the right kind of 'drifts' in small flow-boxes. Even without the constraints of volume- preservation, symplecticity, and so on, it is not wholly straightforward to decide how to arrange for a deformation by a drift with a specific desired outcome. Making sure that those local perturbations can further be arranged to respect the invariance of volume, a symplectic form or a contact form requires additional work, which happens to be of slightly different kinds for these respective categories. Most notably, in the contact case it is natural to drive these flow perturbations by perturbing the contact form.

3.2. *Definitions.* Theorem 1.1 is a consequence of the following more general theorem, which corresponds to [3, Theorem B].

THEOREM 3.1. *Let Λ be a partially hyperbolic set for a flow Φ on a closed manifold M, and let $\mathcal{U}$ be a C^1 neighborhood of Φ. There exist a neighborhood U of Λ and a non-empty open set $\mathcal{O} \subset \mathcal{U}$ such that if $\Psi \in \mathcal{O}$ and $\Delta \subset U$ is a bisaturated partially hyperbolic set for Ψ, then Δ is accessible for Ψ.*

Furthermore, this holds among volume-preserving, symplectic, and contact flows.

We first review notation introduced in [3] before explaining the adaptations that need to be made to prove Theorems 1.1 and 3.1. We will use slightly different notation, because we are using the notation for flows that they use for the charts.

PROPOSITION 3.2. (Adapted charts) *Let M be a smooth manifold with* $\dim(M) = d$. *For each point $p \in M$ there is a chart $f_p \colon B(0, 1) \subset T_pM \to M$ with the following properties.*

(1) *The map $p \mapsto f_p$ is piecewise continuous in the C^1 topology. So there are open sets $U_1, \ldots, U_\ell \subset M$ and*
- *compact sets $K_1, \ldots, K_\ell$ covering M with $K_i \subset U_i$,*

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- *trivializations* $g_i : U_i \times \mathbb{R}^d \to T_{U_i}M$ *such that* $g_i(\{p\} \times B(0, 2))$ *contains the unit ball in* T_pM *for each* $p \in U_i$, *and*
- *smooth maps* $F_i : U_i \times B(0, 2) \to M$,

such that each $p \in M$ *belongs to some* K_i, *with* $f_p = F_i \circ g_i^{-1}$ *on* $B(0, 1) \subset T_pM$.

(2) *When a volume, symplectic, or contact form has been fixed on* M, *this pulls back under* f_p *to a constant (and standard such) form on* T_pM [25, Theorems 5.1.27, 5.5.9, 5.6.6].

Remark 3.3. We note that, given a compact set K with continuous splitting $T_KM = E_1 \oplus E_2$, there is a Riemannian metric with respect to which the norm of the projection from E_2 to E_1 is arbitrarily small. Then the charts F can be chosen so the bundles E_1 and E_2 are lifted in $B^d(0, 3)$ to nearly constant bundles.

Remark 3.4. For a volume, symplectic, or contact form, the standard chart expresses that form, respectively, as

- $dx_1 \wedge \cdots \wedge dx_d$,
- $\sum_{i=1}^{d/2} dx_i \wedge dy_i$,
- $\alpha = dt + \sum_{i=1}^{(d-1)/2} x_i\, dy_i$.

We note that a chart of the latter type is automatically of flow-box type: the Reeb vector field Y of $\alpha = dt + \sum_{i=1}^{(d-1)/2} x_i\, dy_i$ is $\partial/\partial t$ because it is (uniquely) defined by $d\alpha(Y, \cdot) \equiv 0, \alpha(Y) \equiv 1$.

We do not assume dynamical coherence for the partially hyperbolic set Λ (the existence of a foliation tangent to E^c), and so we define approximate center manifolds that will be sufficient.

Definition 3.5. (c-admissible disk) For sufficiently small $\eta > 0$ and $p \in \Lambda$, denote by $B^c(0, \eta)$ the ball around 0 in E^c_p of radius η. The set $V_\eta(p) := f_p(B^c(0, \eta))$ is a *c-admissible disk* with radius $\eta =: r(V_\eta(p))$, and we set $\beta V_\eta(p) \coloneqq V_{\beta\eta}(p)$ for $\beta \in (0, 1)$. A *c-admissible disk family* is a finite collection of pairwise disjoint, c-admissible disks.

Definition 3.6. (Return time) For a subset S of a flow-box [19, Definition 1.1.13] of ‘height’ τ, the return time is $R(S) \coloneqq \inf\{t > \tau \mid \varphi^t(S) \cap S \neq \varnothing\} \in [0, \infty]$.

Note that a flow-box contains no fixed point. If $p \in M$ is not fixed, then $R(B_\eta(p)) \xrightarrow[\eta\to 0]{} \mathrm{per}(p)$ if we agree that $\mathrm{per}(p) = \infty$ if p is not periodic, and $\mathrm{per}(p)$ is the period of p for any periodic p.

For a c-admissible disk family $\mathcal{D}$ and $\beta \in (0, 1)$ we let

$$\beta\mathcal{D} := \{\beta D \ :\ D \in \mathcal{D}\}, \quad |\mathcal{D}| := \bigcup_{D\in\mathcal{D}} D, \quad r(\mathcal{D}) := \sup_{D\in\mathcal{D}} r(D), \quad R(\mathcal{D}) := R(|\mathcal{D}|).$$

3.3. *The Avila–Crovisier–Dolgopyat–Wilkinson arguments.* The proof of accessibility in [3] proceeds in two steps. The first is a general fact for partially hyperbolic sets for diffeomorphisms on the existence of c-admissible disk families that stably meet all unstable and stable leaves as follows.

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Definition 3.7. (Global c-section) We say that a set $X \subset M$ is a *(global) c-section* for (Φ, Λ) if $X \cap \Delta \neq \varnothing$ for every bisaturated subset $\Delta \subset \Lambda$.

Remark 3.8. This terminology alludes to that of a (Poincaré) section for a flow, which meets bunches of orbits; (global) c-sections meet many stable and unstable leaves. Although this is not a defining property, the (global) c-sections we find will be transverse to stable and unstable leaves (Proposition 3.9) and will indeed meet all stable and unstable leaves once accessible (Proposition 3.11).

Via time-t maps, the result on the existence of such families immediately holds in our setting.

PROPOSITION 3.9. [**3**, Proposition 1.4] *Let Λ be partially hyperbolic set for Φ. Then there exists a $\delta > 0$ with the following property. If U is a neighborhood of Λ such that $\overline{U} \subset U_0(\Phi, \Lambda)$ and $T > 0$, then there exist a c-admissible disk family $\mathcal{D}$ and $\sigma > 0$ such that*

(1) $r(\mathcal{D}) < T^{-1}$,
(2) $R(\mathcal{D}) > T$ *(this implies that $|\mathcal{D}|$ contains no fixed point), and*
(3) *if Ψ satisfies $d_1(\Phi, \Psi) < \delta$ and $d_0(\Phi, \Psi) < \sigma$, then for any bisaturated partially hyperbolic set $\Delta \subset U$ for Ψ, the set $|\mathcal{D}|$ is a (global) c-section for (Ψ, Δ).*

Remark 3.10. (Fixed points nowhere dense) This prompts us to note that the set of fixed points of a partially hyperbolic dynamical system is nowhere dense: the set of fixed points of a continuous dynamical system is closed, and the restriction to it is the identity, so the interior is empty by partial hyperbolicity. Thus the c-admissible disk family in Proposition 3.9 can be chosen away from the set of fixed points.

The next step is a result about stable accessibility on center disks. Its proof needs adaptations for flows, the core part of which is Lemma 3.15 below.

PROPOSITION 3.11. [**3**, Proposition 1.3] *If Λ is a partially hyperbolic set for a flow Φ and $\delta > 0$, then (with the notation of Remark 2.3) there exist $T > 0$ and a neighborhood U of Λ such that $\overline{U} \subset U_0(\Phi, \Lambda)$, and if $\mathcal{D}$ is a c-admissible disk family with respect to (Φ, Λ) with $r(\mathcal{D}) < T^{-1}$ and $R(\mathcal{D}) > T$, then for all $\sigma > 0$ there exists $\Psi \in \mathcal{U}_0(\Phi, \Lambda)$ such that*

(1) $d_1(\Phi, \Psi) < \delta$,
(2) $d_0(\Phi, \Psi) < \sigma$,
(3) *if $D \in \mathcal{D}$ and $\Delta \subset U$ is a bisaturated partially hyperbolic set for Ψ, then (Ψ, Δ) is stably accessible on D, and*
(4) *if Φ preserves a volume, symplectic, or contact form, then so does Ψ.*

Theorem 3.1 follows from Propositions 3.9 and 3.11 just as [**3**, Theorem B] follows from [**3**, Propositions 1.3 and 1.4] in [**3**, §1.6], including the preservation of a volume, symplectic, or contact form.

Proof of Proposition 3.11. Much of the proof of this proposition is exactly as in [**3**]. We will explain the ideas in these parts while highlighting the points that need modifications for flows.

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The first step [3, Lemma 2.1] introduces smaller disks that are sufficiently close to, but disjoint from, a c-admissible disk and have the property that there are su-paths at sufficiently small scales connecting any point in the c-admissible disk to one of the smaller disks, not just for the original map, but also for any maps that are sufficiently C^1 close.

For these smaller disks one can then perform perturbations so that the flow is accessible on them. Then one can show that the perturbed flow will be accessible on D (the c-admissible disk for the original flow). By the choice of the c-admissible family we obtain accessibility of the bisaturated set for the perturbed flow.

The existence of the smaller disks does not use perturbations and assumes the existence of a c-admissible disk for a partially hyperbolic map. This holds in our setting by considering time-t maps. We include the statement of the result for completeness and adapt its formulation to flows.

LEMMA 3.12. *There exist $\delta_1, \rho_1 > 0$, $K > 1$ and a neighborhood U_1 of Λ such that for any $\rho \in (0, \rho_1)$, for any c-admissible disk D with radius ρ, centered at $p \in \Lambda$, and for any $\epsilon \in (0, K^{-1}\rho)$, there exist $z_1, \dots, z_\ell \in T_pM$ such that the following assertions hold.*

(1) *The balls $B(z_i, 100d^2\epsilon)$ are in the $K\epsilon$-neighborhood of $f_p^{-1}(D)$.*

(2) *The balls $B(z_i, 100d^2\epsilon)$ are pairwise disjoint.*

(3) *For any $x \in D$, there exists some z_i such that for any Ψ that is δ_1-close to Φ in the C^1 distance and for any bisaturated set $\Delta \subset U_1$ for Ψ:*
  (a) *if $x \in \Delta$, then there is an su-path for Ψ between x and $f_p(B(z_i, \epsilon))$;*
  (b) *if $f_p(B(z_i, \epsilon)) \subset \Delta$, then any point $y \in f_p(B(x, \epsilon/2))$ belongs to an su-path that intersects $f_p(B(z_i, \epsilon))$.*

The idea of the proof of Theorem 1.1 is to create small perturbations of the flow Φ supported near the points $z_1, \dots, z_\ell$ to create accessibility near each z_i. This requires the following notion (which will be used in Lemma 3.14 to show accessibility near the z_i).

Definition 3.13. (θ-accessibility) A pair (Ψ, Δ) of a flow and a bisaturated set is *θ-accessible* on $f_p(B(z, 2d\epsilon))$ if there exist an orthonormal basis $w_1, \dots, w_c$ of E_p^c and for each $j \in \{1, \dots, c\}$ a continuous map

$$H^j : [-1, 1] \times [0, 1] \times f_p^{-1}(\Delta) \cap B(z, 2d\epsilon) \to f_p^{-1}(\Delta) \cap B(0, 2\rho)$$

such that for any $x \in f_p^{-1}(\Delta) \cap B(z, 2d\epsilon)$ and $s \in [-1, 1]$ we have:

(a) $H^j(s, 0, x) = x$;

(b) the map $f_p \circ H^j(s, ., x) : [0, 1] \to \Delta$ is a four-legged su-path (Brin quadrilateral), that is, the concatenation of four curves, each contained in a stable or unstable leaf in alternation;

(c) $\|H^j(s, 1, x) - x\| < \epsilon/10d$; and

(d) $\|H^j(\pm 1, 1x) - (x \pm \theta\epsilon wj)\| < \theta(\epsilon/10d)$.

The second step is that θ-accessibility in a neighborhood of a point implies accessibility on a smaller neighborhood. This is a restatement of [3, Lemma 2.2]. The proof is again almost the same, but we provide it for completeness.

From now on write $d := u + c + s$, where $\dim E^u = u$, $\dim E^c = c$, and $\dim E^s = s$.

LEMMA 3.14. *For any $\theta > 0$, there exist $\delta_2, \rho_2 > 0$ and a neighborhood U_2 of Λ such that*
(1) *for any $p \in \Lambda$, any $z \in B(p, \rho_2) \subset T_pM$ and $\epsilon \in (0, \rho_2)$,*
(2) *for any flow Ψ that is δ_2-close to Φ in the C^1 topology,*
(3) *for any bisaturated set $\Delta \subset U_2$ such that (Ψ, Δ) is θ-accessible on $f_p(B(z, 2d\epsilon))$,*
the pair (Ψ, Δ) is accessible on $f_p(B(z, \epsilon))$.

Proof. We let $v_1, \ldots, v_u$ be an orthonormal basis of E^u_p and $v_{u+c+1}, \ldots, v_d$ an orthonormal basis for E^s_p. We define local flows Φ_i on $f_p^{-1}(\Delta)$ as follows. Let X_i be a vector field along the leaves of $f_p^{-1}(W^j_\Psi)$ where $X_i(x) = D\pi^j_x(x + v_i)$ for $j = u$ if $1 \le i \le u$ and $j = s$ if $u + c + 1 \le j \le d$, and the local flow is defined by the vector field X_i on the set $B(0, 2\rho_1) \cap f_p^{-1}(\Delta)$ and ρ_1 is given by Lemma 3.12. So the orbit of x is the projection by π^j_x on the curve $t \mapsto x + tv_i$ for $|t| < \rho_1$, and the orbits are C^1 curves whose tangent space is arbitrarily close to $\mathbb{R}v_i$ for sufficiently small constants ρ_1, δ_1, and U_1 as in Lemma 3.12.

For ρ_2, δ_2, and U_2 sufficiently small we see that

$$\|\varphi^t_i(x) - (x + t\theta\epsilon v_i)\| < |t|\theta\frac{\epsilon}{10d}. \tag{3.1}$$

We also let $v_{u+j} = w_j$ be an orthonormal basis for the center direction and define inductively

$$\begin{aligned}
\varphi^t_{u+j}(x) &= H^j(t, 1, x) \quad \text{when } t \in [0, 1),\\
\varphi^t_{u+j}(x) &= \varphi^{t-1}_{u+j} \circ \varphi^1_{u+j}(x) \quad \text{when } t > 1,\\
\varphi^t_{u+j}(x) &= \varphi^{t+1}_{u+j} \circ \varphi^{-1}_{u+j}(x) \quad \text{when } t < 0
\end{aligned}$$

where the above holds for t so long as it can be defined. From properties (c) and (d) in the definition of H^j and estimate (3.1) above, $P(t_1, \ldots, t_d) := \varphi^{t_1}_1 \cdots \varphi^{t_d}_d(x_0)$ for $(t_1, \ldots, t_d) \in [-3\theta^{-1}, 3\theta^{-1}]^d$ is a continuous map with $\|P(t_1, \ldots, t_d) - (x_0 + \sum_i t_i\theta\epsilon v_i)\| < 2\epsilon/10$. The image of P contains $B(x_0, 5\epsilon/2)$, and $f_p \circ P$ shows that (Ψ, Δ) is accessible on $B(z, \epsilon)$. □

3.4. *The adaptation to flows.* Having reproduced the parts of the proof of Proposition 3.11 that work just like in discrete time, we now proceed to the perturbation construction, where the adaptations to flows become nontrivial. This result and proof are similar to [3, Lemma 2.3], but in our case our perturbations need to be constructed for flows instead of maps. This is the essential adaptation of the Avila–Crovisier–Dolgopyat–Wilkinson arguments.

LEMMA 3.15. *Consider a partially hyperbolic* flow Φ *generated by a vector field X. With the previous notation, there exist $\eta, \alpha_0 > 0$ such that for any $\alpha \in (0, \alpha_0)$, $p \in \Lambda$, $z \in B(0, 1/4) \subset T_pM$ with $X(f_p(z)) \neq 0$, $r \in (0, 1/4)$ and any unit vector $v \in E^c_p$ there is a vector field Y such that*
(1) *$Y = X$ outside $f_p(B(z, 3r))$,*
(2) *$df_p^{-1}Y = df_p^{-1}X + \alpha\eta v$ on $B(z, 2r)$,*
(3) *Y is α-close to X,*

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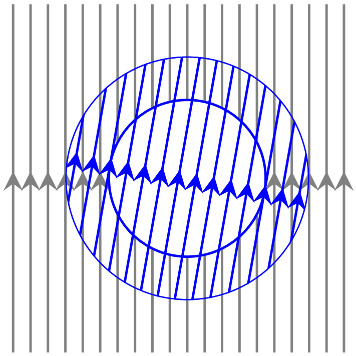

FIGURE 2. The perturbations in Lemma 3.15.

(4) *the flow Ψ defined by Y is $(r/100d^2)$-close to Φ in the C^1 distance, and*
(5) *if Φ preserves a volume, symplectic, or contact form, then so does Ψ.*

Proof. Figure 2 (which utilizes that by Remark 3.10 we are working away from fixed points) illustrates what we would like to achieve: to connect the perturbed vector field (in the smallest circle) to the original one (in the square) by a bump-function interpolation.

For arbitrary vector fields this is all there is. For volume-preserving flows, we invoke the pasting lemma [**35**, Theorem 1] (see also [**1**]) to ensure volume-preservation of the perturbation.

For the symplectic category, take symplectic flow-box charts as in Proposition 3.2 in the rectangle and large circle in Figure 2; they are locally Hamiltonian with constant Hamiltonians on each neighborhood, so we can interpolate the Hamiltonians in the annulus.

For contact flows take a Darboux chart (Remark 3.4) on the rectangle, then put a suitably rotated and scaled version in the large disk. This defines a local contact form whose Reeb field is as desired; interpolate the contact forms in the annulus. (For perturbations in the flow direction, that is, reparameterization, no rotation is needed.) □

Remark 3.16. In a Darboux chart $B(0, r) \times [0, T]$ with contact form $dz + \sum x_i\, dy_i$, an explicit such contact perturbation in the direction of a vector $v = \sum v_i \partial_{x_i} + v_{n+i} \partial_{y_i}$ is

$$\beta = dz + \sum_i x_i\, dy_i + g(z)(v_i f(x_i) y_i - v_{n+i} f(y_i) x_i)\, dz$$

with $g \equiv 1$ outside a neighborhood of $\{0, T\}$ and f constant and supported near 0. Here

$$d\beta = \sum_i dx_i \wedge dy_i + g(z)[v_i(f'(x_i) y_i\, dx_i + f(x_i)\, dy_i) - v_{n+i}(f'(y_i) x_i\, dy_i + f(y_i)\, dx_i)] \wedge dz,$$

and the corresponding Reeb field $R = c\partial_z + \sum_i a_i \partial_{x_i} + b_i \partial_{y_i}$ satisfies

$$0 \equiv \iota_R d\beta = \sum_i \underbrace{[a_i - cg(z)(v_i f(x_i) - v_{n+i} f'(y_i) x_i)]}_{=0} dy_i$$

$$-\underbrace{[b_i - cg(z)(v_{n+i} f(y_i) - v_i f'(x_i) y_i)]}_{=0} \, dx_i$$
$$+g(z)[a_i(v_i f'(x_i) y_i - v_{n+i} f(y_i)) - b_i(v_{n+i} f'(y_i) x_i - v_i f(x_i))] \, dz,$$

so $R = c(\partial_z + g(z) \sum_i [v_i f(x_i) - v_{n+i} f'(y_i) x_i] \partial_{x_i} + [v_{n+i} f(y_i) - v_i f'(x_i) y_i] \partial_{y_i})$ with c determined by $\beta(R) \equiv 1$. Where f is constant, this is $R = c(\partial_z + fgv)$, which 'drifts' in the prescribed direction.

This shows that and how a C^1-small real-valued function produces a C^1 (rather than C^0) perturbation of a Reeb field (and via a C^2-perturbation of the contact form).

To construct the desired flow we will use the above perturbation to establish θ-accessibility for the sets $f_{p_i}(B(z_i, 2d\epsilon))$. We do this by adjusting Brin quadrilaterals using the perturbation above. Before explaining this step we first adjust the neighborhood U and describe the setup we will need.

We first let C_0^s and C_0^u be cone fields in U_0 that are Φ-invariant for $t < 0$ and $t > 0$, respectively. Furthermore, from the choice of δ_3 we know that the cone fields are invariant for any flow that is δ_3-close in the C^1 topology to Φ. For $T > 0$ we let U be a neighborhood of Λ such that

$$\overline{U} \subset U_1 \cap U_2 \cap \bigcap_{|t| \leq T} \varphi^t(U_0). \tag{3.2}$$

We also define $C^u = D\varphi^T(C_0^u)$ and $C^s = D\varphi^T(C_0^s)$ on U. We know there exist $T_1 > 0$ and $\rho_4 > 0$ such that if $T \geq T_1$, $\rho < \rho_4$, and $p \in \Lambda$, then $f_p(B(0, 2\rho)) \subset U$ and the cone fields $Df_p^{-1}(C^s)$ and $Df_p^{-1}(C^u)$ on $B(0, 2\rho)$ are γ-close to E_p^s and E_p^u in T_pM, where $\gamma < \alpha\eta$.

Let $\rho \in (0, \min\{\rho_1, \rho_2, \rho_3, \rho_4\})$ and fix $T \geq T_1$ so that any c-admissible disk D with center $p \in \Lambda$ and $r(D) < T^{-1}$ satisfies $f_p(D) \subset B(0, \rho) \subset T_pM$. We also have U defined by T satisfying (3.2) and the existence of a family $\mathcal{D}$ of c-admissible disks from Proposition 3.9 for some $\sigma > 0$.

Fix $\theta = 2\alpha\eta d$. We now describe the quadrilaterals we will use to establish θ-accessibility for the perturbed flow. For $D \in \mathcal{D}$ we fix the z_i as in Lemma 3.12 and, for sufficiently small ϵ, the sets $f_{p_i}(B(z_i, 100d^2\epsilon)) \subset U$. We can also define subspaces $\mathcal{E}^s$ and $\mathcal{E}^u$ such that

- $Df_{p_i}(z_i)\mathcal{E}^s \subset C^s(f_{p_i}(z_i))$,
- $Df_{p_i}(z_i)\mathcal{E}^u \subset C^u(f_{p_i}(z_i))$,
- $\dim \mathcal{E}^s = \dim E_{p_i}^s$, and
- $\dim \mathcal{E}^u = \dim E_{p_i}^u$.

Let $v_s \in \mathcal{E}^s$ and $v_u \in \mathcal{E}^u$ be unit vectors and fix an orthonormal basis $w_1, \ldots, w_c$ for $E_{p_i}^c$. For the foliations $\mathcal{F}^u$ and $\mathcal{F}^s$ we define the flow Φ_k' that corresponds to the linear flow $(x, t) \mapsto x + tv_k$ projected to the leaves of $\mathcal{F}^k$ for $k \in \{u, s\}$. For each $j \in \{1, \ldots, c\}$ we examine the quadrilaterals given by the composition

$$v_{i,j} = \varphi_s'(-10jd\epsilon) \circ \varphi_u'(-10d\epsilon) \circ \varphi_s'(10jd\epsilon) \circ \varphi_u'(10d\epsilon)$$

and

$$v_{i,-j} = \varphi_s'(10jd\epsilon) \circ \varphi_u'(10d\epsilon) \circ \varphi_s'(-10jd\epsilon) \circ \varphi_u'(-10d\epsilon).$$

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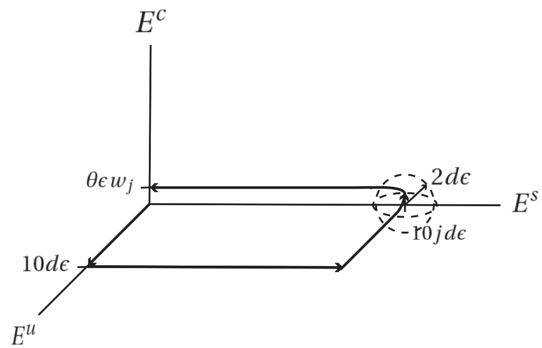

FIGURE 3. Perturbed quadrilateral.

We define

$$R(i, \epsilon) := \max_{1\le j\le c} (\max(\|v_{i,j}\|, \|v_{i,-j}\|)).$$

Because $R(i, \epsilon) \in o(\epsilon)$ [**18**, equation (8)], we have the following lemma.

LEMMA 3.17. *For each z_i there is an $\epsilon_0 > 0$ such that $R(i, \epsilon) < \theta\epsilon/10d$ for $\epsilon \in (0, \epsilon_0)$.*

We let Ψ be the flow generated by a vector field whose restriction to $B(z_i + 10jd\epsilon v_s, 2d\epsilon)$ and $B(z_i - 10jd\epsilon v_s, 2d\epsilon)$ satisfies the conditions in Lemma 3.15.

Then, starting at z_i and using the quadrilaterals above, we see that the new flow coincides with translation by $\theta\epsilon w_j$ from the original flow. Indeed, by construction the first leg of the quadrilateral is left unperturbed by the new flow. Similarly, the second leg is left unperturbed. The third leg is the composition of $x \mapsto x - (10d\epsilon)v_u$ with the translation $(\theta/2)\epsilon w_j$. The fourth leg similarly corresponds to the composition with the linear flow and translation by $(\theta/2)\epsilon w_j$. Then the quadrilateral on $B(z_i, 2d\epsilon)$ corresponds to translation by $\theta\epsilon w_j$; see Figure 3.

Similarly, the quadrilateral associated with $-j$ corresponds to translation by $-\theta\epsilon w_j$ from the original flow. From this we obtain the desired function H^j used to ensure θ-accessibility; see Figure 4.

Furthermore, from the size of the perturbations we see that the flows are at least $(\epsilon/100d)$-close in the C^0 topology. We now show that this flow Ψ is as in Proposition 3.11.

Let $\Delta \subset U$ be a bisaturated set for Ψ. Let $\widetilde{\Psi}$ be C^1-close to Ψ and $\tilde{\Delta} \subset U$ be a bisaturated set for $\widetilde{\Psi}$ contained in a small neighborhood of Δ. By the construction above we know that $(\widetilde{\Psi}, \widetilde{\Delta})$ is θ-accessible on each of the $f_{p_i}(B(z_i, 2d\epsilon))$. Then $(\widetilde{\Psi}, \widetilde{\Delta})$ is accessible on each of the sets $f_{p_i}(z_i, \epsilon)$.

For $D \in \mathcal{D}$ that intersects $\widetilde{\Delta}$ at a point z we know there exists an su-path for $\widetilde{\Psi}$ from z to a point $y \in f_{p_i}(B(z_i, \epsilon))$ for some i from Lemma 3.12. Furthermore, from Lemma 3.12 we know if x is $\epsilon/2$ close to z in D, then there is an su-path to a point $y' \in f_{p_i}(B(z_i, \epsilon))$. Accessibility on $B(z_i, \epsilon)$ implies there is an su-path from y to y'. So any point in the $\epsilon/2$-neighborhood of z contained in D is in the same accessibility class as z. This implies that every point in D is in the same accessibility class for $(\widetilde{\Psi}, \widetilde{\Delta})$ since

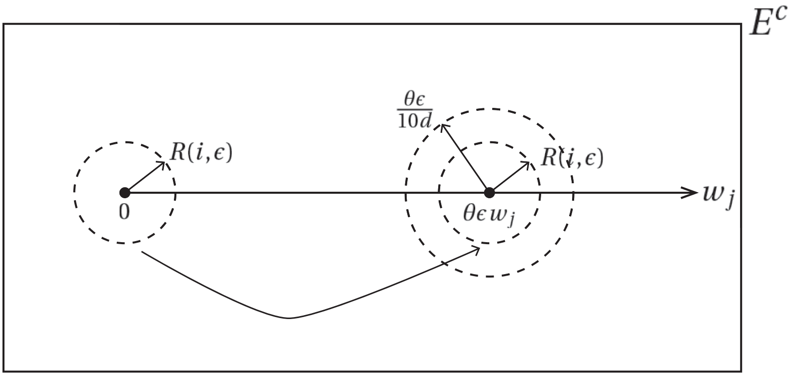

FIGURE 4. θ-accessibility.

D is connected. Then (Ψ, Δ) is stably accessible on any disk $D \in \mathcal{D}$. This concludes the proof of Proposition 3.11. □

4. *Centralizers*

We now adapt arguments by Burslem from the proof of Theorem 2.6 [**14**, Lemmas 5.2, 5.3] in order to prove Theorem 1.2. We will use the following criterion [**4**].

PROPOSITION 4.1. *For $r \geq 0$, a C^r flow Φ has trivial flow-centralizer (Definition 1.7) if Φ has discrete C^r diffeomorphism-centralizer, that is, if for any C^r $f: M \to M$ that commutes with Φ and is sufficiently close to the identity, there is a τ near 0 such that $f = \varphi^\tau$.*

We note that without fixed points, some of the arguments below are a little simpler.

Proof of Theorem 1.2. First, we produce a (non-fixed) closed orbit that is isolated among closed orbits of at most (say) twice its period.

CLAIM 4.2. *There is a non-fixed recurrent point.*

Proof (Crovisier). A partially hyperbolic flow has a non-atomic ergodic invariant measure because the topological entropy is positive [**15**, Theorem 2] while atomic measures have zero entropy. Almost every point is recurrent [**19**, Poincaré Recurrence Theorem 3.2.1] and not fixed (ergodicity). □

This produces a closed orbit by perturbation.

CLAIM 4.3. *A C^1-perturbation has a closed orbit.*

Proof (Rifford). From a non-fixed recurrent point, the Pugh closing lemma gives a C^1-close (partially hyperbolic) flow with a closed orbit; this also works for volume-preserving and symplectic flows [**2**, §3.1] as well as for Reeb flows—this is implicit in [**2**] as follows.

C^1 contact perturbations as in Remark 3.16 (or in the proof of Lemma 3.15) produce the following local connection lemma for a contact form α on a compact manifold: for a

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Darboux chart $C = B(0, \rho) \times [0, T]$ there is a $K > 0$ such that for small enough $\epsilon, r > 0$ and $P, Q \in B(0, r)$ with $d(P, Q) < r\epsilon$, there is a contact form β with $d_{C^1}(R_\alpha, R_\beta) < K\epsilon$ such that $\beta = \alpha$ outside $B(0, r) \times [0, T]$ and the flow of R_β connects $P \times \{0\}$ to $Q \times \{1\}$ in time T.

The Pugh–Mai methods in [2] then imply that for a recurrent point p of R_α and a sufficiently small ϵ, there is an ϵ-perturbation β of α such that p is periodic for R_β. □

Now, by the transversality theorem [25, Theorems A.3.19, 7.2.4 and p. 296], for a C^r-open C^1-dense set of such flows,

- this closed orbit is transverse and hence isolated among closed orbits of up to twice its period, and
- fixed points are transverse and hence finite in number.

Theorem 1.1 then gives a C^1-open dense set of accessible (volume-preserving/symplectic/contact) partially hyperbolic flows Φ with a closed orbit $\mathcal{O}(p)$ that is isolated among closed orbits of at most twice its period. By Proposition 4.1 it suffices to show that their diffeomorphism centralizer is discrete.

If $f: M \to M$ is C^1, commutes with Φ and is sufficiently close to the identity, then f maps closed orbits of Φ to closed orbits of Φ with the same period. Thus, since $\mathcal{O}(p)$ is isolated among orbits of the same period, $f(p) \in \mathcal{O}(p)$ once f is sufficiently close to the identity, and indeed $f(p) = \varphi^\tau(p)$ for some τ near 0. Thus $h := f \circ \varphi^{-\tau}$ fixes p and commutes with Φ. Here, 'sufficiently close' means that $d_{C^0}(h, \text{id}) < \epsilon$, where $\epsilon > 0$ is as in Lemma 4.5 below.

To conclude that $h = \text{id}$, it suffices to verify this on the dense set $\mathcal{M}$ of points that are accessible from p with (su)-paths that avoid fixed stable and unstable leaves, that is, disjoint from $W^u(x)$ and $W^s(x)$ for any fixed point x. (This set is dense because the set of fixed points is finite and their invariant manifolds have positive codimension.)

For any $y \in \mathcal{M}$, recursive application of Lemma 4.4 below to a finite (su)-path from p to y shows that h fixes all vertices of this path and hence y. □

LEMMA 4.4. [14, Lemma 5.3] *If $h(q) = q$ and $W^u(q)$ contains no fixed point, then $h(x) = x$ for all $x \in W^u(q)$. Likewise for $W^s(q)$.*

Proof. Suppose $x \in W^u(q)$; the case $x \in W^s(q)$ is analogous. Then

$$h(x) \in h(W^u(q)) \overset{h \in C^1}{=\!=} W^u(h(q)) \overset{h(q)=q}{=\!=} W^u(q) = W^u(x),$$

while

$$d(\varphi^t(x), \varphi^t(h(x))) = d(\varphi^t(x), h(\varphi^t(x))) < d_{C^0}(h, \text{id}) < \epsilon.$$

This implies $h(x) = x$ by Lemma 4.5 below. □

The proof of [14, Lemma 5.2] applies to leaves containing no fixed points.

LEMMA 4.5. *There is an $\epsilon > 0$ such that if $W^u(x)$ contains no fixed point and $y \in W^u(x)$ satisfies $d(\varphi^t(x), \varphi^t(y)) < \epsilon$ for all $t \in \mathbb{R}$, then $x = y$. Likewise for $W^s(x)$.*

This concludes the proof of Theorem 1.2. □

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