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ADAPTIVE LEARNING IN REGIME-SWITCHING MODELS

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We study adaptive learning in economic environments subject to recurring structural change. Stochastically evolving institutional and policymaking features can be described by regime-switching models with parameters that evolve according to finite state Markov processes. We demonstrate that in nonlinear models of this form, the presence of sunspot equilibria implies two natural schemes for learning the conditional means of endogenous variables: under *mean value learning*, agents condition on a sunspot variable that captures the self-fulfilling serial correlation in the equilibrium, whereas under *vector autoregression learning* (VAR learning), the self-fulfilling serial correlation must be learned. We show that an intuitive condition ensures convergence to a regime-switching rational expectations equilibrium. However, the stability of sunspot equilibria, when they exist, depends on whether agents adopt mean value or VAR learning: coordinating on sunspot equilibria via a VAR learning rule is not possible. To illustrate these phenomena, we develop results for an overlapping-generations model and a New Keynesian model.

Keywords: E-Stability, Adaptive Learning, Regime Switching, Sunspots

1. INTRODUCTION

A given forward-looking macroeconomic model may admit classes of rational expectations equilibria that differ in terms of the sets of state variables that agents use when forming expectations. For example, standard linear stochastic rational expectations models have solutions that depend only on the minimal set of state variables, and may also have solutions that depend on extrinsic random variables (i.e., sunspots). Existence and uniqueness of equilibria are well understood in linear models with constant parameters; however, in a growing area of research that focuses on models with changing parameters, these issues are reemerging.[1]

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In regime-switching models, which constitute the focus of this paper, parameters evolve according to finite state Markov processes.[2] The nonlinear structure of regime-switching rational expectations models prevents a complete characterization of the full class of solutions, though recent papers by Davig and Leeper (2007) and Farmer et al. (2009) suggest that multiplicity of equilibria—some depending on extrinsic sunspot or bubbles processes—can arise in models of interest to applied economists and policymakers.

There has long been interest in indeterminacy, that is, the possibility that an economy can be driven by inefficient, self-fulfilling expectations, i.e., "sunspots." For example, an extensive literature studies whether monetary policy can be designed to prevent coordination on sunspots, and a separate literature examines the extent to which "rational bubbles" equilibria can account for observed movements in asset prices. In this paper, we are interested in a distinct but related question: in nonlinear regime-switching models, which, if any, equilibria are attainable by agents using an adaptive learning rule, such as least squares? Equilibria that are attainable in this manner are said to be "learnable" or "stable under learning."

In regime-switching models, assessing stability under learning is complicated by the fact that the number and nature of the rational expectations equilibria depend on whether agents condition their expectations only on the current realization of the Markov chain capturing the time-varying nature of the model's parameters, or also on the history of its realizations. If agents condition their expectations only on the current regime, we say the economy is in a *regime-dependent equilibrium* (RDE); if agents also condition their expectations on past regimes, we say the economy is in a *history-dependent equilibrium* (HDE). It is well known that conditions guaranteeing uniqueness within the class of RDE may not impinge on the existence of HDE: a model may have a unique RDE and also have multiple HDE. We note that HDE correspond to an indeterminacy in the model, because the dependence on past regimes is self-fulfilling.

The presence of multiple equilibria in our model leads naturally to the question of equilibrium selection, which is the main topic of our paper; and, as mentioned, we propose using stability under adaptive learning as the equilibrium selection mechanism. Following Lucas (1986), we maintain that stability under adaptive learning is a useful metric for identifying empirically relevant equilibria.[3] An equilibrium is plausible or reasonable if, whenever rational expectations are replaced with a standard adaptive learning rule, agents' beliefs converge to those consistent with the rational expectations equilibrium. Based on this assumption, we assess whether regime-switching equilibria are learnable.

Our viewpoint is informed by a large and growing literature that replaces rational expectations with learning rules under which agents are modeled as professional econometricians; that is, they hold forecasting models that share a reduced form with a rational expectations equilibrium, and they adjust the parameters of their model in light of new data. The advantage of this approach is that it places economist and agent on equal footing and avoids the cognitive dissonance inherent in rational expectations models. This approach is particularly compelling in

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regime-switching models because of the coexistence of equilibria in the regime- and history-dependent classes.

Among the stochastic properties of the sunspot equilibria associated with linear models with constant or time-varying parameters is self-fulfilling serial correlation, that is, serial correlation that is present in the equilibrium process only because agents believe it is present. Whether agents know and condition on this serial correlation, or instead must learn about it, implies two distinct learning rules, which may lead to distinct stability outcomes. Under "mean value learning," agents regress on a sunspot variable that includes in its stochastic structure the equilibrium's self-fulfilling serial correlation, whereas under "VAR learning" agents must learn the self-fulfilling serial correlation from the data by employing a (first-order) vector autoregression model.

Our primary results are surprising. When the conditions for a unique RDE are satisfied, the associated equilibrium is stable under learning. Moreover, this condition also governs the stability of HDE under mean value learning; on the other hand, the HDE are not attainable under VAR learning.

The paper is organized as follows: Section 2 provides results for a simple univariate model; Section 3 generalizes the model and defines the classes of equilibria; Section 4 provides the main stability analysis; Section 5 presents applications to an overlapping-generations model with switching preferences, and to a New Keynesian model with switching monetary regimes; Section 6 concludes.

2. THE UNIVARIATE CASE

We begin with a general reduced-form, univariate, regime-switching model,

$$y_t = \beta(s_t) E_t y_{t+1} + \gamma(s_t) r_t, \tag{1}$$

where r_t is a stationary, exogenous AR(1) process, and $s_t \in \{1, 2\}$ is a two-state Markov process, independent of r_t, with transition matrix $P = (p_{ij})$. Although we assume that s_t takes on two values, the analysis provided holds for any finite number of states.

We note that the expectational difference equation (1) might arise from the Fisherian model of inflation in Davig and Leeper (2007). The Fisherian model is given by

$$i_t = E_t \pi_{t+1} + r_t,$$

$$i_t = \alpha(s_t) \pi_t,$$

where $\alpha(s_t) > 0$. The first equation is a (log-linearized) Fisher equation, where i_t is the nominal interest rate, π_t is the inflation rate, and r_t is the real interest rate, taken to be a stationary, exogenous process. The second equation is the policy rule pursued by the central bank in setting nominal interest rates, and it indicates the potentially time-varying nature of policy. By combining equations and redefining variables, we see that the Fisherian model fits the more general model (1). If (1)

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captures the Fisherian model then $\beta(s_t) > 0$; however, in general, $\beta(s_t)$ can be any real number.

2.1. Rational Expectations Equilibria

A rational expectations equilibrium of the regime-switching model is any bounded solution to (1).[4] Because the model (1) is a nonlinear expectational difference equation, it is not, in general, possible to identify the entire collection of rational expectations equilibria; however, it is possible to identify several natural classes of equilibria depending on the conditioning behavior of the agents.

Our first class of equilibria correspond to those studied by Davig and Leeper (2007), and are closely associated with what Davig and Leeper call "the minimal state variable solution" (MSV) of (1), which takes the form[5]

$$y_t = B(s_t) r_t. \tag{2}$$

The dependence of y_t on s_t (and not lags of s_t), and the independence of s_t and r_t, suggests that we consider solutions to (1) of the form $y_t = y_{it} \Leftrightarrow s_t = i$, where y_{1t}, y_{2t} are stochastic processes independent of s_{t+j} for all integers j. We call such a solution a RDE, and note that the MSV is an RDE.[6]

In an RDE, the dependence of y_t on s_t allows simple state-contingent expectations formation so that the y_{it} solve

$$\begin{aligned} y_{1t} &= \beta_1 \left(p_{11} E_t y_{1t+1} + p_{12} E_t y_{2t+1}\right) + \gamma_1 r_t, \\ y_{2t} &= \beta_2 \left(p_{21} E_t y_{1t+1} + p_{22} E_t y_{2t+1}\right) + \gamma_2 r_t, \end{aligned} \tag{3}$$

where, here and in the sequel, we use the notation $\beta_i = \beta(i)$ and $\gamma_i = \gamma(i)$. We refer to (3) as the *stacked system* and conclude that an RDE must satisfy (3). We also note that the MSV solution (2) identified by Davig and Leeper (2007) corresponds to the unique MSV solution to (3) in the sense of McCallum (1983).[7]

The stacked system (3) shows that when expectations are conditioned on current s_t, the univariate nonlinear model (1) is recast into a multivariate linear model, which can then be solved using standard techniques, e.g., Blanchard and Kahn (1980); and, in particular, the number and nature of the RDE can be determined. Recall that a linear model is determinate if there is a unique REE and indeterminate if there are multiple REE. Standard determinacy analysis implies that there is a unique nonexplosive rational expectations equilibrium for (3), and therefore a unique RDE for (1), provided that the matrix

$$\begin{pmatrix} \beta_1 p_{11} & \beta_1 p_{12} \\ \beta_2 p_{21} & \beta_2 p_{22} \end{pmatrix} \tag{4}$$

has eigenvalues inside the unit circle. We refer to the determinacy condition in (4) as the conditionally linear determinacy condition (CLDC); it is analogous to the long-run Taylor principle in Davig and Leeper (2007).[8]

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Because the underlying model (1) is nonlinear, the CLDC does not necessarily guarantee uniqueness when agents also condition their expectations on s_{t-n} for $n > 0$. This point was made by Farmer et al. (2009), who showed that although the CLDC implies uniqueness for the univariate model in the case of positive feedback across regimes (i.e., for all i, $\beta_i > 0$), if some regimes exhibit negative feedback, then multiple equilibria may exist even when the CLDC is satisfied. To illustrate, we first simplify the model by setting $\gamma_i = 0$. In this case, the MSV solution is particularly simple: $y_t = 0$. Now note that a rational expectations equilibrium y_t may be associated with a martingale difference sequence (mds) ξ_t; i.e., $E_{t-1}\xi_t = 0$, so that

$$y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t, \tag{5}$$

and further, any mds ξ_t identifies an REE via (5), provided that the implied process for y_t is bounded. We now construct an mds ξ_t so that the boundedness criterion is met. Assume $|\beta_1| < 1 < |\beta_2|$ (this corresponds to the case in which "one regime is determinate and one regime is indeterminate"). Nondegeneracy requires that regimes are not absorbing; thus let $p_{22} > 0$. Define

$$\xi_t = \begin{cases} -\beta_1^{-1} y_{t-1} + \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{p_{11}}{p_{12}}\beta_1^{-1} y_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ -\beta_2^{-1} y_{t-1} + \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{p_{21}}{p_{22}}\beta_2^{-1} y_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2), \end{cases}$$

where $\delta_{ij} \in \mathbf{R}$ is arbitrary, and ε_t is any martingale difference sequence with uniformly bounded support. Then ξ_t is an mds.[9] The dynamics for y_t implied by plugging ξ_t into (5) follow:

$$y_t = \begin{cases} \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{p_{12}}\beta_1^{-1} y_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{p_{22}}\beta_2^{-1} y_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2). \end{cases} \tag{6}$$

It is straightforward to verify that (6) is a solution to the model by stepping y_t forward one period, taking conditional expectations, and plugging into (1). To show that it is an REE, we have to demonstrate boundedness. We have the following result:[10]

LEMMA 1. *The process y_t, as given by (6), is uniformly bounded if and only if $|\beta_2 p_{22}| > 1$.*

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Intuitively, provided that $|\beta_2 p_{22}| > 1$, the only explosive "state" occurs when $(s_{t-1}, s_t) = (1, 2)$, but then $(s_t, s_{t+1}) = (2, 1)$ or $(s_t, s_{t+1}) = (2, 2)$: either way, the divergence is halted.

It is straightforward to find β_i such that the CLDC is satisfied and (6) is uniformly bounded, and a microfounded example based on an overlapping-generations (OLG) model is provided in Section 5.1. We note that if the CLDC is satisfied and $|\beta_2 p_{22}| > 1$, then β_1 must be positive.[11] Thus, for multiple equilibria to exist when the CLDC is satisfied, it must be the case that $\beta_2 < 0$, so that the univariate model exhibits positive feedback in the determinate regime and negative feedback in the indeterminate regime. This observation will be important in constructing the example in Section 5.1.

When $|\beta_2 p_{22}| > 1$, we refer to the process given by (6) as a HDE, because dynamics explicitly depends on s_t and s_{t-1}. Notice that the indeterminacy of regime 2 spills over across regimes so that there is sunspot dependence in both regimes. This is in contrast to RDE, where y_t switches between two stochastic processes that are independent of the underlying Markov state. In the HDE the value of y_t depends on the current state s_t and also explicitly on the Markov state in the previous period. This dependence is self-fulfilling in the sense that it exists only because of agents' expectations.

2.2. Digression: Expectational Stability in Constant-Parameter Models

Consider the constant-parameter version of (1),

$$y_t = \beta E_t^* y_{t+1} + \gamma r_t, \tag{7}$$

now written with a (possibly) boundedly rational expectations operator E^* and where $r_t = \rho r_{t-1} + \nu_t$, where $0 < \rho < 1$ and ν_t is white noise.

When $|\beta| < 1$ the model is determinate and there exists a unique equilibrium that has the form $y_t = br_t$. To analyze stability under learning, we posit that agents hold a perceived law of motion (PLM, i.e., a forecasting model) whose functional form is consistent with the equilibrium representation:

$$y_t = A + Br_t.$$

Although there is no constant in the equilibrium representation $y_t = br_t$, it is standard to allow agents to consider the possibility that there may be a constant term, i.e., to learn the steady-state values of y as well.

The parameters A and B capture agents' perceptions of the relationship between y and r, and may be estimated using, for example, recursive least squares. Let A_t and B_t be the respective estimates using data up to time t. Agents form forecasts using the PLM to obtain[12]

$$E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t.$$

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Plugging these forecasts into (7) leads to the actual law of motion (ALM),

$$y_t = \beta A_{t-1} + (\beta B_{t-1}\rho + \gamma)r_t.$$

Here we assume that agents know the true process governing r_t. The actual law of motion defines a map $T : \mathbf{R}^2 \to \mathbf{R}^2$ that takes perceived coefficients to actual coefficients:

$$T(A, B) = (\beta A, \beta B\rho + \gamma).$$

Notice that the fixed point of the T-map identifies the unique rational expectations equilibrium of the model. The unique rational expectations equilibrium $y_t = br_t$ is stable under learning if $(A_t, B_t) \to (0, b)$ almost surely.

The asymptotic behavior of (A_t, B_t) often reduces to a fairly simple and intuitive condition known as E-stability: see Evans and Honkapohja (2001). The rational expectations equilibrium is said to be E-stable if it is a locally asymptotically stable fixed point of the ordinary differential equation (o.d.e.)

$$\frac{d(A, B)}{d\tau} = T(A, B) - (A, B).$$

The E-stability principle states that if agents use recursive least squares—or closely related learning algorithms—then E-stable rational expectations equilibria are locally stable under learning.[13] In this simple example, if $(0, b)$ is a locally asymptotically stable fixed point of the o.d.e., then $(A_t, B_t) \to (0, b)$ almost surely.

The economic intuition behind the E-stability principle is simple: reasonable learning algorithms dictate that agents update their parameter estimates in the direction indicated by the forecast errors and $T(A, B) - (A, B)$ is, in a sense, a forecast error. If the rest point of the o.d.e. is stable, then adjusting parameters in the direction indicated by the forecast error will lead the parameters toward the rational expectations equilibrium. The E-stability condition is thus that all eigenvalues of DT have real parts less than one. For the case at hand, the derivatives are given by β and $\rho\beta$.[14] Because the model is determinate by assumption (i.e., $|\beta| < 1$), the rational expectations equilibrium is stable under learning.

When $|\beta| > 1$, the model is indeterminate and there exists a continuum of equilibria. To fix ideas, we assume the model is nonstochastic ($r_t = 0$) and let y_t be an REE corresponding to (7). Then there exists an mds ξ_t so that $y_t = \beta^{-1} y_{t-1} + \xi_t$: the "general form representation" of y_t. Also, there exists a serially correlated process given by $\eta_t = \beta^{-1}\eta_{t-1} + \xi_t$ so that $y_t = \eta_t$: the "common factor representation" of y_t.[15]

We first assume that agents engage in "mean value learning"; that is, they observe the extrinsic process η_t and estimate the conditional mean of y using a forecasting model of the form

$$y_t = A + B\eta_t.$$

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Computing the T-map provides $DT_A = \beta$, $DT_B = 1$, so that the sunspot equilibria are E-stable provided that $\beta < -1$.[16]

Under "VAR learning," agents estimate both the mean and the lag structure of the endogenous variables via

$$y_t = A + By_{t-1} + C\xi_t,$$

where ξ_t is a martingale difference sequence. The primary difference between VAR and mean value learning is that the latter assumes that agents coordinate on the serially correlated sunspot η_t, whereas the former postulates that agents try to detect the self-fulfilling lag structure from the data. Computing the T-map provides the following derivatives:

$$DT_A = \beta(1 + B),$$

$$DT_B = 2\beta B,$$

$$DT_C = \beta B.$$

Because, at the REE, $B = \beta^{-1}$, it follows that that $DT_B = 2$: so if agents employ VAR learning, then the sunspot equilibria are never stable.

2.3. E-Stability in Univariate Regime-Switching Model

We now extend the analysis in the previous subsection to the regime-switching model (1). We analyze the stability of equilibria in each class identified previously.

Without loss of generality, we continue to assume that $\gamma_i = 0$, $i = 1, 2$. Recall that, in case the CLDC holds, there is a unique RDE of the form $y_t = 0$. To analyze stability under learning, given the regime-switching structure, we provide agents with a PLM that, although functionally consistent with the RDE, allows regime-dependent learning:

$$y_t = A(s_t).$$

According to this PLM, the learning agent believes that if the state of the world is given by $s_t = 1$, then $y_t = A(1)$, and if the state of the world is given by $s_t = 2$, then $y_t = A(2)$.

Stepping this equation forward leads to state-contingent expectations

$$E_t\,(y_{t+1}|s_t = j) = p_{j1}A(1) + p_{j2}A(2).$$

Thus, there is a state-contingent T-map given by

$$A(j) \to \beta_j[p_{j1}A(1) + p_{j2}A(2)].$$

Differentiating leads to the Jacobian matrix that governs E-stability:

$$DT = \begin{pmatrix} \beta_1 p_{11} & \beta_1 p_{12} \\ \beta_2 p_{21} & \beta_2 p_{22} \end{pmatrix}.$$

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The condition for E-stability of RDE is that the eigenvalues of DT have real parts less than one. But now notice that the matrix DT coincides with the matrix (4) governing uniqueness of RDE: if the CLDC is satisfied, then the unique regime-dependent equilibrium is E-stable.

We now turn to the stability of the univariate HDE given in (6), and we consider VAR learning. HDE may depend on a constant, lagged y, and an extrinsic sunspot ξ, and the nature of these dependences may vary depending on the state of the world yesterday and today. Consistent with this, we provide agents with a PLM of the form

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t,$$

where ξ_t is an m.d.s., and the parameters A, B, C in the PLM are elements of the real line. We assume that s_t but not contemporaneous y_t is in the information set. Computing conditional forecasts using this PLM, we obtain the following T-map for B:

$$B(i, j) \longrightarrow \beta_j[p_{j1}B(j, 1) + p_{j2}B(j, 2)]B(i, j).$$

Ignoring the boundedness requirement, a fixed point of this map identifies an HDE. The only restrictions, then, are the following:

$$1 = \beta_1\left[p_{11}B(1, 1) + p_{12}B(1, 2)\right] = \beta_2\left[p_{21}B(2, 1) + p_{22}B(2, 2)\right].$$

In particular, there is a two-dimensional continuum of coefficients on lagged y providing fixed points.

Following Farmer et al. (2009), we focus on particular fixed points, given by

$$B(1, 1) = B(2, 1) = 0, \quad B(1, 2) = \frac{\beta_1^{-1}}{p_{12}}, \quad B(2, 2) = \frac{\beta_2^{-1}}{p_{22}}.$$

To analyze stability, we compute the eigenvalues of DT. The T-map for the coefficients B decouples and provides the following Jacobian:

$$DT_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta_1^{-1}\beta_2 p_{21}/p_{12} & \beta_1^{-1}\beta_2 p_{22}/p_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & p_{21}/p_{22} & 2 \end{pmatrix}.$$

The Jacobian has an eigenvalue of 2, which implies that under learning, HDE, when represented this way, are E-unstable.

We now explore the stability of HDE under mean value learning. Agents are, in essence, assumed to know the self-fulfilling serial correlation, but are still required to learn the mean. This assumption implies a PLM of the form

$$y_t = A(s_{t-1}, s_t) + B\eta_t, \tag{8}$$

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where η_t is an extrinsic process given by

$$\eta_t = \begin{cases} \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{p_{12}}\beta_1^{-1}\eta_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{p_{22}}\beta_2^{-1}\eta_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2). \end{cases}$$

Thus agents are assumed to condition their forecasting model for y_t on a history-dependent intercept and on an extraneous exogenous variable η_t that has the required serial correlation properties. Notice that agents' beliefs concerning the mean of y_t are still state-contingent.

The T-map is given by

$$A(i, j) \to \beta_j[p_{j1}A(j, 1) + p_{j2}A(j, 2)]$$

and $B \to B$; this yields the Jacobian matrix

$$DT_A = \begin{pmatrix} \beta_1 p_{11} & \beta_1(1-p_{11}) & 0 & 0 \\ 0 & 0 & \beta_2(1-p_{22}) & \beta_2 p_{22} \\ \beta_1 p_{11} & \beta_1(1-p_{11}) & 0 & 0 \\ 0 & 0 & \beta_2(1-p_{22}) & \beta_2 p_{22} \end{pmatrix},$$

which has the same eigenvalues as the Jacobian matrix for the RDE plus a pair of zero eigenvalues.[17] Therefore, when there is a unique RDE, and agents form their forecasts via the mean-value PLM (8), it is possible for agents to coordinate on either the RDE or the HDE.

Whether HDE exist when the CLDC is satisfied depends on the specific economic model. It is possible to verify that in the univariate case, when β_1, β_2 are restricted to take nonnegative values, the CLDC implies $\beta_2 p_{22} < 1$; thus in the Fisherian model, if there is a unique RDE, then HDE do not exist: the CLDC is sufficient to preclude the presence of HDE. HDE can exist when the CLDC is not satisfied, for example, when $\beta_1 < 1$, $\beta_2 > 1$, and $\beta_2 p_{22} > 1$; however, they are not stable under learning even if agents use mean value learning.

Moving outside the context of the Fisherian interpretation, in a model with negative coefficients, HDE can exist even when the CLDC holds. In Section 5, an OLG model is presented that naturally leads to $\beta_1 > 0$, $\beta_2 < 0$, and in which the CLDC is satisfied; then, whenever $|\beta_2 p_{22}| > 1$, it follows that HDE exist and are stable under mean value learning.

3. REGIME-SWITCHING EQUILIBRIA: THE MULTIVARIATE CASE

Having introduced our results within a univariate model, we now formalize and generalize our findings. We focus on models whose reduced form consists of a

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system of nonlinear expectational difference equations of the form

$$y_t = \beta(s_t) E_t y_{t+1} + \gamma(s_t) r_t, \tag{9}$$

where y_t is an $(n \times 1)$ vector of random variables, and $\beta(s_t)$ and $\gamma(s_t)$ are conformable matrices that depend on s_t, an m-state Markov process taking on values in $\{1, \ldots, m\}$. As before, we use the notation $\beta(s_t) = \beta_i, \gamma(s_t) = \gamma_i \Leftrightarrow s_t = i, i = 1, 2, \ldots, m$. The stochastic matrix P governing the evolution of s_t is taken to be recurrent and aperiodic, so that it has a unique stationary distribution. For simplicity, β_i is taken to be invertible for all i. Finally, $r_t = \rho r_{t-1} + \hat{\varepsilon}_t$ is a $(k \times 1)$ exogenous stationary VAR(1) process independent of s_j for all j. For convenience we assume that r_t has bounded support.

A rational expectations equilibrium of the model is a solution of (9) that also satisfies a boundary condition. We focus on processes satisfying the following property:

DEFINITION. *A stochastic process y_t with initial condition y_0 is uniformly bounded (almost everywhere) or UB if* $\exists M(y_0)$ *such that* $\sup_t ||y_t||_\infty < M(y_0)$, *where* $||\cdot||_\infty$ *is the* L^∞ *or "essential supremum" norm.*

With this definition available, we may define a rational expectations equilibrium:

DEFINITION. *A* rational expectations equilibrium is any UB stochastic process satisfying (9).

Although being UB may appear to be an a priori strong notion of boundedness, it is common in the linear rational expectations literature and is consistent with standard determinacy analysis.

3.1. Regime-Dependent Equilibria

The minimal state variable solution to the model (9) takes the form[18]

$$y_t = B(s_t) r_t, \tag{10}$$

and, as before, this functional form guides our definition of RDE.

DEFINITION. *Let s_t be the Markov process governed by P and taking values in $\{1, 2, \ldots, m\}$. Let y_t be a solution to (9). Then y_t is a RDE if it is uniformly bounded and there exist uniformly bounded stochastic processes $y_{1t}, y_{2t}, \ldots, y_{mt}$, with y_{it} independent of s_{t+j} for all integers j, such that $y_t = y_{it} \Leftrightarrow s_t = i$. Note that the MSV solution is an RDE.*

In an RDE, depending on the realization of s_t, y_t takes on values from one of m stochastic processes, with each process being independent of the Markov state.

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Conditioning (9) on each regime leads to the system

$$\begin{aligned}
y_{1t} &= \beta_1 p_{11} E_t y_{1t+1} + \beta_1 p_{12} E_t y_{2t+1} + \cdots + \beta_1 p_{1m} E_t y_{mt+1} + \gamma_1 r_t, \\
y_{2t} &= \beta_2 p_{21} E_t y_{1t+1} + \beta_2 p_{22} E_t y_{2t+1} + \cdots + \beta_2 p_{2m} E_t y_{mt+1} + \gamma_2 r_t, \\
&\vdots \qquad\qquad\qquad \vdots \\
y_{mt} &= \beta_m p_{m1} E_t y_{1t+1} + \beta_m p_{m2} E_t y_{2t+1} + \cdots + \beta_m p_{mm} E_t y_{mt+1} + \gamma_m r_t,
\end{aligned}$$

which governs dynamics for y_{it} for $i = 1, 2, \ldots, m$. We note that this is a linear system that can be written compactly as

$$\hat{y}_t = M E_t \hat{y}_{t+1} + \gamma r_t, \tag{11}$$

where $M = (\oplus_{j=1}^{m} \beta_j)(P \otimes I_n)$, $\widehat{y_t} = (y_{1t}', y_{2t}', \ldots, y_{mt}')'$ and $\gamma' = (\gamma_1', \ldots, \gamma_m')'$, and where we use the direct sum notation $\oplus_{j=1}^{m} \beta_j = \operatorname{diag}(\beta_1, \beta_2, \cdots, \beta_m)$.

The stacked system (11) is a multivariate linear rational expectations model, and, as before, the MSV solution (10) of the switching model (9) corresponds to the unique MSV solution of the stacked system (11) in the sense of McCallum (1983). The number and nature of solutions to (11) are well known. We are particularly interested in conditions under which there exists a unique UB solution to the stacked system, as this condition will also govern the expectational stability of solutions to the economic system (9). We summarize this in the following remark.

Remark. A necessary and sufficient condition for the existence of a unique uniformly bounded solution to (11) is that the eigenvalues of $(\oplus_{j=1}^{m} \beta_j)(P \otimes I_n)$ lie inside the unit circle. In this case, we say that the CLDC is satisfied.

PROPOSITION 2. *If the CLDC holds then there is a unique RDE that corresponds to the MSV solution.*

This proposition follows from the fact that any RDE solves the stacked system (11).[19] Subsequent sections show a close connection between the conditions for unique RDE and E-stable rational expectations equilibria, and so the CLDC takes on added importance hereafter.

3.2. History-Dependent Equilibria

This subsection formalizes the definition of HDE, that is, equilibria where agents condition their expectations on an expanded state vector that includes s_{t-n}. We focus on the case $n = 1$.

DEFINITION. *Let s_t be the Markov process governed by P, taking values in $\{1, 2, \ldots, m\}$. Let y_t be a solution to (9). Then y_t is a HDE if it is uniformly bounded and its distribution conditional on s_t differs from its distribution conditional on s_t and s_{t-1}: $y_t | s_t \nsim y_t | (s_t, s_{t-1})$.*

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Sunspot solutions of the stacked system (which have constant parameters) can be HDE by having a history-dependent sunspot shock: see the Remark in the Appendix for details. More generally, under conditions identified in the Appendix, there exist HDE with time-varying coefficients on the lagged endogenous variable, and we turn to equilibria of this form now.

Assuming without loss of generality that $\gamma_t = 0$ for all t, these HDE may be represented as

$$y_t = \hat{B}(s_{t-1}, s_t)y_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t, \tag{12}$$

where ξ_t is a uniformly bounded mds, and detailed expressions for the coefficient matrices $\hat{B}$, $\hat{C}$ are provided in the Appendix. Also, just as in the univariate case, it is possible to represent the same HDE in the alternative form

$$y_t = \eta_t, \tag{13}$$

where

$$\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t.$$

The stochastic properties of (13) are equivalent to those of (12); however, these two representations imply different informational assumptions and distinct stability results. As in Section 2, these observations lead to two natural learning rules.

3.3. Equilibrium Representations

The representations of regime-switching rational expectations equilibria preceding, and in Section 2, form the basis for the forecasting rules used by boundedly rational agents when forming expectations. As emphasized in Evans and McGough (2005b), stability under adaptive learning may depend on the functional form of the agents' forecasting model. Therefore, we provide the following summary:

1. There exists an RDE capturing the minimal state variable (MSV) solution to (11), and its representation is given by

$$y_t = B(s_t)r_t.$$

2. HDE are sunspot equilibria that (setting $\gamma_t = 0$ for convenience) can be naturally represented in (at least) two ways: as $y_t = \hat{B}(s_{t-1}, s_t)y_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t$, where ξ_t is an arbitrary, uniformly bounded martingale difference sequence; or as $y_t = \eta_t$, where $\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t$.

4. EXPECTATIONAL STABILITY: THE MULTIVARIATE CASE

In this section we investigate equilibrium stability in our general regime-switching model.

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4.1. E-Stability and the Conditionally Linear Determining Condition

This section demonstrates that the CLDC implies E-stability of RDE. Throughout, we assume that agents observe the current state s_t and know the true transition probabilities, but do not observe y_t. This is consistent with a common convention of the adaptive learning literature that assumes agents observe contemporaneous exogenous variables, but not current values of endogenous variables.[20]

Agents have a PLM of the form

$$y_t = A(s_t) + B(s_t)r_t,$$

where $A(j)$ is $(n \times 1)$, and $B(j)$ is $(n \times k)$. Notice that we assume agents do not know that in equilibrium the $A_i = 0$.

PROPOSITION 3. *If the CLDC holds, then the unique RDE is E-stable.*

The proof is in the Appendix. This result states that an economy described by the main expectational difference equation (9), with expectations formed from the MSV forecasting model and updated using least squares, will converge to the unique RDE.

4.2. E-Stability and Indeterminacy

Now we examine the stability of HDE, and again, for simplicity, we set $\gamma_t = 0$. We begin by considering VAR learning. In this case, the PLM takes the form

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t, \tag{14}$$

where ξ_t is an mds, independent of the Markov states. The PLM makes clear the primary distinction between HDE and the class of RDE solutions, because, here, coefficients depend explicitly on s_t and s_{t-1}, whereas coefficients in the PLM for the RDE depend only on s_t.

General results on the stability of HDE are not available because the set of all fixed point solutions has not been characterized. If $m = 2$, there is, however, a subclass of fixed points that corresponds to multivariate analogues of the equilibria studied by Farmer et al. (2009) in the univariate case. If every eigenvalue of β_2 is larger than $1/p_{22}$ in modulus, then there exists a continuum of HDE such that

$$B(1,1) = B(2,1) = 0,\ B(1,2) = \frac{\beta_1^{-1}}{p_{12}},\ B(2,2) = \frac{\beta_2^{-1}}{p_{22}}. \tag{15}$$

The restriction that the eigenvalues of β_2 are larger, in modulus, than $1/p_{22}$ ensures that the resulting process is uniformly bounded.[21] We have the following result, which is proved in the Appendix.

PROPOSITION 4. *HDE of the form (15) are never stable under VAR learning.*

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Now consider mean value learning. In this case, agents are assumed to condition on a sunspot that captures the self-fulfilling serial correlation in the endogenous vector, and thus only estimate the state-dependent mean. Specifically, assume agents observe an extrinsic noise process that satisfies

$$\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t.$$

Then we take our agent's forecasting model as

$$y_t = A(s_{t-1}, s_t) + B\eta_t.$$

We have the following result, which is proved in the Appendix.

PROPOSITION 5. *Assume the model is parameterized so that HDE exist. If there exists a unique RDE, then there exist HDE that are stable under mean value learning.*

4.3. Discussion

A brief review of our results is in order. First, consider the univariate model. If both regimes exhibit positive feedback, i.e., $\beta_j > 0$, $j = 1, 2$, then the CLDC implies determinacy: there is a unique equilibrium corresponding to the MSV solution and it is stable under learning. HDE do not exist in this case. On the other hand, there exist models with positive feedback in the "determinate" regime, $\beta_j < 1$, and negative feedback in the "indeterminate" regime, $\beta_j < -1$, for which the CLDC does not rule out multiple equilibria. In these cases, HDE are shown to exist even when the CLDC holds, and these HDE are unstable under VAR learning and stable under mean value learning. An economic example that fits this case is the OLG model presented in the next section.

The results for the multivariate case are quite similar to those for the univariate model. The CLDC guarantees a unique RDE, but might be insufficient, in general, to guarantee a unique equilibrium. Models for which the CLDC is satisfied may exhibit HDE that are unstable under VAR learning and stable under mean value learning. An economic example that fits this case is the New Keynesian model presented in the next section.

There are some questions left open by the preceding analysis. We have only studied stability of HDE of the form (15). Regardless of whether the CLDC holds, other HDE may exist, either as different fixed points for the same T-map implied by the PLM (14), or by conditioning on different states (i.e., s_{t-n} for some $n > 1$). We cannot comment on the existence or stability of such HDE.

Finally, we note that if the CLDC does not hold, then there are multiple RDE. Indeed, if the CLDC does not hold, then the stacked system (11) has multiple uniformly bounded solutions corresponding to sunspot equilibria, and provided that the sunspot shock associated with a given equilibrium is independent of the Markov state, that equilibrium will be an RDE.

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5. ECONOMIC EXAMPLES

To gain some insight into the economics necessary to allow stable sunspot equilibria, we consider two examples.

5.1. An Overlapping-Generations Model

This section presents a simple OLG model with money as storage extended to incorporate a preference parameter that follows a two-state Markov process. The basic setup is standard: households live for two periods, they work n_t hours when they are young, and they consume c_{t+1} when they are old. They produce q_t units of a nonstorable good according to the production function $q_t = n_t$. Finally, households can buy and sell goods in exchange for fiat money M at the price p_t.

The representative household solves the following problem:

$$\max \; u(c_{t+1}) - V(q_t)$$
$$p_t q_t = M_t = p_{t+1} c_{t+1}.$$

The household's FOC is given by

$$v'(n_t) = E_t \left[\frac{p_t}{p_{t+1}} u'(c_{t+1}) \right].$$

In equilibrium, $c_t = q_t$; and, assuming the money supply is constant, $p_{t+1}q_{t+1} = p_t q_t$. Combining these observations yields the equilibrium condition $v'(q_t)q_t = E_t u'(q_{t+1})q_{t+1}$. Finally, we assume that v is linear and u is CRRA, with relative risk aversion coefficient σ. Under these assumptions, the nonautarky steady state is $q = 1$ and the linearized model is

$$q_t = (1 - \sigma) E_t q_{t+1}, \tag{16}$$

where variables are now in deviation from steady-state form.

Section 2 demonstrated that if the CLDC is satisfied and HDE of the form (6) exist, then the determinate regime must exhibit positive feedback, and the indeterminate regime must exhibit negative feedback. Thus, assume a two-state Markov process: $\sigma(s_t) \in \{\sigma_1, \sigma_2\}$, where $0 < \sigma_1 < 1 < \sigma_2$. With this assumption, the model (16) fits the reduced-form structure (1); further, it is straightforward to choose parameter values such that the CLDC is satisfied and HDE of the form (6) exist: $\sigma_1 = 0.7$, $\sigma_2 = 2.5$, $p_{11} = 0.5$, $p_{22} = 0.7$ provides an example.

5.2. A New Keynesian Model

There is extensive empirical evidence of regime change in monetary policymaking [see Clarida et al. (2000), Bernanke (2004), and Dennis (2006)]. These findings motivate models that build regime-switching policy rules directly into rational expectations frameworks.

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As an example, Davig and Leeper (2007) and Farmer et al. (2009) construct rational expectations solutions for the New Keynesian model closed with a nominal interest rate rule whose coefficients are subject to occasional regime change. The New Keynesian model is given by (linearized) reduced-form equations for inflation, π, and the output gap, x, such as

$$\pi_t = \delta E_t \pi_{t+1} + \kappa x_t + u_t$$
$$x_t = E_t x_{t+1} - \sigma^{-1}\left(i_t - E_t \pi_{t+1}\right) + g_t,$$

where g_t captures aggregate demand shocks and u_t represents cost-push shocks. The first equation is the New Keynesian Phillips curve, which represents the aggregate supply relation. The second equation is the New Keynesian IS equation, which represents the demand side of the economy.

It is typical to close a New Keynesian model with a nominal interest rate targeting rule along the lines proposed by Taylor (1993). A New Keynesian model with recurring policy change assumes a nominal interest rate rule with time-varying parameters

$$i_t = \alpha(s_t)\pi_t + \gamma(s_t)x_t.$$

To capture recurrent regime change, Davig and Leeper (2007) assume that the parameters $\alpha(s_t)$, $\gamma(s_t)$ in the policy rule follow a two, state Markov chain:

$$\alpha(s_t) = \begin{cases} \alpha_1 & \text{for } s_t = 1 \\ \alpha_2 & \text{for } s_t = 2 \end{cases},$$

and

$$\gamma(s_t) = \begin{cases} \gamma_1 & \text{for } s_t = 1 \\ \gamma_2 & \text{for } s_t = 2 \end{cases}.$$

The random variable s_t follows a finite-state Markov chain with transition probabilities $p_{ij} \equiv \Pr[s_t = j | s_{t-1} = i]$ for $i, j = 1, 2$.

The Taylor Principle dictates that, in a model with constant policy coefficients α, γ, nominal interest rates rise more than one for one with inflation, that is, $\alpha > 1$. Policy that satisfies the Taylor Principle leads to a model with a unique rational expectations equilibrium, whereas when $\alpha < 1$ it is possible for there to exist multiple equilibria that exhibit inefficiently high volatility. With regime-switching policy rules, private-sector expectations build in the possibility of future passive monetary policy and this places a restriction on how active (i.e., to what extent $\alpha > 1$) policy must be to ensure determinacy.

As an example, we parameterize the model as in Davig and Leeper (2007): $\delta = 0.99, \kappa = 0.17, \sigma = 1$. Moreover, we follow Farmer et al. (2010) and set $\alpha_1 = 3.0, \alpha_2 = 0.92, \gamma_1 = \gamma_2 = 0$, and $p_{11} = 0.8, p_{22} = 0.95$. With these parameter values the CLDC is satisfied, as the eigenvalues of the matrix M are 0.98, 0.81, 0.63, 0.63. Thus, there is a unique E-stable RDE. Similarly, it is

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possible to compute the eigenvalues of the matrices β_j, $j = 1, 2$: eigenvalues of β_1 (in absolute value) are $0.81, 0.81$, and for β_2 they are $1.06, 0.81$. Notice that the largest eigenvalue of β_2 is greater than $1/p_{22} = 1.05$. The E-stability results in Propositions 4 and 5 apply to an HDE where both eigenvalues are larger, in modulus, than $1/p_{22}$. However, Farmer et al. (2010) show that there exists a HDE in a form similar to (6). It is possible to extend the E-stability [see the Appendix in Branch et al. (2011)] to this case, and it follows that the HDE are E-stable under mean-value learning but never under VAR learning.

It is surprising that HDE can be stable under mean value learning. In a constant-parameters New Keynesian model, Evans and McGough (2005a) studied the stability of rational expectations equilibria under mean value learning and found that, under a constant-parameter policy rule of the same form as considered in this paper, the "common factor" representation of a sunspot equilibrium is E-unstable.

6. CONCLUSION

We extend the literature on learning to a nonlinear framework to allow for regime shifts and we study the stability under adaptive learning of two classes of equilibria:

- *Regime-Dependent Equilibria*: An RDE is a uniformly bounded process that satisfies the regime-switching expectational difference equation and imposes the restriction that agents do not condition their expectations on lagged regimes (i.e., only the current regime enters the state vector).
- *History-Dependent Equilibria*: An HDE is a process that satisfies the regime-switching expectational difference equation, where agents condition expectations on current and lagged values of the regime (i.e., current and past regimes enter the state vector).

The CLDC ensures the existence of a unique RDE, and further indicates that it is also E-stable. When the CLDC is satisfied, there may still exist sunspot equilibria, and we demonstrate that these equilibria may be learnable, depending on the conditioning set imposed on boundedly rational agents.

We applied our results to three simple models. In the univariate Fisherian model, because feedback is positive, there is a unique equilibrium, and it is selected by E-stability. On the other hand, an OLG model allows negative feedback regimes even in the univariate case, and therefore induces equilibrium multiplicity even when there is a unique RDE. Finally, a benchmark version of the New Keynesian model yields the analogous result in higher dimensions and in a model of applied interest. When there are simulataneously a unique RDE and many HDE, we found that under VAR learning, E-stability selects a unique equilibrium; however, it is possible that if agents condition their expectations on the self-fulfilling serial correlation (e.g., sunspots)—a stronger requirement than VAR learning—then there may exist multiple E-stable equilibria.

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NOTES

1. Some examples of work in this area include Andolfatto and Gomme (2003), Leeper and Zha (2003), Davig (2004), Zampolli (2006), Chung et al. (2007), Davig and Leeper (2007), Svensson and Williams (2007), Benhabib (2009), Farmer et al. (2009). Brainard (1967) is an early example of work on parameter instability.

2. There has been an extensive empirical literature modeling the economy as following a regime-switching process [see Hamilton (1989), Kim and Nelson (1999), and Sims and Zha (2006)].

3. See, in addition, Evans (1985), Bray and Savin (1986), and Marcet and Sargent (1989).

4. The relevant notion of boundedness will be made formal in the next section.

5. A minimal state variable solution of a system of expectational difference equations is a notion introduced by McCallum (1983), and corresponds to a solution that conditions on the smallest possible collection of exogenous and predetermined variables. McCallum (1983) provides an additional "limiting" criterion to select among multiple MSV-like solutions. Within the context of the models studied by McCallum, his limiting criterion identifies a unique MSV solution. In this paper, we will simply refer to the equilibrium of the form (2) as the MSV solution. The multivariate counterpart in Section 3 has the same form.

6. The precise definition of an RDE will be given in the next section.

7. A more complete discussion of the stacked system is given in Section 3.1.

8. Davig and Leeper (2007) showed that, in the univariate case, provided that $|\beta_1| < 1 < |\beta_2|$, the CLDC is equivalent to

$$p_{11}\beta_1(1-\beta_2) + p_{22}\beta_2(1-\beta_1) + \beta_1\beta_2 < 1.$$

9. It is straightforward to verify that ξ_t is an mds by taking one step ahead expectations conditional on the observable states s_{t-n}, $n > 0$.

10. The proof is in the Appendix of the working paper version, Branch et al. (2011).

11. This follows from algebra, with the help of Mathematica.

12. We assume that learning agents may condition on lagged endogenous variables and current exogenous variables, but not on current endogenous variables. This is a standard timing convention in the learning literature. Alternative informational assumptions are available and can affect stability analysis: see Evans and McGough (2005b) for details.

13. The connection between E-stability of a rational expectations equilibrium and its stability under real-time learning is quite deep: see Evans and Honkapohja (2001) for details.

14. Here, and later, we exploit that when the T-map decouples, we can compute derivatives separately.

15. In the model (7), because there is no lag dependence, the serial correlation in the equilibrium y_t is present only because agents expect it to be: thus we call the serial correlation "self-fulfilling." If the reduced-form model (7) also included lagged terms (i.e., y_{t-n} for some $n > 0$), then the sunspot equilibrium would have both self-fulfilling and fundamental serial correlation. In a linear model, the common factor sunspot η_t is required to capture the self-fulfilling part of the model's serial correlation. This requirement is known as the resonance frequency condition—see Evans and Honkapohja (2003) and Evans and McGough (2005b) for details.

16. That $DT_B = 1$ reflects the fact that $T(B) = B$: if η_t is an appropriately serial correlated sunspot, then so too is $B\eta_t$ for any B. For more on this issue, see Evans and McGough (2005b).

17. That the T-map fixes B is a standard result: if η_t is a self-fulfilling sunspot then $B\eta_t$ is as well.

18. The coefficient matrices $B(i)$ are computed in the Appendix.

19. See the Appendix of the working paper version [Branch et al. (2011)] for the proof.

20. This may be a strong assumption, but it can be viewed as the same assumption made under rational expectations.

21. The New Keynesian example in Section 5.2 and the Appendix of the working paper version [Branch et al. (2011)] give an example of an HDE where only one eigenvalue of β_2 is larger, in modulus, than $1/p_{22}$.

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22. If one were to literally use the method of undetermined coefficients, the v in (A.2) would be y_t. However, if v is taken to be a vector of initial conditions chosen to lie on the stable manifold, and if (A.2) is satisfied at $t = 1$, then it will be satisfied for all t.

23. And the set of fixed points can be quite complicated. For example, note that if there is a non-trivial HDE with, say, $B(i, j) \neq 0$, then according to DT_B,

$$\beta_j \left(\sum_{k=1}^{m} p_{jk} B(j, k) \right) = 1$$

so that $C(i, j)$ can be any real number. In particular, the derivative of the T-map has a unit eigenvalue in this dimension. The effect of this unit eigenvalue on the link between E-stability and real-time learning (i.e. the E-stability Principle) is discussed in detail in Evans and Honkapohja (2001).

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APPENDIX

A.1. MSV SOLUTION OF THE MAIN MODEL

Following Davig and Leeper (2007), we take the MSV solution of (9) to have the form $y_t = B(s_t)r_t$, and note that this equilibrium coincides with the MSV solution [in the sense of McCallum (1983)] to the stacked system (11). Using this later insight, we may compute the MSV coefficients. We solve for $B(s_t)$ for $s_t \in \{1, 2, \ldots, m\}$ by using the stacked system: set

$$B = [B(1)', \ldots, B(m)']',$$

which yields $\hat{y}_t = Br_t$, where $\hat{y}_t' = (y_{1t}', \ldots, y_{nt}')$, and

$$\text{vec}(B) = \left[I_{nm} - \rho' \otimes \left(\oplus_{j=1}^{m} \beta_j\right)(P \otimes I_n)\right]^{-1} \text{vec}(\gamma).$$

It is worth remarking at this point that the class of RDE includes the MSV solution to the regime-switching model, and may also include a sunspot equilibria, provided that the sunspot shock is not correlated with the underlying Markov process s_t.

A.2. HDE DETAILS

Farmer et al. (2009) show that there exist multiple uniformly bounded HDE that have the following representation:

$$y_t = \left(\frac{c_{s_{t-1}}}{v_{s_{t-1}}' v_{s_{t-1}}} v_{s_t} v_{s_{t-1}}'\right) y_{t-1} + v_{s_t}\xi_t, \tag{A.1}$$

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provided there exist $c_1, \ldots, c_m$ and $v = (v_1', \ldots, v_m')' \neq 0$ such that $|c_j| \leq 1$ and c and v solve

$$\left\{\left(\oplus_{j=1}^m \beta_j\right)^{-1} - \left[\left(\oplus_{j=1}^m c_j\right) P\right] \otimes I_n\right\} v = 0. \tag{A.2}$$

Here ξ_t is independent of s_{t+n} for all n. The condition (A.2) is essentially derived from the method of undetermined coefficients. When (A.2) is satisfied, solutions to the representation (A.1) are solutions to (9).[22] The construction of the autoregressive parameter in the representation (A.1) is chosen so that, regardless of the history of realizations of s_t, these parameters are bounded in matrix norm and, hence, the solutions are uniformly bounded.

Our definition of HDE—as rational expectations equilibria that exhibit conditional dependence on both s_t and s_{t-1}—allows the identification of a more general class of equilibria than those represented by (A.1). Sticking with the case $\gamma = 0$, we consider equilibria of the form

$$y_t = B(s_{t-1}, s_t) y_{t-1} + C(s_{t-1}, s_t)\xi_t.$$

The coefficients must satisfy

$$\left\{I_n - \beta_j \left[\sum_{k=1}^m p_{jk} B(j, k)\right]\right\} B(i, j) = 0 \tag{A.3}$$

$$\left\{I_n - \beta_j \left[\sum_{k=1}^m p_{jk} B(j, k)\right]\right\} C(i, j) = 0. \tag{A.4}$$

Notice that provided nonzero $B(i, j)$ satisfy (A.3), the $C(i, j)$ are arbitrary. It is straightforward to verify that (A.1) is a solution to (A.3), but as we saw in Section 2, there may be many others.[23]

Remark. The definition of an HDE restricts solutions to the class of uniformly bounded stochastic processes whose conditional density exhibits dependence on s_t and s_{t-1}. Notice that if y_t is an RDE then $y_t|s_t \sim y_t|(s_t, s_{t-1})$. However, by Proposition 2, when the CLDC is not satisfied, there may exist solutions to the stacked system that are not RDE. In particular, when the matrix $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$ has n_s eigenvalues outside the unit circle then for each n_s-dimensional martingale difference sequence ξ_t of forecast errors independent of s_{t-n} for all n, there are a martingale difference sequence $\tilde{\xi}_t$ and an SSE $\hat{y}_t$ with a representation given by

$$\hat{y}_t = b\hat{y}_{t-1} + c r_{t-1} + d(s_{t-1}, s_t)\tilde{\xi}_t, \tag{A.5}$$

where d is any function of s_{t-1} and s_t.

A.3. PROOF OF PROPOSITION 3

Given the PLM $y_t = A(s_t) + B(s_t) r_t$, expectations are state-contingent, where $s_t = j$ implies

$$\begin{aligned} E_t\left(y_{t+1}|s_t = j\right) &= p_{j1}A(1) + p_{j2}A(2) + \cdots + p_{jm}A(m) \\ &\quad + [p_{j1}B(1) + p_{j2}B(2) + \cdots + p_{jm}B(m)]\rho r_t. \end{aligned}$$

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This produces a state-contingent actual law of motion, or, equivalently, a state-contingent T-map

$$A(j) \to \beta_j \left[p_{j1}A(1) + p_{j2}A(2) + \cdots + p_{jm}A(m)\right]$$
$$B(j) \to \beta_j \left[p_{j1}B(1) + p_{j2}B(2) + \cdots + p_{jm}B(m)\right] \rho + \gamma_j.$$

Conveniently, this state-contingent T-map may be stacked, and becomes the T-map associated with the stacked system under the PLM $\hat{y}_t = A + Br_t$, where, as before, $B = [B(1)', \ldots, B(m)']'$, and also $A = [A(1)', \ldots, A(m)']'$. The T-map is given by

$$T(A, B)' = \left[\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\,A, \left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\,B\rho + \gamma\right],$$

and the RDE is a fixed point of $T(A, B)$. Here $T : \mathbf{R}^{(nm\times 1)} \oplus \mathbf{R}^{(nm\times k)} \to \mathbf{R}^{(nm\times 1)} \oplus \mathbf{R}^{(nm\times k)}$.

The eigenvalues of the Jacobian matrices

$$DT_A = \left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)$$
$$DT_B = \rho' \otimes \left[\left(\oplus_{j=1}^{m}\beta_j\right)(P \otimes I_n)\right]$$

govern E-stability; i.e., E-stability requires real parts less than one, so that the E-stability condition is implied by the CLDC.

A.4. PROOF OF PROPOSITION 4

The block of the T-map associated with the perceived parameters B is given by

$$B(i, j) \to \beta_j[p_{j1}B(j, 1) + p_{j2}B(j, 2)]B(i, j).$$

Because this block decouples from the rest, showing that this block of the T-map is unstable is sufficient. The Jacobian is given by

$$\begin{pmatrix} DT_B^1 & DT_B^2 \\ DT_B^3 & DT_B^4 \end{pmatrix},$$

where

$$DT_B^1 = \begin{pmatrix} DT_B^1(1,1) & p_{12}B(1,1)' \otimes \beta_1 \\ 0 & I \otimes \beta_2\left(p_{21}B(2,1) + p_{22}B(2,2)\right) \end{pmatrix},$$

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$$DT_B^2 = \begin{pmatrix} 0 & 0 \\ p_{21}B(1,2)' \otimes \beta_2 & p_{22}B(1,2)' \otimes \beta_2 \end{pmatrix},$$

$$DT_B^3 = \begin{pmatrix} p_{11}B(2,1)' \otimes \beta_1 & p_{12}B(2,1)' \otimes \beta_1 \\ 0 & 0 \end{pmatrix},$$

$$DT_B^4 = \begin{pmatrix} I \otimes \beta_1 \, (p_{11}B(1,1) + p_{12}B(1,2)) & 0 \\ p_{21}B(2,2)' \otimes \beta_2 & DT_B^4(2,2) \end{pmatrix},$$

$$DT_B^1(1,1) = p_{11}B(1,1)' \otimes \beta_1 + I \otimes \beta_1 \, (p_{11}B(1,1) + p_{12}B(1,2)),$$

$$DT_B^4(2,2) = p_{22}B(2,2)' \otimes \beta_2 + I \otimes \beta_2 \, (p_{21}B(2,1) + p_{22}B(2,2)).$$

Inserting

$$B(1,1) = B(2,1) = 0, \quad B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, \quad B(2,2) = \frac{\beta_2^{-1}}{p_{22}}$$

yields repeated unit eigenvalues, plus the eigenvalues of

$$\beta_2^{-1} \otimes \beta_2 + I \otimes I. \qquad \textbf{(A.6)}$$

Now notice that for a given $n \times n$ matrix A, if λ is an eigenvalue of A, then $\lambda + 1$ is an eigenvalue of $A + I_n$. Because $\beta_2^{-1} \otimes \beta_2$ has a unit eigenvalue, we conclude that 2 is an eigenvalue of $(A.6)$, thus implying instability.

A.5. PROOF OF PROPOSITION 5

The corresponding T-map is

$$A(i,j) \to \beta_j \sum_{k=1}^{m} p_{jk} A(j,k) \qquad \textbf{(A.7)}$$

and $T(B) = B$. To prove the proposition, we show that the eigenvalues of the relevant Jacobian matrix DT_A are the mn eigenvalues of $M = (\oplus_{j=1}^{m} \beta_j)\,(P \otimes I_n)$ plus $(m-1)mn$ zeros. Let M_i denote the ith block of n rows of M. Also, let

$$A = (A(1,1)', A(1,2)', \ldots, A(1,m)', A(2,1)', \ldots, A(m,m)')' \in \mathbf{R}^{m^2 n}$$

be the stacked vector of perceived constants. Finally, anticipating the result, define the matrix DT_A as follows:

$$DT_A = \begin{pmatrix} M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ 0 & 0 & \cdots & M_m \\ M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ 0 & 0 & \cdots & M_m \\ \vdots & \vdots & \vdots & \vdots \\ M_1 & 0 & \cdots & 0 \\ 0 & M_2 & \cdots & 0 \\ 0 & 0 & \cdots & M_m \end{pmatrix} \in \mathbf{R}^{m^2n \times m^2n}.$$

Noting that the T-map for the perceived constants, as given by (A.7), may be represented as $A \to DT_A \cdot A$, we conclude that DT_A is the relevant Jacobian matrix.

By construction, the dimension of the kernel of DT_A is $(m-1)mn$, indicating the same number of zero eigenvalues. Now let v be an eigenvector of M, and let λ be the associated eigenvalue. Then, by explicit computation, $w = (v', v', \ldots, v')'$ is an eigenvalue of DT_A associated to eigenvalue λ. Thus the remaining mn eigenvalues of DT_A are the same as the eigenvalues of M.

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