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Envelopes of families of framed surfaces and singular solutions of first-order partial differential equations

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In order to investigate envelopes for singular surfaces, we introduce one- and two-parameter families of framed surfaces and the basic invariants, respectively. By using the basic invariants, the existence and uniqueness theorems of one- and two-parameter families of framed surfaces are given. Then we define envelopes of one- and two-parameter families of framed surfaces and give the existence conditions of envelopes which are called envelope theorems. As an application of the envelope theorems, we show that the projections of singular solutions of completely integrable first-order partial differential equations are envelopes.

Keywords: Envelope; family of framed surfaces; singular solution

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1. Introduction

An envelope of a family of surfaces is a surface that is tangent to each member of the family at some points. If the surfaces are regular, the tangent is well-defined (cf. [**2**, **3**, **5**, **14**]). However, for singular surfaces, the classical definitions of envelopes are vague. In [**13**], the first author clarified the definition of an envelope for r-parameter families of frontals and Legendre mappings in the unit tangent bundle over $\mathbb{R}^{n+1}$. When $r \leqslant n$, the envelope theorem is applicable. This idea can be applied to an envelope of a family of singular surfaces. In this paper, we would like to clarify the definitions of the envelopes for one- and two-parametric surfaces with singular points in $\mathbb{R}^3$. As singular surfaces, we consider framed (base) surfaces. A framed surface in the Euclidean space is a smooth surface with a moving frame (cf. [**4**]). The framed surfaces may have singularities. It is a generalization of not only regular surfaces but also frontals at least locally. For the basic results on the singularity theory see [**1**, **3**, **6**, **9**].

In § 2, we quickly review some definitions and theorems of framed surfaces. In § 3 and 4, we introduce one- and two-parameter families of framed surfaces and the basic invariants, respectively. Then we define envelopes of one- and two-parameter families of framed surfaces. We obtain that the envelopes are also framed base surfaces. As one of the main results, we give the existence conditions of envelopes which are called *envelope theorems* (theorems 3.13 and 4.8). The envelopes are independent of rotated frames, reflected frames and the parameter change of the framed surfaces. Moreover, we demonstrate the relations between envelopes of a classical definition and a family of framed surfaces. As an application of the envelope theorems, we show that the projections of singular solutions of completely integrable first-order partial differential equations are envelopes. In § 5, we consider systems of first-order partial differential equations which correspond to one-parameter families of framed surfaces. In § 6, we also consider single first-order partial differential equations which correspond to two-parameter families of framed surfaces. In [**13**], under a condition, it could be proved that the projection of a singular solution of a single completely integrable first-order partial differential equation is an envelope. However, we can prove without the condition $\Sigma_c(F) = \Sigma_\pi(F)$ (theorem 6.3) in this paper.

All maps and manifolds considered in this paper are differentiable of class C^∞.

2. Framed surfaces

We quickly review some definitions and theorems of framed surfaces. For more details see [**4**].

Let $\mathbb{R}^3$ be the 3-dimensional Euclidean space equipped with the inner product $\boldsymbol{a} \cdot \boldsymbol{b} = a_1b_1 + a_2b_2 + a_3b_3$, where $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The norm of $\boldsymbol{a}$ is given by $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ and the vector product is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \det \begin{pmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix},$$

where $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis on $\mathbb{R}^3$. Let S^2 be the unit sphere in $\mathbb{R}^3$, that is, $S^2 = \{\boldsymbol{a} \in \mathbb{R}^3 | |\boldsymbol{a}| = 1\}$. We denote the set $\{(\boldsymbol{a}, \boldsymbol{b}) \in S^2 \times S^2 | \boldsymbol{a} \cdot \boldsymbol{b} = 0\}$ by Δ. Then Δ is a 3-dimensional smooth manifold.

Let U be a simply connected domain in $\mathbb{R}^2$.

DEFINITION 2.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ is a *framed surface* if $\boldsymbol{x}_{u_i}(u_1, u_2) \cdot \boldsymbol{n}(u_1, u_2) = 0$ for all $(u_1, u_2) \in U$ and $i = 1, 2$, where $\boldsymbol{x}_{u_i}(u_1, u_2) = (\partial \boldsymbol{x}/\partial u_i)(u_1, u_2)$. Moreover, $\boldsymbol{x} : U \to \mathbb{R}^3$ is a *framed base surface* if there exists $(\boldsymbol{n}, \boldsymbol{s}) : U \to \Delta$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a framed surface.

We also say that $(\boldsymbol{x}, \boldsymbol{n}) : U \to \mathbb{R}^3 \times S^2$ is a *Legendre surface* if $\boldsymbol{x}_{u_i}(u_1, u_2) \cdot \boldsymbol{n}(u_1, u_2) = 0$ for all $(u_1, u_2) \in U$ and $i = 1, 2$. Moreover, $\boldsymbol{x}$ is a *frontal* if there exists $\boldsymbol{n} : U \to S^2$ such that $(\boldsymbol{x}, \boldsymbol{n})$ is a Legendre surface.

By definition, a framed base surface is a frontal. At least locally, a frontal is a framed base surface.

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ be a framed surface. We denote $\boldsymbol{t}(u_1, u_2) = \boldsymbol{n}(u_1, u_2) \times \boldsymbol{s}(u_1, u_2)$. Then $\{\boldsymbol{n}(u_1, u_2), \boldsymbol{s}(u_1, u_2), \boldsymbol{t}(u_1, u_2)\}$ is a moving frame along $\boldsymbol{x}(u_1, u_2)$. We have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_{u_1} \\ \boldsymbol{x}_{u_2} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \begin{pmatrix} \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \quad \begin{pmatrix} \boldsymbol{n}_{u_i} \\ \boldsymbol{s}_{u_i} \\ \boldsymbol{t}_{u_i} \end{pmatrix} = \begin{pmatrix} 0 & e_i & f_i \\ -e_i & 0 & g_i \\ -f_i & -g_i & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{2.1}$$

where $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ are smooth functions and we call the functions *basic invariants* of the framed surface. We denote the above matrices in equalities (2.1) by $\mathcal{G}, \mathcal{F}_i, i = 1, 2$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ *basic invariants* of the framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Note that (u_1, u_2) is a singular point of $\boldsymbol{x}$ if and only if $\det \mathcal{G}(u_1, u_2) = 0$.

Since the integrability conditions $\boldsymbol{x}_{u_1 u_2} = \boldsymbol{x}_{u_2 u_1}$ and $\mathcal{F}_{2,u_1} - \mathcal{F}_{1,u_2} = \mathcal{F}_1\mathcal{F}_2 - \mathcal{F}_2\mathcal{F}_1$, the basic invariants should satisfy the following conditions:

$$\begin{cases} a_{1u_2} - b_1 g_2 = a_{2u_1} - b_2 g_1, \\ b_{1u_2} - a_2 g_1 = b_{2u_1} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \end{cases} \quad \begin{cases} e_{1u_2} - f_1 g_2 = e_{2u_1} - f_2 g_1, \\ f_{1u_2} - e_2 g_1 = f_{2u_1} - e_1 g_2, \\ g_{1u_2} - e_1 f_2 = g_{2u_1} - e_2 f_1. \end{cases} \tag{2.2}$$

DEFINITION 2.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{x}}(u_1, u_2) = A(\boldsymbol{x}(u_1, u_2)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u_1, u_2) = A(\boldsymbol{n}(u_1, u_2)), \widetilde{\boldsymbol{s}}(u_1, u_2) = A(\boldsymbol{s}(u_1, u_2))$$

for all $(u_1, u_2) \in U$.

We have the existence and uniqueness theorems for the basic invariants of framed surfaces (cf. [**4**]). For the existence and uniqueness theorems of frontals see [**12**].

THEOREM 2.3 (Existence theorem for framed surfaces). *Let $a_i, b_i, e_i, f_i, g_i : U \to \mathbb{R}, i = 1, 2$ be smooth functions with the integrability conditions (2.2). Then there exists a framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants are a_i, b_i, e_i, f_i, g_i, $i = 1, 2$.*

THEOREM 2.4 (Uniqueness theorem for framed surfaces). *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \to \mathbb{R}^3 \times \Delta$ be framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2)$ coincide.*

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DEFINITION 2.5. We define a smooth mapping $C^F = (J^F, K^F, H^F) : U \to \mathbb{R}^3$ by

$$J^F = \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}, K^F = \det\begin{pmatrix} e_1 & f_1 \\ e_2 & f_2 \end{pmatrix},$$
$$H^F = -\frac{1}{2}\left\{\det\begin{pmatrix} a_1 & f_1 \\ a_2 & f_2 \end{pmatrix} - \det\begin{pmatrix} b_1 & e_1 \\ b_2 & e_2 \end{pmatrix}\right\}.$$

We call $C^F = (J^F, K^F, H^F)$ a *curvature of the framed surface*.

For the properties of the curvature of framed surfaces see [4].

3. One-parameter families of framed surfaces and envelopes

3.1. One-parameter families of framed surfaces

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a smooth mapping, where U is a simply connected domain in $\mathbb{R}^2$ and Λ is an interval in $\mathbb{R}$.

DEFINITION 3.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is a *one-parameter family of framed surfaces* if $(\boldsymbol{x}(\cdot, \lambda), \boldsymbol{n}(\cdot, \lambda), \boldsymbol{s}(\cdot, \lambda))$ is a framed surface for each $\lambda \in \Lambda$.

We denote $\boldsymbol{t}(u_1, u_2, \lambda) = \boldsymbol{n}(u_1, u_2, \lambda) \times \boldsymbol{s}(u_1, u_2, \lambda)$. Then $\{\boldsymbol{n}(u_1, u_2, \lambda), \boldsymbol{s}(u_1, u_2, \lambda), \boldsymbol{t}(u_1, u_2, \lambda)\}$ is a moving frame along $\boldsymbol{x}(u_1, u_2, \lambda)$. For convenient, sometimes we use the notation $u_3 = \lambda$.

We have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_{u_1} \\ \boldsymbol{x}_{u_2} \\ \boldsymbol{x}_{u_3} \end{pmatrix} = \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ c_1 & a_3 & b_3 \end{pmatrix}\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \begin{pmatrix} \boldsymbol{n}_{u_i} \\ \boldsymbol{s}_{u_i} \\ \boldsymbol{t}_{u_i} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & e_i & f_i \\ -e_i & 0 & g_i \\ -f_i & -g_i & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{3.1}$$

where $a_i, b_i, e_i, f_i, g_i, c_1 : U \to \mathbb{R}, i = 1, 2, 3$ are smooth functions and we call the functions *basic invariants* of the one-parameter family of framed surfaces. We denote the above matrices in equalities (3.1) by $\mathcal{G}, \mathcal{F}_i, i = 1, 2, 3$, respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ *basic invariants* of the one-parameter family of framed surfaces $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Since the integrability conditions $\boldsymbol{x}_{u_i u_j} = \boldsymbol{x}_{u_j u_i}$ and $\mathcal{F}_{i,u_k} - \mathcal{F}_{j,u_\ell} = \mathcal{F}_j\mathcal{F}_i - \mathcal{F}_i\mathcal{F}_j$, $i, j, k, \ell = 1, 2, 3$, the basic invariants should be satisfied the following

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conditions:

$$\begin{cases} a_{1u_2} - b_1 g_2 = a_{2u_1} - b_2 g_1, \\ b_{1u_2} - a_2 g_1 = b_{2u_1} - a_1 g_2, \\ a_1 e_2 + b_1 f_2 = a_2 e_1 + b_2 f_1, \\ a_{1u_3} - b_1 g_3 = c_1 e_1 + a_{3u_1} - b_3 g_1, \\ a_1 g_3 + b_{1u_3} = c_1 f_1 + b_{3u_1} + a_3 g_1, \\ \quad -a_1 e_3 - b_1 f_3 = c_{1u_1} - a_3 e_1 - b_3 f_1, \\ a_{2u_3} - b_2 g_3 = c_1 e_2 + a_{3u_2} - b_3 g_2, \\ a_2 g_3 + b_{2u_3} = c_1 f_2 + b_{3u_2} + a_3 g_2, \\ \quad -a_2 e_3 - b_2 f_3 = c_{1u_2} - a_3 e_2 - b_3 f_2, \end{cases} \quad \begin{cases} e_{1u_2} - f_1 g_2 = e_{2u_1} - f_2 g_1, \\ f_{1u_2} - e_2 g_1 = f_{2u_1} - e_1 g_2, \\ g_{1u_2} - e_1 f_2 = g_{2u_1} - e_2 f_1, \\ e_{1u_3} - f_1 g_3 = e_{3u_1} - f_3 g_1, \\ f_{1u_3} - e_3 g_1 = f_{3u_1} - e_1 g_3, \\ g_{1u_3} - e_1 f_3 = g_{3u_1} - e_3 f_1, \\ e_{2u_3} - f_2 g_3 = e_{3u_2} - f_3 g_2, \\ f_{2u_3} - e_3 g_2 = f_{3u_2} - e_2 g_3, \\ g_{2u_3} - e_2 f_3 = g_{3u_2} - e_3 f_2. \end{cases} \tag{3.2}$$

Definition 3.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be one-parameter families of framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as one-parameter families of framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\begin{aligned} \widetilde{\boldsymbol{x}}(u_1, u_2, \lambda) &= A(\boldsymbol{x}(u_1, u_2, \lambda)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u_1, u_2, \lambda) \\ &= A(\boldsymbol{n}(u_1, u_2, \lambda)), \widetilde{\boldsymbol{s}}(u_1, u_2, \lambda) = A(\boldsymbol{s}(u_1, u_2, \lambda)) \end{aligned}$$

for all $(u_1, u_2, \lambda) \in U \times \Lambda$.

We have the existence and uniqueness theorems for the basic invariants of one-parameter families of framed surfaces.

Theorem 3.3 (Existence theorem for one-parameter families of framed surfaces). *Let $a_i, b_i, e_i, f_i, g_i, c_1 : U \times \Lambda \to \mathbb{R}, i = 1, 2, 3$ be smooth functions with the integrability conditions* (3.2). *Then there exists a one-parameter family of framed surfaces $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants are $a_i, b_i, e_i, f_i, g_i, c_1$, $i = 1, 2, 3$.*

Proof. We denote that $M(3)$ is the set of 3×3 matrices and I_3 is the identity matrix. Choose any fixed value $(u_{1_0}, u_{2_0}, u_{3_0}) \in U \times \Lambda$. We consider an initial value problem,

$$F_{u_i}(u_1, u_2, u_3) = \mathcal{F}_i(u_1, u_2, u_3) F(u_1, u_2, u_3),\ i = 1, 2, 3,\ F(u_{1_0}, u_{2_0}, u_{3_0}) = I_3,$$

where $F(u_1, u_2, u_3) \in M(3)$. By the integrability conditions $\mathcal{F}_{i,u_k} - \mathcal{F}_{j,u_\ell} = \mathcal{F}_j \mathcal{F}_i - \mathcal{F}_i \mathcal{F}_j,\ i, j, k, \ell = 1, 2, 3$, we have $F_{u_1 u_2} = F_{u_2 u_1}$, $F_{u_1 u_3} = F_{u_3 u_1}$ and $F_{u_2 u_3} = F_{u_3 u_2}$. Since $U \times \Lambda$ is simply connected, there exists a solution $F(u_1, u_2, u_3)$. Since

$\mathcal{F}_i(u_1, u_2, u_3) \in \mathfrak{o}(3)$, we have

$$\frac{\partial}{\partial u_i}((F^t)F) = \left(\frac{\partial}{\partial u_i}F\right)^t F + (F^t)\frac{\partial}{\partial u_i}F = ((\mathcal{F}_iF)^t)F + (F)^t(\mathcal{F}_iF)$$
$$= (F^t)\big((\mathcal{F}_i)^t + \mathcal{F}_i\big)F = F^t \cdot 0 \cdot F = 0,$$

$i = 1, 2, 3$. It follows that $(F(u_1, u_2, u_3)^t)F(u_1, u_2, u_3)$ is constant. Therefore, we have

$$(F(u_1, u_2, u_3)^t)F(u_1, u_2, u_3) = (F(u_{1_0}, u_{2_0}, u_{3_0})^t)F(u_{1_0}, u_{2_0}, u_{3_0}) = I_3.$$

We set $F(u_1, u_2, u_3) = \big(\boldsymbol{n}(u_1, u_2, u_3), \boldsymbol{s}(u_1, u_2, u_3), \boldsymbol{t}(u_1, u_2, u_3)\big)^t$. Since $(\partial/\partial u_i)$ $(\det F(u_1, u_2, u_3)) = 0$, $i = 1, 2, 3$, we have $\det F(u_1, u_2, u_3) = \det F(u_{1_0}, u_{2_0}, u_{3_0}) = \det I_3 = 1$. Hence, $F(u_1, u_2, u_3)$ is a special orthogonal matrix. Then $\boldsymbol{t}(u_1, u_2, u_3) = \boldsymbol{n}(u_1, u_2, u_3) \times \boldsymbol{s}(u_1, u_2, u_3)$. Next we consider differential equations

$$\boldsymbol{x}_{u_1} = a_1\boldsymbol{s} + b_1\boldsymbol{t},\ \boldsymbol{x}_{u_2} = a_2\boldsymbol{s} + b_2\boldsymbol{t},\ \boldsymbol{x}_{u_3} = c_1\boldsymbol{n} + a_3\boldsymbol{s} + b_3\boldsymbol{t}.$$

By the integrability conditions $\boldsymbol{x}_{u_iu_j} = \boldsymbol{x}_{u_ju_i}, i, j = 1, 2, 3$, there exists a solution $\boldsymbol{x}(u_1, u_2, u_3)$. It follows that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed surfaces whose associated basic invariants are $a_i, b_i, e_i, f_i, g_i, c_1$, $i = 1, 2, 3$. □

Lemma 3.4. *If $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as one-parameter families of framed surfaces, then $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3)$.*

Proof. By definition 3.2 and a direct calculation, we obtain the lemma. □

Lemma 3.5. *If $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3) = (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3)$ and $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})(u_{1_0}, u_{2_0}, u_{3_0}) = (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})(u_{1_0}, u_{2_0}, u_{3_0})$ for a point $(u_{1_0}, u_{2_0}, u_{3_0}) \in U \times \Lambda$, then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$.*

Proof. Firstly, we show $(\boldsymbol{n}, \boldsymbol{s}, \boldsymbol{t}) = (\widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}})$, where $\boldsymbol{t} = \boldsymbol{n} \times \boldsymbol{s}$ and $\widetilde{\boldsymbol{t}} = \widetilde{\boldsymbol{n}} \times \widetilde{\boldsymbol{s}}$. We define a function $f : U \times \Lambda \to \mathbb{R}$ by

$$f(u_1, u_2, u_3) = \boldsymbol{n}(u_1, u_2, u_3) \cdot \widetilde{\boldsymbol{n}}(u_1, u_2, u_3)$$
$$+ \boldsymbol{s}(u_1, u_2, u_3) \cdot \widetilde{\boldsymbol{s}}(u_1, u_2, u_3) + \boldsymbol{t}(u_1, u_2, u_3) \cdot \widetilde{\boldsymbol{t}}(u_1, u_2, u_3).$$

By the definition of the basic invariants, we have

$$f_{u_i} =(e_i - \widetilde{e}_i)(\boldsymbol{s} \cdot \widetilde{\boldsymbol{n}}) + (\widetilde{e}_i - e_i)(\boldsymbol{n} \cdot \widetilde{\boldsymbol{s}}) + (f_i - \widetilde{f}_i)(\boldsymbol{t} \cdot \widetilde{\boldsymbol{n}}) + (\widetilde{f}_i - f_i)(\boldsymbol{n} \cdot \widetilde{\boldsymbol{t}})$$
$$+ (g_i - \widetilde{g}_i)(\boldsymbol{t} \cdot \widetilde{\boldsymbol{s}}) + (\widetilde{g}_i - g_i)(\boldsymbol{s} \cdot \widetilde{\boldsymbol{t}}),$$

$i = 1, 2, 3$. By the assumption $\mathcal{F}_i = \widetilde{\mathcal{F}}_i$, we have $f_{u_i}(u_1, u_2, u_3) = 0$ for all $(u_1, u_2, u_3) \in U \times \Lambda$ and $i = 1, 2, 3$. Moreover, by the assumption $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ $(u_{1_0}, u_{2_0}, u_{3_0}) = (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})(u_{1_0}, u_{2_0}, u_{3_0})$, we have $f(u_{1_0}, u_{2_0}, u_{3_0}) = 3$. It concludes that $f(u_1, u_2, u_3) = 3$ for all $(u_1, u_2, u_3) \in U \times \Lambda$. Hence, we have $\boldsymbol{n} \cdot \widetilde{\boldsymbol{n}} = \boldsymbol{s} \cdot \widetilde{\boldsymbol{s}} = \boldsymbol{t} \cdot \widetilde{\boldsymbol{t}} = 1$. It follows that $\boldsymbol{n}(u_1, u_2, u_3) = \widetilde{\boldsymbol{n}}(u_1, u_2, u_3)$, $\boldsymbol{s}(u_1, u_2, u_3) = \widetilde{\boldsymbol{s}}(u_1, u_2, u_3)$ and $\boldsymbol{t}(u_1, u_2, u_3) = \widetilde{\boldsymbol{t}}(u_1, u_2, u_3)$ for all $(u_1, u_2, u_3) \in U \times \Lambda$.

Next, we show $\boldsymbol{x} = \widetilde{\boldsymbol{x}}$. By the assumption $\mathcal{G} = \widetilde{\mathcal{G}}$, we have $\boldsymbol{x}_{u_1} = a_1\boldsymbol{s} + b_1\boldsymbol{t} = \widetilde{a}_1\widetilde{\boldsymbol{s}} + \widetilde{b}_1\widetilde{\boldsymbol{t}} = \widetilde{\boldsymbol{x}}_{u_1}$, $\boldsymbol{x}_{u_2} = a_2\boldsymbol{s} + b_2\boldsymbol{t} = \widetilde{a}_2\widetilde{\boldsymbol{s}} + \widetilde{b}_2\widetilde{\boldsymbol{t}} = \widetilde{\boldsymbol{x}}_{u_2}$ and $\boldsymbol{x}_{u_3} = c_1\boldsymbol{n} + a_3\boldsymbol{s} + b_3\boldsymbol{t} = \widetilde{c}_1\widetilde{\boldsymbol{n}} + \widetilde{a}_3\widetilde{\boldsymbol{s}} + \widetilde{b}_3\widetilde{\boldsymbol{t}} = \widetilde{\boldsymbol{x}}_{u_3}$. Then, we have $(\boldsymbol{x} - \widetilde{\boldsymbol{x}})_{u_i} = 0$, $i = 1, 2, 3$. Since $\boldsymbol{x}(u_{1_0}, u_{2_0}, u_{3_0}) = \widetilde{\boldsymbol{x}}(u_{1_0}, u_{2_0}, u_{3_0})$, we have $\boldsymbol{x}(u_1, u_2, u_3) = \widetilde{\boldsymbol{x}}(u_1, u_2, u_3)$ for all $(u_1, u_2, u_3) \in U \times \Lambda$. Therefore, we have $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) = (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$. □

THEOREM 3.6 (Uniqueness theorem for one-parameter families of framed surfaces). *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be one-parameter families of framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as one-parameter families of framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3)$ coincide.*

Proof. By lemma 3.4, we have the necessary condition.

Conversely, for a fixed point $(u_{1_0}, u_{2_0}, u_{3_0}) \in U \times \Lambda$, there exist $A \in SO(3)$ and $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})(u_{1_0}, u_{2_0}, u_{3_0}) = (A\boldsymbol{x} + \boldsymbol{a}, A\boldsymbol{n}, A\boldsymbol{s})(u_{1_0}, u_{2_0}, u_{3_0}).$$

By lemmas 3.4 and 3.5, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as one-parameter families of framed surfaces. □

3.2. Envelopes of one-parameter families of framed surfaces

Let $F : W \times \Lambda \to \mathbb{R}$ be a one-parameter family of functions, where W is a simply connected domain in $\mathbb{R}^3$ and Λ is an interval in $\mathbb{R}$. Then one of the classical definitions of an envelope E_I is as follows. For instance see [**2**, **3**].

DEFINITION 3.7. An *envelope* E_I of F is the discriminant set of F, that is, the set of points given by

$$E_I = \{w \in \mathbb{R}^3 \mid \textit{for some } \lambda \in \Lambda,\ F(w, \lambda) = F_\lambda(w, \lambda) = 0\}.$$

If $F(w, \lambda) = F_\lambda(w, \lambda) = 0$, we say that $w \in E_I$ *with respect to* λ.

EXAMPLE 3.8. Let $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, F(w_1, w_2, w_3, \lambda) = (w_1 - \lambda)^3 - w_3^2$ be a one-parameter family of functions. Then $F = 0$ is the image of a cuspidal edge for each fixed $\lambda \in \mathbb{R}$. For the definition and properties of cuspidal edges see [**4**, **11**]. Since $F_\lambda(w_1, w_2, w_3, \lambda) = -3(w_1 - \lambda)^2$, the envelope of the family F is given by $E_I = \{(\lambda, w_2, 0)\} = xy$-plane.

EXAMPLE 3.9. Let $F : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}, F(w_1, w_2, w_3, \lambda) = w_1^3 - (w_3 - \lambda)^2$ be a one-parameter family of functions. Then $F = 0$ is the image of a cuspidal edge for each fixed $\lambda \in \mathbb{R}$. Since $F_\lambda(w_1, w_2, w_3, \lambda) = 2(w_3 - \lambda)$, the envelope of the family F is given by $E_I = \{(0, w_2, \lambda)\} = yz$-plane.

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However, in the sense of the limit tangent plane of the cuspidal edge, yz-plane is not tangent to the cuspidal edges, see [**13**]. Therefore, we would like to distinguish them as envelopes. See examples 3.20 and 3.21.

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3)$. Let $V \subset \mathbb{R}^2$ be an open subset and $e : V \to U \times \Lambda$, $e(p) = (u_1(p), u_2(p), \lambda(p))$ be a smooth mapping. We denote $E = \boldsymbol{x} \circ e : V \to \mathbb{R}^3$.

Definition 3.10. We call E an *envelope* (and e a *pre-envelope*) for the one-parameter family of framed surfaces $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, when the following conditions are satisfied.

(i) The set of regular points of $\lambda : V \to \Lambda$ is dense in V. (The variability condition.)

(ii) $E_{p_i}(p) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p = (p_1, p_2) \in V$ and $i = 1, 2$. (The tangency condition.)

By definition, we have the following result.

Proposition 3.11. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1$, $i = 1, 2, 3$. Suppose that $e : V \to U \times \Lambda$ is a pre-envelope and $E : V \to \mathbb{R}^3$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Then $(E, \boldsymbol{n} \circ e, \boldsymbol{s} \circ e) : V \to \mathbb{R}^3 \times \Delta$ is a framed surface with the basic invariants*

$$\begin{pmatrix} a_{1E} \\ b_{1E} \\ e_{1E} \\ f_{1E} \\ g_{1E} \end{pmatrix}(p) = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}(e(p)) \begin{pmatrix} u_{1p_1} \\ u_{2p_1} \\ \lambda_{p_1} \end{pmatrix}(p), \quad \begin{pmatrix} a_{2E} \\ b_{2E} \\ e_{2E} \\ f_{2E} \\ g_{2E} \end{pmatrix}(p)$$
$$= \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ e_1 & e_2 & e_3 \\ f_1 & f_2 & f_3 \\ g_1 & g_2 & g_3 \end{pmatrix}(e(p)) \begin{pmatrix} u_{1p_2} \\ u_{2p_2} \\ \lambda_{p_2} \end{pmatrix}(p)$$

and the curvature (J_E^F, K_E^F, H_E^F) is given by

$$\begin{aligned} J_E^F(p) =& J^F(e(p)) \det \begin{pmatrix} u_{1p_1} & u_{2p_1} \\ u_{1p_2} & u_{2p_2} \end{pmatrix}(p) \\ &+ \det \begin{pmatrix} a_1 & b_1 \\ a_3 & b_3 \end{pmatrix}(e(p)) \det \begin{pmatrix} u_{1p_1} & \lambda_{p_1} \\ u_{1p_2} & \lambda_{p_2} \end{pmatrix}(p) \\ &+ \det \begin{pmatrix} a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}(e(p)) \det \begin{pmatrix} u_{2p_1} & \lambda_{p_1} \\ u_{2p_2} & \lambda_{p_2} \end{pmatrix}(p), \\ K_E^F(p) =& K^F(e(p)) \det \begin{pmatrix} u_{1p_1} & u_{2p_1} \\ u_{1p_2} & u_{2p_2} \end{pmatrix}(p) \end{aligned}$$

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$$
\begin{aligned}
&+\det\begin{pmatrix} e_1 & f_1 \\ e_3 & f_3 \end{pmatrix}(e(p))\det\begin{pmatrix} u_{1p_1} & \lambda_{p_1} \\ u_{1p_2} & \lambda_{p_2} \end{pmatrix}(p) \\
&+\det\begin{pmatrix} e_2 & f_2 \\ e_3 & f_3 \end{pmatrix}(e(p))\det\begin{pmatrix} u_{2p_1} & \lambda_{p_1} \\ u_{2p_2} & \lambda_{p_2} \end{pmatrix}(p),
\end{aligned}
$$

$$
\begin{aligned}
H_E^F(p) =& H^F(e(p))\det\begin{pmatrix} u_{1p_1} & u_{2p_1} \\ u_{1p_2} & u_{2p_2} \end{pmatrix}(p) \\
&-\frac{1}{2}\left\{\det\begin{pmatrix} a_1 & f_1 \\ a_3 & f_3 \end{pmatrix}(e(p)) - \det\begin{pmatrix} b_1 & e_1 \\ b_3 & e_3 \end{pmatrix}(e(p))\right\}\det\begin{pmatrix} u_{1p_1} & \lambda_{p_1} \\ u_{1p_2} & \lambda_{p_2} \end{pmatrix}(p) \\
&-\frac{1}{2}\left\{\det\begin{pmatrix} a_2 & f_2 \\ a_3 & f_3 \end{pmatrix}(e(p)) - \det\begin{pmatrix} b_2 & e_2 \\ b_3 & e_3 \end{pmatrix}(e(p))\right\}\det\begin{pmatrix} u_{2p_1} & \lambda_{p_1} \\ u_{2p_2} & \lambda_{p_2} \end{pmatrix}(p).
\end{aligned}
$$

Proof. Since E is an envelope, $E_{p_i}(p)\cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$ and $i = 1, 2$. It follows that $(E, \boldsymbol{n}\circ e, \boldsymbol{s}\circ e) : V \to \mathbb{R}^3 \times \Delta$ is a framed surface. By a direct calculation, we have the basic invariants and the curvature of $(E, \boldsymbol{n}\circ e, \boldsymbol{s}\circ e)$. □

Remark 3.12. By the integrability condition (3.2), if $g_1(e(p)) = g_2(e(p)) = g_3(e(p)) = 0$, then $K_E^F(p) = 0$.

As one of the main results, we have the envelope theorem for one-parameter families of framed surfaces.

Theorem 3.13. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a one-parameter family of framed surfaces with the basic invariants* $a_i, b_i, e_i, f_i, g_i, c_1$, $i = 1, 2, 3$. *Suppose that* $e : V \to U \times \Lambda$ *is a smooth mapping satisfying the variability condition. Then the following statements are equivalent:*

(1) $e : V \to U \times \Lambda$ *is a pre-envelope and* $E : V \to \mathbb{R}^3$ *is an envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

(2) $\boldsymbol{x}_\lambda(e(p))\cdot \boldsymbol{n}(e(p)) = 0$ *for all* $p \in V$.

(3) $c_1(e(p)) = 0$ *for all* $p \in V$.

Proof. Suppose that e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. We denote $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{n} = (n_1, n_2, n_3)$. By a direct calculation, we have

$$
\begin{aligned}
E_{p_i}(p) = \big(&x_{1u_1}(e(p))u_{1p_i}(p) + x_{1u_2}(e(p))u_{2p_i}(p) + x_{1\lambda}(e(p))\lambda_{p_i}(p), \\
&x_{2u_1}(e(p))u_{1p_i}(p) + x_{2u_2}(e(p))u_{2p_i}(p) + x_{2\lambda}(e(p))\lambda_{p_i}(p), \\
&x_{3u_1}(e(p))u_{1p_i}(p) + x_{3u_2}(e(p))u_{2p_i}(p) + x_{3\lambda}(e(p))\lambda_{p_i}(p)\big).
\end{aligned}
$$

Since $E_{p_i}(p)\cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$ and $i = 1, 2$, we have

$$
\begin{aligned}
&\big(x_{1u_1}(e(p))n_1(e(p)) + x_{2u_1}(e(p))n_2(e(p)) + x_{3u_1}(e(p))n_3(e(p))\big)u_{1p_i}(p) \\
&+ \big(x_{1u_2}(e(p))n_1(e(p)) + x_{2u_2}(e(p))n_2(e(p)) + x_{3u_2}(e(p))n_3(e(p))\big)u_{2p_i}(p) \\
&+ \big(x_{1\lambda}(e(p))n_1(e(p)) + x_{2\lambda}(e(p))n_2(e(p)) + x_{3\lambda}(e(p))n_3(e(p))\big)\lambda_{p_i}(p) = 0.
\end{aligned}
$$

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By $\boldsymbol{x}_{u_1} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{x}_{u_2} \cdot \boldsymbol{n} = 0$, we have

$$\big(x_{1\lambda}(e(p))n_1(e(p)) + x_{2\lambda}(e(p))n_2(e(p)) + x_{3\lambda}(e(p))n_3(e(p))\big)\lambda_{p_i}(p) = 0,$$

$i = 1, 2$. By the variability condition, we have $\boldsymbol{x}_\lambda(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$. The converse is given by a direct calculation. It follows that (1) and (2) are equivalent.

Moreover, since $\boldsymbol{x}_\lambda(u_1, u_2, \lambda) \cdot \boldsymbol{n}(u_1, u_2, \lambda) = c_1(u_1, u_2, \lambda)$ for all $(u_1, u_2, \lambda) \in U \times \Lambda$, (2) and (3) are equivalent. □

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1,\ i = 1, 2, 3$. For the tangent plane of $\boldsymbol{x}$, spanned by $\boldsymbol{s}$ and $\boldsymbol{t}$, there are other frames by rotations and reflections. We define $(\overline{\boldsymbol{s}}(u_1, u_2, \lambda), \overline{\boldsymbol{t}}(u_1, u_2, \lambda)) \in \Delta$ by

$$\begin{pmatrix} \overline{\boldsymbol{s}}(u_1, u_2, \lambda) \\ \overline{\boldsymbol{t}}(u_1, u_2, \lambda) \end{pmatrix} = \begin{pmatrix} \cos\theta(u_1, u_2, \lambda) & -\sin\theta(u_1, u_2, \lambda) \\ \sin\theta(u_1, u_2, \lambda) & \cos\theta(u_1, u_2, \lambda) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}(u_1, u_2, \lambda) \\ \boldsymbol{t}(u_1, u_2, \lambda) \end{pmatrix},$$

where $\theta : U \times \Lambda \to \mathbb{R}$ is a smooth function. Then $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed surfaces. By a direct calculation, the basic invariants $\overline{a}_i, \overline{b}_i, \overline{e}_i, \overline{f}_i, \overline{g}_i, \overline{c}_1$ of $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}})$ are given by

$$a_i \cos\theta - b_i \sin\theta, a_i \sin\theta + b_i \cos\theta, e_i \cos\theta - f_i \sin\theta,$$
$$e_i \sin\theta + f_i \cos\theta, g_i - \theta_{u_i}, c_1,\ i = 1, 2, 3,$$

where $u_3 = \lambda$. We call the moving frame $\{\boldsymbol{n}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{t}}\}$ a rotated frame along $\boldsymbol{x}$ by θ.

On the other hand, we define $(\widetilde{\boldsymbol{s}}(u_1, u_2, \lambda), \widetilde{\boldsymbol{t}}(u_1, u_2, \lambda)) \in \Delta$ by

$$\begin{pmatrix} \widetilde{\boldsymbol{s}}(u_1, u_2, \lambda) \\ \widetilde{\boldsymbol{t}}(u_1, u_2, \lambda) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \cos\theta(u_1, u_2, \lambda) & -\sin\theta(u_1, u_2, \lambda) \\ \sin\theta(u_1, u_2, \lambda) & \cos\theta(u_1, u_2, \lambda) \end{pmatrix} \begin{pmatrix} \boldsymbol{s}(u_1, u_2, \lambda) \\ \boldsymbol{t}(u_1, u_2, \lambda) \end{pmatrix},$$

where $\theta : U \times \Lambda \to \mathbb{R}$ is a smooth function. Then $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed surfaces. By a direct calculation, the basic invariants $\widetilde{a}_i, \widetilde{b}_i, \widetilde{e}_i, \widetilde{f}_i, \widetilde{g}_i, \widetilde{c}_1$ of $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}})$ are given by

$$a_i \cos\theta - b_i \sin\theta, -a_i \sin\theta - b_i \cos\theta, e_i \cos\theta - f_i \sin\theta,$$
$$- e_i \sin\theta - f_i \cos\theta, -g_i + \theta_{u_i}, c_1,\ i = 1, 2, 3,$$

where $u_3 = \lambda$. We call the moving frame $\{\boldsymbol{n}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}\}$ a reflected frame along $\boldsymbol{x}$ by θ.

PROPOSITION 3.14. *Under the above notations, if $e : V \to U \times \Lambda$ is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, then $e : V \to U \times \Lambda$ is also a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}})$ and $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}})$.*

Proof. By theorem 3.13, we have $c_1(e(p)) = 0$ for all $p \in V$. It follows that $\overline{c}_1(e(p)) = 0$ and $\widetilde{c}_1(e(p)) = 0$ for all $p \in V$. Hence, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}})$ and $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}})$. □

It follows that the envelope is independent of rotated frames and reflected frames of the framed surfaces. Moreover, we demonstrate the envelope is independent of the parameter change of the framed surfaces.

Let $\widetilde{U}$ be a simply connected domain in $\mathbb{R}^2$ and $\widetilde{\Lambda}$ be an interval in $\mathbb{R}$.

Definition 3.15. We say that a map $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ is a *one-parameter family of parameter change* if Φ is a diffeomorphism of the form $\Phi(v_1, v_2, k) = (\phi_1(v_1, v_2, k), \phi_2(v_1, v_2, k), \varphi(k))$.

Proposition 3.16. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a one-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1,\ i = 1, 2, 3$. Suppose that $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ is a one-parameter family of parameter change. Then $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) = (\boldsymbol{x} \circ \Phi, \boldsymbol{n} \circ \Phi, \boldsymbol{s} \circ \Phi) : \widetilde{U} \times \widetilde{\Lambda} \to \mathbb{R}^3 \times \Delta$ is also a one-parameter family of framed surfaces with the basic invariants*

$$\begin{pmatrix} \widetilde{a}_1 & \widetilde{b}_1 & 0 \\ \widetilde{a}_2 & \widetilde{b}_2 & 0 \\ \widetilde{a}_3 & \widetilde{b}_3 & \widetilde{c}_1 \end{pmatrix} = \begin{pmatrix} \phi_{1v_1} & \phi_{2v_1} & 0 \\ \phi_{1v_2} & \phi_{2v_2} & 0 \\ \phi_{1k} & \phi_{2k} & \varphi' \end{pmatrix} \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_1 \end{pmatrix} \circ \Phi,$$

$$\begin{pmatrix} \widetilde{e}_1 & \widetilde{f}_1 & \widetilde{g}_1 \\ \widetilde{e}_2 & \widetilde{f}_2 & \widetilde{g}_2 \\ \widetilde{e}_3 & \widetilde{f}_3 & \widetilde{g}_3 \end{pmatrix} = \begin{pmatrix} \phi_{1v_1} & \phi_{2v_1} & 0 \\ \phi_{1v_2} & \phi_{2v_2} & 0 \\ \phi_{1k} & \phi_{2k} & \varphi' \end{pmatrix} \begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \end{pmatrix} \circ \Phi.$$

Moreover, if $e : V \to U \times \Lambda$ is a pre-envelope, E is an envelope, then $\Phi^{-1} \circ e : V \to \widetilde{U} \times \widetilde{\Lambda}$ is a pre-envelope and E is also an envelope of $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$.

Proof. Since $\widetilde{\boldsymbol{x}}_{v_i}(v_1, v_2, k) = \boldsymbol{x}_{u_1}(\Phi(v_1, v_2, k))\phi_{1v_i}(v_1, v_2, k) + \boldsymbol{x}_{u_2}(\Phi(v_1, v_2, k))\phi_{2v_i}(v_1, v_2, k)$, $\boldsymbol{x}_{u_i}(u_1, u_2, \lambda) \cdot \boldsymbol{n}(u_1, u_2, \lambda) = 0$ for all $(u_1, u_2, \lambda) \in U \times \Lambda$ and $i = 1, 2$, we have $\widetilde{\boldsymbol{x}}_{v_i}(v_1, v_2, k) \cdot \widetilde{\boldsymbol{n}}(v_1, v_2, k) = 0$ for all $(v_1, v_2, k) \in \widetilde{U} \times \widetilde{\Lambda}$ and $i = 1, 2$. Therefore, $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ is a one-parameter family of framed surfaces. By a direct calculation, we have the basic invariants $\widetilde{a}_i, \widetilde{b}_i, \widetilde{e}_i, \widetilde{f}_i, \widetilde{g}_i, \widetilde{c}_1,\ i = 1, 2, 3$.

By the form of the diffeomorphism $\Phi(v_1, v_2, k) = (\phi_1(v_1, v_2, k), \phi_2(v_1, v_2, k), \varphi(k))$, $\Phi^{-1} : U \times \Lambda \to \widetilde{U} \times \widetilde{\Lambda}$ is given by the form $\Phi^{-1}(u_1, u_2, \lambda) = (\psi_1(u_1, u_2, \lambda), \psi_2(u_1, u_2, \lambda), \varphi^{-1}(\lambda))$ for some smooth functions ψ_1 and ψ_2. It follows that $\Phi^{-1} \circ e(p) = (\psi_1(u_1(p), u_2(p), \lambda(p)), \psi_2(u_1(p), u_2(p), \lambda(p)), \varphi^{-1}(\lambda(p)))$, where $e(p) = (u_1(p), u_2(p), \lambda(p))$. Since the set of regular points of $\varphi^{-1} \circ \lambda : V \to \widetilde{\Lambda}$ is dense in V, the variability condition holds. Moreover, we have

$$\begin{aligned} &\widetilde{\boldsymbol{x}}_k(v_1, v_2, k) \cdot \widetilde{\boldsymbol{n}}(v_1, v_2, k) \\ &\quad = (\boldsymbol{x}_{u_1}(\Phi(v_1, v_2, k))\phi_{1_k}(v_1, v_2, k)\boldsymbol{x}_{u_2}(\Phi(v_1, v_2, k))\phi_{2_k}(v_1, v_2, k) \\ &\qquad + \boldsymbol{x}_\lambda(\Phi(v_1, v_2, k))\varphi'(k)) \cdot \boldsymbol{n}(\Phi(v_1, v_2, k)) \\ &\quad = \varphi'(k)\boldsymbol{x}_\lambda(\Phi(v_1, v_2, k)) \cdot \boldsymbol{n}(\Phi(v_1, v_2, k)). \end{aligned}$$

It follows that

$$\widetilde{\boldsymbol{x}}_k(\Phi^{-1} \circ e(p)) \cdot \widetilde{\boldsymbol{n}}(\Phi^{-1} \circ e(p)) = \varphi'(\varphi^{-1}(\lambda(p)))\boldsymbol{x}_\lambda(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$$

for all $p \in V$. By theorem 3.13, $\Phi^{-1} \circ e$ is a pre-envelope of $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$. Therefore, $\widetilde{\boldsymbol{x}} \circ \Phi^{-1} \circ e = \boldsymbol{x} \circ \Phi \circ \Phi^{-1} \circ e = \boldsymbol{x} \circ e = E$ is also an envelope of $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$. □

We give relations between the envelope E_I of a classical definition by using an implicit function (definition 3.7) and the envelope E of a one-parameter family of framed surfaces (definition 3.10).

Proposition 3.17. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a one-parameter family of framed surfaces. Suppose that* $F(w, \lambda) = 0$ *is an implicit function satisfying* $F(\boldsymbol{x}(u_1, u_2, \lambda), \lambda) = 0$ *and* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(u_1, u_2, \lambda), \lambda)$ *is parallel to* $\boldsymbol{n}(u_1, u_2, \lambda)$ *for all* $(u_1, u_2, \lambda) \in U \times \Lambda$. *If* $e : V \to U \times \Lambda$ *is a pre-envelope and* $E : V \to \mathbb{R}^3$ *is an envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, *then* $E(V) \subset E_I$.

Proof. By differentiating $F(\boldsymbol{x}(u_1, u_2, \lambda), \lambda) = 0$ with respect to λ, we have

$$F_{w_1} x_{1\lambda} + F_{w_2} x_{2\lambda} + F_{w_3} x_{3\lambda} + F_\lambda = 0.$$

By the assumption, there exists a smooth function $a : U \times \Lambda \to \mathbb{R}$ such that

$$(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(u_1, u_2, \lambda), \lambda) = a(u_1, u_2, \lambda)\boldsymbol{n}(u_1, u_2, \lambda)$$

for all $(u_1, u_2, \lambda) \in U \times \Lambda$. By theorem 3.13, we have $\boldsymbol{x}_\lambda(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$. It follows that $F_\lambda(\boldsymbol{x}(e(p)), \lambda(p)) = 0$ for all $p \in V$. Therefore, $E(p) \in E_I$ with respect to $\lambda(p)$ for all $p \in V$. □

Proposition 3.18. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a one-parameter family of framed surfaces, and let* $e : V \to U \times \Lambda$ *be a smooth mapping satisfying the variability condition. If* $\operatorname{rank}(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2})(e(p)) = 2$ *for all* $p \in V$ *and trace of* e *lies in the singular set of* $\boldsymbol{x}$, *then* e *is a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ (*and* E *is an envelope*).

Proof. Since trace of e lies in the singular set of $\boldsymbol{x}$, we have the condition

$$\operatorname{rank}\begin{pmatrix} x_{1u_1} & x_{1u_2} & x_{1\lambda} \\ x_{2u_1} & x_{2u_2} & x_{2\lambda} \\ x_{3u_1} & x_{3u_2} & x_{3\lambda} \end{pmatrix}(e(p)) < 3.$$

By the assumption $\operatorname{rank}(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2})(e(p)) = 2$, there exist smooth functions $\alpha, \beta : V \to \mathbb{R}$ such that $\boldsymbol{x}_\lambda(e(p)) = \alpha(p)\boldsymbol{x}_{u_1}(e(p)) + \beta(p)\boldsymbol{x}_{u_2}(e(p))$. It follows that $\boldsymbol{x}_\lambda(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$. Hence, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. □

Proposition 3.19. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a one-parameter family of framed surfaces. Suppose that* $F(w, \lambda) = 0$ *is an implicit function satisfying* $F(\boldsymbol{x}(u_1, u_2, \lambda), \lambda) = 0$ *and* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(u_1, u_2, \lambda), \lambda)$ *is parallel to* $\boldsymbol{n}(u_1, u_2, \lambda)$ *for all* $(u_1, u_2, \lambda) \in U \times \Lambda$, *and* $e : V \to U \times \Lambda$, $e(p) = (u_1(p), u_2(p), \lambda(p))$ *is a smooth mapping satisfying the variability condition. If* $E(p) = \boldsymbol{x} \circ e(p) \in E_I$ *with respect to* $\lambda(p)$, $\operatorname{rank}(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2})(e(p)) = 2$ *and* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(e(p)), \lambda(p)) \neq (0, 0, 0)$ *for all* $p \in V$, *then* e *is a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ (*and* E *is an envelope*).

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Proof. By differentiating $F(\boldsymbol{x}(u_1, u_2, \lambda), \lambda) = 0$ with respect to u_i $(i = 1, 2)$ and λ, we have

$$F_{w_1} x_{1u_i} + F_{w_2} x_{2u_i} + F_{w_3} x_{3u_i} = 0, \;\; F_{w_1} x_{1\lambda} + F_{w_2} x_{2\lambda} + F_{w_3} x_{3\lambda} + F_\lambda = 0.$$

Since $E(p) \in E_I$ with respect to $\lambda(p)$, we have $F_\lambda(\boldsymbol{x}(e(p)), \lambda(p)) = 0$ for all $p \in V$. It follows that

$$\begin{pmatrix} x_{1u_1} & x_{2u_1} & x_{3u_1} \\ x_{1u_2} & x_{2u_2} & x_{3u_2} \\ x_{1\lambda} & x_{2\lambda} & x_{3\lambda} \end{pmatrix} (e(p)) \begin{pmatrix} F_{w_1}(\boldsymbol{x}(e(p)), \lambda(p)) \\ F_{w_2}(\boldsymbol{x}(e(p)), \lambda(p)) \\ F_{w_3}(\boldsymbol{x}(e(p)), \lambda(p)) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

By the assumption $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(e(p)), \lambda(p)) \neq (0, 0, 0)$, we have rank $d\boldsymbol{x}(e(p)) < 3$. It follows that $e(p)$ belongs to the singular set of $\boldsymbol{x}$. By Proposition 3.18, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. □

Example 3.20. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be

$$\begin{aligned} \boldsymbol{x}(u_1, u_2, \lambda) &= (u_1^2 + \lambda, u_2, u_1^3), \boldsymbol{n}(u_1, u_2, \lambda) \\ &= \frac{1}{\sqrt{9u_1^2 + 4}}(-3u_1, 0, 2), \boldsymbol{s}(u_1, u_2, \lambda) = (0, 1, 0). \end{aligned}$$

Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed surfaces and $\boldsymbol{x}$ is a cuspidal edge for each fixed $\lambda \in \mathbb{R}$. By a direct calculation, $\boldsymbol{t}(u_1, u_2, \lambda) = (1/\sqrt{9u_1^2 + 4})(-2, 0, -3u_1)$ and the basic invariants are given by

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & -u_1\sqrt{9u_1^2 + 4} \\ 0 & 1 & 0 \\ -\dfrac{3u_1}{\sqrt{9u_1^2 + 4}} & 0 & -\dfrac{2}{\sqrt{9u_1^2 + 4}} \end{pmatrix},$$

$$\mathcal{F}_1 = \begin{pmatrix} 0 & 0 & \dfrac{6}{9u_1^2 + 4} \\ 0 & 0 & 0 \\ -\dfrac{6}{9u_1^2 + 4} & 0 & 0 \end{pmatrix}, \; \mathcal{F}_2 = \mathcal{F}_3 = 0.$$

Since $\boldsymbol{x}_\lambda(u_1, u_2, \lambda) \cdot \boldsymbol{n}(u_1, u_2, \lambda) = -3u_1/\sqrt{9u_1^2 + 4}$, if we take $e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}$, $e(p_1, p_2) = (0, p_1, p_2)$, then the variability condition holds and $\boldsymbol{x}_\lambda(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p = (p_1, p_2) \in \mathbb{R}^2$. By theorem 3.13, e is a pre-envelope and $E(p) = \boldsymbol{x} \circ e(p) = (p_2, p_1, 0)$ is an envelope. Hence, xy-plane is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, see example 3.8. Moreover, the curvature of the framed surface $(E, \boldsymbol{n} \circ e, \boldsymbol{s} \circ e)$ is given by $(J_E^F, K_E^F, H_E^F) = (-1, 0, 0)$ by proposition 3.11.

EXAMPLE 3.21. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$ be

$$\begin{aligned}\boldsymbol{x}(u_1, u_2, \lambda) &= (u_1^2, u_2, u_1^3 + \lambda), \boldsymbol{n}(u_1, u_2, \lambda)\\ &= \frac{1}{\sqrt{9u_1^2+4}}(-3u_1, 0, 2), \boldsymbol{s}(u_1, u_2, \lambda) = (0, 1, 0).\end{aligned}$$

Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed surfaces and $\boldsymbol{x}$ is a cuspidal edge for each fixed $\lambda \in \mathbb{R}$. By a direct calculation, $\boldsymbol{t}(u_1, u_2, \lambda) = (1/\sqrt{9u_1^2+4})(-2, 0, -3u_1)$ and the basic invariants are given by

$$\mathcal{G} = \begin{pmatrix} 0 & 0 & -u_1\sqrt{9u_1^2+4} \\ 0 & 1 & 0 \\ \dfrac{2}{\sqrt{9u_1^2+4}} & 0 & -\dfrac{3u_1}{\sqrt{9u_1^2+4}} \end{pmatrix},$$

$$\mathcal{F}_1 = \begin{pmatrix} 0 & 0 & \dfrac{6}{9u_1^2+4} \\ 0 & 0 & 0 \\ -\dfrac{6}{9u_1^2+4} & 0 & 0 \end{pmatrix}, \ \mathcal{F}_2 = \mathcal{F}_3 = 0.$$

Since $\boldsymbol{x}_\lambda(u_1, u_2, \lambda) \cdot \boldsymbol{n}(u_1, u_2, \lambda) = 2/\sqrt{9u_1^2+4} \neq 0$ for all $(u_1, u_2, \lambda) \in \mathbb{R}^2 \times \mathbb{R}$, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ does not have the envelope E by theorem 3.13. Hence, yz-plane is not an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, see example 3.9.

4. Two-parameter families of framed surfaces and envelopes

4.1. Two-parameter families of framed surfaces

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a smooth mapping, where U and Λ are simply connected domains in $\mathbb{R}^2$. We denote $u = (u_1, u_2) \in U, \ \lambda = (\lambda_1, \lambda_2) \in \Lambda$.

DEFINITION 4.1. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is a *two-parameter family of framed surfaces* if $(\boldsymbol{x}(\cdot, \lambda), \boldsymbol{n}(\cdot, \lambda), \boldsymbol{s}(\cdot, \lambda))$ is a framed surface for each $\lambda \in \Lambda$.

We denote $\boldsymbol{t}(u, \lambda) = \boldsymbol{n}(u, \lambda) \times \boldsymbol{s}(u, \lambda)$. Then $\{\boldsymbol{n}(u, \lambda), \boldsymbol{s}(u, \lambda), \boldsymbol{t}(u, \lambda)\}$ is a moving frame along $\boldsymbol{x}(u, \lambda)$. For convenient, sometimes we use the notations $u_3 = \lambda_1$ and $u_4 = \lambda_2$.

We have the following systems of differential equations:

$$\begin{pmatrix} \boldsymbol{x}_{u_1} \\ \boldsymbol{x}_{u_2} \\ \boldsymbol{x}_{u_3} \\ \boldsymbol{x}_{u_4} \end{pmatrix} = \begin{pmatrix} 0 & a_1 & b_1 \\ 0 & a_2 & b_2 \\ c_1 & a_3 & b_3 \\ c_2 & a_4 & b_4 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \ \begin{pmatrix} \boldsymbol{n}_{u_i} \\ \boldsymbol{s}_{u_i} \\ \boldsymbol{t}_{u_i} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & e_i & f_i \\ -e_i & 0 & g_i \\ -f_i & -g_i & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{n} \\ \boldsymbol{s} \\ \boldsymbol{t} \end{pmatrix}, \tag{4.1}$$

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where $a_i, b_i, e_i, f_i, g_i, c_1, c_2 : U \to \mathbb{R}, i = 1, 2, 3, 4$ are smooth functions and we call the functions *basic invariants* of the two-parameter family of framed surfaces. We denote the above matrices in equalities (4.1) by $\mathcal{G}, \mathcal{F}_i, i = 1, 2, 3, 4,$ respectively. We also call the matrices $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ *basic invariants* of the two-parameter family of framed surfaces $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

Since the integrability conditions $\boldsymbol{x}_{u_i u_j} = \boldsymbol{x}_{u_j u_i}$ and $\mathcal{F}_{i,u_k} - \mathcal{F}_{j,u_\ell} = \mathcal{F}_j \mathcal{F}_i - \mathcal{F}_i \mathcal{F}_j,\ i, j, k, \ell = 1, 2, 3, 4,$ the basic invariants should be satisfied some conditions. However, we omit them here.

DEFINITION 4.2. Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be two-parameter families of framed surfaces. We say that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are *congruent as two-parameter families of framed surfaces* if there exist a constant rotation $A \in SO(3)$ and a translation $\boldsymbol{a} \in \mathbb{R}^3$ such that

$$\widetilde{\boldsymbol{x}}(u, \lambda) = A(\boldsymbol{x}(u, \lambda)) + \boldsymbol{a}, \widetilde{\boldsymbol{n}}(u, \lambda) = A(\boldsymbol{n}(u, \lambda)), \widetilde{\boldsymbol{s}}(u, \lambda) = A(\boldsymbol{s}(u, \lambda))$$

for all $(u, \lambda) \in U \times \Lambda$.

By the similar methods of the one-parameter families of framed surfaces, we have the existence and uniqueness theorems for the basic invariants of two-parameter families of framed surfaces.

THEOREM 4.3 (Existence theorem for two-parameters families of framed surfaces). *Let $a_i, b_i, e_i, f_i, g_i, c_1, c_2 : U \to \mathbb{R}, i = 1, 2, 3, 4$ be smooth functions with the integrability conditions. Then there exists a two-parameter family of framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ whose associated basic invariants are $a_i, b_i, e_i, f_i, g_i, c_1, c_2,\ i = 1, 2, 3, 4$.*

THEOREM 4.4 (Uniqueness theorem for two-parameters families of framed surfaces). *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}), (\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be two-parameter families of framed surfaces with the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4), (\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3, \widetilde{\mathcal{F}}_4)$, respectively. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ and $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$ are congruent as two-parameter families of framed surfaces if and only if the basic invariants $(\mathcal{G}, \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \mathcal{F}_4)$ and $(\widetilde{\mathcal{G}}, \widetilde{\mathcal{F}}_1, \widetilde{\mathcal{F}}_2, \widetilde{\mathcal{F}}_3, \widetilde{\mathcal{F}}_4)$ coincide.*

4.2. Envelopes of two-parameter families of framed surfaces

Let $F : W \times \Lambda \to \mathbb{R}$ be a two-parameter family of functions, where W and Λ are simply connected domains in $\mathbb{R}^3$ and $\mathbb{R}^2$. Then one of the classical definitions of an envelope E_I is as follows. For instance see [**2**, **3**].

DEFINITION 4.5. An *envelope* E_I of F is the discriminant set of F, that is, the set of points given by

$$E_I = \{w \in \mathbb{R}^3 \mid for\ some\ \lambda = (\lambda_1, \lambda_2) \in \Lambda,$$
$$F(w, \lambda) = F_{\lambda_1}(w, \lambda) = F_{\lambda_2}(w, \lambda) = 0\}.$$

If $F(w, \lambda) = F_{\lambda_1}(w, \lambda) = F_{\lambda_2}(w, \lambda) = 0,$ we say that $w \in E_I$ *with respect to* $\lambda = (\lambda_1, \lambda_2)$.

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Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a two-parameter family of framed surfaces. Let $V \subset \mathbb{R}^2$ be an open subset and $e : V \to U \times \Lambda$, $e(p) = (u_1(p), u_2(p), \lambda_1(p), \lambda_2(p))$ be a smooth mapping. We denote $E = \boldsymbol{x} \circ e : V \to \mathbb{R}^3$.

Definition 4.6. We call E an *envelope* (and e a *pre-envelope*) for the two-parameter family of framed surfaces $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, when the following conditions are satisfied:

(i) The set of regular points of $\lambda : V \to \Lambda$ is dense in V. (The variability condition.)

(ii) $E_{p_i}(p) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p = (p_1, p_2) \in V$ and $i = 1, 2$. (The tangency condition.)

By definition, we have the following result.

Proposition 4.7. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a two-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1, c_2$, $i = 1, 2, 3, 4$. Suppose that $e : V \to U \times \Lambda$ is a pre-envelope and $E : V \to \mathbb{R}^3$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. Then $(E, \boldsymbol{n} \circ e, \boldsymbol{s} \circ e) : V \to \mathbb{R}^3 \times \Delta$ is a framed surface.*

By a direct calculation, we also have the basic invariants and the curvature of the framed surface $(E, \boldsymbol{n} \circ e, \boldsymbol{s} \circ e)$. Here, we omit them.

As one of the main results, we also have the envelope theorem for two-parameter families of framed surfaces.

Theorem 4.8. *Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a two-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1, c_2$, $i = 1, 2, 3, 4$. Suppose that $e : V \to U \times \Lambda$ is a smooth mapping satisfying the variability condition. Then the following statements are equivalent:*

(1) *$e : V \to U \times \Lambda$ is a pre-envelope and $E : V \to \mathbb{R}^3$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.*

(2) *$\boldsymbol{x}_{\lambda_1}(e(p)) \cdot \boldsymbol{n}(e(p)) = \boldsymbol{x}_{\lambda_2}(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$.*

(3) *$c_1(e(p)) = c_2(e(p)) = 0$ for all $p \in V$.*

Proof. We denote $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{n} = (n_1, n_2, n_3)$. Suppose that e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. By a direct calculation, we have

$$\begin{aligned} E_{p_i}(p) = \big(& x_{1u_1}(e(p))u_{1p_i}(p) + x_{1u_2}(e(p))u_{2p_i}(p) + x_{1\lambda_1}(e(p))\lambda_{1p_i}(p) \\ & + x_{1\lambda_2}(e(p))\lambda_{2p_i}(p), x_{2u_1}(e(p))u_{1p_i}(p) + x_{2u_2}(e(p))u_{2p_i}(p) \\ & + x_{2\lambda_1}(e(p))\lambda_{1p_i}(p) + x_{2\lambda_2}(e(p))\lambda_{2p_i}(p), x_{3u_1}(e(p))u_{1p_i}(p) \\ & + x_{3u_2}(e(p))u_{2p_i}(p) + x_{3\lambda_1}(e(p))\lambda_{1p_i}(p) + x_{3\lambda_2}(e(p))\lambda_{2p_i}(p) \big), \end{aligned}$$

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$i = 1, 2$. Since $E_{p_i}(p) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$, we have

$$\begin{aligned}&\big(x_{1u_1}(e(p))n_1(e(p)) + x_{2u_1}(e(p))n_2(e(p)) + x_{3u_1}(e(p))n_3(e(p))\big)u_{1p_i}(p)\\&+\big(x_{1u_2}(e(p))n_1(e(p)) + x_{2u_2}(e(p))n_2(e(p)) + x_{3u_2}(e(p))n_3(e(p))\big)u_{2p_i}(p)\\&+\big(x_{1\lambda_1}(e(p))n_1(e(p)) + x_{2\lambda_1}(e(p))n_2(e(p)) + x_{3\lambda_1}(e(p))n_3(e(p))\big)\lambda_{1p_i}(p)\\&+\big(x_{1\lambda_2}(e(p))n_1(e(p)) + x_{2\lambda_2}(e(p))n_2(e(p)) + x_{3\lambda_2}(e(p))n_3(e(p))\big)\lambda_{2p_i}(p) = 0,\end{aligned}$$

$i = 1, 2$. By $\boldsymbol{x}_{u_1} \cdot \boldsymbol{n} = 0$ and $\boldsymbol{x}_{u_2} \cdot \boldsymbol{n} = 0$, we have

$$\begin{pmatrix}\lambda_{1p_1}(p) & \lambda_{2p_1}(p)\\ \lambda_{1p_2}(p) & \lambda_{2p_2}(p)\end{pmatrix}\begin{pmatrix}\boldsymbol{x}_{\lambda_1}(e(p)) \cdot \boldsymbol{n}(e(p))\\ \boldsymbol{x}_{\lambda_2}(e(p)) \cdot \boldsymbol{n}(e(p))\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

By the variability condition, we have $\boldsymbol{x}_{\lambda_1}(e(p)) \cdot \boldsymbol{n}(e(p)) = \boldsymbol{x}_{\lambda_2}(e(p)) \cdot \boldsymbol{n}(e(p)) = 0$ for all $p \in V$. The converse is given by a direct calculation. Hence, (1) and (2) are equivalent.

Moreover, since $\boldsymbol{x}_{\lambda_1}(u, \lambda) \cdot \boldsymbol{n}(u, \lambda) = c_1(u, \lambda)$ and $\boldsymbol{x}_{\lambda_2}(u, \lambda) \cdot \boldsymbol{n}(u, \lambda) = c_2(u, \lambda)$ for all $(u, \lambda) \in U \times \Lambda$, (2) and (3) are equivalent. □

Let $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ be a two-parameter family of framed surfaces with the basic invariants $a_i, b_i, e_i, f_i, g_i, c_1, c_2$, $i = 1, 2, 3, 4$. For the tangent plane of $\boldsymbol{x}(u, \lambda)$, spanned by $\boldsymbol{s}(u, \lambda)$ and $\boldsymbol{t}(u, \lambda)$, there are other frames by rotations and reflections, where $u = (u_1, u_2)$, $\lambda = (\lambda_1, \lambda_2)$. We define $(\overline{\boldsymbol{s}}(u, \lambda), \overline{\boldsymbol{t}}(u, \lambda)) \in \Delta$ by

$$\begin{pmatrix}\overline{\boldsymbol{s}}(u, \lambda)\\ \overline{\boldsymbol{t}}(u, \lambda)\end{pmatrix} = \begin{pmatrix}\cos\theta(u, \lambda) & -\sin\theta(u, \lambda)\\ \sin\theta(u, \lambda) & \cos\theta(u, \lambda)\end{pmatrix}\begin{pmatrix}\boldsymbol{s}(u, \lambda)\\ \boldsymbol{t}(u, \lambda)\end{pmatrix},$$

where $\theta : U \times \Lambda \to \mathbb{R}$ is a smooth function. Then $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is also a two-parameter family of framed surfaces. By a direct calculation, the basic invariants $\overline{a}_i, \overline{b}_i, \overline{e}_i, \overline{f}_i, \overline{g}_i, \overline{c}_1, \overline{c}_2$ of $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}})$ are given by

$$\begin{gathered}a_i\cos\theta - b_i\sin\theta, a_i\sin\theta + b_i\cos\theta, e_i\cos\theta - f_i\sin\theta,\\ e_i\sin\theta + f_i\cos\theta, g_i - \theta_{u_i}, c_1, c_2,\ i = 1, 2, 3, 4,\end{gathered}$$

where $u_3 = \lambda_1, u_4 = \lambda_2$. We call the moving frame $\{\boldsymbol{n}, \overline{\boldsymbol{s}}, \overline{\boldsymbol{t}}\}$ a rotated frame along $\boldsymbol{x}$ by θ.

On the other hand, we define $(\widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}) \in \Delta$ by

$$\begin{pmatrix}\widetilde{\boldsymbol{s}}(u, \lambda)\\ \widetilde{\boldsymbol{t}}(u, \lambda)\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}\cos\theta(u, \lambda) & -\sin\theta(u, \lambda)\\ \sin\theta(u, \lambda) & \cos\theta(u, \lambda)\end{pmatrix}\begin{pmatrix}\boldsymbol{s}(u, \lambda)\\ \boldsymbol{t}(u, \lambda)\end{pmatrix},$$

where $\theta : U \times \Lambda \to \mathbb{R}$ is a smooth function. Then $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ is also a two-parameter family of framed surfaces. By a direct calculation, the basic invariants $\widetilde{a}_i, \widetilde{b}_i, \widetilde{e}_i, \widetilde{f}_i, \widetilde{g}_i, \widetilde{c}_1, \widetilde{c}_2$ of $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}})$ are given by

$$\begin{aligned}&a_i\cos\theta - b_i\sin\theta, -a_i\sin\theta - b_i\cos\theta, e_i\cos\theta - f_i\sin\theta, -e_i\sin\theta - f_i\cos\theta,\\&- g_i + \theta_{u_i}, c_1, c_2,\ i = 1, 2, 3, 4,\end{aligned}$$

where $u_3 = \lambda_1, u_4 = \lambda_2$. We call the moving frame $\{\boldsymbol{n}, \widetilde{\boldsymbol{s}}, \widetilde{\boldsymbol{t}}\}$ a reflected frame along $\boldsymbol{x}$ by θ. By theorem 4.8, we have the following result.

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Proposition 4.9. *Under the above notations, if* $e : V \to U \times \Lambda$ *is a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$*, then* $e : V \to U \times \Lambda$ *is also a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \overline{\boldsymbol{s}})$ *and* $(\boldsymbol{x}, \boldsymbol{n}, \widetilde{\boldsymbol{s}})$.

It follows that the envelope is independent of rotated frames and reflected frames of the framed surfaces. Moreover, we demonstrate the envelope is independent of the parameter change of the framed surfaces.

Let $\widetilde{U}$ and $\widetilde{\Lambda}$ be simply connected domains in $\mathbb{R}^2$.

Definition 4.10. We say that a map $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ is a *two-parameter family of parameter change* if Φ is a diffeomorphism of the form $\Phi(v, k) = (\phi_1(v, k), \phi_2(v, k), \varphi_1(k), \varphi_2(k))$, where $v = (v_1, v_2),\ k = (k_1, k_2)$.

Then we have the following result by the similar calculation of proposition 3.16.

Proposition 4.11. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a two-parameter family of framed surfaces with the basic invariants* $a_i, b_i, e_i, f_i, g_i, c_1, c_2,\ i = 1, 2, 3, 4$*. Suppose that* $\Phi : \widetilde{U} \times \widetilde{\Lambda} \to U \times \Lambda$ *is a two-parameter family of parameter change. Then* $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}}) = (\boldsymbol{x} \circ \Phi, \boldsymbol{n} \circ \Phi, \boldsymbol{s} \circ \Phi) : \widetilde{U} \times \widetilde{\Lambda} \to \mathbb{R}^3 \times \Delta$ *is also a two-parameter family of framed surfaces with the basic invariants*

$$\begin{pmatrix} \widetilde{a}_1 & \widetilde{b}_1 & 0 \\ \widetilde{a}_2 & \widetilde{b}_2 & 0 \\ \widetilde{a}_3 & \widetilde{b}_3 & \widetilde{c}_1 \\ \widetilde{a}_4 & \widetilde{b}_4 & \widetilde{c}_2 \end{pmatrix} = \begin{pmatrix} \phi_{1v_1} & \phi_{2v_1} & 0 & 0 \\ \phi_{1v_2} & \phi_{2v_2} & 0 & 0 \\ \phi_{1k_1} & \phi_{2k_1} & \varphi_{1k_1} & \varphi_{2k_1} \\ \phi_{1k_2} & \phi_{2k_2} & \varphi_{1k_2} & \varphi_{2k_2} \end{pmatrix} \begin{pmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_1 \\ a_4 & b_4 & c_2 \end{pmatrix} \circ \Phi,$$

$$\begin{pmatrix} \widetilde{e}_1 & \widetilde{f}_1 & \widetilde{g}_1 \\ \widetilde{e}_2 & \widetilde{f}_2 & \widetilde{g}_2 \\ \widetilde{e}_3 & \widetilde{f}_3 & \widetilde{g}_3 \\ \widetilde{e}_4 & \widetilde{f}_4 & \widetilde{g}_4 \end{pmatrix} = \begin{pmatrix} \phi_{1v_1} & \phi_{2v_1} & 0 & 0 \\ \phi_{1v_2} & \phi_{2v_2} & 0 & 0 \\ \phi_{1k_1} & \phi_{2k_1} & \varphi_{1k_1} & \varphi_{2k_1} \\ \phi_{1k_2} & \phi_{2k_2} & \varphi_{1k_2} & \varphi_{2k_2} \end{pmatrix} \begin{pmatrix} e_1 & f_1 & g_1 \\ e_2 & f_2 & g_2 \\ e_3 & f_3 & g_3 \\ e_4 & f_4 & g_4 \end{pmatrix} \circ \Phi.$$

Moreover, if $e : V \to U \times \Lambda$ *is a pre-envelope,* E *is an envelope, then* $\Phi^{-1} \circ e : V \to \widetilde{U} \times \widetilde{\Lambda}$ *is a pre-envelope and* E *is also an envelope of* $(\widetilde{\boldsymbol{x}}, \widetilde{\boldsymbol{n}}, \widetilde{\boldsymbol{s}})$.

We can also give relations between the envelope E_I of a classical definition by using an implicit function (definition 4.5) and the envelope E of a two-parameter family of framed surfaces (definition 4.6). By the similar calculations of propositions 3.17–3.19, we have the following results.

Proposition 4.12. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a two-parameter family of framed surfaces. Suppose that* $F(w, \lambda) = 0$ *is an implicit function satisfying* $F(\boldsymbol{x}(u, \lambda), \lambda) = 0$ *and* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(u, \lambda), \lambda)$ *is parallel to* $\boldsymbol{n}(u, \lambda)$ *for all* $(u, \lambda) \in U \times \Lambda$*. If* $e : V \to U \times \Lambda$ *is a pre-envelope and* $E : V \to \mathbb{R}^3$ *is an envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$*, then* $E(V) \subset E_I$.

Proposition 4.13. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a two-parameter family of framed surfaces, and let* $e : V \to U \times \Lambda$ *be a smooth map satisfying the variability*

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condition. If $\mathrm{rank}(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2})(e(p)) = 2$ *for all* $p \in V$ *and trace of* e *lies in the singular set of* $\boldsymbol{x}$*, then* e *is a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *(and* E *is an envelope).*

PROPOSITION 4.14. *Let* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \times \Lambda \to \mathbb{R}^3 \times \Delta$ *be a two-parameter family of framed surfaces. Suppose that* $F(w, \lambda) = 0$ *is an implicit function satisfying* $F(\boldsymbol{x}(u, \lambda), \lambda) = 0$*,* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(u, \lambda), \lambda)$ *is parallel to* $\boldsymbol{n}(u, \lambda)$ *for all* $(u, \lambda) \in U \times \Lambda$*, and* $e : V \to U \times \Lambda$*,* $e(p) = (u(p), \lambda(p))$ *is a smooth mapping satisfying the variability condition. If* $E(p) = \boldsymbol{x} \circ e(p) \in E_I$ *with respect to* $\lambda(p)$*,* $\mathrm{rank}(\boldsymbol{x}_{u_1}, \boldsymbol{x}_{u_2})(e(p)) = 2$ *and* $(F_{w_1}, F_{w_2}, F_{w_3})(\boldsymbol{x}(e(p)), \lambda(p)) \neq (0, 0, 0)$ *for all* $p \in V$*, then* e *is a pre-envelope of* $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ *(and* E *is an envelope).*

5. Singular solutions of systems of first-order partial differential equations

As an application of the envelope theorem of one-parameter families of framed surfaces, we show that the projection of a singular solution of a system of first-order partial differential equations is an envelope.

We quickly review the theory of systems of first-order partial differential equations. For more details see [7]. We consider implicit function forms as differential equations and $n = 2, d = 2$ in [7]. A system of first-order partial differential equations (or, briefly, an equation) is a submersion germ $(F, G) : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}^2, 0)$ on the 1-jet space of functions of 2-variables. Let θ be a canonical contact 1-form on $J^1(\mathbb{R}^2, \mathbb{R})$ which is given by $\theta = \mathrm{d}y - p_1 \mathrm{d}x_1 - p_2 \mathrm{d}x_2$, where (x_1, x_2, y, p_1, p_2) is the canonical coordinate on $J^1(\mathbb{R}^2, \mathbb{R})$. Let $\pi : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2 \times \mathbb{R}, \pi(x_1, x_2, y, p_1, p_2) = (x_1, x_2, y)$ be the canonical projection. We define a *geometric solution* of $(F, G) = 0$ to be an immersion germ $i : (L, u_0) \to ((F, G)^{-1}(0), z_0)$ such that $i^*\theta = 0$, that is, a Legendre submanifold which is contained in $(F, G)^{-1}(0)$. We say that z_0 is a *contact singular point* if $\theta(T_{z_0}(F, G)^{-1}(0)) = 0$. By a direct calculation, z_0 is a contact singular point if and only if

$$\mathrm{rank}\begin{pmatrix} F_{x_1} + p_1 F_y & F_{x_2} + p_2 F_y & F_{p_1} & F_{p_2} \\ G_{x_1} + p_1 G_y & G_{x_2} + p_2 G_y & G_{p_1} & G_{p_2} \end{pmatrix} < 2$$

at z_0, see [7, proposition 1.2]. We denote the set of contact singular points by $\Sigma_c(F, G)$. We say that an equation $(F, G) = 0$ is *involutory* at $z \in ((F, G)^{-1}(0), z_0)$ if there is a Legendrian submanifold L tangent to $((F, G)^{-1}(0), z_0)$ at z. We say that an equation $(F, G) = 0$ is *involutory* if it is involutory at any point of $((F, G)^{-1}(0), z_0)$. Then $(F, G) = 0$ is involutory if and only if

$$\begin{aligned}[F, G] = FG_y - GF_y + F_{x_1} G_{p_1} - F_{p_1} G_{x_1} + F_{x_2} G_{p_2} - F_{p_2} G_{x_2} \\ + p_1(F_y G_{p_1} - G_y F_{p_1}) + p_2(F_y G_{p_2} - G_y F_{p_2}) = 0\end{aligned}$$

for any $z \in (F, G)^{-1}(0)$, see [7, 10].

Since single equations are automatically involutory, the notion of involutory is essential for overdetermined systems of first-order partial differential equations (cf. [10]).

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An equation $(F, G) = 0$ is said to be *completely integrable* at z_0 if there exists a foliation by geometric solutions on $(F, G)^{-1}(0)$ around z_0, that is, there exists an immersion germ $\Gamma : (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0)) \to ((F, G)^{-1}(0), z_0)$ such that $\Gamma(\cdot, c)$ is a geometric solution of $F = 0$ for each $c \in (\mathbb{R}, c_0)$. This means that Γ is a one-parameter family of Legendre immersions. In this case, such a foliation is called a *complete solution* of $(F, G) = 0$ at z_0.

A geometric solution $i : (L, u_0) \to ((F, G)^{-1}(0), z_0)$ of $(F, G) = 0$ is called a *singular solution* of $(F, G) = 0$ at z_0 if for any representative $\widetilde{i} : U \to (F, G)^{-1}(0)$ of i and any open subset $\widetilde{U} \subset U$, $\widetilde{i}|_{\widetilde{U}}$ is not contained in a leaf of any complete solutions of $(F, G) = 0$.

Then we have the following results.

THEOREM 5.1 [7]. *Let $(F, G) : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}^2, 0)$ be a system of first-order partial differential equation germs. Then $(F, G) = 0$ is completely integrable at z_0 if and only if $(F, G) = 0$ is involutory, and $\Sigma_c(F, G) = \emptyset$ or $\Sigma_c(F, G)$ is a 2-dimensional submanifold around z_0. Moreover, if $\Sigma_c(F, G) \neq \emptyset$, then $\Sigma_c(F, G)$ is a singular solution of $F = 0$ at z_0.*

PROPOSITION 5.2. *Let $(F, G) : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}^2, 0)$ be a system of first-order partial differential equation germs. Suppose that $\Gamma : (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0)) \to ((F, G)^{-1}(0), z_0)$ is a complete solution of $(F, G) = 0$ at z_0. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0)) \to \mathbb{R}^3 \times \Delta$ is a one-parameter family of framed surfaces, where*

$$\begin{aligned}
\boldsymbol{x}(u_1, u_2, c) &= \pi \circ \Gamma(u_1, u_2, c), \\
\boldsymbol{n}(u_1, u_2, c) &= \frac{(-p_1(u_1, u_2, c), -p_2(u_1, u_2, c), 1)}{\sqrt{p_1^2(u_1, u_2, c) + p_2^2(u_1, u_2, c) + 1}}, \\
\boldsymbol{s}(u_1, u_2, c) &= \frac{(0, 1, p_2(u_1, u_2, c))}{\sqrt{1 + p_2^2(u_1, u_2, c)}}
\end{aligned}$$

and

$$\Gamma(u_1, u_2, c) = (x_1(u_1, u_2, c), x_2(u_1, u_2, c), y(u_1, u_2, c), p_1(u_1, u_2, c), p_2(u_1, u_2, c)).$$

Proof. Since Γ is a complete solution of $(F, G) = 0$, we have $\Gamma_{u_1}^* \theta = 0$ and $\Gamma_{u_2}^* \theta = 0$ for fixed $c \in (\mathbb{R}, c_0)$, that is,

$$y_{u_i}(u_1, u_2, c) - p_1(u_1, u_2, c) x_{1u_i}(u_1, u_2, c) - p_2(u_1, u_2, c) x_{2u_i}(u_1, u_2, c) = 0,$$

$i = 1, 2$. It follows that $\boldsymbol{x}_{u_i}(u_1, u_2, c) \cdot \boldsymbol{n}(u_1, u_2, c) = 0$ for all $(u_1, u_2, c) \in (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0))$ and $i = 1, 2$. By definition, $\boldsymbol{n}(u_1, u_2, c) \cdot \boldsymbol{s}(u_1, u_2, c) = 0$ for all $(u_1, u_2, c) \in (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0))$. Hence, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a one-parameter family of framed surfaces. □

By using the envelope theorem (theorem 3.13), we have the following result.

THEOREM 5.3. *Under the same assumptions in proposition 5.2, $e : (\mathbb{R}^2, q_0) \to (\mathbb{R}^2 \times \mathbb{R}, (u_0, c_0))$ is a smooth mapping satisfying the variability condition. Then e is a pre-envelope and $E = \boldsymbol{x} \circ e$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ if and only if $E(q) \in \pi(\Sigma_c(F, G))$ for all $q \in (\mathbb{R}^2, q_0)$.*

Proof. By theorem 5.1, $\Sigma_c(F,G)$ is a 2-dimensional manifold around z_0 and a singular solution of $(F,G)=0$ at z_0. Since (F,G) is a submersion at $z_0 \in \Sigma_c(F,G)$, $F_y \neq 0$ or $G_y \neq 0$ at z_0. Without loss of generality, we may consider the following cases:

(1) $F_y \neq 0, \ G_{x_1} \neq 0$

(2) $F_y \neq 0, \ G_{p_1} \neq 0$

at z_0.

For the first case (1), we assume that

$$F(x_1,x_2,y,p_1,p_2) = -y + f(x_2,p_1,p_2), \ G(x_1,x_2,y,p_1,p_2) = -x_1 + g(x_2,p_1,p_2)$$

by using implicit function theorem. Then we have

$$\begin{aligned} y(u_1,u_2,c) &= f(x_2(u_1,u_2,c),p_1(u_1,u_2,c),p_2(u_1,u_2,c)),\\ x_1(u_1,u_2,c) &= g(x_2(u_1,u_2,c),p_1(u_1,u_2,c),p_2(u_1,u_2,c)). \end{aligned} \tag{5.1}$$

It follows that

$$y_c = f_{x_2}x_{2c} + f_{p_1}p_{1c} + f_{p_2}p_{2c}, \ x_{1c} = g_{x_2}x_{2c} + g_{p_1}p_{1c} + g_{p_2}p_{2c}.$$

Then we have

$$\begin{aligned} \boldsymbol{x}_c \cdot \boldsymbol{n} &= \frac{1}{\sqrt{p_1^2+p_2^2+1}} (-p_1(g_{x_2}x_{2c} + g_{p_1}p_{1c} + g_{p_2}p_{2c}) \\ &\quad -p_2x_{2c} + f_{x_2}x_{2c} + f_{p_1}p_{1c} + f_{p_2}p_{2c}) \\ &= \frac{1}{\sqrt{p_1^2+p_2^2+1}} ((-p_1g_{x_2} - p_2 + f_{x_2})x_{2c} \\ &\quad +(-p_1g_{p_1} + f_{p_1})p_{1c} + (-p_1g_{p_2} + f_{p_2})p_{2c}). \end{aligned}$$

If e is a pre-envelope and $E = \boldsymbol{x} \circ e$ is an envelope of $(\boldsymbol{x},\boldsymbol{n},\boldsymbol{s})$, then

$$((-p_1g_{x_2} - p_2 + f_{x_2})x_{2c} + (-p_1g_{p_1} + f_{p_1})p_{1c} + (-p_1g_{p_2} + f_{p_2})p_{2c})(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$ by theorem 3.13. Moreover, since Γ is a complete solution and equations (5.1), we have

$$\begin{aligned} & y_{u_1} = p_1x_{1u_1} + p_2x_{2u_1}, \ y_{u_2} = p_1x_{1u_2} + p_2x_{2u_2}, \\ & y_{u_1} = f_{x_2}x_{2u_1} + f_{p_1}p_{1u_1} + f_{p_2}p_{2u_1}, \ y_{u_2} = f_{x_2}x_{2u_2} + f_{p_1}p_{1u_2} + f_{p_2}p_{2u_2}, \\ & x_{1u_1} = g_{x_2}x_{2u_1} + g_{p_1}p_{1u_1} + g_{p_2}p_{2u_1}, \ x_{1u_2} = g_{x_2}x_{2u_2} + g_{p_1}p_{1u_2} + g_{p_2}p_{2u_2}. \end{aligned}$$

It follows that

$$\begin{aligned} & (-p_1g_{x_2} - p_2 + f_{x_2})x_{2u_1} + (-p_1g_{p_1} + f_{p_1})p_{1u_1} + (-p_1g_{p_2} + f_{p_2})p_{2u_1} = 0, \\ & (-p_1g_{x_2} - p_2 + f_{x_2})x_{2u_2} + (-p_1g_{p_1} + f_{p_1})p_{1u_2} + (-p_1g_{p_2} + f_{p_2})p_{2u_2} = 0. \end{aligned}$$

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Hence, we have

$$\begin{pmatrix} x_{2u_1} & p_{1u_1} & p_{2u_1} \\ x_{2u_2} & p_{1u_2} & p_{2u_2} \\ x_{2c} & p_{1c} & p_{2c} \end{pmatrix} \left(-p_1 g_{x_2} - p_2 + f_{x_2} - p_1 g_{p_1} + f_{p_1} - p_1 g_{p_2} + f_{p_2}\right)(e(q))$$
$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ is an immersion, we have

$$(-p_1 g_{x_2} - p_2 + f_{x_2})(e(q)) = 0, (-p_1 g_{p_1} + f_{p_1})(e(q)) = 0, (-p_1 g_{p_2} + f_{p_2})(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$. By the definition of the contact singular set $\Sigma_c(F, G)$, if $z \in \Sigma_c(F, G)$, then

$$\operatorname{rank} \begin{pmatrix} -p_1 & f_{x_2} - p_2 & f_{p_1} & f_{p_2} \\ -1 & g_{x_2} & g_{p_1} & g_{p_2} \end{pmatrix} (z) < 2.$$

It follows that $E(q) \in \pi(\Sigma_c(F, G))$ for all $q \in (\mathbb{R}^2, q_0)$.

Conversely, if $E(q) \in \pi(\Sigma_c(F, G))$ for all $q \in (\mathbb{R}^2, q_0)$, then $\boldsymbol{x}_c(e(q)) \cdot \boldsymbol{n}(e(q)) = 0$ for all $q \in (\mathbb{R}^2, 0)$. By theorem 3.13, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$.

For the cases (2), we assume that

$$F(x_1, x_2, y, p_1, p_2) = -y + f(x_1, x_2, p_2),\ G(x_1, x_2, y, p_1, p_2) = -p_1 + g(x_1, x_2, p_2)$$

by using implicit function theorem. Then we have

$$\begin{aligned} y(u_1, u_2, c) &= f(x_1(u_1, u_2, c), x_2(u_1, u_2, c), p_2(u_1, u_2, c)), \\ p_1(u_1, u_2, c) &= g(x_1(u_1, u_2, c), x_2(u_1, u_2, c), p_2(u_1, u_2, c)). \end{aligned} \tag{5.2}$$

It follows that

$$y_c = f_{x_1} x_{1c} + f_{x_2} x_{2c} + f_{p_2} p_{2c}.$$

Then we have

$$\begin{aligned} \boldsymbol{x}_c \cdot \boldsymbol{n} &= \frac{1}{\sqrt{p_1^2 + p_2^2 + 1}} \left(-p_1 x_{1c} - p_2 x_{2c} + f_{x_1} x_{1c} + f_{x_2} x_{2c} + f_{p_2} p_{2c}\right) \\ &= \frac{1}{\sqrt{p_1^2 + p_2^2 + 1}} \left((-p_1 + f_{x_1}) x_{1c} + (-p_2 + f_{x_2}) x_{2c} + f_{p_2} p_{2c}\right). \end{aligned}$$

If e is a pre-envelope and $E = \boldsymbol{x} \circ e$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, then

$$\left((-p_1 + f_{x_1}) x_{1c} + (-p_2 + f_{x_2}) x_{2c} + f_{p_2} p_{2c}\right)(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$ by theorem 3.13. Moreover, since Γ is a complete solution and equations (5.2), we have

$$\begin{aligned} &y_{u_1} = p_1 x_{1u_1} + p_2 x_{2u_1},\ y_{u_2} = p_1 x_{1u_2} + p_2 x_{2u_2}, \\ &y_{u_1} = f_{x_1} x_{1u_1} + f_{x_2} x_{2u_1} + f_{p_2} p_{2u_1},\ y_{u_2} = f_{x_1} x_{1u_2} + f_{x_2} x_{2u_2} + f_{p_2} p_{2u_2}. \end{aligned}$$

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It follows that

$$(-p_1 + f_{x_1})x_{1u_1} + (-p_2 + f_{x_2})x_{2u_1} + f_{p_2}p_{2u_1} = 0,$$
$$(-p_1 + f_{x_1})x_{1u_2} + (-p_2 + f_{x_2})x_{2u_2} + f_{p_2}p_{2u_2} = 0.$$

Hence, we have

$$\begin{pmatrix} x_{1u_1} & x_{2u_1} & p_{2u_1} \\ x_{1u_2} & x_{2u_2} & p_{2u_2} \\ x_{1c} & x_{2c} & p_{2c} \end{pmatrix} \begin{pmatrix} -p_1 + f_{x_1} - p_2 + f_{x_2} \\ f_{p_2} \end{pmatrix} (e(q)) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ is an immersion, we have

$$(-p_1 + f_{x_1})(e(q)) = 0, (-p_2 + f_{x_2})(e(q)) = 0, f_{p_2}(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$. By definition of the contact singular set $\Sigma_c(F,G)$, if $z \in \Sigma_c(F,G)$, then

$$\text{rank} \begin{pmatrix} f_{x_1} - p_1 & f_{x_2} - p_2 & 0 & f_{p_2} \\ g_{x_1} & g_{x_2} & -1 & g_{p_2} \end{pmatrix} (z) < 2.$$

It follows that $E(q) \in \pi(\Sigma_c(F,G))$ for all $q \in (\mathbb{R}^2, q_0)$.

Conversely, if $E(q) \in \pi(\Sigma_c(F,G))$ for all $q \in (\mathbb{R}^2, q_0)$, then $\boldsymbol{x}_c(e(q)) \cdot \boldsymbol{n}(e(q)) = 0$ for all $q \in (\mathbb{R}^2, 0)$. By theorem 3.13, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. $\square$

Example 5.4. Let $(F,G) : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2$ be

$$F(x_1, x_2, y, p_1, p_2) = -y + \left(\frac{2}{3}p_1\right)^3, \ G(x_1, x_2, y, p_1, p_2) = p_2.$$

Then a complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R} \to (F,G)^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c) = \left(u_1^2 + c, u_2, u_1^3, \frac{3}{2}u_1, 0\right).$$

By proposition 5.2, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u_1, u_2, c) = (u_1^2 + c, u_2, u_1^3), \boldsymbol{n}(u_1, u_2, c)$$
$$= \frac{1}{\sqrt{9u_1^2 + 4}}(-3u_1, 0, 2), \boldsymbol{s}(u_1, u_2, c) = (0, 1, 0)$$

is a one-parameter family of framed surfaces. In this case, $\Sigma_c(F,G) = \{(x_1, x_2, 0, 0, 0)\}$ is a 2-dimensional manifold. By theorem 5.3, xy-plane is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, see example 3.20.

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EXAMPLE 5.5. Let $(F, G) : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}^2$ be

$$F(x_1, x_2, y, p_1, p_2) = -x_1 + \left(\frac{2}{3}p_1\right)^2, \; G(x_1, x_2, y, p_1, p_2) = p_2.$$

Then a complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R} \to (F, G)^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c) = \left(u_1^2, u_2, u_1^3 + c, \frac{3}{2}u_1, 0\right).$$

By proposition 5.2, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^3 \times \Delta$,

$$\begin{aligned}\boldsymbol{x}(u_1, u_2, c) &= (u_1^2, u_2, u_1^3 + c), \boldsymbol{n}(u_1, u_2, c)\\ &= \frac{1}{\sqrt{9u_1^2 + 4}}(-3u_1, 0, 2), \boldsymbol{s}(u_1, u_2, c) = (0, 1, 0)\end{aligned}$$

is a one-parameter family of framed surfaces. In this case, $\Sigma_c(F, G) = \emptyset$. Hence, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ does not have the envelope E by theorem 5.3, see example 3.21.

6. Singular solutions of single first-order partial differential equations

As an application of the envelope theorem of two-parameter families of framed surfaces, we show that the projection of a singular solution of a single first-order partial differential equation is an envelope. In [**13**], under a condition, it could be proved that the projection of a singular solution of a single completely integrable first-order partial differential equation is an envelope. However, we can prove without the condition $\Sigma_c(F) = \Sigma_\pi(F)$ in this paper.

We quickly review the theory of first-order partial differential equations. For more details see [**8**].

An equation is a submersion germ $F : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$. We define a *geometric solution* of $F = 0$ to be an immersion germ $i : (L, u_0) \to (F^{-1}(0), z_0)$ of a 2-dimensional manifold such that $i^*\theta = 0$, that is, a Legendre submanifold which is contained in $F^{-1}(0)$. We say that z_0 is a *contact singular point* if $\theta(T_{z_0}F^{-1}(0)) = 0$. It is easy to see that z_0 is a contact singular point if and only if $F = F_{p_1} = F_{p_2} = F_{x_1} + p_1F_y = F_{x_2} + p_2F_y = 0$ at z_0. We also say that z_0 is a π-*singular point* if $F = F_{p_1} = F_{p_2} = 0$ at z_0. We denote the set of contact singular points by $\Sigma_c(F)$, the set of π-singular points by $\Sigma_\pi(F)$.

An equation $F = 0$ is said to be *completely integrable* at z_0 if there exists a foliation by geometric solution on $F^{-1}(0)$ around z_0, that is, there exists an immersion germ $\Gamma : (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0)) \to (F^{-1}(0), z_0)$ such that $\Gamma(\cdot, c_1, c_2)$ is a geometric solution of $F = 0$ for each $(c_1, c_2) \in (\mathbb{R}^2, c_0)$. This means that Γ is a two-parameter family of Legendre immersions. In this case, such a foliation is called a *complete solution* of $F = 0$ at z_0.

A geometric solution $i : (L, u_0) \to (F^{-1}(0), z_0)$ of $F = 0$ is called a *singular solution* of $F = 0$ at z_0 if for any representative $\widetilde{i} : U \to F^{-1}(0)$ of i and any open subset $\widetilde{U} \subset U$, $\widetilde{i}|_{\widetilde{U}}$ is not contained in a leaf of any complete solutions of $F = 0$.

Then we have the following results.

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THEOREM 6.1 [8]. *Let $F : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ be a first-order partial differential equation germs. Then $F = 0$ is completely integrable at z_0 if and only if $\Sigma_c(F) = \emptyset$ or $\Sigma_c(F)$ is a 2-dimensional submanifold around z_0. Moreover, if $\Sigma_c(F) \neq \emptyset$, then $\Sigma_c(F)$ is a singular solution of $F = 0$ at z_0.*

PROPOSITION 6.2. *Let $F : (J^1(\mathbb{R}^2, \mathbb{R}), z_0) \to (\mathbb{R}, 0)$ be a first-order partial differential equation germs. Suppose that $\Gamma : (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0)) \to (F^{-1}(0), z_0)$ is a complete solution of $F = 0$ at z_0. Then $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0)) \to \mathbb{R}^3 \times \Delta$ is a two-parameter family of framed surfaces, where*

$$
\begin{aligned}
\boldsymbol{x}(u_1, u_2, c_1, c_2) &= \pi \circ \Gamma(u_1, u_2, c_1, c_2),\\
\boldsymbol{n}(u_1, u_2, c_1, c_2) &= \frac{(-p_1(u_1, u_2, c_1, c_2), -p_2(u_1, u_2, c_1, c_2), 1)}{\sqrt{p_1^2(u_1, u_2, c_1, c_2) + p_2^2(u_1, u_2, c_1, c_2) + 1}},\\
\boldsymbol{s}(u_1, u_2, c_1, c_2) &= \frac{(0, 1, p_2(u_1, u_2, c_1, c_2))}{\sqrt{1 + p_2^2(u_1, u_2, c_1, c_2)}}
\end{aligned}
$$

and

$$
\begin{aligned}
\Gamma(u_1, u_2, c_1, c_2) = (&x_1(u_1, u_2, c_1, c_2), x_2(u_1, u_2, c_1, c_2), y(u_1, u_2, c_1, c_2),\\
&p_1(u_1, u_2, c_1, c_2), p_2(u_1, u_2, c_1, c_2)).
\end{aligned}
$$

Proof. Since Γ is a complete solution of $F = 0$, we have $\Gamma^*_{u_1}\theta = 0$ and $\Gamma^*_{u_2}\theta = 0$ for fixed $(c_1, c_2) \in (\mathbb{R}^2, c_0)$, that is,

$$
\begin{aligned}
&y_{u_i}(u_1, u_2, c_1, c_2) - p_1(u_1, u_2, c_1, c_2)x_{1u_i}(u_1, u_2, c_1, c_2)\\
&\quad - p_2(u_1, u_2, c_1, c_2)x_{2u_i}(u_1, u_2, c_1, c_2) = 0,
\end{aligned}
$$

$i = 1, 2$. It follows that $\boldsymbol{x}_{u_i}(u_1, u_2, c_1, c_2) \cdot \boldsymbol{n}(u_1, u_2, c_1, c_2) = 0$ for all $(u_1, u_2, c_1, c_2) \in (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0))$ and $i = 1, 2$. By definition, $\boldsymbol{n}(u_1, u_2, c_1, c_2) \cdot \boldsymbol{s}(u_1, u_2, c_1, c_2) = 0$ for all $(u_1, u_2, c_1, c_2) \in (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0))$. Hence, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a two-parameter family of framed surfaces. □

By using the envelope theorem (theorem 4.8), we have the following result.

THEOREM 6.3. *Under the same assumptions in proposition 6.2, suppose that $e : (\mathbb{R}^2, q_0) \to (\mathbb{R}^2 \times \mathbb{R}^2, (u_0, c_0))$ is a smooth mapping satisfying the variability condition. Then e is a pre-envelope and $E = \boldsymbol{x} \circ e$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ if and only if $E(q) \in \pi(\Sigma_c(F))$ for all $q \in (\mathbb{R}^2, q_0)$.*

Proof. By theorem 6.1, $\Sigma_c(F)$ is a 2-dimensional manifold around z_0 and a singular solution of $F = 0$ at z_0. Since F is a submersion at $z_0 \in \Sigma_c(F)$, $F_y \neq 0$ at z_0. Therefore, we may assume that

$$F(x_1, x_2, y, p_1, p_2) = -y + f(x_1, x_2, p_1, p_2)$$

by implicit function theorem. Since

$$
\begin{aligned}
y(u_1, u_2, c_1, c_2) = f(&x_1(u_1, u_2, c_1, c_2), x_2(u_1, u_2, c_1, c_2),\\
&p_1(u_1, u_2, c_1, c_2), p_2(u_1, u_2, c_1, c_2)),
\end{aligned}
\tag{6.1}
$$

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we have $y_{c_i} = f_{x_1}x_{1c_i} + f_{x_2}x_{2c_i} + f_{p_1}p_{1c_i} + f_{p_2}p_{2c_i}$, and hence

$$\boldsymbol{x}_{c_i} \cdot \boldsymbol{n} = \frac{1}{\sqrt{p_1^2 + p_2^2 + 1}} \left((-p_1 + f_{x_1})x_{1c_i} + (-p_2 + f_{x_2})x_{2c_i} + f_{p_1}p_{1c_i} + f_{p_2}p_{2c_i}\right),$$

$i = 1, 2$. If e is a pre-envelope and $E = \boldsymbol{x} \circ e$ is an envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, then

$$\left((-p_1 + f_{x_1})x_{1c_i} + (-p_2 + f_{x_2})x_{2c_i} + f_{p_1}p_{1c_i} + f_{p_2}p_{2c_i}\right)(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$, $i = 1, 2$ by theorem 4.8. Moreover, since Γ is a complete solution and equation (6.1), we have

$$\begin{aligned} y_{u_1} &= p_1 x_{1u_1} + p_2 x_{2u_1}, \ y_{u_2} = p_1 x_{1u_2} + p_2 x_{2u_2}, \\ y_{u_1} &= f_{x_1}x_{1u_1} + f_{x_2}x_{2u_1} + f_{p_1}p_{1u_1} + f_{p_2}p_{2u_1}, \\ y_{u_2} &= f_{x_1}x_{1u_2} + f_{x_2}x_{2u_2} + f_{p_1}p_{1u_2} + f_{p_2}p_{2u_2}. \end{aligned}$$

It follow that

$$(-p_1 + f_{x_1})x_{1u_i} + (-p_2 + f_{x_2})x_{2u_i} + f_{p_1}p_{1u_i} + f_{p_2}p_{2u_i} = 0,$$

$i = 1, 2$. Hence, we have

$$\begin{pmatrix} x_{1u_1} & x_{2u_1} & p_{1u_1} & p_{2u_1} \\ x_{1u_2} & x_{2u_2} & p_{1u_2} & p_{2u_2} \\ x_{1c_1} & x_{2c_1} & p_{1c_1} & p_{2c_1} \\ x_{1c_2} & x_{2c_2} & p_{1c_2} & p_{2c_2} \end{pmatrix} \begin{pmatrix} -p_1 + f_{x_1} - p_2 + f_{x_2} \\ f_{p_1} \\ f_{p_2} \end{pmatrix} (e(q)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Since Γ is an immersion, we have

$$(-p_1 + f_{x_1})(e(q)) = 0, (-p_2 + f_{x_2})(e(q)) = 0, f_{p_1}(e(q)) = 0, f_{p_2}(e(q)) = 0$$

for all $q \in (\mathbb{R}^2, q_0)$. It follows that $E(q) \in \pi(\Sigma_c(F))$ for all $q \in (\mathbb{R}^2, q_0)$.

Conversely, if $E(q) \in \pi(\Sigma_c(F))$ for all $q \in (\mathbb{R}^2, q_0)$, then $\boldsymbol{x}_{c_i}(e(q)) \cdot \boldsymbol{n}(e(q)) = 0$ for all $q \in (\mathbb{R}^2, q_0)$, $i = 1, 2$. By theorem 4.8, e is a pre-envelope of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$. □

For concrete examples of completely integrable first-order partial differential equations and their envelopes see [**13**]. However, these examples satisfied the condition $\Sigma_c(F) = \Sigma_\pi(F) = \{z \in J^1(\mathbb{R}^2, \mathbb{R}) | F(z) = F_{p_1}(z) = F_{p_2}(z) = 0\}$. Here we give an example that $\Sigma_c(F) \neq \Sigma_\pi(F)$.

Example 6.4. Let $F : J^1(\mathbb{R}^2, \mathbb{R}) \to \mathbb{R}$ be $F(x_1, x_2, y, p_1, p_2) = -y + p_1^n = 0$, where $n \geqslant 2$ is a natural number. By a direct calculation, we have

$$\Sigma_c(F) = \{(x_1, x_2, 0, 0, 0)\} \subset \Sigma_\pi(F) = \{(x_1, x_2, 0, 0, p_2)\}.$$

A complete solution $\Gamma : \mathbb{R}^2 \times \mathbb{R}^2 \to F^{-1}(0)$ is given by

$$\Gamma(u_1, u_2, c_1, c_2) = \left(\frac{n}{n-1}u_1^{n-1} + c_1, c_2, u_1^n, u_1, u_2\right).$$

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By proposition 6.2, $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^3 \times \Delta$,

$$\boldsymbol{x}(u_1, u_2, c_1, c_2) = \left(\frac{n}{n-1} u_1^{n-1} + c_1, c_2, u_1^n \right),$$
$$\boldsymbol{n}(u_1, u_2, c_1, c_2) = \frac{(-u_1, -u_2, 1)}{\sqrt{u_1^2 + u_2^2 + 1}},$$
$$\boldsymbol{s}(u_1, u_2, c_1, c_2) = \frac{(0, 1, u_2)}{\sqrt{1 + u_2^2}}$$

is a two-parameter family of framed surfaces. Since $\boldsymbol{x}_{c_1} \cdot \boldsymbol{n} = \boldsymbol{x}_{c_2} \cdot \boldsymbol{n} = 0$ if and only if $u_1 = u_2 = 0$, we take a smooth mapping $e : \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2, e(q_1, q_2) = (0, 0, q_1, q_2)$. Then e satisfies the variability condition and hence a pre-envelope by theorem 4.8. It follows that the envelope $E : \mathbb{R}^2 \to \mathbb{R}^3$ of $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is given by $E(q_1, q_2) = \boldsymbol{x} \circ e(q_1, q_2) = (q_1, q_2, 0) \in \pi(\Sigma_c(F))$, see theorem 6.3.

Conflict of interest

The authors declare that there is no conflicts of interests in this work.

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