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Computing a categorical Gromov–Witten invariant

Andrei Căldăraru and Junwu Tu

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Abstract

We compute the $g = 1$, $n = 1$ B-model Gromov–Witten invariant of an elliptic curve E directly from the derived category $\mathsf{D}^b_{\mathsf{coh}}(E)$. More precisely, we carry out the computation of the categorical Gromov–Witten invariant defined by Costello using as target a cyclic $\mathscr{A}_\infty$ model of $\mathsf{D}^b_{\mathsf{coh}}(E)$ described by Polishchuk. This is the first non-trivial computation of a positive-genus categorical Gromov–Witten invariant, and the result agrees with the prediction of mirror symmetry: it matches the classical (non-categorical) Gromov–Witten invariants of a symplectic 2-torus computed by Dijkgraaf.

Contents

1. Introduction

1.1 The initial form of mirror symmetry, as described in 1991 by Candelas, de la Ossa, Green and Parkes [CdGP91], centered on the surprising prediction that the genus-zero Gromov–Witten invariants of a quintic 3-fold $\check{X}$ could be computed by solving a differential equation governing the variation of Hodge structure associated to another space, the so-called *mirror quintic* X. Many other such mirror pairs $(X, \check{X})$ were later found in physics, satisfying similar relationships between the genus-zero Gromov–Witten invariants of $\check{X}$ and the variation of Hodge structure of X.

1.2 A far-reaching generalization of mirror symmetry was proposed several years later by Kontsevich [Kon95] in his address to the 1994 International Congress of Mathematicians. He conjectured that the more fundamental relationship between the spaces X and $\check{X}$ in a mirror pair should be the existence of a derived equivalence between the derived category $\mathsf{D}^b_{\mathsf{coh}}(X)$ of coherent sheaves on X and the Fukaya category $\mathsf{Fuk}(\check{X})$ of $\check{X}$. This statement became known as the homological mirror symmetry conjecture.

1.3 Implicit in Kontsevich's proposal was the idea that the equality of numerical invariants predicted by the original version of mirror symmetry should follow tautologically from the homological mirror symmetry conjecture. To achieve this one needs to construct *categorical Gromov–Witten invariants*: invariants associated to an $\mathscr{A}_\infty$ triangulated category $\mathscr{C}$, with the property that they recover the classical Gromov–Witten invariants of the space $\check{X}$ when the target category $\mathscr{C}$ is taken to be $\mathsf{Fuk}(\check{X})$. Once one has such invariants, evaluating them on $\mathsf{D}^b_{\mathsf{coh}}(X)$ yields new invariants of X, the so-called *B-model Gromov–Witten invariants* of X. These invariants are defined for *any* genus, not just for genus zero. (The genus-zero B-model invariants are expected to match the data of the variation of Hodge structures used before.) The categorical nature of the construction automatically implies, for a pair of spaces $(X, \check{X})$ which satisfies homological mirror symmetry, that the B-model invariants of X match the Gromov–Witten invariants of $\check{X}$.

1.4 Genus-zero categorical Gromov–Witten invariants satisfying the desired properties were defined in 2015 by Ganatra, Perutz and Sheridan [GPS15] following ideas of Saito [Sai83a, Sai83b] and Barannikov [Bar01]. However, according to the authors, this approach does not extend to positive genus.

For arbitrary genus Costello [Cos09] proposed a definition of categorical invariants associated to a cyclic $\mathscr{A}_\infty$-algebra (or category), following ideas of Kontsevich and Soibelman [KS09]. Unfortunately many details of [Cos09] were left open, and computing explicit examples turned out to be a difficult task. Costello (unpublished) computed one example where the target algebra is the ground field (corresponding to the case where the target space X is a point). No other explicit computations of Costello's invariants exist. Costello and Li [CL12] wrote:

> A candidate for the B-model partition function associated to a Calabi–Yau category was proposed in [Cos07, Cos05, KS09] based on a classification of a class of 2-dimensional topological field theories. Unfortunately, it is extremely difficult to compute this B-model partition function.

1.5 In this paper we compute the $g = 1$, $n = 1$ B-model categorical invariant of an elliptic curve E_τ, starting from Costello's definition and using as input an $\mathscr{A}_\infty$ model of the derived category $\mathsf{D}^b_{\mathsf{coh}}(E_\tau)$ proposed by Polishchuk [Pol11]. It is the first computation of a categorical Gromov–Witten invariant with non-trivial target and positive genus.

More precisely, for a complex number τ in the upper-half plane $\mathbb{H}$, let E_τ denote the elliptic curve of modular parameter τ, $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. For each such τ we compute a complex number $F^{\mathsf{B}}_{1,1}(\tau)$, the corresponding B-model categorical invariant. Regarding the result as a function of τ, we obtain the so-called *B-model Gromov–Witten potential*, a complex-valued function on the upper half plane.

1.6 Mirror symmetry predicts the result of the above computation. There is a standard way to collect the classical $g = 1$, $n = 1$ Gromov–Witten invariants of an elliptic curve in a generating power series, the *A-model Gromov–Witten potential* $F^{\mathsf{A}}_{1,1}(q)$. The result is known through work of Dijkgraaf [Dij95]:

$$F^{\mathsf{A}}_{1,1}(q) = -\tfrac{1}{24}E_2(q).$$

Here E_2 is the standard Eisenstein holomorphic, quasi-modular form of weight 2, expanded at $q = \exp(2\pi \mathrm{i}\tau)$. In this computation, the inserted class at the one puncture is $[\mathsf{pt}]^{\mathsf{PD}}$, the Poincaré dual class of a point.

The prediction of mirror symmetry is that the A- and B-model potentials should match after the Kähler and complex moduli spaces are identified via the mirror map, which in the case of elliptic curves takes the form

$$q = \exp(2\pi \mathrm{i}\tau).$$

Thus the prediction of mirror symmetry is that the B-model potential should equal

$$F^{\mathsf{B}}_{1,1}(\tau) = -\tfrac{1}{24}E_2(\tau).$$

1.7 To get our computation off the ground we need a cyclic $\mathscr{A}_\infty$-algebra model of the derived category $\mathsf{D}^b_{\mathsf{coh}}(E_\tau)$. Such an algebra was described by Polishchuk [Pol11], using structure constants that are modular, almost holomorphic forms. We denote this $\mathscr{A}_\infty$-algebra by A_τ. We will use both Polishchuk's original algebra, and a gauge-equivalent modification of it whose structure constants are quasi-modular, holomorphic forms. The interplay between calculations in these two models, via the Kaneko–Zagier theory of quasi-modular forms, will form a central part of our final computation.

1.8 Like in the classical Gromov–Witten calculation, in order to get a non-trivial answer in the B-model computation we need to insert a certain Hochschild class $[\xi] \in HH_{-1}(A_\tau)$ at the puncture, mirror dual to $[\mathsf{pt}]^{\mathsf{PD}}$. This class will be represented by the Hochschild chain in $A_\tau^{\otimes 1}$,

$$\xi = \frac{1}{\tau - \bar{\tau}} \, d\bar{z}.$$

(The identification of E_τ with $\mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$ yields a well-defined class $d\bar{z}$ in $H^1(E_\tau, \mathscr{O}_{E_\tau})$. This group is a direct summand of A_τ. Therefore ξ is a well-defined element of homological degree (-1) of the algebra A_τ, and as such it gives rise to a class in $HH_{-1}(A_\tau)$.)

The following theorem is the main result of this paper.

THEOREM 1.9. *With insertion of the class $[\xi]$, Costello's categorical Gromov–Witten invariant of A_τ at $g = 1$, $n = 1$ equals*

$$F^{\mathsf{B}}_{1,1}(\tau) = -\tfrac{1}{24}E_2(\tau).$$

1.10 We interpret this result in two ways. On the one hand, we think of it as confirmation of the mirror symmetry prediction at $g = 1$ as in § 1.6. On the other hand, through the prism of homological mirror symmetry we can view our result as a statement about the Fukaya category of the family $\check{E}_\rho$ which is the mirror to the family E_τ of elliptic curves. Indeed, from work of Polishchuk and Zaslow [PZ98] we know that homological mirror symmetry holds for elliptic curves. The authors construct an equivalence

$$\mathsf{D}^b_{\mathsf{coh}}(E_\tau) \cong \mathsf{Fuk}(\check{E}_\rho),$$

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where $\check{E}$ is the 2-torus mirror to E_τ, endowed with a certain complexified Kähler class ρ. Therefore our computation, which is *a priori* about $\mathsf{D}^b_{\mathsf{coh}}(E_\tau)$, can be reinterpreted as a calculation about $\mathsf{Fuk}(\check{E}_\rho)$. From this perspective we regard Theorem 1.9 as verification of the prediction that Costello's categorical Gromov–Witten invariants of the Fukaya category agree (in this case) with the classical ones of the underlying space, as computed by Dijkgraaf.

1.11 There is one important aspect of Costello's work that we have suppressed in the above discussion. In order to extract an actual Gromov–Witten potential from a cyclic $\mathscr{A}_\infty$-algebra A (as opposed to a line in a certain Fock space) we need to choose a splitting of the Hodge filtration on the periodic cyclic homology of A. The correct splitting is forced on us by mirror symmetry. The Hochschild and cyclic homology of A_τ agree with those of E_τ, as they are derived invariants. Under this identification, a splitting of the Hodge filtration is the choice of a splitting of the natural projection

$$H^1_{\mathsf{dR}}(E_\tau) \to H^1(E_\tau, \mathscr{O}_{E_\tau}) = HH_{-1}(A_\tau).$$

Mirror symmetry imposes the requirement that the lift of $[\xi]$ must be invariant under monodromy around the cusp, which in turn uniquely determines the lifting. It is with this choice that we carry out the computations in Theorem 1.9. See § 9 for more details.

1.12 There is another approach to higher-genus invariants in the B-model, due to Costello and Li [CL12, Li11, Li12, Li16], inspired by the Bershadsky–Cecotti–Ooguri–Vafa (BCOV) construction in physics [BCOV94].

These other invariants also depend on a choice of splitting of the Hodge filtration. In their works Costello and Li analyzed the BCOV-type invariants of elliptic curves obtained from *arbitrary* splittings of the Hodge filtration and showed that these invariants satisfy the Virasoro constraints. Moreover, they studied a family of splittings depending on a parameter $\sigma \in \mathbb{H}$ and proved the modularity of the corresponding BCOV potentials. The monodromy invariant splitting that we consider corresponds to the limiting splitting $\sigma \to \mathtt{i}\infty$. We have learned the idea that this is the correct one for mirror symmetry from conversations with Costello and Li.

1.13 The BCOV-type invariants have the advantage that they give a more geometric definition of B-model Gromov–Witten invariants for Calabi–Yau spaces, and are also more easily computed than the original categorical ones of Costello [Cos09]. In fact, for elliptic curves Li was able to establish mirror symmetry at arbitrary genus for BCOV B-model invariants, and to directly compute the potential functions in any genus and for arbitrary insertions.

However, the BCOV-type invariants are fundamentally different from the ones we study in this paper, in that they are not *a priori* categorical: knowing homological mirror symmetry does not allow one to conclude the equality of the A- and B-model invariants. Moreover, the BCOV approach does not immediately generalize to other non-geometric situations wherein one only has a category, and not an underlying space.

1.14 Outline of the paper
Section 2 outlines Dijkgraaf's computation in the classical setting. Section 3 discusses mirror symmetry in the geometric setting. The next two sections review modular forms, Kaneko–Zagier theory, and Polishchuk's $\mathscr{A}_\infty$-algebra. Costello's general formalism is outlined in § 6, and the next section contains a roadmap for the computation for elliptic curves. Section 8 describes a computation, essentially due to Costello, of the string vertices for $\chi = -1$. The last two sections

present two different ways to compute the Gromov–Witten invariant $F_{1,1}^{\mathsf{B}}$ that we want. The first method involves reducing the problem to a very large linear algebra computation, which is then solved by computer. The second method, presented in § 10, gives a purely mathematical deduction of the result, using a comparison between computations in the holomorphic and modular gauges, respectively.

1.15 Standing assumptions

We work over the field of complex numbers $\mathbb{C}$. Throughout the paper we will need to use various comparison results between algebraic homology theories (Hochschild, cyclic) and geometric ones (Hodge, de Rham). Most of these comparison results are in the literature; however, some appear to be known only to specialists but are not published. In particular, we have not been able to find in the literature a comparison between the algebraic Getzler–Gauss–Manin connection and the classical geometric one. We tacitly assume that they agree, but this should be considered a conjectural result. (A similar assumption is made in [GPS15].)

2. The classical invariants

In this section we outline Dijkgraaf's computation [Dij95] of the classical $g = 1$, $n = 1$ Gromov–Witten invariants of elliptic curves.

2.1 Let $\check{E} = \mathbb{R}^2/\mathbb{Z}^2$ denote the two-dimensional torus, endowed with any complex structure making it into an elliptic curve. (The specific choice of complex structure will not matter.) We upgrade $\check{E}$ to a symplectic manifold by choosing any symplectic form ω in $H^2(\check{E}, \mathbb{R})$ whose area is one.

We also fix a point $P_0 \in \check{E}$, which we think of as determining the origin of the group structure of $\check{E}$.

2.2 For any $\beta \in H_2(\check{E}, \mathbb{Z})$, let $\overline{M}_{1,1}(\check{E}, \beta)$ be the moduli space of stable maps from a stable genus-one curve with one marked point to $\check{E}$ with homology class β. It has virtual real dimension two. If $\beta = d \cdot [\check{E}]$ for some integer $d > 0$ then

$$\overline{M}_{1,1}(\check{E}, \beta) = M_{1,1}(\check{E}, \beta)$$

and the virtual dimension agrees with the actual dimension. In fact the moduli space $M_{1,1}(\check{E}, \beta)$ parametrizes in this case the pairs (f, P) where $f : E \to \check{E}$ is a degree-d isogeny onto $\check{E}$ from another elliptic curve E, and P is any point on E. (There are only finitely many such isogenies possible, therefore $\dim_{\mathbb{R}} M_{1,1}(X, \beta) = 2$.)

2.3 To compute an actual numerical Gromov–Witten invariant we need to insert a cohomology class at the marked point, so that the integrand is a 2-form. Inserting a ψ-class from $M_{1,1}$ gives zero, so the only choice left is to pull-back a 2-form α from $\check{E} = T^2$ via the evaluation map

$$\mathsf{ev}_1 : M_{1,1}(\check{E}, \beta) \to \check{E}.$$

The natural choice used is to take $\alpha = [\mathsf{pt}]^{\mathsf{PD}}$, the Poincaré dual class to a point. If we think of this point as being P_0, then the associated Gromov–Witten invariant for $\beta = d \cdot [\check{E}]$ will compute the number of isogenies $f : E \to \check{E}$, up to isomorphism. (Inserting α has the effect of requiring P to map to P_0.) This number is well known – it equals the sum $(\sum_{k \mid d} k)$ of divisors of d. This can be seen by counting the number of matrices with integer coefficients, of determinant d, up to $\mathsf{SL}(2, \mathbb{Z})$ conjugation.

2.4 One encapsulates the result of this computation into the classical Gromov–Witten potential of the A-model

$$F_{1,1}^{\mathsf{A}}(q) = \sum_{\beta \in H_2(\check{E}, \mathbb{Z})} \langle \alpha \rangle_{1,1}^{\check{E}, \beta} q^{\langle \beta, \omega \rangle} = -\frac{1}{24} + \sum_{d \geqslant 1} \sum_{k \mid d} k q^d.$$

(The leading coefficient of $-1/24$ arises from a computation on $\overline{M}_{1,1}$ which we will omit.) This formula equals $-\frac{1}{24} E_2(q)$ under the change of variables $\tau = (1/2\pi\mathtt{i})\log(q)$, where E_2 is the Eisenstein holomorphic, quasi-modular normalized form of weight 2; see §4 for details on modular forms.

3. Geometric mirror symmetry

In this section we discuss the identifications that mirror symmetry prescribes between structures of the A- and the B-models for elliptic curves. We place ourselves in the classical geometric context, where one deals with spaces and not with categories.

3.1 At its core mirror symmetry is an identification between two families of geometric structures, the A-model and the B-model. The A-model is usually a trivial family of complex manifolds, endowed with a varying *complexified Kähler class* (this notion is a generalization of the usual Kähler class; see below). The B-model family is a varying family of complex manifolds. The *mirror map* is an isomorphism

$$\mathscr{M}^{\mathsf{Kähler}} \cong \mathscr{M}^{\mathsf{cx}}$$

between the moduli space $\mathscr{M}^{\mathsf{Kähler}}$ of complexified Kähler classes and the moduli space $\mathscr{M}^{\mathsf{cx}}$ of complex structures. This isomorphism is not defined everywhere, but only in the neighborhood of certain limit points of these spaces, the so-called *large volume* and *large complex structure* limit points.

3.2 In the case of elliptic curves both the A- and the B-model families have descriptions in terms of familiar structures. We begin by describing the A-model family.

Let $\check{E}$ denote the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ endowed with some complex structure, as in §2. The moduli space of complexified Kähler structures on $\check{E}$ is defined to be

$$\mathscr{M}^{\mathsf{Kähler}}(\check{E}) = (H^2(\check{E}, \mathbb{R})/H^2(\check{E}, \mathbb{Z})) \oplus \mathtt{i} \cdot \left\{ A \in H^2(\check{E}, \mathbb{R}) \,\middle|\, \int_{\check{E}} A > 0 \right\}.$$

In other words, a complexified Kähler class ρ on $\check{E}$ can be written as $\rho = b + \mathtt{i}A$ where b is a form in $H^2(\check{E}, \mathbb{R})/H^2(\check{E}, \mathbb{Z})$ and $A \in H^2(\check{E}, \mathbb{R})$ represented by a symplectic form with positive area.

The moduli space $\mathscr{M}^{\mathsf{Kähler}}$ is naturally isomorphic to $\mathbb{H}/\mathbb{Z}$. The correspondence $\rho \leftrightarrow \tau$ thus identifies a neighborhood of $\mathtt{i} \cdot \infty$ in $\mathscr{M}^{\mathsf{Kähler}}(\check{E})$ (the large volume limit point) with a neighborhood of the cusp in the moduli space of complex structures of elliptic curves (the large complex structure limit point).

The map $\rho \leftrightarrow \tau$ is the mirror map for elliptic curves described by Polishchuk and Zaslow [PZ98]. We have used it in order to identify the A- and B-model potentials in §1.6.

Due to the periodicity of b and $\mathsf{Re}(\tau)$ and the positivity of A and $\mathsf{Im}(\tau)$, it often makes more sense to use exponential coordinates on $\mathscr{M}^{\mathsf{Kähler}}$ and $\mathscr{M}^{\mathsf{cx}}$. We will write $q = \exp(2\pi\mathtt{i}\rho)$ or $q = \exp(2\pi\mathtt{i}\tau)$ depending on the context, with the hope that this will not cause any confusion.

3.3 On the A-model side the trivial bundle over $\mathscr{M}^{\mathsf{Kähler}}$ with fiber $H^{1-\bullet}(\check{E}, \mathbb{C})$ carries the structure of a graded variation of polarized semi-infinite Hodge structures (VSHS), in the sense of [Bar01, GPS15]. We shall only need parts of this structure: a graded fiberwise pairing (the Poincaré pairing $\langle -, - \rangle_{\mathsf{P}}$ given by wedge and integrate) and a flat connection, the *Dubrovin connection* ∇^{D} (see [GPS15]),

$$\nabla_{q\partial_q}(x) = q\partial_q(x) - x \cup \omega,$$

where $\omega \in H^2(\check{E}, \mathbb{C})$ is the unique symplectic form such that $\int_{\check{E}} \omega = 1$.

3.4 We have a similar structure of polarized VSHS over $\mathscr{M}^{\mathsf{cx}}$ coming from the B-model. The fiber over $\tau \in \mathscr{M}^{\mathsf{cx}}$ of the underlying vector bundle is the graded vector space $HH_{\bullet-1}(E_\tau)$. The fiberwise pairing is given by the *Mukai pairing* $\langle -, - \rangle_{\mathsf{M}}$; see [Căl05]. We recall the formula for the Mukai pairing. The Hochschild–Kostant–Rosenberg isomorphism gives an identification

$$HH_\bullet(E_\tau) \cong \bigoplus_{q-p=\bullet} H^p(E_\tau, \Omega^q_{E_\tau}).$$

The Mukai pairing then becomes

$$\langle u, v \rangle_{\mathsf{M}} = \frac{1}{2\pi\mathtt{i}} \int_{E_\tau} (-1)^{|u|} u \wedge v,$$

where $|u| = q$ for $u \in H^p(E_\tau, \Omega^q_{E_\tau})$. See Ramadoss [Ram08] for an explanation of this sign. The factor of $1/2\pi\mathtt{i}$ arises from the comparison of the Serre duality pairing (the residue pairing) with the wedge-and-integrate pairing. Note that this factor is missing in the comparison [GPS15, Conjecture 1.14]. Also note that the Todd class of an elliptic curve is trivial, so there is no correction from it as in [Căl05].

The connection is the Gauss–Manin connection ∇^{GM} after identifying $HH_\bullet(E_\tau) \cong \bigoplus_{q-p=\bullet} H^p(E_\tau, \Omega^q_{E_\tau})$ further with $H^\bullet_{\mathsf{dR}}(E_\tau)$ as $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces; see [Gri68]. (We are using the fact that the Hodge–de Rham spectral sequence degenerates on E_τ.)

3.5 The mirror map gives an isomorphism between the A-model polarized VSHS and the B-model one. The following theorem identifies the classes $[\Omega] \in HH_1(E_\tau)$ and $[\xi] \in HH_{-1}(E_\tau)$ which correspond to the classes $1 \in H^0(\check{E}_\rho)$ and $[\mathsf{pt}]^{\mathsf{PD}} \in H^2(\check{E}_\rho)$, respectively, under this isomorphism.

PROPOSITION 3.6. *Under the mirror map $\rho \leftrightarrow \tau$ the class $1 \in H^0(\check{E}_\rho)$ corresponds to the class of the global holomorphic volume form*

$$[\Omega] = [2\pi\mathtt{i} \cdot dz] \in HH_1(E_\tau).$$

Similarly, the class $[\mathsf{pt}]^{\mathsf{PD}} \in H^2(\check{E}_\rho)$ corresponds to the class

$$[\xi] = \frac{1}{\tau - \bar{\tau}}[d\bar{z}] \in HH_{-1}(E_\tau).$$

Proof. Because we are only interested in what happens in a neighborhood of the large complex limit point we have well-defined forms dz and $d\bar{z}$ which give bases of $H^0(E_\tau, \Omega^1_{E_\tau})$ and $H^1(E_\tau \mathscr{O}_{E_\tau})$, respectively, on each elliptic curve E_τ. The classes $[\Omega]$ and $[\xi]$ are therefore pointwise multiples of these forms. The goal is to identify which multiples they are.

In the A-model we have

$$\langle 1, \nabla^{\mathsf{D}}_{q\partial_q}(1) \rangle_{\mathsf{P}} = \langle 1, -[\omega] \rangle_{\mathsf{P}} = -1.$$

The relationships $q = \exp(2\pi\mathrm{i}\rho)$ and $\rho \leftrightarrow \tau$ force the identification

$$q\partial_q = \frac{1}{2\pi\mathrm{i}}\partial_\tau.$$

Therefore in the B-model we require

$$\langle \Omega, \nabla^{\mathsf{GM}}_{(1/2\pi\mathrm{i})\partial_\tau}\Omega \rangle_{\mathsf{M}} = -1.$$

A straightforward calculation with periods shows that

$$\nabla^{\mathsf{GM}}_{\partial_\tau}(dz) = \frac{1}{\tau - \bar{\tau}}(dz - d\bar{z})$$

and this forces the class $[\Omega]$ to equal $[2\pi\mathrm{i} \cdot dz]$.

Similarly, the A-model identity

$$\langle 1, [\mathsf{pt}]^{\mathsf{PD}} \rangle_{\mathsf{P}} = 1$$

forces the equality

$$\langle [\Omega], [\xi] \rangle_{\mathsf{M}} = 1,$$

which in turn implies

$$[\xi] = \frac{1}{\tau - \bar{\tau}}[d\bar{z}]$$

as stated in § 1.8. $\qquad\square$

3.7 We conclude this section with a discussion of the splitting of the Hodge filtration on the de Rham cohomology $H^1_{\mathsf{dR}}(E_\tau)$. For an elliptic curve E_τ the Hodge filtration is expressed by the short exact sequence

$$0 \to H^0(E_\tau, \Omega^1_{E_\tau}) \to H^1_{\mathsf{dR}}(E_\tau) \to H^1(E_\tau, \mathscr{O}_{E_\tau}) \to 0,$$

or, equivalently, by the sequence

$$0 \to HH_1(E_\tau) \to H^1_{\mathsf{dR}}(E_\tau) \to HH_{-1}(E_\tau) \to 0.$$

We have natural basis vectors $[dz]$ and $[\xi]$ of the first and last vector spaces in this sequence. Thus choosing a splitting of the Hodge filtration means picking a lift $[\xi + f(\tau)\,dz]$ of $[\xi]$ from

$$HH_{-1}(E_\tau) = H^1(E_\tau, \mathscr{O}_{E_\tau})$$

to $H^1_{\mathsf{dR}}(E_\tau)$ for every τ.

Mirror symmetry prescribes that the correct lift must be *invariant* under monodromy around the cusp. The following lemma characterizes this lift.

LEMMA 3.8. *The following conditions are equivalent for a family $[\tilde{\xi}]^{\mathsf{geom}}$ of lifts of the family of Hochschild classes $[\xi]$:*

(i) *the lift $[\tilde{\xi}]^{\mathsf{geom}}$ is invariant under monodromy around the cusp for all $\tau \in \mathbb{H}$;*

(ii) *the family $[\tilde{\xi}]^{\mathsf{geom}}$ is flat with respect to the Gauss–Manin connection.*

Proof. Fix an identification of E_τ with $\mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\tau$. This determines a basis A, B of cycles in $H_1(E_\tau)$ corresponding to the paths from 0 to 1 and from 0 to τ, respectively. Under the monodromy $\tau \mapsto \tau + 1$ around the cusp the basis (A, B) maps to $(A, A + B)$.

Let A^*, B^* denote the basis in $H^1_{\mathsf{dR}}(E_\tau)$ dual to the basis (A, B). Then under the same monodromy the pair (A^*, B^*) maps to $(A^* - B^*, B^*)$ (the inverse transpose matrix). We have

$$dz = A^* + \tau B^*,$$
$$d\bar{z} = A^* + \bar{\tau} B^*.$$

It follows that the invariant cocycle B^* is expressed in the $(dz, d\bar{z})$ basis as

$$B^* = \frac{1}{\tau - \bar{\tau}}(dz - d\bar{z}).$$

For a class $[\tilde{\xi}] \in H^1_{\mathsf{dR}}(E_\tau)$ to be invariant under monodromy around the cusp it must be a multiple of B^*. And indeed there exists a unique monodromy invariant lift of $[\xi]$ from

$$HH_{-1}(E_\tau) = H^1_{\mathsf{dR}}(E_\tau)/HH_1(E_\tau)$$

to $H^1_{\mathsf{dR}}(E_\tau)$, namely

$$-B^* = \frac{1}{\tau - \bar{\tau}}(d\bar{z} - dz).$$

By definition of the Gauss–Manin connection, since $-B^*$ is a topological cohomology class, it is ∇^{GM}-flat and this condition uniquely identifies the family in (ii). $\square$

3.9 We will also need in §10 the explicit form of the Kodaira–Spencer class,

$$\mathsf{KS}(\partial_\tau) = -\frac{1}{\tau - \bar{\tau}}\frac{\partial}{\partial z}\, d\bar{z} \in H^1(E_\tau, T_{E_\tau}).$$

This follows from the relationship between the Kodaira–Spencer class and the Gauss–Manin connection, given by the formula

$$\nabla^{\mathsf{GM}}_{\partial_\tau}([\Omega])\,\mathsf{mod}\,H^{1,0}(E_\tau) = \mathsf{KS}(\partial_\tau)\lrcorner[\Omega].$$

4. Quasi-modular forms and Kaneko–Zagier theory

Before we can describe Polishchuk's $\mathscr{A}_\infty$ model for the derived category of an elliptic curve we need to review the theory of quasi-modular, holomorphic forms and its relationship to the theory of almost holomorphic, modular forms. Their interplay is described by Kaneko–Zagier theory; see [KZ15, 5.1] and [KZ95].

4.1 The ring of holomorphic, modular forms for the group $\Gamma = \mathsf{SL}(2, \mathbb{Z})$ is isomorphic to $\mathbb{C}[E_4, E_6]$, where E_4 and E_6 are the Eisenstein modular forms of weights 4 and 6, respectively. (Since there are competing systems of notation in the literature we reserve the notation E_k to mean the corresponding normalized form, in the sense that the function has been rescaled to satisfy $\lim_{\tau\to i\cdot\infty} E_k(\tau) = 1$.)

4.2 For the purposes of this paper we need to consider certain functions on $\mathbb{H}$ which are not at the same time holomorphic and modular. Of particular interest to us is the Eisenstein form E_2: it is still holomorphic, but it does not satisfy the usual transformation law with respect to Γ; see [KZ15, 4.1]. We will call elements of the ring $\widetilde{M}(\Gamma) = \mathbb{C}[E_2, E_4, E_6]$ *quasi-modular* holomorphic forms; the weight of such a form is defined by declaring the weight of E_{2k} to be $2k$.

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4.3 While the form E_2 is not modular, the simple modification

$$E_2^*(\tau) = E_2(\tau) - \frac{3}{\pi^2}\frac{2\pi\mathtt{i}}{\tau - \bar{\tau}}$$

is; it satisfies the same transformation rule as regular modular forms of weight 2 with respect to the group Γ. However, E_2^* is no longer holomorphic. The functions in the ring $\mathscr{O}(\mathbb{H})[1/(\tau - \bar{\tau})]$ which are modular in the above sense (for arbitrary weights) and satisfy a suitable growth condition at the cusp form a ring $\widehat{M}(\Gamma)$. Its elements will be called *almost holomorphic* modular forms. One can show that

$$\widehat{M}(\Gamma) = \mathbb{C}[E_2^*, E_4, E_6].$$

4.4 The rings $\widetilde{M}(\Gamma)$ and $\widehat{M}(\Gamma)$ are closed under the actions of certain differential operators. The former is closed under the action of ∂_τ, while the latter is closed under

$$\widehat{\partial}_\tau = \partial_\tau + \frac{\mathsf{wt}}{\tau - \bar{\tau}},$$

where wt denotes the weight of the form on which $\widehat{\partial}_\tau$ acts.

The following theorem is the main result of Kaneko–Zagier theory [KZ95].

THEOREM 4.5. *The operator*

$$\varphi : \widehat{M}(\Gamma) \to \widetilde{M}(\Gamma)$$

which maps an almost holomorphic, modular form

$$F(\tau, \bar{\tau}) = \sum_{m \geqslant 0} \frac{F_m(\tau)}{(\tau - \bar{\tau})^m}$$

to its 'constant term' part $F_0(\tau)$ is a differential ring isomorphism. Its inverse will be denoted by

$$\mathsf{KZ} : \widetilde{M}(\Gamma) \to \widehat{M}(\Gamma).$$

5. Polishchuk's algebra and its holomorphic modification

In this section we review Polishchuk's cyclic $\mathscr{A}_\infty$-algebra A_τ associated to an elliptic curve E_τ. This presentation uses almost holomorphic, modular forms. The main result of this section asserts that the $\mathscr{A}_\infty$-algebra A_τ^{hol} obtained by replacing these structure constants by their images under the homomorphism φ of Theorem 4.5 yields an algebra which is *gauge equivalent* to A_τ and varies holomorphically with τ.

5.1 For $\tau \in \mathbb{H}$, let E_τ be the corresponding elliptic curve. We will denote its origin by P_0. The derived category $\mathsf{D}^b_{\mathsf{coh}}(E_\tau)$ of coherent sheaves on the elliptic curves E_τ is compactly generated by the object $\mathscr{F} = \mathscr{O} \oplus L$, where L is the degree-one line bundle $L = \mathscr{O}(P_0)$. Therefore the differential graded (dg) algebra

$$A_\tau^{\mathsf{dg}} = \mathsf{RHom}^\bullet_{E_\tau}(\mathscr{F}, \mathscr{F})$$

is derived equivalent to E_τ. By homological perturbation we can transfer the dg-algebra structure from A_τ^{dg} to a quasi-equivalent $\mathscr{A}_\infty$-algebra structure on

$$A_\tau = H^\bullet(A_\tau^{\mathsf{dg}}) = \mathsf{Ext}^\bullet_{E_\tau}(\mathscr{F}, \mathscr{F}).$$

The graded vector space A_τ is concentrated in cohomological degrees zero and one, and

$$\dim A_\tau^0 = \dim A_\tau^1 = 3.$$

5.2 Polishchuk [Pol11] explicitly computes an $\mathscr{A}_\infty$ structure on A_τ using a particular choice of homotopy between A_τ and A_τ^{dg}. In order to express his result Polishchuk chooses a basis for the six-dimensional algebra A_τ consisting of:

- the identity morphisms $\mathsf{id}_\mathscr{O} : \mathscr{O} \to \mathscr{O}$ and $\mathsf{id}_L : L \to L$ in A_τ^0;
- the morphisms $\xi : \mathscr{O} \to \mathscr{O}[1]$ and $\xi_L : L \to L[1]$ in A_τ^1 given by $\xi = d\bar{z}$, and similarly for ξ_L;
- the degree-zero morphism $\theta : \mathscr{O} \to L$ in A_τ^0 given by the theta function $\theta = \theta(z, \tau)$;
- an explicit dual morphism $\eta : L \to \mathscr{O}[1]$ in A_τ^1 given by the formula in [Pol11, 2.2].

5.3 For our purposes it will be more convenient to work with a slightly modified basis for A_τ, obtained by dividing the elements of degree one in Polishchuk's basis (η, ξ and ξ_L) by the factor $\tau - \bar{\tau}$. We leave the degree-zero elements unchanged. The rescaled elements will still be denoted by η, ξ, ξ_L; they differ by a factor of $1/2\pi\mathtt{i}$ from the ones that Polishchuk considers in [Pol11, 2.5, Remark 2]).

With this rescaling the new element ξ agrees with the expression in § 1.8.

5.4 At this point we need to decide which sign conventions to use for the $\mathscr{A}_\infty$ axioms. There are (at least) two such conventions in the literature. In this paper we will use the so-called *shifted* signs, which are more common in the symplectic geometry literature and are easier to use. See [Cho08] for a discussion of these two conventions and of ways to translate between them. Note, however, that Polishchuk [Pol11] uses in the unshifted sign convention; we will make the change to the shifted convention as we go. Following [Cho08], we will denote by $|x|'$ the shifted degree of an element $x \in A_\tau$, $|x|' = |x| - 1$.

5.5 Once we have fixed a global holomorphic volume form Ω on E_τ as in Proposition 3.6 we get an induced symmetric non-degenerate pairing on A_τ arising from Serre duality. Correcting it by the shifted sign conventions, we obtain a skew-symmetric pairing on A_τ given by

$$\begin{aligned}
\langle \xi, \mathsf{id}_\mathscr{O} \rangle = -1, \quad &\langle \xi_L, \mathsf{id}_L \rangle = -1, \\
\langle \mathsf{id}_\mathscr{O}, \xi \rangle = 1, \quad &\langle \mathsf{id}_L, \xi_L \rangle = 1, \\
\langle \theta, \eta \rangle = 1, \quad &\langle \eta, \theta \rangle = -1.
\end{aligned}$$

All other pairings between basis elements are zero. This pairing will give the *cyclic* structure on A_τ.

5.6 We will now describe Polishchuk's formulas for the multiplications

$$\mu_k : A[1]^{\otimes k} \to A[2].$$

These operations are non-trivial only when k is even. The first product μ_2 is the usual Yoneda product on the Ext algebra A_τ, with a sign adjustment: the only non-zero products (apart from the obvious ones involving id and id_L) are $\theta\eta = -\xi$ and $\eta\theta = \xi_L$. (We will use reverse notation for composition of morphisms, so that $\theta\eta$ means $\eta \circ \theta$. The same convention shall be used for the higher products, in line with Polishchuk's conventions.)

5.7 An important feature of Polishchuk's formulas is that they respect cyclic symmetry with respect to the inner product on A_τ. We will say that a homogeneous map (of some degree)

$$c_k : A^{\otimes k} \to \mathbb{C}$$

is cyclic if it is invariant under cyclic rotation of its arguments, up to a Koszul sign determined by the shifted degrees of its arguments. The products μ_k will be cyclic in the sense that the tensors

$$c_k(x_1, \ldots, x_k) = \langle \mu_{k-1}(x_1, \ldots, x_{k-1}), x_k \rangle$$

are cyclic in the above sense. An $\mathscr{A}_\infty$-algebra endowed with a pairing of degree d which makes the maps c_k cyclic will be said to be Calabi–Yau of degree d, or CY$[d]$, or cyclic of degree d.

5.8 All the higher multiplications of A_τ can be deduced from a single one using this cyclic symmetry. The complete list is presented in [Pol11, Theorem 2.5.1], but for reference we list the formula for only one such multiplication. Explicitly, let a, b, c, d be non-negative integers, and let $s = a + b + c + d$. Then for s odd we have

$$\mu_k(\xi_L^a, \eta, \xi^b, \theta, \xi_L^c, \eta, \xi^d) = (-1)^{a+b+\binom{s+1}{2}} \frac{1}{a!b!c!d!} \cdot \frac{1}{(2\pi\mathrm{i})^{s+1}} \cdot g_{a+c,b+d} \cdot \eta,$$

and zero otherwise. In this formula $k = s + 3$, and

$$g_{a+c,b+d} \in \widehat{M}(\Gamma)_{s+1}$$

is a certain almost holomorphic modular form of weight $s + 1$ defined by Polishchuk in [Pol11, 1.1].

5.9 We associate a notion of weight to the basis elements of A_τ as follows:

$$\mathsf{wt}(\mathsf{id}) = \mathsf{wt}(\mathsf{id}_L) = 0,$$
$$\mathsf{wt}(\theta) = \mathsf{wt}(\eta) = 1/2,$$
$$\mathsf{wt}(\xi) = \mathsf{wt}(\xi_L) = 1.$$

Note that with respect to these assignments of weights the multiplications μ_k are of total weight zero, and the inner product $\langle -, - \rangle$ is of total weight -1.

5.10 Even though the dependence on $\tau \in \mathbb{H}$ of the family of elliptic curves E_τ is holomorphic, the associated family of $\mathscr{A}_\infty$-algebras A_τ does not depend holomorphically on τ: the structure constants $g_{a+c,b+d}$ are not holomorphic. We think of the multiplications $\{\mu_k\}$ as giving a 2-cocycle μ^* in the Hochschild cochain $CC^\bullet(A_\tau)$. (The differential in this complex is $[\mu^*, -]$.) The failure of holomorphicity is measured by the 2-cochain $\bar{\partial}\mu^*$.

The following result shows that the anti-holomorphic dependence of this family is nevertheless trivial to first order (i.e. it is zero in Hochschild cohomology). The proof is a straightforward computation.

THEOREM 5.11. *Let $CC^\bullet(A_\tau)$ denote the Hochschild cochain complex of A_τ, with differential $[\mu^*, -]$. Let $\psi_3 : A_\tau[1]^{\otimes 3} \to A_\tau[1]$ be the cyclic Hochschild 1-cochain defined by*

$$\psi_3(\xi, \theta, \xi_L) = -\frac{1}{2\pi\mathrm{i}} \cdot \frac{1}{(\tau - \bar{\tau})^2} \cdot \theta$$

and its cyclic rotations with respect to the pairing. Then we have

$$\bar{\partial}\mu^* = [\mu^*, \psi_3].$$

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5.12 The deformation induced by $\bar{\partial}\mu$ is trivial not just to first order: the bounding element ψ_3 can be integrated to an $\mathscr{A}_\infty$ quasi-isomorphism

$$f : A_\tau \to A_\tau^{\mathsf{hol}},$$

from A_τ to a *holomorphic* family A_τ^{hol} of $\mathscr{A}_\infty$-algebras over $\mathbb{H}$. The process is similar to integrating pseudo-isotopies of $\mathscr{A}_\infty$-algebras; see [Fuk10, Proposition 9.1]. We will now present the construction of the algebras A_τ^{hol} and of the $\mathscr{A}_\infty$ quasi-isomorphism f.

The holomorphic family A_τ^{hol} has the same underlying vector space as A_τ and the same basis. Its structure constants are obtained simply by applying the map φ of Theorem 4.5 to the structure constants of A_τ, so that, for example, the multiplications of A_τ^{hol} are obtained by cyclic symmetry from

$$\mu_k^{\mathsf{hol}}(\xi_L^a, \eta, \xi^b, \theta, \xi_L^c, \eta, \xi^d) = (-1)^{a+b+\binom{s+1}{2}} \frac{1}{a!b!c!d!} \cdot \frac{1}{(2\pi\mathtt{i})^{s+1}} \cdot \varphi(g_{a+c,b+d}) \cdot \eta.$$

Since φ is a ring map, the maps $\{\mu_k^{\mathsf{hol}}\}$ also satisfy the $\mathscr{A}_\infty$ relations.

5.13 One can explicitly compute $\varphi(g_{a,b})$ using the recurrence formulas in [Pol11, Proposition 2.6.1(ii)]. For example, we have

$$\varphi(g_{1,0}) = \varphi(e_2^*) = \varphi\left(e_2 - \frac{2\pi\mathtt{i}}{\tau - \bar{\tau}}\right) = e_2,$$
$$\varphi(g_{2,1}) = \varphi\left(-g_{1,0}^2 + \frac{5}{6}g_{3,0}\right) = -e_2^2 + \frac{5}{6} \cdot 3! \cdot e_4 = -e_2^2 + 5e_4.$$

In general, $g_{a,b}$ is a polynomial expression with rational coefficients in e_{2k}^*; $\varphi(g_{a,b})$ is obtained by replacing e_2^* by e_2 in the same expression, leaving all the other terms unchanged.

The following theorem describes recursively the construction of the maps f_n which assemble to give a quasi-equivalence $A_\tau \cong A_\tau^{\mathsf{hol}}$. The proof is a straightforward (though tedious) inductive calculation.

THEOREM 5.14. *Inductively define multilinear maps*

$$f_n : A_\tau[1]^{\otimes n} \to A_\tau^{\mathsf{hol}}[1]$$

by setting $f_1 = \mathsf{id}$ and

$$f_n := \sum_{i\geqslant 1, j\geqslant 1, k\geqslant 1, i+j+k=n} \int \psi_3(f_i \otimes f_j \otimes f_k)\, d\bar{\tau}.$$

Here the integration symbol $\int (\ldots)\, d\bar{\tau}$ is formally applied to the coefficients of the tensors, and is defined by

$$\int \frac{1}{(\tau - \bar{\tau})^m}\, d\bar{\tau} := \frac{1}{(m-1)} \frac{1}{(\tau - \bar{\tau})^{m-1}} \quad \text{for } m \geqslant 2.$$

Then the maps $\{f_n\}_{n\geqslant 1}$ form a cyclic $\mathscr{A}_\infty$ quasi-isomorphism

$$f : A_\tau \to A_\tau^{\mathsf{hol}}.$$

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6. Costello's formalism

In this section we review Costello's general definition of categorical Gromov–Witten invariants. In the next section we will specialize this construction to the case of the $g = 1$, $n = 1$ invariant of an elliptic curve.

6.1 Let A be a cyclic $\mathscr{A}_\infty$-algebra whose pairing is of degree d as in § 5.7, and let $\mathsf{HH} = \mathsf{HH}_\bullet(A)[d]$ denote the shifted Hochschild homology of A. The categorical Gromov–Witten potential that we ultimately want to construct is an element

$$F^{\mathsf{cat}} \in \mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])[\![\lambda]\!].$$

Here λ is a formal variable used to keep track of the Euler characteristics of curves and u is the usual formal variable of degree -2 which appears in the definition of cyclic homology. It is a placeholder for keeping track of insertions of ψ classes.

The invariant F^{cat} depends not only on the algebra A, but also on a further choice: a splitting of the Hodge filtration on the periodic cyclic homology $\mathsf{HP}_\bullet(A)$ of A. The construction of F^{cat} proceeds in two stages. First one constructs an abstract invariant F^{abs} which only depends on A. It is a state in the homology of a certain Fock space associated to A. The choice of splitting of the Hodge filtration is then used to identify the homology of the Fock space with $\mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])[\![\lambda]\!]$ and hence to extract F^{cat} from F^{abs}.

6.2 The potential thus constructed is invariant under cyclic quasi-equivalence. If A' is another cyclic $\mathscr{A}_\infty$-algebra and f is a cyclic quasi-equivalence $A \cong A'$, then f induces an isomorphism $\mathsf{HP}_\bullet(A) \cong \mathsf{HP}_\bullet(A')$ of filtered vector spaces. The original splitting s of the Hodge filtration on $\mathsf{HP}_\bullet(A)$ then determines a splitting s' of the Hodge filtration on $\mathsf{HP}_\bullet(A')$. The invariance of the Gromov–Witten invariants means that the potential associated to (A, s) is the same as that associated to (A', s') under the obvious identifications.

6.3 Individual Gromov–Witten invariants can be read off from F^{cat} as coefficients of its power series expansion. For example, in the next section we will want to compute the $g = 1$, $n = 1$ invariant of Polishchuk's algebra A_τ. The Euler characteristic of the curves in $M_{1,1}$ is $\chi = -1$; thus we are interested in the $\lambda^{-\chi} = \lambda^1$ part of F^{cat}. We want to insert the Hochschild class $[\xi]$ (and no ψ-classes). This means that the Gromov–Witten invariant we are interested in is the coefficient of $[\Omega]u^{-1}\lambda$ in the power series expansion of F^{cat}: here $[\Omega]$ is the class in $HH_1(A_\tau)$ which is Mukai dual to $[\xi]$ (Proposition 3.6), and we have used u^{-1} to denote no ψ-class insertions (one ψ class would give u^{-2}, etc.).

We begin by describing the structures on the cyclic chain complex of A that will be needed for the construction of F^{abs}.

6.4 The circle action on Hochschild chains

Let V denote the shifted Hochschild cochain complex of A,

$$V = (CC_\bullet(A), b)[d],$$

whose homology is HH. Connes' B operator is a degree-one operator on V. It gives a homological circle action on the dg vector space V. We use it to form the periodic cyclic complex

$$V_{\mathsf{Tate}} = (CC_\bullet(A)(\!(u)\!), b + uB)[d]$$

and its negative cyclic subcomplex

$$V^{hS^1} = (CC_\bullet(A)\llbracket u \rrbracket, b + uB)[d].$$

The homology of these complexes are the (shifted by d) periodic and negative cyclic homology of A, respectively.

6.5 The Mukai pairing

The Mukai pairing [CW10] is a non-degenerate symmetric pairing of degree $2d$ on HH. Shklyarov [Shk13] and Sheridan [She20] construct a lift of the Mukai pairing to the chain level. It is a symmetric bilinear form of degree $2d$ on V,

$$\langle - , - \rangle_{\mathsf{M}} : V \otimes V \to \mathbb{C}[2d].$$

We will discuss this pairing further in § 7.4.

6.6 The higher residue pairing

The chain-level Mukai pairing on V induces a $\mathbb{C}((u))$ sesquilinear pairing of degree $2d$ on V_{Tate},

$$\langle - , - \rangle_{\mathsf{hres}} : V_{\mathsf{Tate}} \times V_{\mathsf{Tate}} \to \mathbb{C}((u))[2d],$$

the *higher residue pairing*[1] (see [She20, Definition 2 and § 5.42] for details). For $\alpha, \beta \in CC_\bullet(A)$, $i, j \in \mathbb{Z}$ the higher residue pairing is given by

$$\langle \alpha u^i, \beta u^j \rangle_{\mathsf{hres}} = (-1)^i \langle \alpha, \beta \rangle_{\mathsf{M}} \cdot u^{i+j}.$$

Laurent power series only have finitely many negative powers of u, hence the above formula extends by sesquilinearity to all of V_{Tate}.

The higher residue pairing is a chain map with respect to $b + uB$, which makes it descend to a pairing on periodic cyclic homology. The space V_{Tate} is filtered by powers of u, and the pairing respects this filtration. In particular, for $x, y \in \mathsf{HC}_\bullet^-(A)$ we have $\langle x, y \rangle_{\mathsf{hres}} \in \mathbb{C}\llbracket u \rrbracket$.

The constant term of the power series $\langle x, y \rangle_{\mathsf{hres}}$ equals the Mukai pairing $\langle \overline{x}, \overline{y} \rangle_{\mathsf{M}}$ of the reductions of x and y to Hochschild homology. However, the power series $\langle x, y \rangle_{\mathsf{hres}}$ can have higher powers of u which are not computed by the above homology Mukai pairing.

6.7 The ordinary residue pairing

We will also consider the *residue pairing*, the $\mathbb{C}$-valued pairing of degree $2d-2$ on V_{Tate} obtained by taking the coefficient of u^{-1} (the residue) in the higher residue pairing. It is skew-symmetric, and restricts to zero on the dg subspace V^{hS^1}. We will think of V^{hS^1} as a Lagrangian subspace of the symplectic vector space V_{Tate}.

6.8 Weyl algebra and Fock module

We associate a Weyl algebra $\mathcal{W}$ to the symplectic vector space V_{Tate} and a Fock space $\mathscr{F}$ to its Lagrangian subspace V^{hS^1}. Explicitly, $\mathcal{W}$ is defined as

$$\mathcal{W} = T^\bullet(V_{\mathsf{Tate}})/([x, y] - \langle x, y \rangle_{\mathsf{res}}),$$

where $[x, y]$ is the graded commutator of $x, y \in T^\bullet(V_{\mathsf{Tate}})$, and $\mathscr{F}$ is the quotient of $\mathcal{W}$ by the left ideal generated by $V^{hS^1} \subset V^{\mathsf{Tate}}$. It is an irreducible left $\mathcal{W}$-module.

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[1] This is called higher residue pairing since the pairing yields possibly higher powers of u.

6.9 String vertices

Sen and Zwiebach [SZ94] note that there are certain distinguished singular chains $S_{g,n}$, the *string vertices*, on the uncompactified moduli spaces of curves $M_{g,n}$. They play the role of approximations of the fundamental classes of the compactified spaces $\overline{M}_{g,n}$. The string vertices can be defined by recursive relations encapsulated in a differential equation known as the *quantum master equation*; see [Cos09, Theorem 1]. We will discuss string vertices in § 8.

Moreover, Kontsevich and Soibelman argue that chains on moduli spaces of curves act on the cyclic chain complex of any Calabi–Yau $\mathscr{A}_\infty$-algebra. This action will be further explained in § 7.4.

6.10 Deformed Fock module

Costello's main observation in [Cos09] is that the action of the string vertices $S_{g,n}$ on cyclic chains can be used to deform the module structure on $\mathscr{F}$. The result is a $\mathcal{W}[\![\lambda]\!]$-module $\mathscr{F}^{\mathsf{def}}$ which is a deformation of $\mathscr{F}$ over $\mathbb{C}[\![\lambda]\!]$.

The deformed module $\mathscr{F}^{\mathsf{def}}$ is constructed as follows. The standard Fock module $\mathscr{F}$ is the quotient of $\mathcal{W}$ by the left ideal generated by the relations

$$\alpha u^k = 0 \quad \text{for all } \alpha \in V, k \geqslant 0.$$

The module $\mathscr{F}^{\mathsf{def}}$ is obtained by taking the quotient of $\mathcal{W}$ by the left ideal I^{def} generated by the deformed relations

$$\alpha u^k = \sum_{g,n} \rho(\iota(S_{g,n}))(\alpha u^k)\lambda^{2g-2+n}$$

for all $\alpha \in V$, $k \geqslant 0$. Here ι is a certain operator on chains that will be described in § 8, and ρ denotes the action of singular chains on cyclic chains.

Costello argues that for reasonable $\mathscr{A}_\infty$-algebras, for which the Hodge filtration splits in the sense of Definition 6.13 below, the homology $H_\bullet(\mathscr{F}^{\mathsf{def}})$ is a flat deformation of the Fock module $H_\bullet(\mathscr{F})$ over the classical (non-dg) Weyl algebra $H_\bullet(\mathcal{W})$.

6.11 Fock modules over classical Weyl algebras are rigid, and thus the above deformation must be trivial at the level of homology. This implies that there is an isomorphism

$$H_\bullet(\mathscr{F}^{\mathsf{def}}) \xrightarrow{\sim} H_\bullet(\mathscr{F})[\![\lambda]\!]$$

of $H_\bullet(\mathcal{W})[\![\lambda]\!]$-modules, unique up to multiplication by a power series in $\mathbb{C}[\![\lambda]\!]$ which begins with 1.

By its construction the module $H_\bullet(\mathscr{F}^{\mathsf{def}})$ has a canonical generator $\mathbb{1}$ (it is a quotient of $H_\bullet(\mathcal{W}[\![\lambda]\!])$ by a certain ideal). Costello defines the abstract Gromov–Witten potential

$$F^{\mathsf{abs}} \in H_\bullet(\mathscr{F})[\![\lambda]\!]$$

to be the image of $\mathbb{1}$ under the above isomorphism. The ambiguity from the above power series will not play a role for us because we only care about the leading term of F^{abs} (the λ^1 term).

6.12 Splitting of the Hodge filtration

In order to extract a concrete categorical Gromov–Witten potential

$$F^{\mathsf{cat}} \in \mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])[\![\lambda]\!]$$

from the abstract one we need to choose a splitting of the Hodge filtration, a notion we now make precise.

Endow the graded vector space HH with a trivial circle operator and define $\mathsf{HH}_{\mathsf{Tate}} = \mathsf{HH}(\!(u)\!)$. The homology level Mukai pairing on HH induces a higher residue pairing on $\mathsf{HH}_{\mathsf{Tate}}$ defined by the same formula as the higher residue pairing. The higher residue pairing on $\mathsf{HH}_{\mathsf{Tate}}$ respects the *grading* with respect to powers of u. This grading induces a decreasing filtration on $\mathsf{HH}_{\mathsf{Tate}}$, similar to the one on $H_{\bullet}(V_{\mathsf{Tate}})$.

DEFINITION 6.13. A splitting of the Hodge filtration on the periodic cyclic homology of A is an isomorphism of filtered vector spaces

$$\mathsf{HH}_{\mathsf{Tate}} \xrightarrow{\sim} H_{\bullet}(V_{\mathsf{Tate}})$$

which respects the higher residue pairings.

6.14 Choosing a splitting of the Hodge filtration is equivalent to assigning to each $x \in \mathsf{HH}_i(A)$ a lift $\tilde{x} \in \mathsf{HC}_i^-(A)$. This assignment is required to satisfy the property that for $x, y \in \mathsf{HH}_{\bullet}(A)$ we have

$$\langle \tilde{x}, \tilde{y} \rangle_{\mathsf{hres}} = \langle x, y \rangle_{\mathsf{M}}.$$

In other words, we impose the condition that the higher residue pairing evaluated on lifts of elements in HH must have no higher powers of u.

6.15 Endowing the graded vector space $\mathsf{HH}_{\mathsf{Tate}}$ with the residue Mukai pairing makes it into a symplectic vector space, and HH^{hS^1} is a Lagrangian subspace. This yields a corresponding Weyl algebra $\mathcal{W}_H$ and Fock space $\mathscr{F}_H$. The latter can be identified with $\mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])$ as a $\mathcal{W}_H$-module via the action on the generator $\mathbb{1}$: negative powers of u act by multiplication, non-negative powers of u act by differentiation.

A choice of splitting of the Hodge filtration induces an isomorphism of Weyl algebras

$$\mathcal{W}_H \xrightarrow{\sim} H_{\bullet}(\mathcal{W})$$

and a corresponding isomorphism of Fock modules

$$S : \mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])[\![\lambda]\!] \cong \mathscr{F}_H \xrightarrow{\sim} H_{\bullet}(\mathscr{F})[\![\lambda]\!].$$

The categorical Gromov–Witten potential F^{cat} that we ultimately want is defined to be

$$F^{\mathsf{cat}} = S^{-1}(F^{\mathsf{abs}}),$$

the preimage under S of the abstract Gromov–Witten potential.

7. A roadmap to the computation

We specialize Costello's formalism described in the previous section to the computation of the $g = 1$, $n = 1$ invariant of elliptic curves. We complete the main calculation of Theorem 1.9, assuming a few intermediate computations which we put off to §§ 8–10.

7.1 Fix a complex number $\tau \in \mathbb{H}$ and consider the cyclic $\mathscr{A}_\infty$-algebra A_τ discussed in §5, whose pairing has degree $d = 1$. We are interested in computing the coefficient of $[\Omega]u^{-1}\lambda$ in the potential F^{cat} obtained from A_τ, using a splitting of the Hodge filtration on the periodic cyclic homology of A_τ which matches at the geometric level the one described in Lemma 3.8.

As explained above, the computation proceeds in three steps.

(i) We first compute the string vertices $S_{0,3}$ and $S_{1,1}$. (These are the only pairs (g, n) with Euler characteristic -1.) The answer will be expressed as certain linear combinations of ribbon graphs, using a model for chains on decorated moduli spaces of curves due to Kontsevich and Soibelman [KS09].

(ii) We then find an explicit splitting of the Hodge filtration on $\mathsf{HP}_\bullet(A)$ by choosing a particular lift $[\tilde{\xi}] \in \mathsf{HC}_{-1}^-(A)$ of the class $[\xi] \in \mathsf{HH}_{-1}(A)$. This splitting is chosen to match the geometric one in Lemma 3.8.

(iii) We complete the computation by combining the results of (i) and (ii) into a calculation with Fock modules.

In this section we will outline these steps, leaving the details of the explicit computations to §§8–10.

7.2 The string vertices

We want to represent singular chains on $M_{g,n}$ by linear combinations of ribbon graphs, using classical results by Strebel [Str84] and Penner [Pen87] that compare the ribbon graph complex and the singular chain complex $C_\bullet(M_{g,n})$. For technical reasons explained below, the classical ribbon graph complex is not flexible enough and we need a generalization described by Kontsevich and Soibelman in [KS09, §11.6]. The details will be discussed in §8.

The main reason this generalization is needed is so that we can write a combinatorial model for the quantum master equation, which at the level of singular chains was described in [Cos09]. The combinatorial equation can then be solved recursively, degree by degree in λ. The first few terms of the solution are as follows:

$$\iota(S_{0,3}) = \frac{1}{2} \quad u^{-1} \overset{\times u^{-1}}{\underset{u^{-1}}{\rule{0pt}{1.5em}}}$$

and

$$\iota(S_{1,1}) = \frac{1}{24} \left(\rightarrow \times u^{-2} \right) - \frac{1}{4} \left(u^{-1} \times \right).$$

We refer to §8 for the meaning of these diagrams. The operator ι which turns an output into an input will also be discussed in §8.

7.3 The correct lift

The class $[\Omega] \in \mathsf{HH}_1(A_\tau)$ has a canonical lift to cyclic homology, therefore the only information needed to specify a splitting of the Hodge filtration for A_τ is to choose a lift $[\tilde{\xi}] \in \mathsf{HC}_{-1}^-(A_\tau)$ of the Hochschild homology class $[\xi] \in \mathsf{HH}_{-1}(A_\tau)$ defined in Proposition 3.6. Such a lift will have the form

$$\tilde{\xi} = \xi + \alpha \cdot u + O(u^2)$$

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for some chain $\alpha \in CC_1(A_\tau)$ that satisfies

$$b(\alpha) = -B(\xi) = -1 \otimes \xi.$$

For degree reasons any choice of α satisfying this condition will uniquely determine a lift $\tilde{\xi}$: the higher-order terms can always be filled in to satisfy $(b + uB)(\tilde{\xi}) = 0$, and these further choices do not change the class $[\tilde{\xi}]$ in $\mathsf{HP}_{-1}(A_\tau)$.

We will argue in § 9 that the correct lift is characterized by the system of equations

$$\begin{cases} b(\alpha) = -1 \otimes \xi, \\ b^{1|1}(\partial_\tau \mu^* \mid \alpha) = 0. \end{cases}$$

Here $b^{1|1}$ is the operator defined in [She20, 3.11] and $\partial_\tau \mu^*$ is the derivative of the multiplications $\{\mu_k\}$ of A_τ, viewed as a Hochschild 2-cocycle in $\mathsf{HH}^2(A_\tau)$.

These equations are linear in α; we will argue in § 9 that large-scale computer calculations allow us to solve this system explicitly. Alternatively, in the same section we will argue by theoretical means that this system admits a solution, and this solution is unique up to a b-exact term.

### 7.4 The action of singular chains on cyclic chains

Kontsevich–Soibelman construct an action of their generalized ribbon graphs (thought of as chains on moduli spaces of curves) on the Hochschild chain complex of an $\mathscr{A}_\infty$-algebra A. Specifically, to a ribbon graph Γ with m inputs and n outputs they associate a map

$$\rho(\Gamma) : CC_\bullet(A)^{\otimes m} \to CC_\bullet(A)^{\otimes n}$$

of a certain degree depending on Γ. For example, the chain-level Mukai pairing is the degree-$2d$ map associated to the following graph of genus zero, two inputs and no outputs:
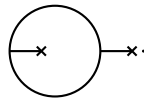
Similarly, the coproduct on the shifted Hochschild homology of a cyclic $\mathscr{A}_\infty$-algebra (induced by the product on Hochschild cohomology, under the Calabi–Yau identification of cohomology with shifted homology) is given by the following graph with one input and two outputs:
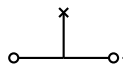
7.5 The Fock space computation

We are interested in the coefficient of $[\Omega] u^{-1} \lambda^1$ in the series

$$F^{\mathsf{cat}} \in \mathsf{Sym}(u^{-1} \mathsf{HH}[u^{-1}])\llbracket \lambda \rrbracket.$$

Since the subspace $\mathsf{HH}[[u]] \subset \mathcal{W}_H$ of the Weyl algebra acts by derivation on the Fock module $\mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])\llbracket \lambda \rrbracket$, this coefficient can be recovered as the constant coefficient of λ^1 in $[\xi] \cdot F^{\mathsf{cat}}$, where we have denoted by $\cdot$ the action of $\mathcal{W}_H$.

Since F^{cat} is the preimage of the generator $\mathbb{1}$ of $H_\bullet(\mathscr{F}^{\mathsf{def}})$ under the isomorphism of $\mathcal{W}_H$-modules

$$S : \mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])\llbracket \lambda \rrbracket \cong H_\bullet(\mathscr{F})\llbracket \lambda \rrbracket \xrightarrow{\sim} H_\bullet(\mathscr{F}^{\mathsf{def}}),$$

the element $[\xi] \cdot F^{\mathsf{cat}}$ is the unique element in $\mathsf{Sym}(u^{-1}\mathsf{HH}[u^{-1}])\llbracket \lambda \rrbracket$ satisfying

$$S([\xi] \cdot F^{\mathsf{cat}}) = [\tilde{\xi}] \cdot \mathbb{1} = [\tilde{\xi}].$$

(We have used here the fact that the isomorphism $\mathcal{W}_H \cong H_\bullet(\mathcal{W})$ induced by the splitting maps $[\xi] \in \mathcal{W}_H$ to $[\tilde{\xi}]$, and the second $\cdot$ above refers to the action of $H_\bullet(\mathcal{W})$.)

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7.6 Write the lift $[\tilde{\xi}]$ of $[\xi]$ as

$$\tilde{\xi} = \xi + \alpha u + \beta u^2$$

by collecting in βu^2 the terms with powers of u greater than two. Consider $\tilde{\xi} = \tilde{\xi} u^0$ as a chain-level element in $\mathscr{F}^{\mathsf{def}}$. In this module the deformed relations (6.10) give the equality

$$\tilde{\xi} = \tilde{\xi} u^0 = (\tilde{T}_1(\tilde{\xi}) + \tilde{T}_2(\tilde{\xi}) + \tilde{T}_3(\tilde{\xi}))\lambda + O(\lambda^2),$$

where T_1, T_2, T_3 are the results of inserting $\tilde{\xi}$ into the three ribbon graphs that appear in the expressions for the string vertices $S_{0,3}$ and $S_{1,1}$ as in §7.2:

$$
\begin{aligned}
\tilde{T}_1(\tilde{\xi}) &= \frac{1}{2}\ \rho\!\left(u^{-1}\ \overset{\times u^{-1}}{\underset{\circ\ \ \ \circ}{\rule{0pt}{1em}}}\ u^{-1} \right)\!(\tilde{\xi}) = \frac{1}{2}\ \rho\!\left(u^{-1}\ \overset{\times}{\underset{\circ\ \ \ \circ}{\rule{0pt}{1em}}}\ u^{-1} \right)\!(\xi), \\[2mm]
\tilde{T}_2(\tilde{\xi}) &= \frac{1}{24}\ \rho\!\left(\overset{\frown}{\underset{\smile}{\to \times u^{-2}}} \right)\!(\tilde{\xi}) = \frac{1}{24}\ \rho\!\left(\overset{\frown}{\underset{\smile}{\to \times}} \right)\!(\alpha), \\[2mm]
\tilde{T}_3(\tilde{\xi}) &= -\frac{1}{4}\ \rho\!\left(\overset{\frown}{\underset{\times}{u^{-1}}} \right)\!(\tilde{\xi}) = -\frac{1}{4}\ \rho\!\left(\overset{\frown}{\underset{\times}{\rule{0pt}{1em}}} \right)\!(\xi).
\end{aligned}
$$

Since $\tilde{T}_2(\tilde{\xi})$ only depends on the chain α, we will sometimes denote it by $T_2(\alpha)$.

7.7 Having explicit formulas for the operations μ_k of the algebra A_τ and for the chain α allows us to compute the values of the three expressions above. This will be done with a computer calculation in §9, and the result is that $T_2(\tilde{\xi}) = T_3(\tilde{\xi}) = 0$ and

$$T_1(\tilde{\xi}) = \tfrac{1}{2}(\xi u^{-1})(\xi u^{-1}) \in \mathcal{W}.$$

Thus in $\mathscr{F}^{\mathsf{def}}$ we have the equality

$$\tilde{\xi} = \tfrac{1}{2}(\xi u^{-1})(\xi u^{-1})\lambda + O(\lambda^2).$$

7.8 As written above, the expression on the right-hand side is not explicitly in the image of S, because the classes ξ are not $(b + uB)$-closed while the classes in the image of S always are. However, we know abstractly that this must be the case, because the map S is an isomorphism of dg modules. So all we need to do is to rewrite the expression $(\xi u^{-1})(\xi u^{-1})$ using the Weyl algebra relations to make it appear as the image under S of an element in $\mathcal{W}_H$.

Note that $(\tilde{\xi} u^{-1})(\tilde{\xi} u^{-1})$ is the image under S of $([\xi]u^{-1})([\xi]u^{-1})$. For degree reasons the terms α and βu are in the left ideal $I^{\mathsf{def}} \bmod \lambda^2$ (inserting them into any of the ribbon graphs in $S_{0,3}$ and $S_{1,1}$ gives zero). Thus we have the equality in $\mathscr{F}^{\mathsf{def}}$

$$
\begin{aligned}
(\xi u^{-1})(\xi u^{-1}) &= (\tilde{\xi} u^{-1} - \alpha - \beta u)(\tilde{\xi} u^{-1} - \alpha - \beta u) \\
&= (\tilde{\xi} u^{-1})(\tilde{\xi} u^{-1}) - (\alpha + \beta u)(\tilde{\xi} u^{-1}) \\
&= (\tilde{\xi} u^{-1})(\tilde{\xi} u^{-1}) - \langle \alpha, \xi \rangle_{\mathsf{M}}.
\end{aligned}
$$

Here we have used the Weyl algebra relation

$$(\alpha + \beta u)(\tilde{\xi} u^{-1}) = (\tilde{\xi} u^{-1})(\alpha + \beta u) + \langle \alpha + \beta u, \tilde{\xi} u^{-1} \rangle_{\mathsf{res}} = \langle \alpha, \xi \rangle_{\mathsf{M}}.$$

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7.9 We put together all the previous computations to get

$$[\xi] \cdot F^{\mathsf{cat}} = S^{-1}(\tilde{\xi}) = \tfrac{1}{2}([\xi]u^{-1})([\xi]u^{-1})\lambda - \tfrac{1}{2}\langle \alpha, \xi \rangle_{\mathsf{M}} \lambda + O(\lambda^2),$$

and therefore the categorical Gromov–Witten invariant we want (the constant coefficient of λ in $[\xi] \cdot F^{\mathsf{cat}}$) is

$$F^{\mathsf{B}}_{1,1}(\tau) = -\tfrac{1}{2}\langle \alpha, \xi \rangle_{\mathsf{M}}.$$

We will argue in § 9 that

$$\langle \alpha, \xi \rangle_{\mathsf{M}} = \tfrac{1}{12} E_2(\tau),$$

thus completing the computation of Theorem 1.9:

$$F^{\mathsf{B}}_{1,1}(\tau) = -\tfrac{1}{24} E_2(\tau).$$

8. String vertices

In this section we discuss the quantum master equation on chains on moduli spaces of curves, and the construction of string vertices as linear combinations of ribbon graphs with rational coefficients.

8.1 Let $M^{\mathsf{fr}}_{g,n}$ denote the coarse moduli space of smooth, genus-g curves with n framed (parametrized), ordered outgoing boundaries. The wreath product group $(S^1)^n \ltimes \Sigma_n$ acts on the space $M^{\mathsf{fr}}_{g,n}$, where the circles $(S^1)^n$ act by rotation of the framing, and the symmetric group Σ_n acts by permutation of the boundaries. Denote by $M_{g,n} = M^{\mathsf{fr}}_{g,n}/((S^1)^n \ltimes \Sigma_n)$ the quotient space, parametrizing smooth, genus-g curves with n unframed, unordered outgoing boundaries. Let $C_\bullet(M_{g,n})$ be the normalized singular chain complex of $M_{g,n}$ with rational coefficients.

There is a degree-one operator

$$\Delta : C_\bullet(M_{g,n}) \to C_{\bullet+1}(M_{g+1,n-2}),$$

defined as the sum of all the possible ways of sewing pairs of boundary components, with a full S^1 twist. Twist-sewing also defines an odd bracket

$$\{-, -\} : C_\bullet(M_{g_1,n_1}) \otimes C_\bullet(M_{g_2,n_2}) \to C_{\bullet+1}(M_{g_1+g_2,n_1+n_2-2}),$$

where the gluing is performed by choosing one boundary from the first surface and one from the second surface.

THEOREM 8.2 (Costello [Cos09, § 4, Theorem 1]). *There exists, for each pair of integers $g \geqslant 0$, $n > 0$ with $2 - 2g - n < 0$, a chain $S_{g,n} \in C_\bullet(M_{g,n})$ of degree $6g - 6 + 2n$ such that:*

(i) *$S_{0,3}$ is a 0-chain of degree $\frac{1}{3!}$;*

(ii) *for all $g, n \geqslant 0$ and $2 - 2g - n > 0$ we have*

$$\partial S_{g,n} + \sum_{\substack{g_1+g_2=g \\ n_1+n_2=n+2}} \frac{1}{2}\{S_{g_1,n_1}, S_{g_2,n_2}\} + \Delta S_{g-1,n+2} = 0.$$

Furthermore, such a collection $\{S_{g,n}\}$ is unique up to homotopy in the sense of [Cos09, § 8].

8.3 The equation in part (ii) of the theorem is a form of the *quantum master equation* for chains on moduli spaces; see [Cos09]. It was first discovered by Sen and Zwiebach [SZ94]. The singular chains $S_{g,n}$ solving the quantum master equation are called *string vertices*.

8.4 In order to perform explicit computations we need to use a combinatorial model for chains on moduli spaces of curves for which the twist-sewing operations make sense. Such a combinatorial model is described by Kontsevich and Soibelman [KS09, § 11.6]. Their construction gives a chain model $C_\bullet^{\mathsf{comb}}(M_{g,m,n}^{\mathsf{fr}})$ for the moduli space $M_{g,m,n}^{\mathsf{fr}}$ of genus-g curves with m inputs and n outputs which are ordered and framed. The moduli space $M_{g,m,n} = M_{g,m,n}^{\mathsf{fr}}/((S^1)^n \ltimes \Sigma_{m,n})$ is defined as before. In the following, we shall make the identification of $M_{g,n}$ with $M_{g,0,n}$.

However, the ribbon graph model only exists for $m \geqslant 1$ – ribbon graphs must have at least one incoming boundary component. To remedy this, Costello notes that there is a map

$$\iota : C_\bullet(M_{g,0,n}) \to C_\bullet(M_{g,1,n-1}),$$

obtained by switching the designation of one of the outgoing boundaries in $M_{g,0,n}$ to 'incoming' and summing over all such choices of outgoing boundary.

8.5 We can describe the chains $S_{g,n}$ combinatorially, via the map ι. Taking coinvariants of the $(S^1)^n \ltimes \Sigma_{1,n-1}$ action on $C_\bullet^{\mathsf{comb}}(M_{g,1,n-1}^{\mathsf{fr}})$ produces a combinatorial model of $C_\bullet(M_{g,1,n-1})$ of the form

$$C_\bullet^{\mathsf{comb}}(M_{g,1,n-1}) = ((u_1^{-1}, \ldots, u_n^{-1}) C_\bullet^{\mathsf{comb}}(M_{g,1,n-1}^{\mathsf{fr}})[u_1^{-1}, \ldots, u_n^{-1}])_{\Sigma_{1,n-1}}.$$

More details can be found in [Cos09, § 5].

Thus, a chain in $C_\bullet^{\mathsf{comb}}(M_{g,1,n-1})$ is a decorated ribbon graph whose input and outputs are labeled by negative powers of u. For example, the chains $\iota(S_{0,3})$ and $\iota(S_{1,1})$ that we have described in § 7.2 are elements in $C_\bullet(M_{0,1,2})$ and $C_\bullet(M_{1,1,0})$, respectively. The vertices labeled with a cross are input vertices, which for Kontsevich–Soibelman are in the set V_{in}. The white vertices (little circles) in $\iota(S_{0,3})$ denote output vertices in V_{out}.

8.6 We now explain where the coefficients that appear in the chains defined in (7.2) come from. For

$$\iota(S_{0,3}) = \frac{1}{2} \quad \text{}$$

the coefficient is $\frac{1}{2}$ instead of $\frac{1}{3!}$ because when applying the map ι there are three choices of outgoing boundary to turn to an incoming one.

Next we compute $\iota(\Delta(S_{0,3})) \in C_\bullet^{\mathsf{comb}}(M_{1,1,0})$ in the moduli space of genus-one curves with one unparametrized input boundary component and zero outgoing boundary components. Since $\binom{3}{2} = 3$, the chain $\Delta(S_{0,3})$ is $\frac{1}{2} = 3 \cdot \frac{1}{6}$ times the result of self-sewing of any two boundary components. Self-sewing of two outgoing boundary components can be described combinatorially:

$$\iota\big(\Delta(S_{0,3})\big) = \frac{1}{2} \quad \text{} \quad \in C_1^{\mathsf{comb}}(M_{1,1,0}).$$

8.7 The string vertex $\iota(S_{1,1})$, being a degree-two chain in $C_2^{\mathsf{comb}}(M_{1,1,0})$, must be a linear combination of the form

$$\iota(S_{1,1}) = a \;\cdot\; \overset{\times u^{-2}}{\bigcirc} \;+\; b \;\cdot\; \overset{u^{-1}}{\bigcirc}$$

for some $a, b \in \mathbb{Q}$. (These are all the possible ribbon graphs of even degree in $C_\bullet^{\mathsf{comb}}(M_{1,1,0})$.)

Denote by ∂ the boundary operator on $C_\bullet^{\mathsf{comb}}(M_{1,1,0})$, and by D the circle operator which is of degree one. The total differential on the equivariant complex $C_\bullet^{\mathsf{comb}}(M_{1,1,0})$ is given by $(\partial + uD)$. Then the quantum master equation for $n = 1$, $g = 1$ becomes

$$(\partial + uD)S_{1,1} + \Delta(S_{0,3}) = 0.$$

One can check that

$$\partial\left(\overset{u^{-1}}{\bigcirc} \right) = 2 \; \overset{\times u^{-1}}{\bigcirc} + \overset{\times u^{-1}}{\bigcirc},$$

$$uD\left(\overset{\times u^{-2}}{\bigcirc} \right) = 6 \; \overset{\times u^{-1}}{\bigcirc}$$

and all other terms are zero. Putting these together with the quantum master equation, we conclude that the coefficients a, b satisfy

$$\begin{cases} 2b + \frac{1}{2} = 0, \\ 6a + b = 0. \end{cases}$$

Solving this gives the combinatorial model of the string vertex $S_{1,1}$:

$$\iota(S_{1,1}) = \frac{1}{24} \; \overset{\times u^{-2}}{\bigcirc} \;-\; \frac{1}{4} \; \overset{u^{-1}}{\bigcirc}.$$

8.8 This computation is essentially due to Costello (unpublished). He used it to compute the categorical Gromov–Witten invariant at $g = 1$, $n = 1$ of a point (we take the $\mathscr{A}_\infty$-algebra A to be $\mathbb{C}$). At $g = 1$, $n = 1$ the insertion of the ψ-class gives the coefficient $1/24$ in the first term of $\iota(S_{1,1})$. This agrees with the geometric computation that

$$\int_{\overline{M}_{1,1}} \psi = \frac{1}{24}.$$

9. The computer calculation

In §7 we have argued that the computation of Theorem 1.9 can be reduced to finding the correct chain α which gives the lift $\tilde{\xi}$, and computing the values $T_2(\tilde{\xi}) = T_3(\tilde{\xi}) = 0$,

$$T_1(\tilde{\xi}) = \tfrac{1}{2}(\xi u^{-1})(\xi u^{-1}),$$

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and

$$\langle \alpha, \xi \rangle_{\mathsf{M}} = \tfrac{1}{12} E_2(\tau).$$

In this section we describe our initial approach to computing these values, by reducing the question of computing them to a large linear algebra problem which can be solved by computer. In the next section we will give a purely mathematical deduction of Theorem 1.9.

9.1 The fact that $T_3(\tilde{\xi}) = 0$ and

$$T_1(\tilde{\xi}) = \tfrac{1}{2}(\xi u^{-1})(\xi u^{-1})$$

follows easily from the Kontsevich–Soibelman definition of the action of ribbon graphs on cyclic chains of A_τ. Roughly speaking, one inserts Hochschild chains at each face of a graph (marked by a cross), labels each half edges by a basis element of A_τ, then contracts these tensors using the inner product on A_τ for every edge, and the cyclic A_∞ structure on every vertex.

For example, the fact that

$$\rho\left(\left(\vphantom{\bigg|}\quad\times\quad\right)\right)(\xi) = 0$$

follows form the fact that for any choice of basis vectors x, y of the algebra A_τ we have

$$c_5(\xi, x, y, x^\vee, y^\vee) = 0,$$

where $x^\vee$, $y^\vee$ are dual basis vectors of x, y with respect to the pairing of the algebra A_τ. (This corresponds to the fact that we need to label the leg of the graph labeled with a cross with ξ, label the other half-edges of the graph by basis elements of the algebra, and evaluate the corresponding cyclic operations at the vertices and the duals of the pairing at the edges.)

Similarly, for the ribbon graph that appears in $S_{0,3}$ the input edge is labeled ξ; the other two half edges adjacent to the trivalent vertex will be labeled $\mathsf{id}_\mathscr{O}$ in order to get a non-trivial cyclic c_3, and then the output is read at the output vertices as two copies of ξu^{-1}.

9.2 We next need to understand the conditions that single out the correct lift

$$\tilde{\xi} = \xi + \alpha u + O(u^2) \in \mathsf{HC}_{-1}^-(A_\tau)$$

of the class $\xi \in \mathsf{HH}_{-1}(A_\tau)$. As mentioned before, for degree reasons the only choice to be made is that of the chain $\alpha \in CC_1(A_\tau)$. This chain must satisfy

$$b(\alpha) = -B(\xi) = -1 \otimes \xi,$$

and any choice of α satisfying this equation can be extended to a class $[\tilde{\xi}] \in \mathsf{HC}_{-1}^-(A_\tau)$ which is uniquely determined by α. Therefore we will talk about the lift $[\tilde{\xi}]$ determined by α.

The following proposition will specify the correct choice of α to match the geometric splitting in Lemma 3.8.

PROPOSITION 9.3. *Fix $\tau_0 \in \mathbb{H}$ and consider the set of chains $\alpha \in CC_1(A_{\tau_0})$ satisfying*

$$b(\alpha) = -B(\xi) = -1 \otimes \xi.$$

(i) *Among these chains there exists a unique one (up to b-exact chains)* α^{GM} *with the property that the chain*

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha^{\mathsf{GM}}) \in CC_{-1}(A_{\tau_0})$$

is b-exact, where $b^{1|1}$ *is the operator defined in* [She20, 3.11] *and* $\partial_\tau \mu^*$ *is the derivative of the operations* $\{\mu_k\}$, *viewed as a Hochschild 2-cocycle in* $\mathsf{HH}^2(A_{\tau_0})$.

(ii) *Let* $[\tilde{\xi}]$ *be the lift of* $[\xi]$ *to periodic cyclic homology corresponding to the unique chain in* (i). *Then its image under the Hochschild–Kostant–Rosenberg (HKR) isomorphism matches the class* $[\tilde{\xi}]^{\mathsf{geom}}$ *of Lemma* 3.8.

Proof. (i) In the proof of Lemma 3.8 we constructed a family $[\xi]$ of classes in $\mathsf{HH}_{-1}(E_\tau)$ and a family of lifts $[\tilde{\xi}]^{\mathsf{geom}} \in H^1_{\mathsf{dR}}(E_\tau)$, both parametrized by $\tau \in \mathbb{H}$. This family of lifts is uniquely characterized by the local condition that it is Gauss–Manin flat.

Locally around $\tau_0 \in \mathbb{H}$ pick representatives $\tilde{\xi}$ of the images $[\tilde{\xi}]$ in $\mathsf{HC}^-_{-1}(A_{\tau_0})$ of the geometric classes $[\tilde{\xi}]^{\mathsf{geom}} \in H^1_{\mathsf{dR}}(E_\tau)$ under the inverse HKR isomorphism. Each such $\tilde{\xi}$ will be of the form

$$\tilde{\xi} = \xi + \alpha^{\mathsf{GM}} \cdot u + O(u^2)$$

for some chain $\alpha^{\mathsf{GM}} \in CC_1(A_\tau)$ which depends on τ. Since $[\tilde{\xi}]$ depends smoothly on τ we can choose α^{GM} to also depend smoothly on τ.

We have assumed that the geometric Gauss–Manin connection ∇^{GM} is identified with the algebraic Getzler–Gauss–Manin connection ∇^{GGM} under HKR. The chain-level Getzler–Gauss–Manin connection applied to a family of chains $x \in CC_\bullet(A_\tau)$ is given by the following formula (see [She20, Definition 3.32]):

$$\nabla^{\mathsf{GGM}}_{\partial_\tau}(x) = \partial_\tau(x) - u^{-1} b^{1|1}(\partial_\tau \mu^* \mid x) - B^{1|1}(\partial_\tau \mu^* \mid x).$$

Thus $\nabla^{\mathsf{GGM}}(\tilde{\xi})$ has the form

$$\nabla^{\mathsf{GGM}}(\tilde{\xi}) = -b^{1|1}(\partial_\tau \mu^* \mid \alpha^{\mathsf{GM}}) + O(u).$$

The flatness of the family $[\tilde{\xi}]^{\mathsf{geom}}$ implies that $b^{1|1}(\partial_\tau \mu^* \mid \alpha^{\mathsf{GM}})$ is b-exact.

This shows that a family α with the required property exists, at least locally around each τ_0. We will now show uniqueness. Let α and α' be two chains satisfying the desired properties. Their difference is a Hochschild cycle,

$$b(\alpha - \alpha') = -1 \otimes \xi + 1 \otimes \xi = 0,$$

so it gives a class $[\alpha - \alpha'] \in HH_1(A_\tau)$ for each τ. The chain $\partial_\tau \mu^* \in CC^2(A_{\tau_0})$ is a cocycle, and its class in Hochschild cohomology is the negative of the Kodaira–Spencer class

$$\mathsf{KS}(\partial_\tau) = -[\partial_\tau \mu^*] \in \mathsf{HH}^2(A_{\tau_0}).$$

(Note that the comparison between the algebraic and geometric Kodaira–Spencer classes has a factor of -1 built in; see [She20, 3.32].) Moreover, under the comparison assumption (§ 1.15), at homology level the class

$$[b^{1|1}(\partial_\tau \mu^* \mid \alpha - \alpha')] \in \mathsf{HH}_{-1}(A_{\tau_0})$$

agrees with the contraction $-\mathsf{KS}(\partial_\tau) \lrcorner [\alpha - \alpha']$. By assumption the former is b-exact.

Therefore we have

$$\mathsf{KS}(\partial_\tau) \lrcorner [\alpha - \alpha'] = 0$$

in $\mathsf{HH}_{-1}(A_{\tau_0})$. But for elliptic curves, contraction with $\mathsf{KS}(\partial_\tau)$ is an isomorphism $\mathsf{HH}_1(E_\tau) \xrightarrow{\sim} \mathsf{HH}_{-1}(E_\tau)$. We conclude that $[\alpha - \alpha'] = 0$ in $\mathsf{HH}_{-1}(E_\tau)$, and thus α^{GM} is unique up to b-exact chains. (This uniqueness argument was suggested to us by Sheridan.)

(ii) This follows immediately from the arguments above for the proof of (i) and the uniqueness of $[\tilde{\xi}]^{\mathsf{geom}}$. $\qquad\square$

9.4 For a fixed value of $\tau \in \mathbb{H}$ the conditions

$$b(\alpha) = -1 \otimes \xi$$

and

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha) = 0$$

are linear in α. The structure constants of the multiplications μ_k of A_τ can be computed using the formulas in [Pol11, 1.2], and their derivatives with respect to τ can be computed explicitly as well. For example, we have

$$\begin{aligned} g_{21} &= \tfrac{5}{6}g_{30} - g_{10}^2, \\ g_{41} &= \tfrac{7}{10}g_{50} - 4g_{10}g_{30}, \\ \partial_\tau g_{10} &= \frac{g_{21}}{2\pi \mathtt{i}} - \frac{2g_{10}}{\tau - \bar{\tau}}. \end{aligned}$$

These are then used to write the linear operators b and $b^{1|1}$ in the basis for A_τ described in (5.3). The resulting matrices allow us to express the equations $b(\alpha) = -1 \otimes \xi$ and $b^{1|1}(\partial_\tau \mu^* \mid \alpha) = 0$, and a solution α can be found in $A_\tau^{\otimes 9} \oplus A_\tau^{\otimes 7} \oplus \cdots A_\tau^{\otimes 3}$. We then use L^1 optimization techniques to find such a solution with a small number of non-zero terms; the best such solution we found has 92 non-zero terms.

9.5 To complete the calculation of the potential we need to follow the procedure outlined in [KS09, §11.6] to compute the result of inserting the various chains into the corresponding ribbon graphs.

Evaluating the insertion of α is a large-scale computation which needs to be done by computer, but follows the same outline as the ones above. The results are

as required, completing the proof of Theorem 1.9.

9.6 In the next section we will need to prove that the potential function $F_{1,1}^{\mathsf{B}}(\tau)$ admits a finite limit at the cusp (i.e. as $\tau \to \mathtt{i} \cdot \infty$). Since the structure constants of Polishchuk's algebra A_τ do extend to the cusp (in our basis, see [Pol11, 2.5, Remark 2]), the result would follow immediately if we could argue that a choice for the chain α can be found which also has a finite limit at the cusp.

Unfortunately we will not be able to prove directly the existence of such a chain. Instead we will use an auxiliary lift $[\tilde{\xi}']$ of the family $[\xi]$ of Hochschild homology classes. This lift will *not*

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be Gauss–Manin flat, but by its very construction it will have a finite limit at the cusp. Using the lift $[\tilde{\xi}']$ instead of the correct Gauss–Manin flat lift in the computations will not compute the correct Gromov–Witten potential. However, comparing the result of this 'wrong' computation with the correct one will allow us to conclude that $F_{1,1}^{\mathsf{B}}(\tau)$ extends to the cusp.

9.7 The desired family of chains

$$\alpha' \in A_\tau^{\otimes 3} \oplus A_\tau^{\otimes 5} \oplus A_\tau^{\otimes 7}$$

which determines the lifts $\tilde{\xi}'$ can be written explicitly. Taking

$$
\begin{aligned}
\alpha' = {}& \mathsf{id}_L \theta\eta + \frac{1}{4} \cdot \mathsf{id}_\mathscr{O}\theta\eta + \frac{1}{2E_4(\tau)} \left(9E_2(\tau) \cdot \mathsf{id}_L \eta\xi\xi\theta\right. \\
& + 60 \cdot (\theta\eta\xi\xi\theta\eta\xi + \eta\xi\theta\eta\xi\xi\theta) - 12 \cdot (\xi\theta\eta\xi\xi\theta\eta + \xi\xi\theta\eta\xi\theta\eta + \mathsf{id}_\mathscr{O}\xi\theta\eta\xi\xi\xi) \\
& + 36 \cdot \mathsf{id}_\mathscr{O}\xi\xi\theta\eta\xi\xi - 24 \cdot \mathsf{id}_\mathscr{O}\xi\xi\xi\theta\eta\xi - 60 \cdot \mathsf{id}_L \eta\xi\xi\xi\xi\theta),
\end{aligned}
$$

one can check by direct calculation that we have

$$b(\alpha') = -1 \otimes \xi = -B(\xi).$$

It is obvious from the above formulas that the chain α' extends to the cusp: its coefficients have finite limit at $\tau = \mathtt{i} \cdot \infty$.

10. Proof of the main theorem

In this final section we give a purely mathematical proof of Theorem 1.9, without relying on computer calculations. The main idea is to carry out the same computation in two different gauges, Polishchuk's modular gauge and the holomorphic gauge described in § 5, and with three different splittings of the Hodge filtration. Comparing the results of these computations will allow us to determine the Gromov–Witten potential $F_{1,1}^{\mathsf{B}}(\tau)$.

10.1 In a nutshell the computation of the categorical Gromov–Witten invariant $F_{1,1}^{\mathsf{B}}(\tau)$ and of some of its variants can be reduced to the following three steps.

(i) Fix an $\mathscr{A}_\infty$ model A_τ or A_τ^{hol} of the derived category of the elliptic curve E_τ.

(ii) Construct a splitting of the Hodge filtration by finding a solution α of the equation

$$b(\alpha) = -B(\xi).$$

(This α determines a lift $[\tilde{\xi}]$ of $[\xi]$ as in § 9.)

(iii) Compute the final invariant as

$$F_{1,1}^{\mathsf{B}}(\tau, \alpha) = T_2(\alpha) - \tfrac{1}{2}\langle \alpha, \xi \rangle_{\mathsf{M}},$$

where

$$T_2(\alpha) = \frac{1}{24}\rho\left(\!\!\vcenter{\hbox{}}\!\!\right)(\alpha)$$

and

$$\langle \alpha, \xi \rangle_{\mathsf{M}} = \rho\left(\!\!\vcenter{\hbox{}}\!\!\right)(\alpha, \xi).$$

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(Since we do not want to rely on computer calculations, we cannot assume that $T_2(\alpha) = 0$ as we have done before.)

In this section we will use three different choices for the chain α in the above process, which give three different splittings of the Hodge filtration and consequently three different potentials (only one of which is of interest from the point of view of mirror symmetry; see below). We remark that in our case, any lift of $[\xi]$ to a negative cyclic class $[\tilde{\xi}]$ satisfies the symplectic condition of Definition 6.13 for degree reasons.

10.2 The first choice of α is the chain α^{GM} described in Proposition 9.3. For this choice the corresponding lifting $[\tilde{\xi}]$ is Gauss–Manin flat. The chain α^{GM} is characterized by the condition

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha^{\mathsf{GM}}) = 0.$$

Using it in the above procedure yields the correct Gromov–Witten potential we are interested in:

$$F_{1,1}^{\mathsf{B}}(\tau) = F_{1,1}^{\mathsf{B}}(\tau, \alpha^{\mathsf{GM}}).$$

In a sense that will be made precise below, the chain α^{GM} depends in a holomorphic, but not modular fashion on τ. Thus using it with the holomorphic $\mathscr{A}_\infty$ model A_τ^{hol} will allow us to deduce that $F_{1,1}^{\mathsf{B}}(\tau)$ is holomorphic in τ.

10.3 The second choice of α is the 11-term chain α' described in § 9.7. The resulting potential $F_{1,1}^{\mathsf{B}}(\tau, \alpha')$ has the property that it extends to the cusp. Comparing it with $F_{1,1}^{\mathsf{B}}(\tau)$ will allow us to deduce that this potential also admits a finite limit at the cusp.

10.4 Finally, the third choice of chain α is a modular version α^{mod} of α^{GM}. It is a solution of the equation

$$b^{1|1}(\widehat{\partial}_\tau \mu^* \mid \alpha^{\mathsf{GM}}) = 0,$$

where $\widehat{\partial}_\tau$ is the natural differential operator on almost holomorphic, modular forms described in § 4.4. Using it in the modular model A_τ yields a new potential

$$F_{1,1}^{\mathsf{B,mod}}(\tau) = F_{1,1}^{\mathsf{B}}(\tau, \alpha^{\mathsf{mod}}).$$

By its very construction it will be obvious that $F_{1,1}^{\mathsf{B,mod}}$ is modular.

In Proposition 10.15 we will argue that we have

$$F_{1,1}^{\mathsf{B}}(\tau) - F_{1,1}^{\mathsf{B,mod}}(\tau) = \frac{1}{4\pi\mathtt{i}(\tau - \bar{\tau})}.$$

The following theorem is a refinement of the statement of our main result, Theorem 1.9.

THEOREM 10.5. *The Gromov–Witten potential function $F_{1,1}^{\mathsf{B}} : \mathbb{H} \to \mathbb{C}$ satisfies the following properties:*

(a) *$F_{1,1}^{\mathsf{B}}$ is holomorphic;*
(b) *$F_{1,1}^{\mathsf{B}}$ extends to the cusp;*
(c) *the function*

$$F_{1,1}^{\mathsf{B,mod}}(\tau) = F_{1,1}^{\mathsf{B}}(\tau) - \frac{1}{4\pi\mathtt{i}(\tau - \bar{\tau})}$$

is modular of weight 2.

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Therefore we have

$$F_{1,1}^{\mathsf{B}}(\tau) = -\tfrac{1}{24} E_2(\tau).$$

Proof. The proof of the fact that the potential $F_{1,1}^{\mathsf{B}}$ satisfies conditions (a)–(c) above is the content of Propositions 10.10, 10.11, and 10.15. The final statement of the theorem follows from the fact that $-\tfrac{1}{24} E_2(\tau)$ is the unique function on $\mathbb{H}$ that satisfies conditions (a)–(c). □

10.6 In the remainder of this section we give precise arguments for the fact that the potential $F_{1,1}^{\mathsf{B}}(\tau)$ satisfies conditions (a)–(c) of Theorem 10.5. We begin with a lemma which shows that the term $T_2(\alpha)$ used in the definition of the potential is not affected by changing α by any multiple of the class $[\Omega] \in HH_1(A_\tau)$ defined in Proposition 3.6.

Lemma 10.7. *We have*

$$\rho\left(\!\!\begin{array}{c}\includegraphics{circle1}\end{array}\!\!\right)([\Omega]) = 0$$

Proof. The Calabi–Yau structure of A_τ gives an identification

$$HH^\bullet(A_\tau) \cong HH_{1-\bullet}(A_\tau).$$

This turns the former into a graded commutative Frobenius algebra with pairing given by the Mukai pairing on homology. In turn this yields a two-dimensional topological field theory.

Moreover, under this identification the class $[\Omega] \in HH_1(A_\tau)$ is matched with a multiple of $1 \in HH^0(A_\tau)$. The field theory interpretation implies that

$$\rho\left(\!\!\begin{array}{c}\includegraphics{circle2}\end{array}\!\!\right)([\Omega]) = \text{ a multiple of } \mathsf{Tr}_{HH^\bullet(A_\tau)}(1)$$

The term $\mathsf{Tr}_{HH^\bullet(A_\tau)}(1)$ is the supertrace of multiplication by 1 in the Hochschild cohomology ring $HH^\bullet(A_\tau)$, which is zero (the Euler characteristic of the graded vector space $\mathsf{HH}^\bullet(A_\tau)$). □

10.8 Let $\alpha_1(\tau), \alpha_2(\tau)$ be two families of solutions of the equation $b(\alpha) = -B(\xi)$. Then for every τ the chain $\alpha_1(\tau) - \alpha_2(\tau)$ is b-closed, and therefore it defines a class $[\alpha_1(\tau) - \alpha_2(\tau)] \in \mathsf{HH}_1(A_\tau)$ which must be a multiple $c(\tau) \cdot [\Omega]$ of $[\Omega]$.

Corollary 10.9. *The difference between the potentials obtained from α_1 and α_2 is*

$$F_{1,1}^{\mathsf{B}}(\tau, \alpha_1) - F_{1,1}^{\mathsf{B}}(\tau, \alpha_2) = -\frac{c(\tau)}{2}.$$

Proof. We have

$$
\begin{aligned}
F_{1,1}^{\mathsf{B}}(\tau, \alpha_1) - F_{1,1}^{\mathsf{B}}(\tau, \alpha_2) &= T_2(\alpha_1 - \alpha_2) - \frac{1}{2}\langle \alpha_1 - \alpha_2, \xi \rangle_{\mathsf{M}} \\
&= T_2(c(\tau) \cdot [\Omega]) - \frac{1}{2}\langle c(\tau) \cdot [\Omega], [\xi] \rangle_{\mathsf{M}} \\
&= -\frac{c(\tau)}{2}.
\end{aligned}
$$
□

PROPOSITION 10.10. *The categorical Gromov–Witten invariant $F_{1,1}^{\mathsf{B}}(\tau)$ is a holomorphic function on $\mathbb{H}$. Moreover, it is an element of weight 2 in the graded ring $\widetilde{M}(\Gamma)_{(0)}$ obtained by localizing $\widetilde{M}(\Gamma)$ at the multiplicative set of elements of homogeneous weight.*

Proof. By (6.2) and Theorem 5.14 it will be enough to prove the statement when using the algebra A_τ^{hol} that we defined in (5.12) as the chosen $\mathscr{A}_\infty$ model of the derived category of the elliptic curve E_τ. Therefore we fix the algebra $A = A_\tau^{\mathsf{hol}}$.

In Proposition 9.3 we have argued that the equations defining the chain α^{GM} can always be solved locally around every $\tau_0 \in \mathbb{H}$, and the solution is unique up to b-exact chains. Moreover, the corresponding lift of $[\xi]$ will be Gauss–Manin flat. The proof of this result did not depend on the specific algebra we used, only on the fact that the section ξ of the family $\{A_\tau\}_{\tau \in \mathbb{H}}$ was flat with respect to differentiation by ∂_τ. Using any local choice α^{GM} of such α allows us to define $F_{1,1}^{\mathsf{B}}$ locally on $\mathbb{H}$.

The action of ribbon graphs on $CC_\bullet(A)$ is a chain map, in the sense that it intertwines the ∂ operator on ribbon graphs and the b operator on chains. Both ribbon graphs involved in the definition of the potential are trivalent, therefore they are sent to zero by ∂. This implies that modifying α^{GM} by a b-exact chain does not change the result of its insertion in these two graphs. We conclude that the locally defined potentials glue to a well-defined global potential on the entire upper half plane $\mathbb{H}$.

The structure constants of the algebra A_τ^{hol} are holomorphic, quasi-modular forms in the ring $\widetilde{M}(\Gamma)$. All the operations used to compute the Gromov–Witten potential (differentiating, solving linear systems, inserting chains into ribbon graphs) always keep us inside $\widetilde{M}(\Gamma)_{(0)}$. (We need to allow for divisions by homogeneous weight elements of $\widetilde{M}(\Gamma)$ in order to solve linear systems.) We conclude that $F_{1,1}^{\mathsf{B}}(\tau)$ is an element of $\widetilde{M}(\Gamma)_{(0)}$, and an easy computation shows that its weight is 2. □

PROPOSITION 10.11. *The potential $F_{1,1}^{\mathsf{B}}(\tau)$ extends to the cusp: the limit*

$$\lim_{\tau \to \mathbf{i} \cdot \infty} F_{1,1}^{\mathsf{B}}(\tau)$$

exists and is finite.

Proof. We use the modular gauge A_τ. The structure constants of this family of $\mathscr{A}_\infty$-algebras extend to the cusp [Pol11, 2.5, Remark 2]. The chain α' defined in (9.7) also extends to the cusp. The terms $T_2(\alpha')$ and $\langle \alpha', \xi \rangle_{\mathsf{M}}$ are (very complicated) polynomial expressions in the structure constants of the algebra and the coefficients of α'. Therefore the potential $F_{1,1}^{\mathsf{B}}(\tau, \alpha')$ obtained from the chain α' extends to the cusp.

The chains α' and α^{GM} both satisfy $b(\alpha) = -B(\xi)$, thus we are in the setup of Corollary 10.9. If we argue that the function $c(\tau)$ defined by the equality

$$[\alpha^{\mathsf{GM}} - \alpha'] = c(\tau) \cdot [\Omega]$$

extends to the cusp, we will be able to conclude that

$$F_{1,1}^{\mathsf{B}}(\tau) = F_{1,1}^{\mathsf{B}}(\tau, \alpha') - \frac{c(\tau)}{2}$$

also extends to the cusp, which is what we need to prove.

The expression $b^{1|1}(\partial_\tau \mu^* \mid \alpha')$ is a polynomial expression in the structure constants of A_τ, their derivatives, and the coefficients of α', all of which have limits at the cusp; therefore this expression also extends to the cusp. Using the defining property of α^{GM} that

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha^{\mathsf{GM}}) = 0,$$

we conclude that $b^{1|1}(\partial_\tau \mu^* \mid \alpha' - \alpha^{\mathsf{GM}})$ also extends to the cusp.

By HKR we conclude that

$$-\mathsf{KS}(\partial_\tau) \lrcorner\, c(\tau) \cdot [\Omega]$$

extends to the cusp. The class $\mathsf{KS}(\partial_\tau)$ was computed in §3.9,

$$\mathsf{KS}(\partial_\tau) = -\frac{1}{\tau - \bar\tau} \frac{\partial}{\partial z}\, d\bar z$$

and therefore

$$-\mathsf{KS}(\partial_\tau) \lrcorner [\Omega] = \frac{1}{\tau - \bar\tau} \frac{\partial}{\partial z}\, d\bar z \lrcorner [2\pi\mathtt{i} \cdot dz] = \frac{2\pi\mathtt{i}}{\tau - \bar\tau}\, d\bar z = 2\pi\mathtt{i} \cdot \xi.$$

It follows that $c(\tau)$ admits a finite limit at the cusp. $\qquad\square$

10.12 From now on we will position ourselves in the modular gauge – we take Polishchuk's algebra A_τ as the $\mathscr{A}_\infty$ model for the elliptic curve.

Fix any $\tau \in \mathbb{H}$. We have already shown that the equation

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha) = 0$$

along with $b(\alpha) = -B(\xi)$ defines a unique (up to b-exact terms) chain α^{GM} which we used to define the potential $F^{\mathsf{B}}_{1,1}(\tau)$.

However, the resulting chain α^{GM} is not modular. The point where modularity breaks down is in the computation of $\partial_\tau \mu^*$: even though the coefficients of μ^* are in the ring $\widehat{M}(\Gamma)$ of almost holomorphic modular forms, the coefficients of $\partial_\tau \mu^*$ are no longer modular.

10.13 This observation suggests how to modify the above computation so that all the intermediate computations (and the result) stay in $\widehat{M}(\Gamma)$. Namely, if we replace the equation above with

$$b^{1|1}(\widehat{\partial}_\tau \mu^* \mid \alpha^{\mathsf{mod}}) = 0,$$

where $\widehat{\partial}_\tau$ is the natural differential operator on $\widehat{M}(\Gamma)$,

$$\widehat{\partial}_\tau = \partial_\tau + \frac{\mathsf{wt}}{\tau - \bar\tau},$$

the solution α^{mod} will depend in a modular, almost holomorphic way on τ.

Note that locally around every point in $\mathbb{H}$ such a solution α^{mod} exists and is unique up to a b-exact chain because the operator KZ of Theorem 4.5 induces an isomorphism between solutions α^{mod} of the above equation and solutions α of the original equation,

$$b^{1|1}(\partial_\tau \mu^* \mid \alpha).$$

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10.14 We use the solution α^{mod} to define a new potential,

$$F_{1,1}^{\mathsf{B,mod}}(\tau) = F_{1,1}^{\mathsf{B}}(\tau, \alpha^{\mathsf{mod}}).$$

As the operator KZ is a differential ring isomorphism, it follows that we have

$$F_{1,1}^{\mathsf{B,mod}} = \mathsf{KZ}(F_{1,1}^{\mathsf{B}}),$$

and, in particular, $F_{1,1}^{\mathsf{B,mod}}$ is a modular form in $\widehat{M}(\Gamma)_{(0)}$ of weight 2.

PROPOSITION 10.15. *We have*

$$F_{1,1}^{\mathsf{B}}(\tau) - F_{1,1}^{\mathsf{B,mod}} = \frac{1}{4\pi\mathtt{i}(\tau - \bar{\tau})}.$$

Therefore $F_{1,1}^{\mathsf{B}}(\tau)$ *satisfies condition (c) of Theorem 10.5.*

Proof. Consider the difference

$$\delta = \alpha^{\mathsf{GM}} - \alpha^{\mathsf{mod}}.$$

It is a b-closed chain in $CC_1(A_\tau)$, thus it gives a class $[\delta]$ in $\mathsf{HH}_1(A_\tau)$. We will show that

$$[\delta] = -\frac{1}{2\pi\mathtt{i}(\tau - \bar{\tau})}[\Omega].$$

Corollary 10.9 then implies the result.

The chains α^{GM} and α^{mod} define lifts of ξ which we denote by $\tilde{\xi}$ and $\tilde{\xi}^{\mathsf{mod}}$, respectively. The former is flat with respect to the Gauss–Manin connection. The main idea is to calculate

$$\nabla_{\partial_\tau}^{\mathsf{GM}}([\xi]^{\mathsf{mod}})$$

in two different ways, and to compare the results.

The definition of the Getzler–Gauss–Manin connection

$$\nabla_{\partial_\tau}^{\mathsf{GGM}}(x) = \partial_\tau(x) - u^{-1}b^{1|1}(\partial_\tau\mu^* \mid x) - B^{1|1}(\partial_\tau\mu^* \mid x)$$

implicitly assumes that we have chosen a connection on the family A_τ of $\mathscr{A}_\infty$-algebras over $\mathbb{H}$ with the property that the basis elements of A_τ are flat for this connection. This allows us to write expressions like $\partial_\tau(x)$, by which we mean that we write x in this chosen basis and differentiate the coefficients.

However there is no reason to insist on the use of such a connection – *any* connection ∇ will work to define the Getzler–Gauss–Manin connection, at least at the level of homology; see [She20, Corollary 3.36] for a proof of this fact. We just need to replace the formula above by

$$\nabla_{\partial_\tau}^{\mathsf{GGM}}(x) = \nabla_{\partial_\tau}(x) - u^{-1}b^{1|1}(\nabla_{\partial_\tau}\mu^* \mid x) - B^{1|1}(\nabla_{\partial_\tau}\mu^* \mid x),$$

where now we think of applying ∇_{∂_τ} to arbitrary tensors.

We will use two different connections ∇^{std} and ∇^{mod} on the bundle A_τ over $\mathbb{H}$. The first, ∇^{std}, is the standard connection for which the basis vectors $\mathsf{id}_{\mathscr{O}}, \mathsf{id}_L, \theta, \eta, \xi, \xi_L$ are flat. Computing $\nabla^{\mathsf{GGM}}\partial_\tau(\tilde{\xi})$ using ∇^{std} yields

$$\begin{aligned}
\nabla_{\partial_\tau}^{\mathsf{GGM}}(\tilde{\xi}^{\mathsf{mod}}) &= \nabla_{\partial_\tau}^{\mathsf{std}}(\xi) - b^{1|1}(\nabla_{\partial_\tau}^{\mathsf{std}}\mu^* \mid \alpha^{\mathsf{mod}}) + O(u) \\
&= -b^{1|1}(\partial_\tau\mu^* \mid \alpha^{\mathsf{mod}}) + O(u).
\end{aligned}$$

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Define a second connection, ∇^{mod}, on the bundle A_τ of $\mathscr{A}_\infty$-algebras by the formula

$$\nabla^{\mathsf{mod}}_{\partial_\tau}(x) = -\frac{\mathsf{wt}(x)}{\tau - \bar{\tau}}x$$

for any basis vector x of A_τ, where the weight of basis elements was defined in §5.9. A straightforward computation shows that

$$\nabla^{\mathsf{mod}}_{\partial_\tau}\mu_k = \widehat{\partial}_\tau\mu_k,$$

where the left-hand side refers to the differentiation of the tensor μ_k with respect to the connection ∇, while the right-hand side refers to the differentiation of the structure constants of the same tensor using the operator $\widehat{\partial}_\tau$.

Computing $\nabla^{\mathsf{GGM}}_{\partial_\tau}(\tilde{\xi}^{\mathsf{mod}})$ using the connection ∇^{mod} yields

$$\begin{aligned}
\nabla^{\mathsf{GGM}}_{\partial_\tau}(\tilde{\xi}^{\mathsf{mod}}) &= \nabla^{\mathsf{mod}}_{\partial_\tau}(\xi) - b^{1|1}(\nabla^{\mathsf{mod}}_{\partial_\tau}\mu^* \mid \alpha^{\mathsf{mod}}) + O(u) \\
&= -\frac{1}{\tau - \bar{\tau}}\xi - b^{1|1}(\widehat{\partial}_\tau\mu^* \mid \alpha^{\mathsf{mod}}) + O(u) \\
&= -\frac{1}{\tau - \bar{\tau}}\xi + O(u).
\end{aligned}$$

Equating the two calculations above yields

$$b^{1|1}(\partial_\tau\mu^* \mid \delta) = b^{1|1}(\partial_\tau\mu^* \mid \alpha^{\mathsf{GM}} - \alpha^{\mathsf{mod}}) = -\frac{1}{\tau - \bar{\tau}}\xi + O(u).$$

Passing to Hochschild homology, we get

$$[\partial_\tau\mu^*] \lrcorner [\delta] = -\frac{1}{\tau - \bar{\tau}}[\xi].$$

Since $[\delta] = c(\tau) \cdot [\Omega]$ for some $c(\tau)$, under the HKR isomorphism this becomes

$$-\mathsf{KS}(\partial_\tau) \lrcorner (c(\tau) \cdot [\Omega]) = -2\pi\mathtt{i} \cdot c(\tau) \cdot \mathsf{KS}(\partial_\tau) \lrcorner [dz] = -\frac{1}{(\tau - \bar{\tau})^2}\,d\bar{z},$$

which forces $c(\tau)$ to equal

$$c(\tau) = -\frac{1}{2\pi\mathtt{i}(\tau - \bar{\tau})}. \qquad \square$$

Remark 10.16. It is worth noting that the modular splitting obtained from the chain α^{mod} corresponds to the complex conjugate splitting from complex algebraic geometry. This follows from a direct computation.

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