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Nonuniqueness of the Equilibrium in Lewis and Schultz's Model

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Lewis and Schultz (2003) develop a statistical signaling model to deal with international conflicts or bargaining situations in which states have private information about their payoffs. They claim that they can confirm that there always exists a unique equilibrium in their model. In this paper, I show that Lewis and Schultz's claim is not true and their model admits multiple equilibria under some parameter settings. Monte Carlo analysis shows that when there are multiple equilibria, the parameter estimates may not converge to their true values even if the number of observations increases.

1 Introduction

As more and more theoretical arguments in international relations emphasize the importance of strategic interactions between states, many scholars have been trying to build corresponding empirical models to test such arguments validly. In particular, there have been several efforts to develop game theoretic models and estimate the model parameters directly to capture the strategic behavior (Smith 1999; Signorino 1999, 2003; Lewis and Schultz 2003).

Although previous statistical models deal with complete information games, Lewis and Schultz (2003) develop a statistical signaling game in which players have private information about their payoffs. In their model, two states, the challenger and the target, dispute over some good, whereas each state does not know how attractive the war outcome is to the other state. Their model is very appealing since information plays a key role in most international conflicts or bargaining situations. Many interstate disputes cannot be explained without private information (Fearon 1995; Goemans 2000; Powell 2004) and therefore, to understand them, the empirical model should also be able to incorporate the information structure between states. In addition, their model is applicable to many situations and can be easily solved.

Lewis and Schultz solve the game for the perfect Bayesian equilibrium (PBE) and claim that they can confirm that there always exists a unique equilibrium in the model (Lewis and Schultz 2003, 351). The uniqueness of equilibrium is an important issue since the estimator is guaranteed to be consistent only when there is a unique solution in the model. The PBE is a natural solution concept for private information games, but it is well known that there often exist multiple PBE.

In this paper, I present a counterexample, which shows that Lewis and Schultz's claim is not true and their model admits multiple equilibria under some parameter settings. In fact, there are an infinite number of parameter settings under which the game has multiple equilibria. I also conduct Monte Carlo analysis to illustrate how the existence of multiple equilibria can cause a problem. When there is more than one equilibrium, since we do not know from which equilibrium the data are generated, estimation procedure may be problematic. The analysis shows that the estimates may not get closer to their true values even if the number of observations increases.

Author's note: I thank Curtis Signorino for helpful comments. I also thank the coeditor and the two anonymous reviewers for insightful suggestions. All remaining errors are mine. For replication code for the simulations in this article, see Jo (2011).

As the existence of multiple equilibria may cause estimation failure or inconsistent estimates, the usefulness of the model might be limited. Even if the model specification is correct and there are a large number of observations, the model may not be able to give us the right answer to important research questions. First of all, inconsistent parameter estimates make it impossible to recover the distribution of terminal node payoffs. Also, we cannot learn how theoretically interesting covariates (e.g., relative military and economic power and regime type) affect states' preferences over war outcomes. Next, the model cannot predict dispute outcomes correctly even if all necessary information is given. Moreover, hypothesis testing is invalid. For example, if we want to know whether there are audience costs when a state is caught bluffing, we cannot draw a valid conclusion either way even if the estimated audience cost is statistically significantly negative or positive. Finally, although Lewis and Schultz highlight that their model can capture the complicated signaling and learning dynamics between the challenger and the target states, the target's posterior beliefs about the challenger's type cannot be calculated precisely and thus, we cannot learn how the challenger's action affects the target's action through the updating of its beliefs.

The analysis proceeds as follows. The next section briefly describes the model. The nonuniqueness of the equilibrium is shown in Section 3 and the results of the Monte Carlo analysis are presented in Section 4. Then, I conclude.

2 Summary of the Model

In this section, I sketch the model developed by Lewis and Schultz (2003). I follow the notation used in the original paper. Interested readers should consult the original paper for more details.

There are two states, A and B. Both desire to possess some good that currently belongs to B. The sequence of moves is as follows. First, A decides whether to challenge B for the good. If A chooses not to challenge, the status quo (SQ) prevails and each state $i \in \{A, B\}$ gets a payoff of S_i. If A challenges, then B has to decide whether to resist the demand. If B chooses not to resist, it concedes the good to A (CD). The payoffs for A and B in this case are denoted by V_A and C_B, respectively. If B resists, then A has to choose whether or not to fight. If A stands firm (SF) and decides to fight, a war occurs and each state $i \in \{A, B\}$ gets a payoff of W_i, which is the expected value of war to i. If A backs down (BD) and chooses not to fight, then B does not lose the good and gets a payoff of V_B, whereas A gets a payoff of a.

Each state has some private information about its payoffs. In particular, both states' payoffs from war, W_A and W_B, and A's payoff from backing down, a, have some random components:

$$W_A = \overline{W}_A + \epsilon_A,$$

$$W_B = \overline{W}_B + \epsilon_B,$$

$$a = \overline{a} + \epsilon_a,$$

where ϵ_i, $i \in \{A, B, a\}$, is drawn independently from a normal distribution with mean 0 and standard deviation σ. The disturbance terms are known only to the corresponding state, whereas the other payoffs and $\overline{W}_A$, $\overline{W}_B$ and $\overline{a}$ are commonly known. The game is depicted in Fig. 1.

3 Nonuniqueness of the Equilibrium

3.1 *Equilibrium Conditions*

Lewis and Schultz solve this game for a PBE, which requires that each state play sequentially rational strategies given its beliefs and that the beliefs should be calculated from the equilibrium strategies according to Bayes' rule.

Let p_F be the probability that A chooses to fight at the final node, p_R be the probability that B resists at the second node, and p_C be the probability that A makes a challenge at the first node. Each equilibrium

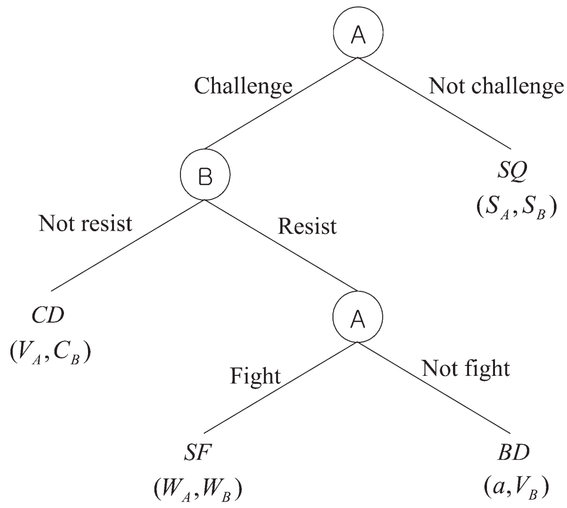

Fig. 1 Lewis and Schultz's crisis signaling game.

choice probability can be calculated as follows: First, B resists if the expected payoff from resisting is greater than the payoff from conceding, C_B. That is,

$$p_R = \Pr[p_F W_B + (1 - p_F)V_B > C_B]. \tag{1}$$

Given p_R, A decides to challenge if it expects a higher payoff from challenging than the status quo payoff, S_A. Formally,

$$\begin{aligned} p_C &= \Pr[p_R \max(a, W_A) + (1 - p_R)V_A > S_A] \\ &= \Pr[\max(a, W_A) > c^*(p_R)], \end{aligned} \tag{2}$$

where $c^*(p_R) = \frac{S_A-(1-p_R)V_A}{p_R}$. After making a challenge, A chooses to fight at the final node if the payoff from war, W_A, is greater than the payoff from backing down, a. Thus,

$$p_F = \Pr[W_A > a | \max(a, W_A) > c^*(p_R)]. \tag{3}$$

Since the errors are distributed normally with mean 0 and standard deviation σ, equations (1), (2), and (3) can be rewritten as follows:

$$p_R = \Phi\left[\frac{p_F \overline{W}_B + (1 - p_F)V_B - C_B}{p_F \sigma}\right] \equiv f(p_F), \tag{4}$$

$$p_C = 1 - \Phi\left(\frac{c^*(p_R) - \overline{W}_A}{\sigma}\right)\Phi\left(\frac{c^*(p_R) - \overline{a}}{\sigma}\right) \equiv g(c^*(p_R)) \text{ and} \tag{5}$$

$$p_F = \Phi_2\left(\frac{\overline{W}_A - \overline{a}}{\sigma\sqrt{2}}, \frac{\overline{W}_A - c^*(p_R)}{\sigma}, \frac{1}{\sqrt{2}}\right) \Big/ p_C \equiv h(c^*(p_R))/p_C. \tag{6}$$

The system of equations in (4), (5), and (6) characterizes the equilibrium choice probabilities of the model. By substitution,

$$\begin{aligned} p_R &= f(p_F) \\ &= f \circ (h(c^*(p_R))/p_C) \\ &= f \circ (h(c^*(p_R))/g(c^*(p_R))) \\ &= f \circ j(p_R), \end{aligned}$$

where $j(x) = h(c^*(x))/g(c^*(x))$. Define a function $F : (0, 1) \to Y \subseteq \mathbb{R}$ as[1]

$$F(p_R) = p_R - f \circ j(p_R). \quad (7)$$

Then, finding a solution to (4), (5), and (6) is identical to finding a root of the function F.

One can see that F has at least one root. Since $f(x) \in (0, 1)$ for all x,

$$\lim_{p \downarrow 0} F(p) = 0 - \lim_{p \downarrow 0} f \circ j(p) < 0 \quad \text{and} \quad \lim_{p \uparrow 1} F(p) = 1 - \lim_{p \uparrow 1} f \circ j(p) > 0. \quad (8)$$

As F is a continuous function, the inequalities (8) mean that there always exists an x such that $F(x) = 0$, that is, there is an equilibrium.

Lewis and Schultz (2003, 351) argue that there always exists a unique solution to (4), (5), and (6) without providing any proofs. If their argument is true, F should have a unique root. Unfortunately, this claim is incorrect. A counterexample is provided in the next subsection.

To get some intuition for the counterexample, consider F defined in (7). By (8), F cannot be globally decreasing. Thus, F being an increasing function on (0, 1) is sufficient for uniqueness of the root of F. Under the reasonable assumption that $V_B > C_B$,[2] the function $f(p_F)$ decreases in p_F. Therefore, if $j(p_R)$ is increasing, the root of F is unique. To understand the shape of the function j, look at Fig. 2,

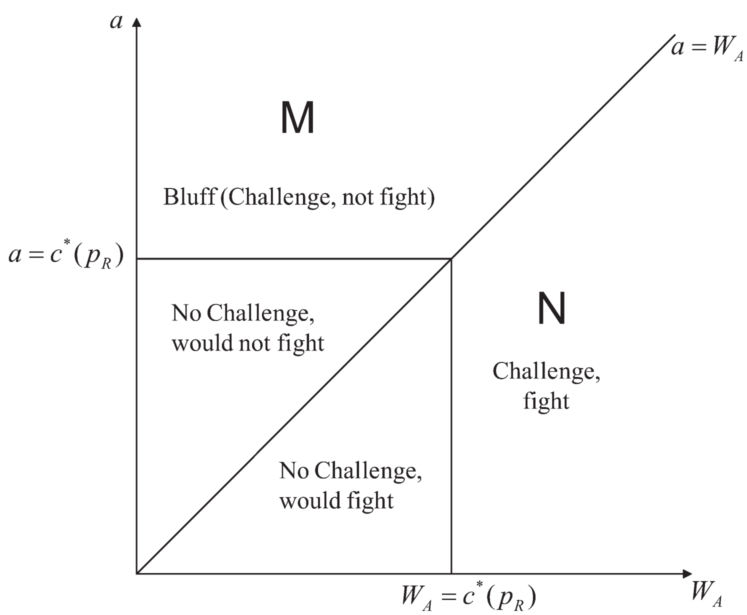

Fig. 2 A's best response for given p_R.

[1]The open interval (0, 1) is used instead of [0, 1] because 0 or 1 cannot be an equilibrium choice probability for B to resist. One can check this by considering $p_R = 0$ or 1 in the game.

[2]Since V_B is B's payoff for A backing down and C_B for B conceding the good to A, the assumption $V_B > C_B$ is reasonable.

which depicts A's best response for given p_R. It is easy to see that A makes a challenge if (W_A, a) falls in M or N since either W_A or a is greater than c^* in these regions and that it chooses to fight if (W_A, a) is under the 45° degree line since $W_A > a$. Thus, the conditional probability that A will fight, given that it has made a challenge, can be written as follows:

$$p_F = j(p_R) = \frac{\Pr[(W_A, a) \in N]}{\Pr[(W_A, a) \in M] + \Pr[(W_A, a) \in N]}.$$

If p_R increases, then the line $a = c^*(p_R)$ moves up and the line $W_A = c^*(p_R)$ moves to the right. Since both M and N shrink as p_R increases, the shape of j depends entirely on the choice of the distribution of the random components in a and W_A, meaning that j might be nonmonotonic. Thus, the uniqueness of the equilibrium is not guaranteed in general.

3.2 *A Counterexample*

In this subsection, I present a counterexample in which there are three equilibria even under a reasonable parameter setting. Recall that the number of the roots of F is the number of PBE. Therefore, I only need to show that F may admit multiple roots.

Figure 3 shows that the function F has three roots when $\overline{W}_A = -1.9$, $\overline{W}_B = -2.8$, $\overline{a} = -1.2$, $V_A = V_B = 1$, $S_A = C_B = 0$, and $\sigma = 1$. Under this parameter setting, there are three equilibria: (1) $p_R = 0.3$, $p_C = 0.96$, $p_F = 0.31$, (2) $p_R = 0.47$, $p_C = 0.58$, $p_F = 0.27$, and (3) $p_R = 0.87$, $p_C = 0.18$, $p_F = 0.2$. It is worth mentioning that this parameter setting is not unreasonable at all. It describes a situation in which each state gets a higher payoff when it possesses the contended good than when it does not, each gets negative payoff from war since war is costly, and there are some audience costs when the challenging state backs down from its threat. Under this situation, however, there are three possible equilibria in which both states behave quite differently. For example, in the first equilibrium, B resists with relatively low probability while it resists with quite high probability in the third equilibrium. Knowing this, A almost always makes a challenge in the first equilibrium while it does so with very low probability in the third equilibrium.

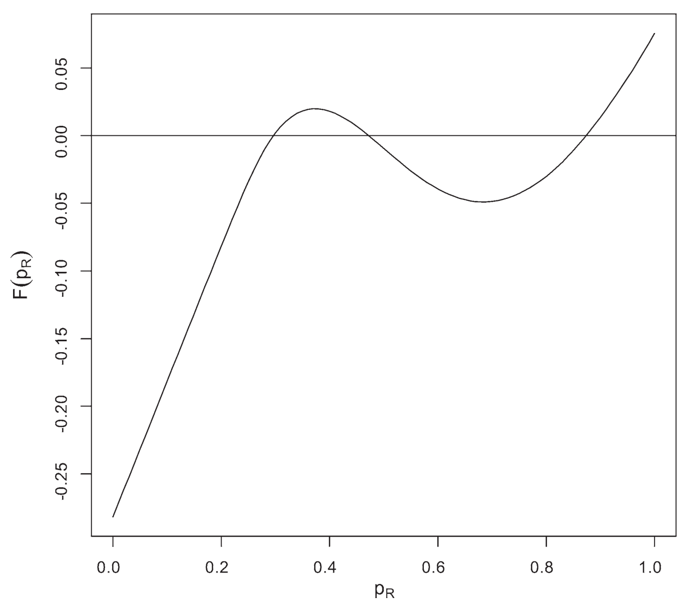

Fig. 3 Roots of function F when $\overline{W}_A = -1.9$, $\overline{W}_B = -2.8$, $\overline{a} = -1.2$, $V_A = V_B = 1$, $S_A = C_B = 0$, and $\sigma = 1$ (equilibrium p_R solves $F(p_R) = 0$).

It should be noted that this is not a single exceptional case of multiple equilibria. Since F is continuous in each parameter, there are an *infinite* number of parameter settings under which the game has multiple equilibria.

4 Monte Carlo Analysis

In this section, I conduct Monte Carlo analysis to illustrate how the existence of multiple equilibria can cause a problem.[3] The existence of multiple equilibria is very common in private information games and may not be a big problem in many situations. For example, we still can use the model to explain some historical events. Or we can learn something from comparative statics and so on.

However, if we are interested in estimating the model parameters given outcome data, the existence of multiple equilibria can be problematic in that it may cause estimation failure or inconsistent parameter estimates. Specifically, in the Maximum Likelihood Estimation (MLE) procedure, one equilibrium has to be chosen to calculate the likelihood value for a given set of parameters, which means that when there are multiple equilibria, depending on the chosen equilibrium, we will get different estimation results.

In the simulation, I use two sets of parameters to generate data. One is a set of parameters under which there exists a unique equilibrium, whereas the other produces multiple equilibria. For each parameter setting, three sets of data of different sizes (500, 2000, and 10, 000) are generated. The purpose of this simulation is to show that as the number of observations increases, the estimated parameters are getting closer to the true values in the former case, but they may not in the latter.

I employ one regressor x in B's payoff from war, W_B. Formally,

$$\overline{W}_B = W_{B1} + W_{B2}x.$$

The regressor x is generated from a uniform distribution on [0, 1]. To solve identification problems, I normalize payoff parameters so that each player gets a payoff of zero when it does not possess the good. Formally, $S_A = C_B = 0$. Additionally, I set the standard deviation of the error terms, σ, to be 1. The true parameter specifications used in the simulation are presented in Table 1. As shown in Fig. 4, for all x in [0, 1], there is always a unique equilibrium under the first parameter setting, whereas there are always three equilibria under the second setting. In the data-generating process, after generating x, if there are three equilibria, I sort them by the magnitude of the probability of B resisting, p_R. I call it the first equilibrium if it has the smallest p_R among the three. Likewise, I call it the second equilibrium if it has the second smallest p_R and the third equilibrium if its p_R is the largest. And for comparison, I generate three sets of outcome data, each corresponding to the first, the second, and the third equilibrium. In the estimation procedure, a common root–finding algorithm is used to find an equilibrium for a given set of parameter values as if there is only one equilibrium.[4]

Table 1 True parameter specification

	Setting 1 *(unique)*	*Setting* 2 *(multiple)*
V_A	1	1
$\overline{a}$	−1.2	−1.2
$\overline{W_A}$	−1.7	−1.9
V_B	1	1
W_{B1}	−2	−2.9
W_{B2}	0.1	0.1
σ	1	1

[3] For replication code, see Jo (2011).

[4] I use Alan Miller's "zeroin" function, which is a slightly modified version of the Algorithmic Language (ALGOL) procedure "zero" given by Brent (1973).

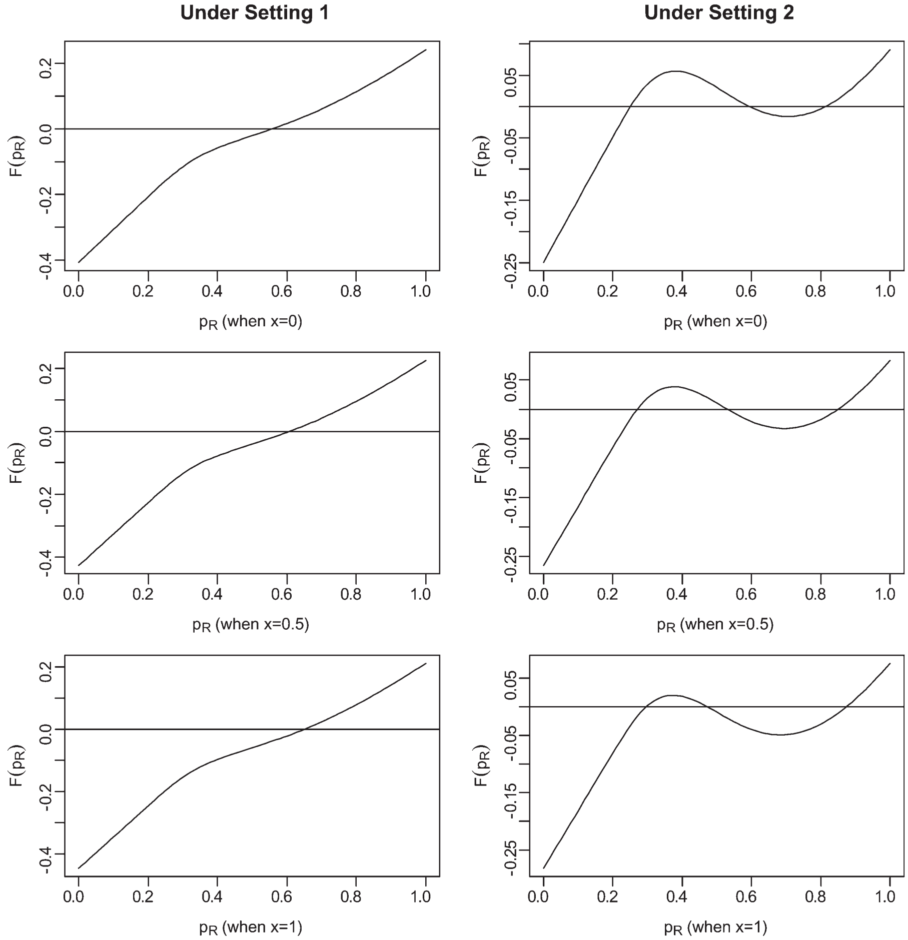

Fig. 4 Equilibrium p_R under the parameter settings in Table 1 (equilibrium p_R solves $F(p_R) = 0$).

The whole simulation procedure from data generation to estimation is iterated 2000 times for each data set. The simulation results show that the model performs well when there is a unique equilibrium, but it may fail to recover the true parameter values when there are multiple equilibria. Tables 2 and 3 present the mean of the estimates under Setting 1 and Setting 2, respectively. The density of the estimates is shown in Figs. 5–8 for the cases in which the outcome data were generated from the unique equilibrium under Setting 1 and from the first, the second, and the third equilibrium under Setting 2, respectively. In the figures, the solid line denotes the true value and the dashed line denotes the mean of the estimates. All the plots for the same parameter share the same scale. From both the tables and the figures, it is easy to see that when there is a unique equilibrium, the estimates get closer to their true values as the number of observations increases. However, it may not be the case when there are multiple equilibria. Specifically, when the outcome data are generated from the first and the second equilibrium, the performance does not

Table 2 Mean of estimates under Setting 1

	V_A	$\overline{a}$	$\overline{W}_A$	V_B	W_{B1}	W_{B2}
True parameter (setting 1)	1	−1.2	−1.7	1	−2	0.1
$N = 500$	0.996	−1.199	−1.691	0.983	−1.949	0.137
$N = 2000$	1.009	−1.206	−1.703	0.978	−1.950	0.116
$N = 10,000$	0.998	−1.198	−1.697	0.997	−1.988	0.105

Note. N: the number of observations.

Table 3 Mean of estimates under Setting 2

	V_A	$\overline{a}$	$\overline{W}_A$	V_B	W_{B1}	W_{B2}
True parameter (Setting 2)	1	−1.2	−1.9	1	−2.9	0.1
The first equilibrium						
$N = 500$	0.976	−1.132	−1.793	0.948	−2.659	0.118
$N = 2000$	0.984	−1.163	−1.832	0.945	−2.697	0.110
$N = 10,000$	0.976	−1.138	−1.824	0.916	−2.677	0.108
The second equilibrium						
$N = 500$	0.965	−1.199	−1.801	1.073	−2.598	−0.020
$N = 2000$	0.957	−1.207	−1.797	1.063	−2.531	−0.019
$N = 10,000$	0.957	−1.213	−1.801	1.038	−2.461	−0.021
The third equilibrium						
$N = 500$	0.991	−1.209	−1.872	0.989	−2.596	0.142
$N = 2000$	1.000	−1.202	−1.890	1.002	−2.813	0.108
$N = 10,000$	1.000	−1.200	−1.898	1.001	−2.890	0.101

Note. N: the number of observations. Under Setting 2, there are three equilibria. I call it the first equilibrium if it has the smallest p_R among the three, the second equilibrium if it has the second smallest p_R, and the third equilibrium if its p_R is the largest. The result for the i th equilibrium means that it is the estimation result when the outcome data were generated from that equlibrium in the simulation.

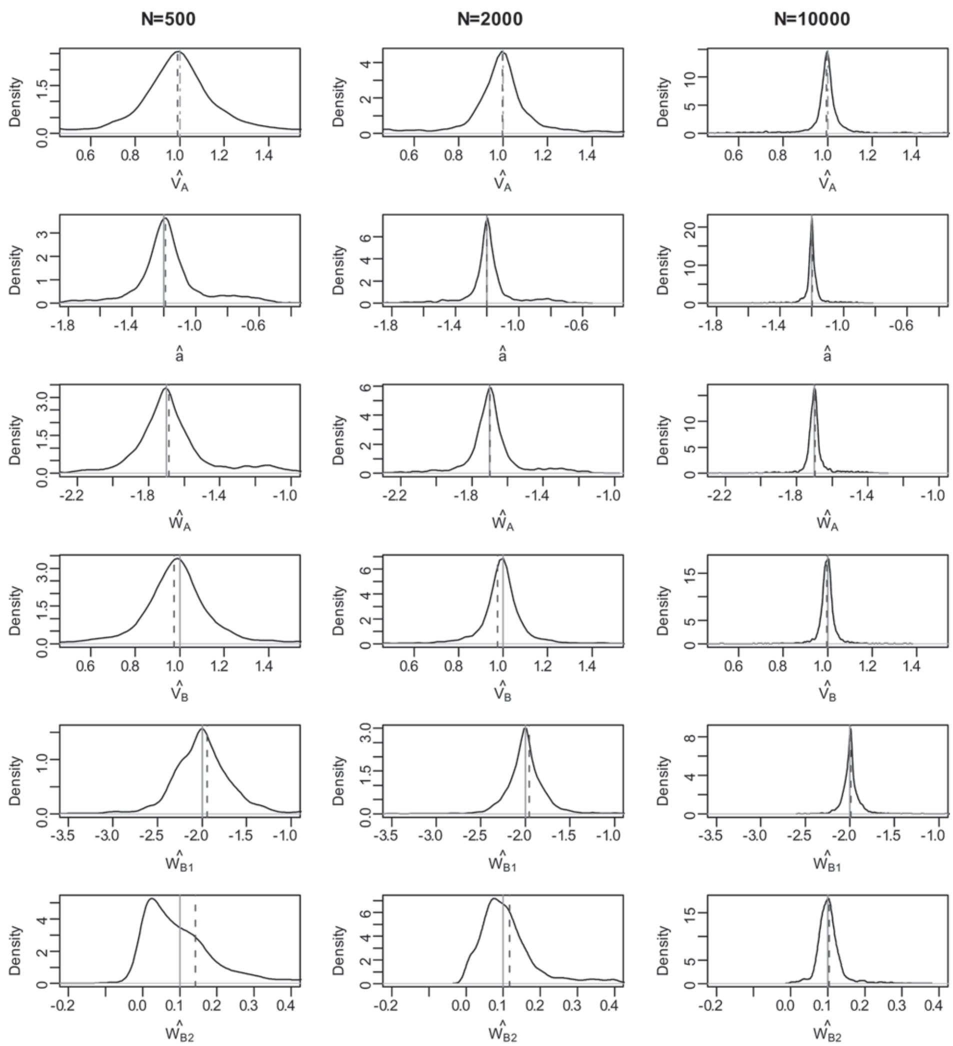

Fig. 5 Kernel density plot of estimated parameters under Setting 1 (the solid line denotes the true value and the dashed line denotes the mean of the estimates N: the number of observations).

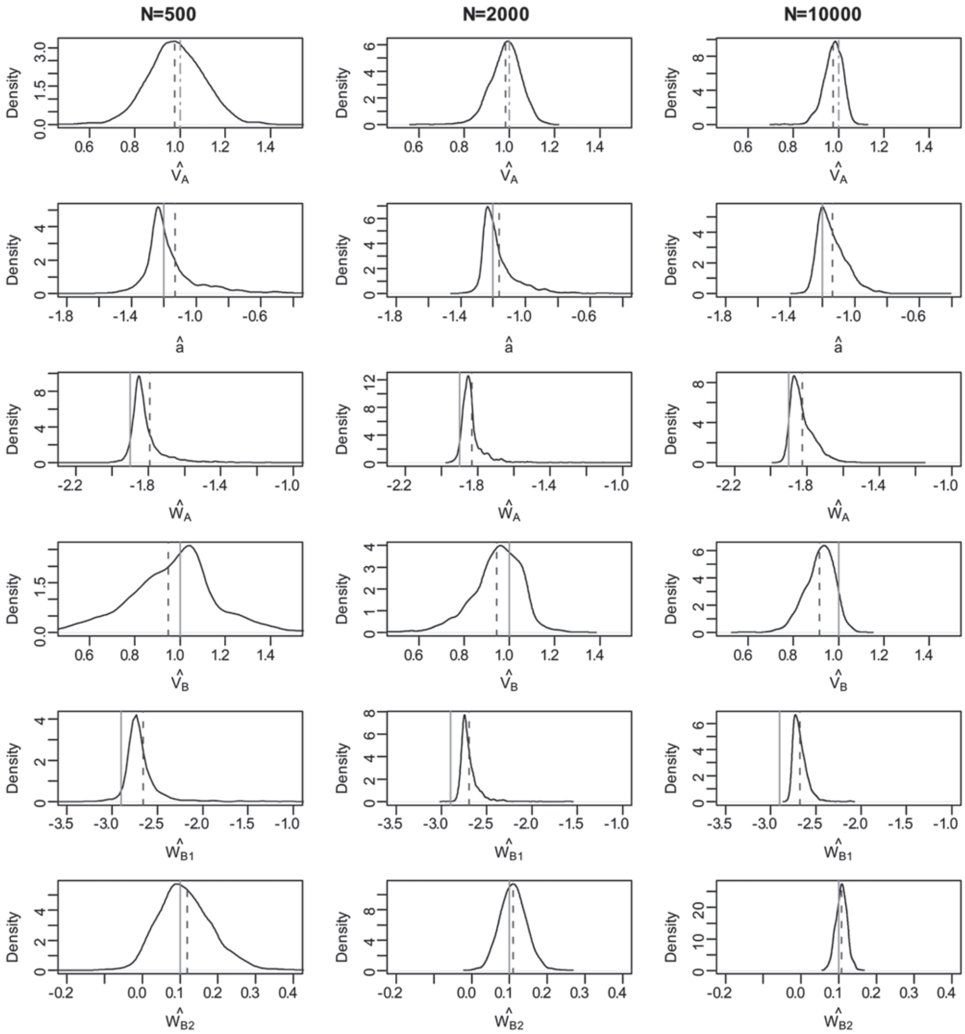

Fig. 6 Kernel density plot of estimated parameters from the data generated from the first equilibrium under Setting 2 (Under Setting 2, there are three equilibria. I call it the first equilibrium if it has the smallest p_R among the three. The solid line denotes the true value and the dashed line denotes the mean of the estimates. N: the number of observations).

improve with an increase in the number of observations. Many parameters are overestimated or underestimated in these cases. In particular, when the outcome data are generated from the second equilibrium, the effect of x on B's payoff from war, W_{B2}, is not estimated correctly. Its true value is positive. But it is estimated to be negative even when the number of observations is 10, 000. Thus, for instance, if x measures the relative military power between the two states, the model may not be able to capture that B is more likely to prefer war as it has more military power.

When there are multiple equilibria, the estimation results might be incorrect because the likelihood value is miscalculated whenever the root-finding algorithm chooses a wrong equilibrium. For instance, there are three equilibria when $x = 0.5$ under Setting 2 (see the right middle panel in Fig. 4). The three equilibria predict different dispute outcomes. Under the first equilibrium in which B is not likely to resist ($p_R = 0.27$), the probability of observing CD (after A makes a challenge, B concedes the good to A) is 0.72. On the other hand, under the second equilibrium in which B is more likely to resist ($p_R = 0.53$), the probability of ending up at CD is 0.22. Finally, under the third equilibrium in which B is highly likely to resist ($p_R = 0.85$), the probability of observing CD is 0.03. Assume that the data are generated from the first equilibrium. If we have outcome CD in the data and the third equilibrium is chosen in the

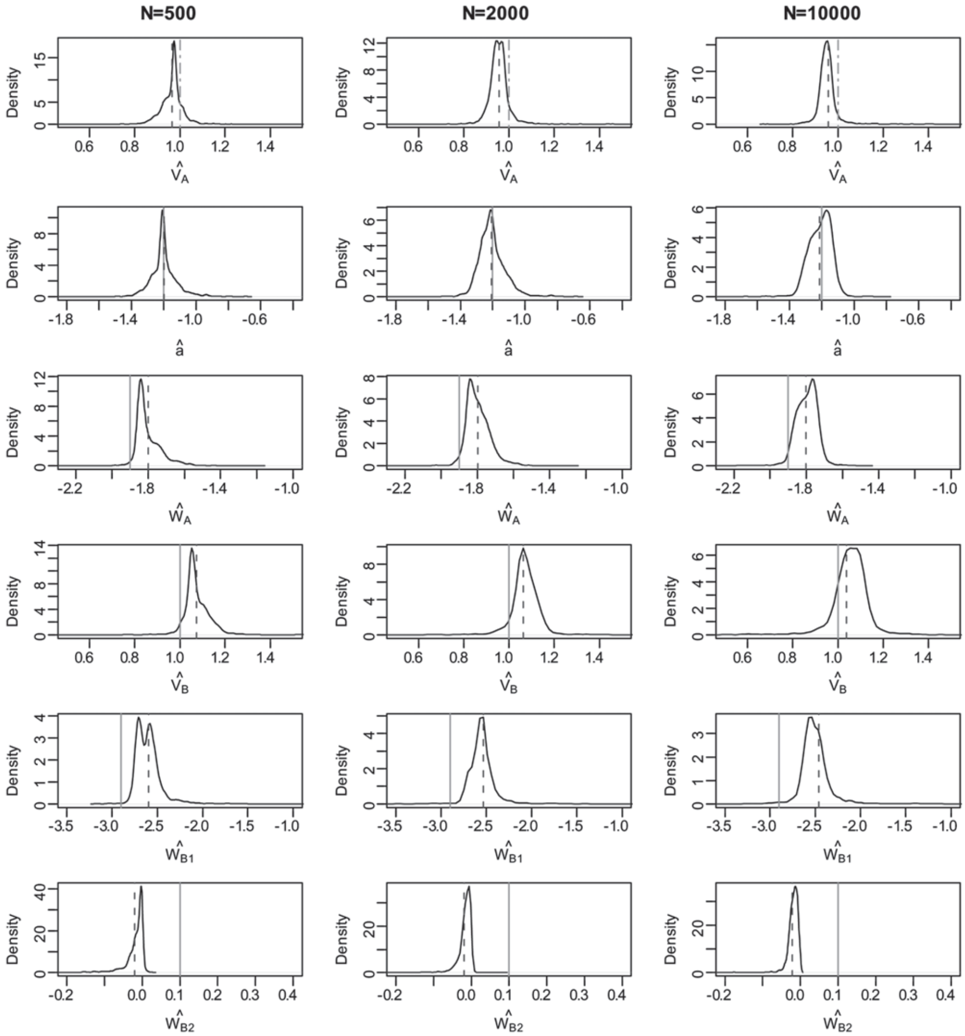

Fig. 7 Kernel density plot of estimated parameters from the data generated from the second equilibrium under Setting 2 (Under Setting 2, there are three equilibria. I call it the second equilibrium if it has the second smallest p_R among the three. The solid line denotes the true value and the dashed line denotes the mean of the estimates. N : the number of observations).

estimation process, the likelihood value will be much lower than it should be. Or if we observe that A backs down after making a challenge in the data and the third equilibrium is chosen in the estimation process, the likelihood value will be lower again than it should be since the true probability of such an outcome is 0.18 while it is 0.13 under the third equilibrium. As the likelihood value at the true parameter values may be underevaluated when the wrong equilibrium is chosen in the estimation process, there is a chance that the likelihood function is optimized at the wrong values, which might be far from the true values.

On the other hand, the simulation results show that when the third equilibrium is chosen to generate the outcome data, the estimates converge to the true parameter values. This is because, when lucky, a root-finding algorithm may pick up the true (the third) equilibrium very often in the estimation process and the estimation procedure may work fine.

Thus, depending on the root-finding algorithm used in the estimation, we may get different results. The problem is that there is no way to know whether we are lucky enough to get the right answer to our research questions. The estimates may not converge to the true values, even with a large number of observations, when the right equilibrium does not get chosen in the estimation procedure. Even if we

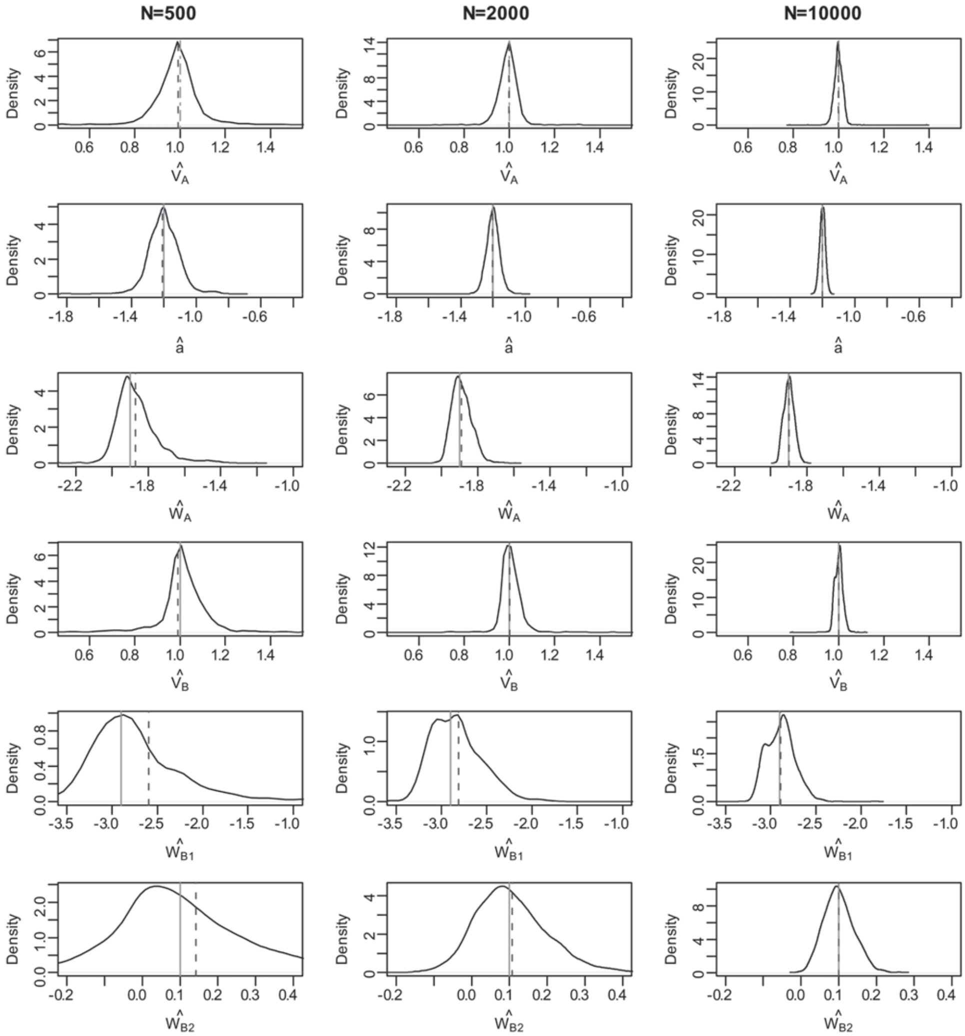

Fig. 8 Kernel density plot of estimated parameters from the data generated from the third equilibrium under Setting 2 (Under Setting 2, there are three equilibria. I call it the third equilibrium if it has the largest p_R among the three. The solid line denotes the true value and the dashed line denotes the mean of the estimates. N: the number of observations).

use a grid search method to find all the equilibria in the estimation procedure, it is not enough to help the situation because we do not know what the true data-generating process is. The model does not provide any information about which equilibrium generated the data, which is essential to estimating the parameters correctly.

5 Conclusions

Despite many nice features, Lewis and Schultz's model has multiple equilibria under some parameter settings. This paper show that multiple equilibria exist under reasonable parameter settings and it may cause inconsistent parameter estimates. Without solving this problem, the usefulness of the model might be limited.

One easy way to minimize the problem would be to introduce refinements. For example, whenever there are multiple equilibria in the estimation procedure, one may choose the equilibrium, which maximizes the sum of the players' expected payoffs. Or the equilibrium that maximizes A's payoff may be used. Another way is to find a distribution of the error terms, which guarantees the uniqueness of the equilibrium. Finally, one may restrict the ranges of the parameters so that there always exists a unique equilibrium. With these additional restrictions, the model would be able to provide more insight into states' strategic behavior.

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