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Weighted Brianchon-Gram Decomposition

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Abstract. We give in this note a weighted version of Brianchon and Gram's decomposition for a simple polytope. We can derive from this decomposition the weighted polar formula of Agapito and a weighted version of Brion's theorem, in a manner similar to Haase, where the unweighted case is worked out. This weighted version of Brianchon and Gram's decomposition is a direct consequence of the ordinary Brianchon–Gram formula.

1 Introduction

Let $A \subset \mathbb{R}^n$ be a closed convex subset. The characteristic function $\mathbf{1}_A$ of A is the function $\mathbf{1}_A \colon \mathbb{R}^n \to \mathbb{C}$ given by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

Let $\mathcal{K}(\mathbb{R}^n)$ be the complex vector space spanned by the functions $\mathbf{1}_A$. Thus, a function $f \in \mathcal{K}(\mathbb{R}^n)$ is a linear combination

$$f = \sum_{i=1}^{m} \alpha_i \mathbf{1}_{A_i},$$

where the A_i are closed convex sets in $\mathbb{R}^n$ and the α_i are complex numbers.

Among the elements of $\mathcal{K}(\mathbb{R}^n)$ there are three well known decomposition formulas: the Brianchon–Gram decomposition [Br, G] (see also [B, S]), which determines the characteristic function of any polytope as a signed sum of characteristic functions of cones associated to its faces, the polar decomposition of a simple polytope P [L, V], which uses the notion of polarization[1] and Brianchon and Gram's formula in order to write the characteristic function of P in terms of the characteristic functions of cones based on the vertices of P only, and the Brion decomposition of a polytope [Br1], which is also a direct consequence of the Brianchon–Gram formula. We can put *weights* to the faces of P in a meaningful way and get a new element of $\mathcal{K}(\mathbb{R}^n)$. For instance, let q be any complex number and let $[a, b]$ be any interval. We can write (see Figure 1)

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[1] A polarizing vector is a generic element of $(\mathbb{R}^n)^*$ which is nonconstant on each edge of P. (See [L, V] and compare with [A].)

Figure 1: Decomposition of an interval.

That is, as an element of $\mathcal{K}(\mathbb{R})$, the function $(q-1)\mathbf{1}_{\{a\}} + (q-1)\mathbf{1}_{\{b\}} + \mathbf{1}_{[a,b]}$ defines a weighted characteristic function over $[a,b]$. We denote it by $\mathbf{1}^q_{[a,b]}$. In general, we can assign arbitrary complex values to the facets of a polytope P and construct a new element $\mathbf{1}^w_P$ of $\mathcal{K}(\mathbb{R}^n)$. (See (3).)

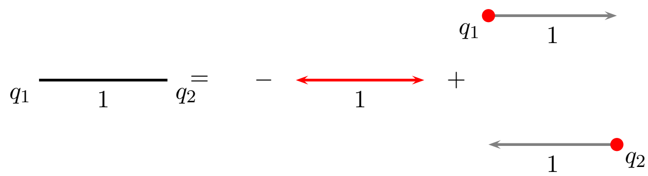

Figure 2: Weighted Brianchon–Gram decomposition for an interval.

When we assign the same value q to the facets of P, we also denote $\mathbf{1}^w_P$ by $\mathbf{1}^q_P$. The weighted polar decomposition formula of [A] expresses $\mathbf{1}^q_P$ as an alternating sum of weighted characteristic functions of cones based on the vertices of P. For example, in the case of $\mathbf{1}^q_{[a,b]}$, we have $\mathbf{1}^q_{[a,b]} = \mathbf{1}^q_{[a,\infty)} - \mathbf{1}^{1-q}_{(-\infty,b]}$. The purpose of this note is to show that $\mathbf{1}^w_P$ (defined in (3)) satisfies a weighted version of the Brianchon–Gram formula, from which it readily follows the weighted polar formula of [A] and a weighted version of Brion's theorem [Br1]. The relationship among these formulas has already been pointed out by Haase [H] in the unweighted case. Our main result is the weighted version of the Brianchon–Gram formula as stated in (4). (See Figures 2, 3 and 4 for illustrations of this formula.)

2 The Weighted Formula

Let P be a d-dimensional polyhedron in $\mathbb{R}^d$ (for standard definitions on polyhedra we refer to [B]). We can write it as the intersection of a finite number of half-spaces

$$P = H_1 \cap \cdots \cap H_N, \tag{1}$$

where $H_i = \{x \mid \langle u_i, x\rangle + \mu_i \geq 0\}$, with $\mu_i \in \mathbb{R}$ and $u_i \in (\mathbb{R}^d)^*$ for $1 \leq i \leq N$. Note that $\mathbb{R}^d = \{x \mid \langle 0, x\rangle + 0 \geq 0\}$, hence $\mathbb{R}^d$ is trivially a polyhedron. It follows that P is a closed convex set. We assume that P is obtained with the smallest possible N. The facets of P are $\sigma_i = P \cap \partial H_i$ for $i = 1, \ldots, N$. If the intersection (1) is bounded then P is a polytope. We say that a d-dimensional polyhedron P is simple if every vertex of P belongs (when it exists) to exactly d facets of P. In the case of a polyhedron without vertices, we assume that this condition is trivially satisfied.

Let P be any d-dimensional polyhedron in $\mathbb{R}^d$. For each $i = 1, \dots, N$, we assign arbitrary complex numbers q_i to the facets σ_i of P. Each non-trivial face F of P ($F \neq \phi, P$) can be uniquely described as an intersection of facets

$$F = \bigcap_{i \in I_F} \sigma_i, \tag{2}$$

where I_F denotes the set of all facets of P containing F. When P is simple, the number of elements in I_F is equal to the codimension of F.

To each non-trivial face F we assign the value $\prod_{i \in I_F} q_i$. When $F = P$, we give it the value 1. This amounts to defining the weighted function $w \colon P \to \mathbb{C}$ by $w(x) = \prod_{i \in I_F} q_i$, where F is the face of P of smallest dimension containing x. If x is in the interior of P, we set $w(x) = 1$. We extend this definition to all $\mathbb{R}^d$ and get the weighted characteristic function

$$\mathbf{1}_P^w(x) = \begin{cases} w(x) & \text{if } x \in P, \\ 0 & \text{if } x \notin P. \end{cases} \tag{3}$$

Now, let F be any face of P. The tangent cone to P at F is

$$\mathbf{C}_F = \{y + r(x - y) \mid r \geq 0,\ y \in F,\ x \in P\}.$$

It follows that $\mathbf{C}_F$ is also a polyhedron. For example, when $F = P$, we have $\mathbf{C}_P = \mathbb{R}^d$ and $\mathbf{1}_{\mathbf{C}_P}^w = \mathbf{1}_{\mathbb{R}^d}$.

Theorem 1 (Weighted Brianchon–Gram) *Let P be a simple polytope of dimension d in $\mathbb{R}^d$. We have*

$$\mathbf{1}_P^w = \sum_{F \preceq P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}^w, \tag{4}$$

where the sum is over all faces F of P.

When $q_1 = \cdots = q_N = 1$, we have the ordinary Brianchon–Gram formula. We illustrate this theorem for a triangle in Figure 3. (See also Figure 2.)

Proof Let Σ be the set of facets of P and let F be a proper face of P. Let $I_F = \{\sigma \in \Sigma \mid F \subset \sigma\}$. Then $F = \bigcap_{i \in I_F} \sigma_i$. Since P is simple, the cardinality of I_F is equal to the codimension of F and for all possible non-empty subsets J of I_F, the face $\bigcap_{i \in J} \sigma_i$ of P contains F. A straightforward computation shows that

$$\prod_{i \in I_F} q_i = 1 + \sum_{\phi \neq J \subset I_F} \prod_{i \in J} (q_i - 1), \tag{5}$$

where the q_i are arbitrary complex numbers assigned to the facets σ_i of P. We decompose P into all its faces F (including $F = P$). By (3) and (5), we have

$$\mathbf{1}_P^w = \mathbf{1}_P + \sum_{F \neq P} \prod_{i \in I_F} (q_i - 1) \mathbf{1}_F. \tag{6}$$

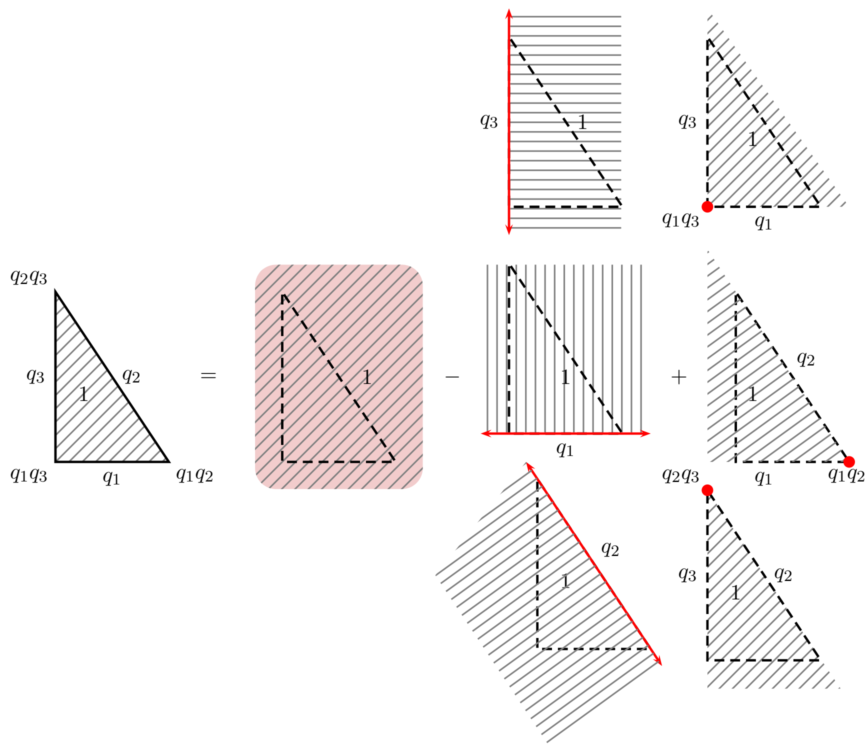

Figure 3: Weighted Brianchon–Gram decomposition for a triangle.

We can apply the ordinary Brianchon–Gram formula to each F and get

$$\mathbf{1}_P^w = \sum_{F \preceq P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F} + \sum_{F \neq P} \prod_{i \in I_F} (q_i - 1) \sum_{G \preceq F} (-1)^{\dim G} \mathbf{1}_{\mathbf{C}_G}. \tag{7}$$

Formula (6) also holds for the tangent cone $\mathbf{C}_F$ of P at F; that is

$$\mathbf{1}_{\mathbf{C}_F}^w = \mathbf{1}_{\mathbf{C}_F} + \sum_{H \neq \mathbf{C}_F} \prod_{i \in I_H} (q_i - 1) \mathbf{1}_H. \tag{8}$$

The proper faces H of the tangent cone $\mathbf{C}_F$ are in turn tangent cones of lower dimension associated to faces G of other faces of P. By regrouping the characteristic functions in (7) according to (8), we obtain

$$\mathbf{1}_P^w = \sum_{F \preceq P} (-1)^{\dim F} \mathbf{1}_{\mathbf{C}_F}^w. \qquad \blacksquare$$

Theorem 1 can be extended to non-simple polytopes where the only faces F which are *non-generic* (*i.e., F* has dimension f but $|I_F| \neq d - f$) are vertices. Indeed, in

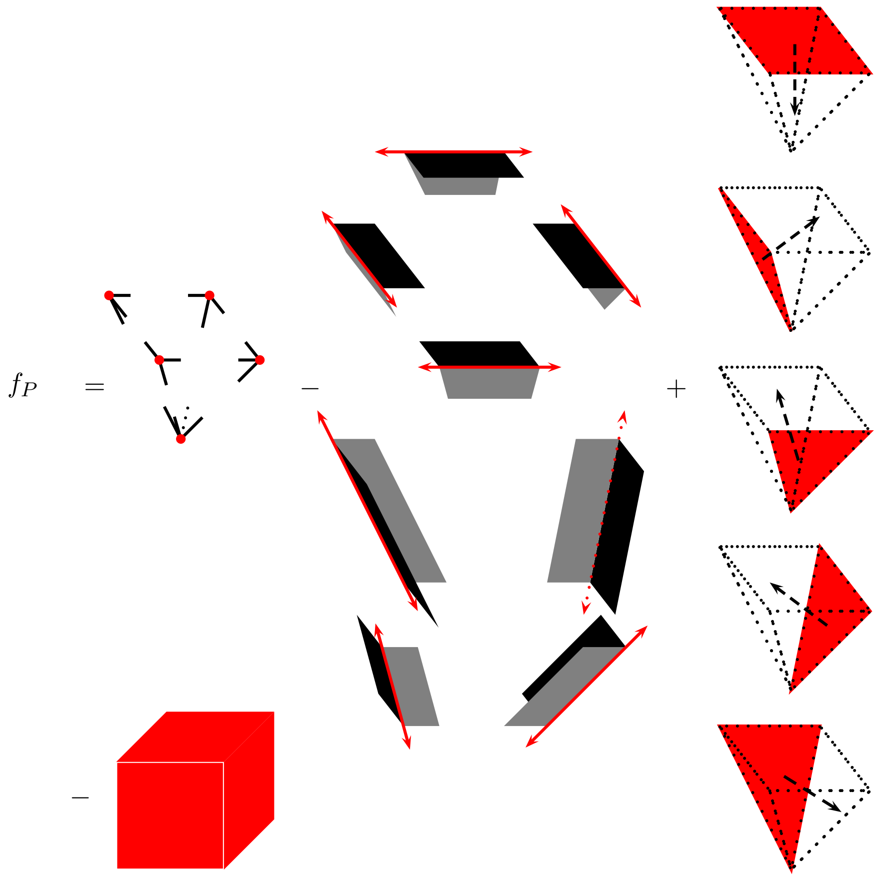

Figure 4: Weighted Brianchon–Gram decomposition for a pyramid.

dimension three, only vertices may be non-generic for an arbitrary non-simple polytope. In higher dimensions though, there may be other faces which are the intersection of *too many* facets. We illustrate this special extension for a pyramid in Figures 4–7.

Let P be any non-simple polytope of dimension d in $\mathbb{R}^d$ whose non-generic faces are only vertices. Let $V_{ns}(P)$ be the set of non-simple vertices of P. If $v \in V_{ns}(P)$, then $|I_v| > d$. (Recall that in the simple case $|I_v| = d$ for all vertices of P.) We chop off all the non-simple vertices v of P by taking hyperplanes σ_v very close to these vertices. We orient the σ_v away from v and denote the corresponding half-space associated to σ_v by H_v. We assign the constant value 1 to the hyperplanes σ_v. We obtain a simple polytope P_s for which formula (4) holds; that is

$$\mathbf{1}^w_{P_s} = \sum_{F_s \preceq P_s} (-1)^{\dim F_s} \mathbf{1}^w_{\mathbf{C}_{F_s}}, \tag{9}$$

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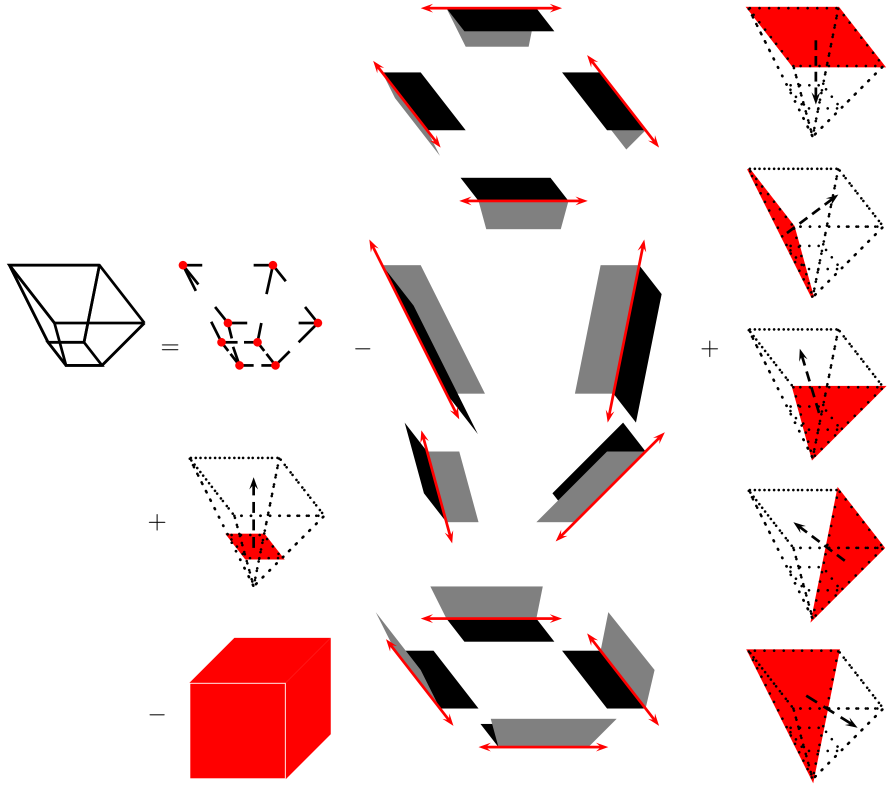

Figure 5: Weighted Brianchon–Gram for a truncated pyramid.

where the sum is over all the faces F_s of P_s. On the other hand, we set

$$f_P = \sum_{F \preceq P} (-1)^{\dim F} \mathbf{1}^w_{\mathbf{C}_F}, \tag{10}$$

where the sum is now over all the faces F of the non-simple polytope P.

We can clearly see that $\mathbf{1}^w_{\mathbf{C}_F} = \mathbf{1}^w_{\mathbf{C}_{F_s}}$ for all the faces of P and P_s which are not the non-simple vertices $v \in V_{ns(P)}$ and are not the faces of P_s contained in the various σ_v. Thus, we have

$$\mathbf{1}^w_{P_s} - f_P = \sum_{v \in V_{ns}(P)} \Big(\sum_{F_s \subset \sigma_v} (-1)^{\dim F_s} \mathbf{1}^w_{\mathbf{C}_{F_s}} - \mathbf{1}^w_{\mathbf{C}_v} \Big). \tag{11}$$

Notice that all the cross sections of $\mathbf{C}_v$ contained in H_v and parallel to σ_v, are simple polytopes (of the same type) in $\mathbb{R}^{d-1}$. This is a consequence of the imposed condition on P, that its only non-generic faces be vertices. We can apply the weighted Brianchon–Gram formula to these cross sections and get weighted characteristic functions over its faces, which are equal in absolute value but with opposite signs,

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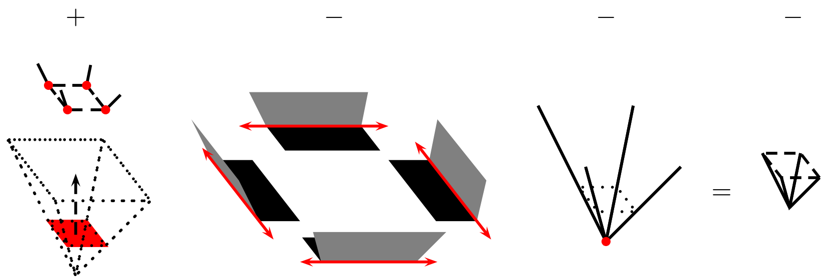

Figure 6: Key difference $\mathbf{1}^w_{P_s} - f_P$ for a pyramid.

to the weighted characteristic functions over the faces of the corresponding cross sections of the cones $\mathbf{C}_{F_s}$, where $F_s \subset \sigma_v$, for all $v \in V_{ns}(P)$. Then, we can write (11) as

$$\mathbf{1}^w_{P_s} - f_P = -\sum_{v \in V_{ns}(P)} \mathbf{1}^w_{\mathbf{C}_v \setminus H_v}. \tag{12}$$

On the other hand, it can be easily checked that

$$\mathbf{1}^w_{P_s} - \mathbf{1}^w_P = -\sum_{v \in V_{ns}(P)} \mathbf{1}^w_{\mathbf{C}_v \setminus H_v}. \tag{13}$$

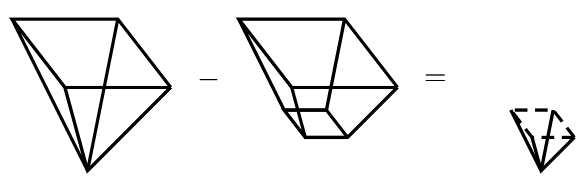

Figure 7: Difference of the pyramid and the truncated pyramid.

Therefore, we conclude that

$$\mathbf{1}^w_P = \sum_{F \preceq P} (-1)^{\dim F} \mathbf{1}^w_{\mathbf{C}_F}. \tag{14}$$

As an immediate consequence of Theorem 1 we obtain a weighted version of Brion's theorem[2] [Br1]. (See also [Br2, p. 82].)

[2]Brion already proved a more general weighted version of his formula in [Br1].

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Corollary 1 *For any simple polytope P, we have*

$$\mathbf{1}_P^w = g + \sum_v \mathbf{1}_{\mathbf{C}_v}^w,$$

where g is a linear combination of characteristic functions of cones with straight lines and the sum is over all vertices v of P.

This readily follows from grouping together all tangent cones in (4) that contain straight lines, just as in [H].

When P is a simple polyhedron and the same value $q \in \mathbb{C}$ is assigned to all its facets, the weighted characteristic function (3) gets the form

$$\mathbf{1}_P^w(x) = \mathbf{1}_P^q(x) = \begin{cases} q^{\operatorname{codim}(F)} & \text{if } x \in P, \\ 0 & \text{if } x \notin P, \end{cases} \tag{15}$$

where F is the face of P of smallest dimension containing x. Theorem 1 implies the weighted polytope decomposition of [A]. To show this, let ξ be a polarizing vector in $(\mathbb{R}^d)^*$ and let Δ be a simple polytope in $\mathbb{R}^d$. (We now follow the definitions and notation of [A].) We put together the faces of P according to where they achieve their minimum in the ξ-direction. We obtain

$$(-1)^{\#v}\mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v} = \sum_{\substack{v \preceq F \preceq P \\ \xi(v) \leq \xi(F)}} (-1)^{\dim F}\mathbf{1}_{\mathbf{C}_F}^q,$$

where $\#v$ denotes the number of edges of $\mathbf{C}_v$ *flipped* according to ξ and where $\mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v}$ is the weighted characteristic function of the ξ-polarized tangent cone $\mathbf{C}_v^\sharp$ defined in [A]. (To agree with the weighted formulas in [A] we use the substitution $q = 1/(1+y)$.) Then

$$\sum_v (-1)^{\#v}\mathbf{1}_{\mathbf{C}_v^\sharp}^{w_v} = \sum_v \sum_{\substack{v \preceq F \preceq P \\ \xi(v) \leq \xi(F)}} (-1)^{\dim F}\mathbf{1}_{\mathbf{C}_F}^q = \sum_F (-1)^{\dim F}\mathbf{1}_{\mathbf{C}_F}^q = \mathbf{1}_P^q.$$

Thus, the weighted polar decomposition of [A] is a direct consequence of Theorem 1, which can be proved exactly in the same fashion as in [L].

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