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THE LOCAL CYCLICITY PROBLEM: MELNIKOV METHOD USING LYAPUNOV CONSTANTS

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Abstract In 1991, Chicone and Jacobs showed the equivalence between the computation of the first-order Taylor developments of the Lyapunov constants and the developments of the first Melnikov function near a non-degenerate monodromic equilibrium point, in the study of limit cycles of small-amplitude bifurcating from a quadratic centre. We show that their proof is also valid for polynomial vector fields of any degree. This equivalence is used to provide a new lower bound for the local cyclicity of degree six polynomial vector fields, so $\mathcal{M}(6) \geq 44$. Moreover, we extend this equivalence to the piecewise polynomial class. Finally, we prove that $\mathcal{M}_p^c(4) \geq 43$ and $\mathcal{M}_p^c(5) \geq 65$.

Keywords: Melnikov theory; Lyapunov constants; local cyclicity

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1. Introduction

In the last century, Hilbert presented a list of problems that almost all of them are solved. One problem that remains open consists in determining the maximal number $\mathcal{H}(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree n. This problem is known as the second part of the 16th Hilbert's problem. In the year of 1977, Arnol'd in [4] proposed a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonian systems.

In this work, we are interested in another local version, that consists in providing the maximum number $\mathcal{M}(n)$ of small amplitude limit cycles bifurcating from an elementary centre or an elementary focus, clearly $\mathcal{M}(n) \leq \mathcal{H}(n)$. In other words, $\mathcal{M}(n)$ is an upper bound of the (local) cyclicity of such equilibrium points. For more details, see [49]. For $n = 2$, Bautin proved that $\mathcal{M}(2) = 3$, see [5]. For $n = 3$, the family of cubic systems without quadratic terms was studied in [7, 51]. The proof of $\mathcal{M}_h(3) = 5$ can be found in [54]. Żołądek in [55] shows the first evidence that $\mathcal{M}(3) \geq 11$. However, this problem has been recently revisited by himself in [57]. The first proof of this fact was obtained some years later by Christopher in [14], studying first-order perturbations of another cubic centre also provided by Żołądek in [56].

In 2012, Giné conjectures that $\mathcal{M}(n) = n^2 + 3n - 7$, see [25, 26]. This suggests a higher value for $\mathcal{M}(n)$ for polynomial vector fields of low degree. Gouveia and Torregrosa in [30] show that $\mathcal{M}(5) \geq 33$, $\mathcal{M}(7) \geq 61$, $\mathcal{M}(8) \geq 76$, and $\mathcal{M}(9) \geq 88$. The first evidence that this conjecture fails is given in [53]. Recently, a complete proof that $\mathcal{M}(3) \geq 12$ is provided by Giné, Gouveia, and Torregrosa in [27]. Moreover, in the same paper, they show that $\mathcal{M}(4) \geq 21$. For $n = 6$, the best lower bound was given in [42], proving that $\mathcal{M}(6) \geq 40$. The first main result of this paper updates this value.

Theorem 1.1. *The local cyclicity of monodromic equilibrium points for polynomial vector fields of degree* $n = 6$ *is greater than or equal to* 44*. That is,* $\mathcal{M}(6) \geq 44$.

In this work, we are also interested in piecewise polynomial vector fields. Andronov, in [2] was the first to study such class of systems. In the last years, they have been widely studied, since many problems of engineering, physics, and biology can be modelled by them, see [1, 6]. One of the most studied situations in the plane is given by two vector fields defined in two half-planes separated by a straight line. As in the case of the classical qualitative theory of polynomial systems, the study of the number and location of the isolated periodic orbits, also called limit cycles, have received special attention, see for example [11, 17, 24, 34, 41, 43]. In particular, it can be seen as an extension of the 16th-Hilbert problem for planar piecewise polynomial vector fields, see [38].

Here, we study limit cycles bifurcating from the origin in the class of piecewise differential equations of the form

$$\begin{cases} (\dot{x}, \dot{y}) & = & (P^+(x, y, \lambda), Q^+(x, y, \lambda)), \text{ when } y \geq 0, \\ (\dot{x}, \dot{y}) & = & (P^-(x, y, \lambda), Q^-(x, y, \lambda)), \text{ when } y < 0, \end{cases}$$

where $P^{\pm}(x, y, \lambda)$ and $Q^{\pm}(x, y, \lambda)$ are polynomials in x and y. The straight line $\Sigma = \{y = 0\}$ divides the plane in two half-planes, $\Sigma^{\pm} = \{(x, y) : \pm y > 0\}$, and the trajectories on Σ are defined following the Filippov convention, see [20]. Here we will consider only limit cycles of crossing type, that is, when both vector fields point out in the same direction on the separation line Σ. Then, we denote by $\mathcal{H}_p^c(n)$ the number of limit cycles of crossing type for piecewise polynomial vector fields of degree n. Similarly, $\mathcal{M}_p^c(n)$ counts the ones of small amplitude bifurcating from a monodromic equilibrium point. The upper index c means for crossing limit cycles and the subscript p for piecewise class. Clearly, $\mathcal{M}(n) \leq \mathcal{H}(n)$ and $\mathcal{M}_p^c(n) \leq \mathcal{H}_p^c(n)$.

For $n = 1$, Huan and Yang in [37] show a numerical evidence that $\mathcal{H}_p^c(1) \geq 3$. Llibre and Ponce in [43] provide an analytical proof of this property. Using the averaging technique, this lower bound was reobtained by Buzzi, Pessoa, and Torregrosa in [11]. Freire, Ponce, and Torres also obtained the same number in [21]. In this last work, the three limit cycles are explained by studying the full return map, two appear near the origin and the other one far from it. But the three limit cycles appear nested and surrounding one sliding segment. In fact, the two limit cycles of small amplitude appearing from an equilibrium

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point provide the lower bound $\mathcal{M}_p^c(1) \geq 2$. This value can be proved with the results in [21] and we will recall it in the next section. Recently in [23], the three limit cycles have been obtained near infinity in a Hopf type bifurcation.

For $n = 2$, using averaging theory of fifth-order, and perturbing the linear centre, Llibre and Tang in [44] proved that $\mathcal{H}_p^c(2) \geq 8$. Recently, da Cruz, Novaes, and Torregrosa in [18] improve this lower bound and, using local developments of the difference map, Gouveia and Torregrosa in [29] provide the first high lower bounds for piecewise polynomial classes of degrees three, four and five: $\mathcal{M}_p^c(3) \geq 26$, $\mathcal{M}_p^c(4) \geq 40$, and $\mathcal{M}_p^c(5) \geq 58$. Here we update two of these lower bounds using averaging theory in our second main result.

Theorem 1.2. *The local cyclicity for piecewise polynomial vector fields of degree four and five is* $\mathcal{M}_p^c(4) \geq 43$ *and* $\mathcal{M}_p^c(5) \geq 65$, *respectively.*

In this paper, we are interesting also in differential equations containing a privileged ε that acts as a small perturbation of a periodic behaviour. Lagrange, in his study about the three-body problem, formulated the idea of averaging. During some years, this method was used in many fields without people bothering about proofs validity. In 1928, Fatou in [19] gave the first analytic proof. Nowadays, in planar differential systems, the averaging theory is also used for proving the existence of limit cycles which appear, after perturbation, from a fully periodic neighbourhood. A similar mechanism, when the differential equation is nonautonomous, used for the same purpose is the Melnikov method. Moreover, it is an excellent tool for studying global bifurcations that occur near homoclinic or heteroclinic loops in one-parameter families.

Let us consider the perturbation of a Hamiltonian system

$$\begin{cases} x' = -H_y + \varepsilon\, P(x, y, \varepsilon, \lambda), \\ y' = H_x + \varepsilon\, Q(x, y, \varepsilon, \lambda). \end{cases} \tag{1}$$

Then, the first Melnikov function writes as

$$M(h) = \int_{\Gamma_h} Q(x, y, 0, \lambda)dx - P(x, y, 0, \lambda)dy, \tag{2}$$

where $\Gamma_h = \{H(x,\, y) = h^2\}$ are closed ovals. This first-order analysis is based on the Implicit Function Theorem. In fact, for ε small enough, the simple zeros of $M(h)$ correspond to limit cycles of the perturbed system (1). That is, if h^* satisfies that $M(h^*) = 0$ and $M'(h^*) \neq 0$ then, there exists an hyperbolic limit cycle of (1) that goes to Γ_{h^*} when h goes to h^*.

This result can be generalized also for the perturbation of centres, $(\dot{x},\, \dot{y}) = (P_c,\, Q_c)$, having an inverse integrating factor $V(x,\, y)$. Then after a time rescaling the perturbed system

$$\begin{cases} \dot{x} = P_c(x, y) + \varepsilon\, P(x, y, \varepsilon, \lambda), \\ \dot{y} = Q_c(x, y) + \varepsilon\, Q(x, y, \varepsilon, \lambda), \end{cases} \tag{3}$$

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can be written as

$$\begin{cases} x' = -H_y + \varepsilon \dfrac{P(x,y,\varepsilon,\lambda)}{V(x,y)}, \\ y' = H_x + \varepsilon \dfrac{Q(x,y,\varepsilon,\lambda)}{V(x,y)}. \end{cases} \tag{4}$$

So, the corresponding generalized first Melnikov function of (4), or also of (3), is

$$M(h) = \int_{\Gamma_h} \frac{Q(x,y,0,\lambda)dx - P(x,y,0,\lambda)dy}{V(x,y)}. \tag{5}$$

In this paper, we only consider the above Melnikov function for the study of perturbations of periodic orbits near planar autonomous differential systems. In this case, there are some works explaining the equivalence between Melnikov studies and the averaging theory, see [9, 35]. For more details in this theory, we refer the reader to [10, 33, 50, 52].

This bifurcation mechanism can be used also in piecewise differential equations, see [45–47]. Although this problem can be treated in a more general way, in this work, we will consider a simplified version of it. We will study that only the perturbation in (3) is defined by piecewise functions. This perturbation will be defined in two zones separated by a straight line, that is, the differential equation to consider can be written as

$$(\dot{x},\dot{y}) = \begin{cases} (P_c(x,y), Q_c(x,y)) + \varepsilon\,(P^+(x,y,\varepsilon,\lambda), Q^+(x,y,\varepsilon,\lambda)), & \text{if } y \geq 0, \\ (P_c(x,y), Q_c(x,y)) + \varepsilon\,(P^-(x,y,\varepsilon,\lambda), Q^-(x,y,\varepsilon,\lambda)), & \text{if } y < 0. \end{cases} \tag{6}$$

Therefore, the piecewise generalized first Melnikov function of (6) is

$$M_p(h) = \frac{1}{H_x^+(x_0,0)} M^+(h) + \frac{1}{H_x^-(x_0,0)} M^-(h), \tag{7}$$

where

$$M^{\pm}(h) = \int_{\Gamma_h^{\pm}} \frac{Q^{\pm}(x,y,0,\lambda)dx - P^{\pm}(x,y,0,\lambda)dy}{V(x,y)}$$

with $\Gamma_h^{\pm} = \Gamma_h \cap \{\pm y > 0\}(x_0,0)\Gamma_h y = 0 x_0 > 0.\Gamma_h^{\pm} = \Gamma_h \cap \{\pm y > 0\}(x_0,0)\Gamma_h y = 0 x_0 > 0.\Gamma_h^{\pm} = \Gamma_h \cap \{\pm y > 0\}(x_0,0)\Gamma_h y = 0 x_0 > 0.\Gamma_h^{\pm} = \Gamma_h \cap \{\pm y > 0\}(x_0,0)\Gamma_h y = 0 x_0 > 0.\Gamma_h^{\pm} = \Gamma_h \cap \{\pm y > 0\}(x_0,0)\Gamma_h y = 0 x_0 > 0$. Clearly, in both piecewise regions, we have the same centre having $V(x, y)$ as an inverse integrating factor. The main results of this work could be extended to perturbing piecewise centres but this is not the aim of this paper.

From the mechanism described above, the functions (2), (5), and (7) provide the number of limit cycles bifurcating from the period annulus up to first-order analysis. Another tool for obtaining limit cycles bifurcating from the origin comes from the study of the Taylor development of the return map, the so-called *Lyapunov constants*. In this context, the perturbation terms of the non-degenerate linear centre, $(\dot{x}, \dot{y}) = (-y, x)$, usually have degree at least two because in this case, the Lyapunov constants are polynomials in the perturbation parameters, [15]. Another small limit cycle bifurcates from the origin using the classical Hopf bifurcation when the linear coefficients are added in the perturbation

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terms and the trace parameter change sign adequately. This bifurcation phenomenon is also known as the degenerated Hopf bifurcation. More details on it can be seen in [3]. In piecewise differential equations, a similar bifurcation can also be introduced, but adding two limit cycles instead of only one. The linear terms are responsible for one of them and the other is due to the existence of what is known as the sliding segment that appears when the constant terms are added. The latter is known as the *pseudo-Hopf bifurcation* and they are summarized in the next result.

Proposition 1.3 (Gouveia and Torregrosa [28]). *Consider the perturbed system*

$$\begin{cases} \dot{x} = -(1+c^2)y + \sum_{k+\ell=2}^{\infty} a^+_{k\ell} x^k y^\ell, \\ \dot{y} = x + 2cy + \sum_{k+\ell=2}^{\infty} b^+_{k\ell} x^k y^\ell, \end{cases} \qquad \begin{cases} \dot{x} = -y + \sum_{k+\ell=2}^{\infty} a^-_{k\ell} x^k y^\ell, \\ \dot{y} = d + x + \sum_{k+\ell=2}^{\infty} b^-_{k\ell} x^k y^\ell, \end{cases}$$

for $y \geq 0$ and $y < 0$, respectively. If $a^+_{11} - a^-_{11} + 2b^+_{02} - 2b^-_{02} + b^+_{20} - b^-_{20} \neq 0$ then there exist c and d small enough such that two crossing limit cycles of small amplitude bifurcate from the origin.

We notice that when $c = d = 0$, the expression $a^+_{11} - a^-_{11} + 2b^+_{02} - 2b^-_{02} + b^+_{20} - b^-_{20}$ defines the stability of the origin. So, we could define it as the first Lyapunov constant for the piecewise perturbation of the linear centre. More details on the pseudo-Hopf bifurcation are collected in [12, 21, 22]. In [14, 27, 29, 30], we can see that a good and simple tool for studying the number of limit cycles of small amplitude bifurcating from the origin is the first-order development of the Lyapunov constants. We notice that, in this case, the Jacobian matrix of the vector field at the origin must be of non-degenerate monodromic type.

Then, a natural question arises. *Are there any relation between both mechanisms?* Chicone and Jacobs in 1991 provided a positive answer for quadratic families of vector fields, see [13]. But, their proof is also valid for polynomial vector fields of any degree. Hence, we can say that the next result comes from their original work. These ideas appear also in the works of Han and Yu, see [33, 36], where the Melnikov function in the Hopf bifurcation is studied.

Theorem 1.4 (Chicone and Jacobs [13]). *Let $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$ be a polynomial vector field of degree n, with a non degenerated centre at the origin. Consider the perturbed systems, in the class of polynomial vector fields of degree n,*

$$(\dot{x}, \dot{y}) = \left(P_c(x,y) + \sum_{k+l=2}^{n} a_{kl} x^k y^l, Q_c(x,y) + \sum_{k+l=2}^{n} b_{kl} x^k y^l \right) \tag{8}$$

and

$$(\dot{x}, \dot{y}) = \left(P_c(x,y) + \varepsilon \sum_{k+l=2}^{n} a_{kl} x^k y^l + O(\varepsilon^2), Q_c(x,y) + \varepsilon \sum_{k+l=2}^{n} b_{kl} x^k y^l + O(\varepsilon^2) \right). \tag{9}$$

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If we denote by $L_k^{[1]}$ the first-order expansion, with respect to the parameters a_{kl}, b_{kl}, for the Taylor series of the Lyapunov constant L_k associated with (8) then, for ρ small, the first Melnikov function of (9) is

$$M(\rho) = \sum_{k=1}^{N} L_k^{[1]} \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0}\rho^j\right) \rho^{2k+2}, \tag{10}$$

with the Bautin ideal $\langle L_1, \ldots, L_N, \ldots\rangle = \langle L_1, \ldots, L_N\rangle$.

We remark that the perturbations in the above result are taken without linear terms. This restriction is due to the fact that the elements in the Bautin ideal should be polynomials. A direct consequence is the first part of the next corollary. For the second part, the classical Hopf bifurcation is necessarily to be used. As usual, $O(\varepsilon^2)$ denotes the terms of degree at least 2 in the perturbative parameter ε.

Corollary 1.5. *Let A_m be the matrix corresponding to $(L_1^{[1]}, \ldots, L_m^{[1]})$ with respect to the parameters $(a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots) \in \mathbb{R}^{n^2+3n-4}$, where each $L_k^{[1]}$ is the linear k-Lyapunov constant of system (8) and $m \le N$. Then, if $\operatorname{rank} A_m = \ell$ and for ε small enough, system (9) has $\ell - 1$ hyperbolic limit cycles of small amplitude bifurcating from the origin. Additionally, adding the trace parameter, there are polynomial perturbations of the centre $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$, exhibiting ℓ hyperbolic limit cycles of small amplitude in a small enough neighbourhood of the origin.*

Recently, see [27, 29, 30], a detailed study of the return map of a differential equation near a non-degenerated monodromic point located at the origin has been done. The authors focus their attention not only on first-order Taylor developments, with respect to parameters, of the coefficients of the return map with respect to the initial condition. These papers are a continuation of the work given by Christopher in [14]. The degenerated Hopf bifurcation technique applies when $L_k^{[1]} = u_k$ for $k = 1, \ldots, N$, and all u_k are new independent parameters. Then, the existence of N limit cycles of small amplitude bifurcating from the origin is guaranteed. This approach fails when the rank does not increase in each step. That is when at least one $L_k^{[1]}$ is a linear combination of the previous ones. The main advantage of Theorem 1.4 is that we can compute easier the expressions of $L_k^{[1]}$ than the coefficients of the series expansion of M. In [32], the study of the multiplicity of zeros is also taken into account and moreover it provides upper bounds for the number of limit cycles up to this perturbation order.

This paper is structured as follows. In § 2, for completeness, we prove Theorem 1.1 recovering the original proof for quadratic vector fields obtained in [13]. As a natural application of it, we also prove Corollary 1.5. Then, we can get limit cycles of small amplitude bifurcating from a centre equilibrium point. Finally, we extend Theorem 1.4 and Corollary 1.5 to the class of piecewise polynomial vector fields. In § 3, we show, with a simple example of a polynomial vector field of degree 6, that the maximal computed rank of the linear developments does not coincide with the subscript of the corresponding Lyapunov constant. Then, the result using only linear parts given by Christopher in [14] does not apply. But using Theorem 1.4, we provide more limit cycles of small amplitude.

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In § 4, we prove our first main result, Theorem 1.1, and we study the local cyclicity problem for some vector fields of degrees 7, 8, and 10, proving that $\mathcal{M}(7) \geq 60$, $\mathcal{M}(8) \geq 70$, and $\mathcal{M}(10) \geq 97$. Finally, § 5 is devoted to piecewise polynomial vector fields of degrees 3, 4, and 5, proving our second main result Theorem 1.2. We also provide a new proof of $\mathcal{M}_p^c(3) \geq 26$, different from the one published in [29], using now only first-order developments.

2. The proof of Chicone–Jacobs' result

This section is devoted to the proof of Theorem 1.4 and Corollary 1.5. The proof of Theorem 1.4 for quadratic vector fields can be found in [13]. You can see it partially also in [48]. Here we reproduce it but extended for every polynomial vector field of degree n.

Proof of Theorem 1.4. To simplify notation, we rewrite the perturbed parameters in (8) and (9) as

$$(a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots) = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m, \text{ with } m = n^2 + 3n - 4.$$

After a linear change if necessary, Equations (8) and (9), in the usual polar coordinates, write as

$$\frac{dr}{d\theta} = \frac{r^2 R_3(\theta) + \cdots + r^n R_{n+1}(\theta)}{1 + r\Theta_3(\theta) + \cdots + r^{n-1}\Theta_{n+1}(\theta)} \tag{11}$$

where R_i and Θ_i are homogeneous trigonometric polynomials in $\sin\theta$, $\cos\theta$ of degree i. We define $r(\theta;\rho)$ the solution of the initial value problem defined by (11) with initial condition $r(0;\rho) = \rho$. Then, from the first return map $r(2\pi;\rho)$, we can define the displacement function associated with (11) as

$$\Delta(\rho) = r(2\pi;\rho) - \rho.$$

Clearly, the zeros of this function provide periodic orbits of the corresponding system in Cartesian coordinates.

The Taylor series in ρ of the displacement function Δ_1 associated with (8) is

$$\Delta_1(\rho, \lambda) = \sum_{k=1}^{N} L_k(\lambda)\rho^{2k+1}\left(1 + \sum_{j=1}^{\infty} \beta_{kj}(\lambda)\rho^j\right). \tag{12}$$

with β_{kj} polynomials vanishing at zero. Similarly, the Taylor series in ε of the displacement function Δ_2 associated with (9) is

$$\Delta_2(\rho, \varepsilon) = \sum_{k=1}^{\infty} d_k(\rho)\varepsilon^k = \sum_{k=1}^{\infty} \frac{1}{k!}\left(\frac{\partial^k d(\rho,\varepsilon)}{\partial\varepsilon^k}\bigg|_{\varepsilon=0}\right)\varepsilon^k. \tag{13}$$

We notice that both Taylor series representation are only local, but with the Global Bifurcation Lemma, see [13], the coefficients $d_k(\rho)$ are defined and being analytic on the ρ-domain corresponding to the portion of the $x-$axis cut by the periodic trajectories surrounding the centre at the origin of the system. We observe that the idea of Melnikov theory is determining $d_k(\rho)$ under the condition $d_j(\rho) = 0$ for all $j < k$.

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Writing the parameters λ as their Taylor series expansion in terms of a privileged parameter ε, $\lambda_l(\varepsilon) = \sum_{j=1}^{\infty} \lambda_{jl}\varepsilon^j$, in (12), we have for each k and i,

$$L_k(\lambda(\varepsilon)) = \sum_{j=1}^{\infty} L_k^{(j)}(\lambda(\varepsilon))\varepsilon^j \quad \text{and} \quad \beta_{ki}(\lambda(\varepsilon)) = \sum_{j=1}^{\infty} \beta_{kij}\varepsilon^j.$$

Rearranging the Taylor series (12), for ε and ρ small enough, it follows that

$$\Delta_1(\rho, \lambda(\varepsilon)) = \sum_{k=1}^{N}\sum_{j=1}^{\infty} L_k^{(j)}\varepsilon^j \left(1 + \sum_{i=0}^{\infty}\sum_{j=1}^{\infty} \beta_{kij}\rho^j\varepsilon^i\right)\rho^{2k+1}. \tag{14}$$

As the first-orders in ε of Equation (8), changing λ by $\varepsilon\lambda$, and Equation (9) coincide, the coefficients of degree one in ε for (14) and (13) are equal. So,

$$d_1(\rho) = \sum_{k=1}^{N} L_k^{[1]}\left(1 + \sum_{j=1}^{\infty} \beta_{kj0}\rho^j\right)\rho^{2k+1}.$$

It is well known that in the above expression the odd exponents in ρ corresponds to the property that the first non vanishing Lyapunov constant has always an odd index, see [3]. For a proof that the Lyapunov constants of even indices are in the ideal of the odd ones, we refer the reader to [16, 31]. Applying the ideas in [13, Sect. 4] to differential system (4), we can write

$$d_1(\rho) = \frac{1}{H_x(\rho, 0)}M(\rho),$$

where the function M is defined in (5).

The statement follows because, in our case, $H(x, y) = (x^2 + y^2)/2 + \cdots$. □

Proof of Corollary 1.5. We study first the case when the trace parameter is zero. In this case, it is well known that the Lyapunov constants are polynomials in the perturbation parameters, see [15]. Then we can compute the linear terms of the Lyapunov constants with, for example, the classical mechanism described in [42]. Consequently, Theorem 1.4 gets the expression of the first Melnikov function (10). Hence, if the rank of the computed expressions is ℓ, we have $\ell - 1$ simple zeros near $\rho = 0$ and there exists a perturbed system exhibiting $\ell - 1$ limit cycles of small amplitude. Finally, adding the trace parameter as in the classical Hopf bifurcation (see [40]), we get the total ℓ limit cycles as it is stated in the statement. □

We notice that the above two results can be easily generalized for piecewise vector fields. Let us consider the piecewise version of the perturbed differential systems (8) and

(9) given by

$$(\dot{x}, \dot{y}) = \left(P_c(x,y) + \sum_{k+l=2}^{n} a_{kl}^{\pm} x^k y^l, Q_c(x,y) + \sum_{k+l=2}^{n} b_{kl}^{\pm} x^k y^l\right) \tag{15}$$

and

$$(\dot{x}, \dot{y}) = \left(P_c(x,y) + \varepsilon \sum_{k+l=2}^{n} a_{kl}^{\pm} x^k y^l + O(\varepsilon^2), Q_c(x,y) + \varepsilon \sum_{k+l=2}^{n} b_{kl}^{\pm} x^k y^l + O(\varepsilon^2)\right). \tag{16}$$

They are defined for the parameters a_{kl}^{+}, b_{kl}^{+} and a_{kl}^{-}, b_{kl}^{-}, in the regions $y \geq 0$ and $y < 0$, respectively. Here, as previously, the unperturbed system $(\dot{x},\, \dot{y}) = (P_c(x,\, y),\, Q_c(x,\, y))$ has a non-degenerated centre at the origin.

The next corollary provides the piecewise extension of Theorem 1.4. So, its proof follows closely to the proof of Theorem 1.4. In this case, the return map is also analytic in ρ and all the steps are similar. The main difference is that the Taylor series of d_1 in ρ have all powers starting in 2, because now we have not the property of symmetry that vanishes all the even terms in the developments. We remark that the perturbation in (16) has no constant nor linear terms.

Corollary 2.1. *Let $L_k^{[1]}$ be the linear terms of the Lyapunov constants of a polynomial piecewise vector field (15) near a monodromic equilibrium point. Then the corresponding first Melnikov function (5) of (16) writes as*

$$M(\rho) = \sum_{k=2}^{N} L_k^{[1]} \left(1 + \sum_{j=1}^{\infty} \alpha_{kj0} \rho^j\right) \rho^{k+1},$$

being the Bautin ideal $\langle L_2,\, \ldots,\, L_N,\, \ldots\rangle = \langle L_2,\, \ldots,\, L_N\rangle$.

Finally, similarly as for analytic perturbations here also the rank of the matrix provides a lower bound for the number of limit cycles bifurcating from a monodromic equilibrium point. But, using piecewise perturbations, we have two extra limit cycles of small amplitude as we have shown in Proposition 1.3. Then we have at least as many limit cycles as the number of linearly independent linear parts plus one.

Corollary 2.2. *Let A_m be the matrix corresponding to $(L_2^{[1]},\, \ldots,\, L_m^{[1]})$ with respect to the parameters $(a_{20}^{\pm},\, a_{11}^{\pm},\, a_{02}^{\pm},\, \ldots,\, b_{20}^{\pm},\, b_{11}^{\pm},\, b_{02}^{\pm},\, \ldots) \in \mathbb{R}^{2n^2+6n-8}$, where each $L_k^{[1]}$ is the linear part of the k-Lyapunov constant of system (15). Then, if $\operatorname{rank} A_m = \ell$, for ε small enough, there exist perturbation parameters, $(a^{\pm},\, b^{\pm})$, such that system (16) has $\ell - 1$ limit cycles of small amplitude bifurcating from the origin. Additionally, adding the trace parameter, there are piecewise polynomial perturbations of $(\dot{x},\, \dot{y}) = (P_c(x,\, y),\, Q_c(x,\, y))$ exhibiting $\ell + 1$ hyperbolic limit cycles of small amplitude in a small enough neighbourhood of the origin.*

We remark that the definition of the matrix A_m for Corollaries 1.5 and 2.2 does not coincide, because of the different definitions of the subscripts in the nonvanishing coefficients of the respectively Taylor developments in ρ. But the important property for the number of the limit cycles of small amplitude is which is the value of its rank.

3. A first but not trivial example

In this section, we show the explicit linear developments of the Lyapunov constants in a simple example provided by [39]. In the proof of the next result, we can see how Corollary 1.5 is better than the direct use of the Lyapunov constants. We remark that, most probably, the number of limit cycles presented will be the maximum using only first-order analysis. We have chosen a vector field of a high degree to see better the effectiveness of our results.

Proposition 3.1. *Let us consider the system*

$$(\dot{x}, \dot{y}) = (-y + x^5 y, x + x^4 y^2). \tag{17}$$

Then, there exist polynomial perturbations of degree 6 *such that from the origin of the centre* (17) *bifurcate* 16 *limit cycles of small amplitude.*

Proof. System (17) has a time-reversible centre at the origin. Because it is invariant by the change $(x, y, t) \to (x, -y, -t)$. Additionally, it has also a rational first integral obtained from its corresponding inverse integrating factor. They are, respectively,

$$H(x, y) = \frac{(x^2 + y^2)^5}{(1 - x^5)^2} \quad \text{and} \quad V(x, y) = \frac{x^{10} - 2x^5 + 1}{(x^2 + y^2)^{3/2}}.$$

The next step is the computation of the linear terms of the Taylor developments of the Lyapunov constants. We have computed $L_k^{[1]}$ for $k = 1, \ldots, 40$ with, for example, the algorithm described in [30]. As in the previous section, let A_n be the matrix corresponding to $(L_1^{[1]}, \ldots, L_n^{[1]})$ with respect to the parameters $(a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots) \in \mathbb{R}^{50}$. Then, the proof of the statement follows using Corollary 1.5 because $\operatorname{rank} A_{20} = 16$.

We point out that we have checked that $\operatorname{rank} A_k = k$ for $k = 1, \ldots, 12$, but $\operatorname{rank} A_{13} = 12$. Additionally, that $\operatorname{rank} A_{14} = 13$, $\operatorname{rank} A_k = 14$, for $k = 15, \ldots, 18$, $\operatorname{rank} A_{19} = 15$, and $\operatorname{rank} A_k = 16$, for $k = 20, \ldots, 40$. From these values, we can say that, up to $k = 40$, there are no more limit cycles than the ones in the statement. Moreover, without Corollary 1.5, that is with the results in [14], only 12 limit cycles of small amplitude can bifurcate from the origin.

The explicit expression of the necessary Lyapunov constants to get the statement are:

$$L_1^{[1]} = \frac{2}{3}(3a_{30} + a_{12} + b_{21} + 3b_{03}),$$
$$L_2^{[1]} = \frac{2}{5}(b_{41} + a_{32} + b_{23} + 5b_{05} + a_{14} + 5a_{50}),$$

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$$
\begin{aligned}
L_3^{[1]} &= \frac{2}{35}(33a_{20} + 7a_{02} - b_{11}),\\
L_4^{[1]} &= \frac{2}{315}(161a_{40} - 21b_{31} - 9b_{13} + 29a_{22} + 21a_{04}),\\
L_5^{[1]} &= \frac{2}{231}(21a_{60} - 7b_{33} - 5b_{15} + 5a_{24} + 7a_{06} + 7a_{42} - 21b_{51}),\\
L_6^{[1]} &= -\frac{2}{715}(2661b_{03} + 873b_{21} + 901a_{12} + 2773a_{30}),\\
L_7^{[1]} &= -\frac{2}{10725}(42775b_{05} + 8639a_{14} + 8755a_{32} + 8447b_{23} + 8051b_{41} + 44259a_{50}),\\
L_8^{[1]} &= -\frac{2}{2127125}(-265929b_{11} + 1931573a_{02} + 9055937a_{20}),\\
L_9^{[1]} &= \frac{2}{121246125}(-40561101b_{31} - 18052569b_{13} + 42963501a_{04} + 320497961a_{40}\\
&\quad + 58729949a_{22}),\\
L_{10}^{[1]} &= -\frac{2}{24249225}(12806957a_{60} + 4471569a_{42} + 4628869a_{06} - 12806957b_{51}\\
&\quad - 4471569b_{33} + 3270585a_{24} - 3270585b_{15}),\\
L_{11}^{[1]} &= \frac{2}{663966875}(9725859717a_{12} + 29642231001a_{30} + 28891397801b_{03}\\
&\quad + 9538151417b_{21}),\\
L_{12}^{[1]} &= \frac{2}{16599171875}(770680333485b_{05} + 789987106585a_{50} + 155267074197a_{14}\\
&\quad + 156791990497a_{32} + 147654869097b_{41} + 152709627597b_{23}),\\
L_{13}^{[1]} &= \frac{2}{3137243484375}(158564044415887a_{20} + 33790154009423a_{02}\\
&\quad - 4680685675639b_{11}),\\
L_{14}^{[1]} &= \frac{2}{10108895671875}(44434290786711a_{04} + 332135401856251a_{40}\\
&\quad + 60777619220839a_{22} - 186928552178l9b_{13} - 421368784695l1b_{31}),\\
L_{15}^{[1]} &= \frac{2}{3418644718125}(23966540375301a_{60} + 8360451650717a_{42}\\
&\quad + 6115824649905a_{24} - 23966540375301b_{51} - 8360451650717b_{33}\\
&\quad - 6115824649905b_{15} + 8657859935137a_{06}),\\
L_{16}^{[1]} &= -\frac{2}{3357597491015625}(3245412818709921797b_{03} + 1075093752985900449b_{21}\\
&\quad + 1088751590673845749a_{12} + 3300044169461702997a_{30}),
\end{aligned}
$$

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$$L_{17}^{[1]} = -\frac{2}{430461216796875}(4417434206393301995b_{05} + 8570379723356479 9b_{41} + 87765192673441699b_{23} + 89435736465275399a_{32} + 88812279725145499a_{14} + 449630929414297095a_{50}),$$

$$L_{18}^{[1]} = -\frac{2}{56468685076171875}(6343162117243756 4093a_{20} - 1883311935706466021b_{11} + 1350376537347630 8497a_{02}),$$

$$L_{19}^{[1]} = -\frac{2}{6211555358378906 25}(464272204405895182111a_{40} + 84747082997571253579a_{22} - 59162027050841642971b_{31} - 26101104368457380559b_{13} + 61867863575213428971a_{04}),$$

$$L_{20}^{[1]} = -\frac{2}{3565432775709492 1875}(59336225834427613088 61a_{60} + 15030422421675922027 05a_{24} - 20596844144438867875 37b_{33} + 21254673915587464470 37a_{06} - 15030422421675922027 05b_{15} - 59336225834427613088 61b_{51} + 20596844144438867875 37a_{42}). \qquad \square$$

4. Local cyclicity for systems of degrees 6, 7, 8, and 10

In this section, we use Corollary 1.5 to study lower bounds for the cyclicity in some centres of low degree, $n = 6$, 7, 8, and 10. Next result proves Theorem 1.1. The other propositions do not provide better lower bounds for the local cyclicity in the indicated degrees. But we have decided to be added here because the centres and their perturbation studies are new. From the proofs, it seems that the number of limit cycles that bifurcate from the origin will be the maximum values for first-order studies. We have computed some more terms than the necessary to prove the statements. In all cases, we show that only studying first-order developments and using the technique described in [14] the results on lower bounds for the cyclicity are worst. We have not studied degrees $n = 3$, 4, 5, 9 because we do not have new centres having this special property. Unfortunately, we have not improved the lower bounds previously obtained in [27, 30].

Proposition 4.1. *There are polynomial perturbations of degree* 6 *such that from the origin of the centre*

$$\begin{cases} \dot{x} = -y + \dfrac{128}{15}x^6 - \dfrac{128}{15}x^5y - \dfrac{416}{45}x^4y^2 + \dfrac{448}{45}x^3y^3 - \dfrac{256}{15}x^2y^4 + \dfrac{256}{45}xy^5 + \dfrac{8}{9}y^6, \\ \dot{y} = 2x - \dfrac{896}{45}x^5y - \dfrac{1664}{45}x^4y^2 + \dfrac{96}{5}x^3y^3 - \dfrac{512}{45}x^2y^4 + \dfrac{112}{45}xy^5 + \dfrac{32}{15}y^6, \end{cases} \tag{18}$$

bifurcate at least 44 *limit cycles of small amplitude.*

Proof. The vector field in the statement and the proof that it has a centre at the origin is given by Giné in [25]. The proof of the statement follows from Corollary 1.5

computing $L_k^{[1]}$, for $k = 1, \ldots, 80$, and checking that $\operatorname{rank} A_k = 44$, for $k = 52, \ldots, 80$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 32$. Because of the size, we only show the linear developments of the first three Lyapunov constants:

$$\begin{aligned} L_1^{[1]} &= \frac{4}{3}(2a_{12} + b_{21}) + 4(a_{30} + b_{03}), \\ L_2^{[1]} &= \frac{4}{5}(2a_{32} + 4a_{14} + b_{41} + 2b_{23}) + 4(a_{50} + 4b_{05}), \\ L_3^{[1]} &= \frac{512}{1575}(488a_{02} - 86a_{11} + 236a_{20} - 118b_{02} + 172b_{11} - 61b_{20}). \end{aligned}$$ □

Remark 4.2. We notice that for the centre (18) we have also computed 44 Lyapunov constants up to fourth-order and, using [30], we can prove only the existence of 41 limit cycles of small amplitude bifurcating from the origin.

Proposition 4.3. *There are polynomial perturbations of degree* 7 *such that from the origin of the centre*

$$\begin{cases} \dot{x} = -\dfrac{2527}{3}x^6y - \dfrac{2968}{3}x^5y^2 - \dfrac{4186}{3}x^4y^3 - \dfrac{2800}{3}x^3y^4 - 553x^2y^5 + 56xy^6 \\ \qquad + \dfrac{184}{3}x^3y + \dfrac{88}{3}x^2y^2 + 48xy^3 - y, \\ \dot{y} = 672x^7 + 1484x^6y + \dfrac{2219}{3}x^5y^2 + \dfrac{5684}{3}x^4y^3 - \dfrac{742}{3}x^3y^4 + \dfrac{1148}{3}x^2y^5 \\ \qquad - 315xy^6 - 28y^7 - 58x^4 - 44x^3y - \dfrac{104}{3}x^2y^2 - \dfrac{44}{3}xy^3 + 10y^4 + x, \end{cases}$$

bifurcate at least 60 *limit cycles of small amplitude.*

Proof. The system in the statement has a centre at the origin because it has a rational first integral of the form $H(x(x^2 + y^2), y(x^2 + y^2))$, where

$$H(x, y) = \frac{(42x - 7y - 1)^3\psi(x, y)}{(448x^2 + 336xy + 63y^2 - 44x - 12y + 1)^3(1183x^2 - 68x + 1)} \tag{19}$$

with $\psi(x, y) = 10752x^3 + 29568x^2y + 17640xy^2 + 3024y^3 - 1600x^2 - 2760xy - 576y^2 + 74x + 57y - 1$. The original cubic polynomial vector field having a centre at the origin is given in [8].

The proof follows also from Corollary 1.5 computing $L_k^{[1]}$, for $k = 1, \ldots, 80$, and checking that $\operatorname{rank} A_{63} = 60$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 55$. We show only the first three Lyapunov constants:

$$\begin{aligned} L_1^{[1]} &= \frac{1}{3}(a_{12} + b_{21}) + a_{30} + b_{03}, \\ L_2^{[1]} &= \frac{1}{45}(1664a_{02} + 352a_{11} + 9a_{14} + 3952a_{20} + 9a_{32} + 45a_{50} + 968b_{02} + 45b_{05} \\ &\quad + 1064b_{11} + 616b_{20} + 9b_{23} + 9b_{41}), \end{aligned}$$

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$$L_3^{[1]} = \frac{2192}{105}(a_{04} + 176a_{13} + 15a_{16} + 1392a_{22} + 528a_{31} + 9a_{34} + 6800a_{40} + 15a_{52} + 105a_{70} + 880b_{04} + 105b_{07} + 912b_{13} + 528b_{22} + 15b_{25} + 1072b_{31} + 880b_{40} + 9b_{43} + 15b_{61}). \qquad \square$$

Proposition 4.4. *There are polynomial perturbations of degree* 8 *such that from the origin of the centre*

$$\begin{cases} \dot{x} = \frac{54}{175}x^8 + \frac{18}{35}x^7y - \frac{54}{175}x^6y^2 + \frac{894}{175}x^5y^3 - 2x^4y^4 \\ \qquad + \frac{66}{25}x^3y^5 - \frac{26}{35}x^2y^6 - \frac{342}{175}xy^7 + \frac{16}{25}y^8 - y, \\ \dot{y} = -\frac{198}{175}x^7y - \frac{1254}{175}x^6y^2 - \frac{586}{175}x^5y^3 - \frac{258}{35}x^4y^4 \\ \qquad - \frac{22}{5}x^3y^5 + \frac{18}{25}x^2y^6 - \frac{382}{175}xy^7 + \frac{162}{175}y^8 + x, \end{cases}$$

bifurcate at least 70 *limit cycles of small amplitude.*

Proof. We consider the centre with quartic homogeneous nonlinearities given in [25] but written in polar coordinates, $(x, y) = (r\cos\theta, r\sin\theta)$. Then, with the change $R = r^{3/5}$ and recovering again new Cartesian coordinates, we obtain the system in the statement of degree 8. Then it has also a centre at the origin. The proof of the statement follows again from Corollary 1.5 computing $L_k^{[1]}$, for $k = 1, \ldots, 130$, and checking that $\operatorname{rank} A_{87} = 70$. We notice that $\operatorname{rank} A_k = k$, for $k = 1, \ldots, 45$. The first three Lyapunov constants are

$$L_1^{[1]} = \frac{1}{3}(a_{12} + b_{21}) + a_{30} + b_{03},$$

$$L_2^{[1]} = \frac{1}{5}(a_{14} + a_{32} + 5a_{50} + 5b_{05} + b_{23} + b_{41}),$$

$$L_3^{[1]} = \frac{1}{35}(5a_{16} + 3a_{34} + 5a_{52} + 35a_{70} + 35b_{07} + 5b_{25} + 3b_{43} + 5b_{61}). \qquad \square$$

Proposition 4.5. *There are polynomial perturbations of degree* 10 *such that from the origin of the centre*

$$\begin{cases} \dot{x} = \frac{6}{25}x^{10} + \frac{2}{5}x^9y + \frac{8}{25}x^8y^2 + \frac{152}{25}x^7y^3 - \frac{28}{25}x^6y^4 + \frac{44}{5}x^5y^5 - \frac{8}{5}x^4y^6 \\ \qquad + \frac{24}{25}x^3y^7 + \frac{6}{25}x^2y^8 - \frac{54}{25}xy^9 + \frac{16}{25}y^{10} - y, \\ \dot{y} = -\frac{6}{5}x^9y - \frac{182}{25}x^8y^2 - \frac{104}{25}x^7y^3 - \frac{352}{25}x^6y^4 - \frac{164}{25}x^5y^5 - \frac{28}{5}x^4y^6 \\ \qquad - \frac{136}{25}x^3y^7 + \frac{48}{25}x^2y^8 - \frac{46}{25}xy^9 + \frac{18}{25}y^{10} + x, \end{cases}$$

bifurcate at least 97 *limit cycles of small amplitude.*

Proof. The centre in the statement is obtained following the same procedure than in the proof of Proposition 4.4 but with the change $R = r^{3/9}$. The proof of the existence of such limit cycles of small amplitude follows again from Corollary 1.5 computing $L_k^{[1]}$, for $k = 1, \ldots, 130$, and checking that rank $A_{126} = 97$. We notice that rank $A_k = k$, for $k = 1, \ldots, 58$. Curiously, the expressions of the first three Lyapunov constants coincide with the ones of the previous proposition. □

5. Perturbing piecewise systems of degrees 3, 4, and 5

This section is devoted to proving Theorem 1.2. We also show that there are other cubic centres having cyclicity also higher or equal than 26 as it was proved in [29] but using second-order developments. The following result only uses first-order analysis. The best lower bound for the local cyclicity, up to our knowledge, for piecewise quadratic vector fields is [18]. As we have no new nor better results for $n = 2$ we have concentrated our efforts to high degrees. But, the computational difficulties arise for $n \geq 6$ even using the parallelization algorithms. This is why we have stopped in degree five polynomial piecewise differential equations. So we do not write them here again.

As in the previous section, we have computed some more Lyapunov constants than the strictly necessary to get the proofs. With the aim to convince ourselves that we have obtained the optimal results for first-order studies. From the proofs, it can be seen also that the classical study using only the return map and the Lyapunov constants provide worse results. See again [29].

The general perturbed system considered in this section is

$$\begin{cases} (\dot{x}, \dot{y}) = \left(P_c(x,y) + \sum_{k+\ell=0}^{n} a^+_{k\ell} x^k y^\ell, Q_c(x,y) + \sum_{k+\ell=0}^{n} b^+_{k\ell} x^k y^\ell \right) & \text{for } y \geq 0, \\ (\dot{x}, \dot{y}) = \left(P_c(x,y) + \sum_{k+\ell=0}^{n} a^-_{k\ell} x^k y^\ell, Q_c(x,y) + \sum_{k+\ell=0}^{n} b^-_{k\ell} x^k y^\ell \right) & \text{for } y < 0. \end{cases} \tag{20}$$

Proposition 5.1. *There exist polynomial piecewise perturbations of degree $n = 3$ as (20) such that 26 crossing limit cycles of small amplitude bifurcate from the origin of system*

$$\begin{cases} \dot{x} = -y + \frac{168}{125}x^2 + \frac{8252}{125}xy - \frac{2968}{125}y^2 \\ \qquad - \frac{44436}{625}x^3 - \frac{533631}{625}x^2y + \frac{592508}{625}xy^2 + \frac{69552}{625}y^3, \\ \dot{y} = x - \frac{4974}{125}x^2 + \frac{9164}{125}xy + \frac{2874}{125}y^2 \\ \qquad + \frac{232848}{625}x^3 - \frac{910392}{625}x^2y + \frac{385231}{625}xy^2 + \frac{407064}{625}y^3. \end{cases}$$

Proof. The system in the statement is the Bondar–Sadovskiĭ cubic centre system given by the rational first integral (19), see again [8], but rotated with the matrix

$$\begin{pmatrix} 3/5 & -4/5 \\ 4/5 & 3/5 \end{pmatrix}.$$

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The proof of the statement follows from Corollary 2.2 computing $L_k^{[1]}$, for $k = 2, \ldots, 32$, and checking that $\operatorname{rank} A_{28} = 25$. We notice that $\operatorname{rank} A_k = k - 1$, for $k = 2, \ldots, 24$. Because of the size, we only show the linear developments of the first three Lyapunov constants:

$$
\begin{aligned}
L_2^{[1]} &= \frac{2}{3}(a_{11}^{+} - a_{11}^{-} + b_{20}^{+} - b_{20}^{-} + 2b_{02}^{+} - 2b_{02}^{-}),\\
L_3^{[1]} &= \frac{1}{9000}(126000\pi a_{02}^{-} - 25200\pi a_{11}^{-} + 1125\pi a_{12}^{-} + 163800\pi a_{20}^{-} + 3375\pi a_{30}^{-}\\
&\quad + 126000\pi a_{02}^{+} - 25200\pi a_{11}^{+} + 1125\pi a_{12}^{+} + 163800\pi a_{20}^{+} + 3375\pi a_{30}^{+}\\
&\quad - 135900\pi b_{02}^{-} + 3375\pi b_{03}^{-} + 18900\pi b_{11}^{-} - 85500\pi b_{20}^{-} + 1125\pi b_{21}^{-} - 135900\pi b_{02}^{+}\\
&\quad + 3375\pi b_{03}^{+} + 18900\pi b_{11}^{+} - 85500\pi b_{20}^{+} + 1125\pi b_{21}^{+} - 577664a_{11}^{-} + 577664a_{11}^{+}\\
&\quad - 1155328b_{02}^{-} - 577664b_{20}^{-} + 1155328b_{02}^{+} + 577664b_{20}^{+}),\\
L_4^{[1]} &= \frac{1}{2812500}(4738650000\pi a_{02}^{-} - 947730000\pi a_{11}^{-} + 42309375\pi a_{12}^{-}\\
&\quad + 6160245000\pi a_{20}^{-} + 126928125\pi a_{30}^{-} + 4738650000\pi a_{02}^{+} - 947730000\pi a_{11}^{+}\\
&\quad + 42309375\pi a_{12}^{+} + 6160245000\pi a_{20}^{+} + 126928125\pi a_{30}^{+} - 5110972500\pi b_{02}^{-}\\
&\quad + 126928125\pi b_{03}^{-} + 710797500\pi b_{11}^{-} - 3215512500\pi b_{20}^{-} + 42309375\pi b_{21}^{-}\\
&\quad - 5110972500\pi b_{02}^{+} + 126928125\pi b_{03}^{+} + 710797500\pi b_{11}^{+} - 3215512500\pi b_{20}^{+}\\
&\quad + 42309375\pi b_{21}^{+} + 1547032000a_{02}^{-} - 84000000a_{03}^{-} - 14072769976a_{11}^{-}\\
&\quad + 3736000a_{12}^{-} + 2604141664a_{20}^{-} - 37356000a_{21}^{-} + 41772000a_{30}^{-}\\
&\quad - 1547032000a_{02}^{+} + 84000000a_{03}^{+} + 14072769976a_{11}^{+} - 3736000a_{12}^{+}\\
&\quad - 2604141664a_{20}^{+} + 37356000a_{21}^{+} - 41772000a_{30}^{+} - 27939203952b_{02}^{-}\\
&\quad + 68208000b_{03}^{-} + 628004832b_{11}^{-} + 4644000b_{12}^{-} - 14413096320b_{20}^{-}\\
&\quad + 42424000b_{21}^{-} + 9390000b_{30}^{-} + 27939203952b_{02}^{+} - 68208000b_{03}^{+}\\
&\quad - 628004832b_{11}^{+} - 4644000b_{12}^{+} + 14413096320b_{20}^{+} - 42424000b_{21}^{+} - 9390000b_{30}^{+}).
\end{aligned}
$$
□

We remark that, in the above result, we have rotated because, without it, the local cyclicity takes a lower value, up to first-order analysis. We have not proved that this rotation plays any special role. Except that the entries of the matrix are small rational numbers. With other rotation matrices, the result will be generically the same. This example shows how the separation straight line change the local cyclicity of the origin.

Proposition 5.2. *There exist polynomial piecewise perturbations of degree $n = 4$ as (20) such that 43 crossing limit cycles of small amplitude bifurcate from the origin of system*

$$
\begin{cases}
\dot{x} = x^4 - 6x^2y^2 + y^4 + x^3 - 3xy^2 + x^2 - y^2 - y,\\
\dot{y} = 4x^3y - 4xy^3 + 3x^2y - y^3 + 2xy + x.
\end{cases}
$$

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Proof. The above vector field has a centre at the origin because it is the holomorphic system of degree four $\dot{z} = \mathrm{i}\, z + z^2 + z^3 + z^4$, but written in Cartesian coordinates, $z = x + \mathrm{i}\, y$. The proof follows also from Corollary 2.2 computing $L_k^{[1]}$, for $k = 2, \ldots, 60$, and checking that rank $A_{60} = 42$. We notice that rank $A_k = k - 1$, for $k = 2, \ldots, 38$. Because of the size, we only show the linear developments of the first three Lyapunov constants:

$$\begin{aligned}
L_2^{[1]} &= \frac{2}{3}(a_{11}^+ - a_{11}^- + b_{20}^+ - b_{20}^-),\\
L_3^{[1]} &= \frac{1}{8}\pi(a_{02}^+ + a_{02}^- + a_{12}^+ + a_{12}^- + 3a_{30}^+ + 3a_{30}^- + 3b_{03}^+ + 3b_{03}^-\\
&\quad - 4b_{20}^+ - 4b_{20}^- + b_{21}^+ + b_{21}^-),\\
L_4^{[1]} &= 8b_{20}^+ - 8b_{20}^- + 4a_{02}^+ - 4a_{02}^- - 6b_{21}^+ + 6b_{21}^- + 4b_{22}^+ - 4b_{22}^- - 2a_{11}^+\\
&\quad + 2a_{11}^- + 4a_{12}^+ - 4a_{12}^- + 4a_{13}^+ - 4a_{13}^- - 16a_{20}^+ + 16a_{20}^- + 6a_{30}^+\\
&\quad - 6a_{30}^- + 6a_{31}^+ - 6a_{31}^- - 4b_{03}^+ + 4b_{03}^- + 16b_{04}^+ - 16b_{04}^- - 14b_{11}^+ + 14b_{11}^-.
\end{aligned}$$

□

Proposition 5.3. *There exist polynomial piecewise perturbations of degree $n = 5$ as (20) such that 65 crossing limit cycles of small amplitude bifurcate from the origin of system*

$$\begin{cases}\dot{x} &= x^5 - 10x^3y^2 + 5xy^4 + x^4 - 6x^2y^2 + y^4 + x^3 - 3xy^2 + x^2 - y^2 - y,\\ \dot{y} &= 5x^4y - 10x^2y^3 + y^5 + 4x^3y - 4xy^3 + 3x^2y - y^3 + 2xy + x.\end{cases}$$

Proof. The vector field in the statement is, written in Cartesian coordinates ($z = x + \mathrm{i}\, y$), the holomorphic system of degree five $\dot{z} = \mathrm{i}\, z + z^2 + z^3 + z^4 + z^5$. The proof follows again from Corollary 2.2 computing $L_k^{[1]}$, for $k = 2, \ldots, 80$, and checking that rank $A_{72} = 64$. We notice that rank $A_k = k - 1$, for $k = 2, \ldots, 58$. Because of the size, we only show the linear developments of the first three Lyapunov constants:

$$\begin{aligned}
L_2^{[1]} &= \frac{2}{3}(a_{11}^+ - a_{11}^- + 2(b_{02}^+ - 2b_{02}^-) + b_{20}^+ - b_{20}^-),\\
L_3^{[1]} &= \frac{1}{8}\pi(a_{12}^+ + a_{12}^- + 3(a_{30}^+ + a_{30}^-) - b_{02}^+ - b_{02}^- + 3(b_{03}^+ + b_{03}^-)\\
&\quad - 4(b_{20}^+ + b_{20}^-) + b_{21}^+ + b_{21}^-),\\
L_4^{[1]} &= \frac{2}{15}(3b_{40}^+ - 3b_{40}^- + 3b_{21}^- - 3b_{21}^+ + 4b_{20}^+ - 4b_{20}^- + 2b_{03}^- - 2b_{03}^+ - 7b_{11}^+ + 7b_{11}^-\\
&\quad - 3a_{30}^- + 3a_{30}^+ - 8b_{04}^- + 8b_{04}^+ - 8a_{20}^+ + 8a_{20}^- + 3a_{31}^+ - 3a_{31}^- + 2a_{13}^+ - 2a_{13}^-\\
&\quad - 6b_{02}^- + 6b_{02}^+ + 2a_{12}^+ - 2a_{12}^- + 2b_{22}^+ - 2b_{22}^- - a_{11}^+ + a_{11}^- - 12a_{02}^+ + 12a_{02}^-).
\end{aligned}$$

□

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