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APPROXIMATION NUMBERS OF COMPOSITION OPERATORS ON WEIGHTED BESOV SPACES OF ANALYTIC FUNCTIONS

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Abstract Li et al. [A spectral radius type formula for approximation numbers of composition operators, J. Funct. Anal. 267(12) (2014), 4753-4774] proved a spectral radius type formula for the approximation numbers of composition operators on analytic Hilbert spaces with radial weights and on H^p spaces, $p \geq 1$, involving Green capacity. We prove that their formula holds for a wide class of Banach spaces of analytic functions and weights.

Keywords: composition operators; approximation numbers; Besov spaces; condenser capacity; Bagby points

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1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc and let $\phi : \mathbb{D} \mapsto \mathbb{D}$ be a non-constant analytic self-map of the unit disc. The composition operator with symbol ϕ is defined by $C_\phi(f) := f \circ \phi$, for every analytic function f on $\mathbb{D}$. The properties of composition operators acting on several analytic function spaces on $\mathbb{D}$ have been studied extensively. The main interest is the connection between the operator theoretic behaviour of C_ϕ and the function theoretic behaviour of the symbol ϕ. We refer the interested reader in the books [5, 18] and the references therein for more information on composition operators and function theory. In the present paper, we will study the approximation numbers of composition operators.

Let X and Y be two Banach spaces and let $T : X \mapsto Y$ be a bounded linear operator. The approximation numbers $a_n(T)$, $n \in \mathbb{N}$, of T are defined by

$$a_n(T) = \inf_R \|T - R\|,$$

where $\|\cdot\|$ denotes the operator norm and the infimum is taken over all linear operators $R : X \mapsto Y$ with $\text{rank}(R) := \dim(R(X)) < n$. For the general theory of approximation numbers, see e.g. [23, Section III.G]. Here we will study composition operators acting on weighted Besov spaces of analytic functions on $\mathbb{D}$.

Let $w : \mathbb{D} \mapsto (0, +\infty]$ be a lower semicontinuous function on $L^1(\mathbb{D})$. For $p > 1$, the weighted Besov space B^p_w is the family of analytic functions f in $\mathbb{D}$ satisfying

$$\|f\|_{B^p_w} := |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^p w(z) dA(z) \right)^{1/p} < +\infty.$$

For $w(z) = (1 - |z|^2)^{p-2}$, we obtain the standard Besov space $B^p = B^p_w$, which is an important Möbius invariant space of analytic functions whose properties have been investigated extensively; see e.g [24]. For $p > 0$, the Hardy space H^p consists of the family of analytic functions f on $\mathbb{D}$ satisfying

$$\left(\sup_{r \in (0,1)} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < +\infty.$$

We will also use the norm

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|,$$

for the space H^∞ of bounded analytic functions on $\mathbb{D}$.

Our study was initiated by the work of Li et al. [14], where a spectral radius type formula was proved for the approximation numbers of composition operators on analytic Hilbert spaces with radial weights and on H^p spaces, $p \geq 1$, involving condenser capacity. There are several (equivalent) ways to define condenser capacity; here we will use the logarithmic energy integrals.

A condenser is a pair (E, F) where E and F are non-empty disjoint compact subsets of $\mathbb{C}$. We will denote by $S(E, F)$ the family of signed measures $\sigma = \sigma_E - \sigma_F$, where σ_E and σ_F are Borel probability measures supported on E and F, respectively. The energy of a measure $\sigma \in S(E, F)$ is defined by

$$I(\sigma) := \iint \log \frac{1}{|z - w|} d\sigma(z) d\sigma(w).$$

Although both the measure and the integrand in the above energy integral are not positive, it is true that $I(\sigma) > 0$, for every $\sigma \in S(E, F)$; see e.g. [13, p. 80]. Following Bagby [2], we define the equilibrium energy of (E, F) by

$$I(E, F) := \inf_{\sigma \in S(E,F)} I(\sigma)$$

and the capacity of (E, F) is given by

$$\mathrm{Cap}(E, F) := \frac{2\pi}{I(E, F)}.$$

We note that, in [14], the authors define condenser capacity to be the reciprocal of the equilibrium energy. This will explain the slight deviation by a factor 2π in our statements from the statements of the results in [14]. For other definitions, using Green energy

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integrals or Dirichlet integrals, and for more information about condenser capacity, we refer to [2, 6, 13].

In [14], the authors considered the case $p = 2$ and studied the Hilbert spaces B^2_w, for weights $w \in L^1(\mathbb{D})$ that satisfy the following additional properties:

(P1) w is continuous on $\mathbb{D}$,

(P2) w is radial; that is, $w(z) = w(|z|)$, $z \in \mathbb{D}$.

In particular, the well-known Dirichlet-type spaces corresponding to the weights $w(z) = (1-|z|^2)^s$, $s \in (-1, +\infty)$, and containing as special cases the standard Bergman space ($s = 2$), Hardy space ($s = 1$) and Dirichlet space ($s = 0$), are covered in the family B^2_w, for weights w satisfying (P1) and (P2). They proved the following equalities for the approximation numbers of composition operators.

Theorem A (Li *et al.* [14]). *Let $w \in L^1(\mathbb{D})$ be a weight satisfying the properties* (P1) *and* (P2) *and let $\phi : \mathbb{D} \mapsto \mathbb{D}$ be a non-constant analytic function in X, where X is either B^2_w or H^p, $p \in [1, +\infty)$. Then the following hold for the approximation numbers of $C_\phi : X \mapsto X$.*

(1) If $\|\phi\|_\infty < 1$,

$$\lim_{n\to\infty} (a_n(C_\phi))^{1/n} = \exp(-2\pi/\mathrm{Cap}(\partial\mathbb{D}, \overline{\phi(\mathbb{D})})), \tag{1.1}$$

(2) if $\|\phi\|_\infty = 1$,

$$\lim_{n\to\infty} (a_n(C_\phi))^{1/n} = 1. \tag{1.2}$$

Our purpose in this paper is to show that Equations (1.1) and (1.2) hold for a wider class of Banach spaces of analytic functions on $\mathbb{D}$. We note that, for $p \neq 2$, the weighted Besov spaces B^p_w and in particular the standard Besov spaces B^p are not covered in Theorem A. Even in the case $p = 2$, there are important weighted Hilbert spaces of analytic functions with weights that do not satisfy the properties (P1) or (P2) mentioned above. For example, the harmonically weighted Dirichlet spaces are obtained by weights of the form

$$w(z) = \int_{\partial\mathbb{D}} \frac{1-|z|^2}{|\zeta - z|^2} d\mu(\zeta),$$

where μ is a finite positive Borel measure on $\partial\mathbb{D}$, which in general does not satisfy (P2). In particular, the weights obtained by unit Dirac measures $\mu = \delta_\zeta$, $\zeta \in \partial\mathbb{D}$, which generate the well-known local Dirichlet spaces, are not radial; see e.g. [4, 15–17] and the book [7, Chapter 7] for more information about harmonically weighted Dirichlet spaces. More generally, for weights w that are positive superharmonic functions on $\mathbb{D}$, neither (P1) nor (P2) are satisfied in general; see e.g. [1, 3, 8, 19] for more information about Dirichlet spaces with superharmonic weights. We note that, by definition, superharmonic functions are lower semicontinuous but in general may not be continuous.

In [14], the authors are expressing the norm of the weighted analytic Hilbert spaces by an infinite series, using the assumption that the weights considered are radial. Then, they use results of Widom [22] on rational approximation of bounded analytic functions, to approximate the truncated power series expansions of the functions in the Hilbert space, on the image of the symbol of the composition operator. Here we adopt the arguments in the proof of Widom's result to approximate functions in B^p_w, using the rational functions corresponding to the Bagby points (see § 2) of the condenser $(\partial\mathbb{D}, \overline{\phi(\mathbb{D})})$. These rational functions have simple poles $\{a_i\}$ and simple zeros $\{b_i\}$, $i = 1, \ldots, n$, which are well separated (see [10, 11]) in the sense that

$$\min_{i\neq j} |a_i - a_j| \geq \frac{C}{n^2} \qquad \text{and} \qquad \min_{i\neq j} |b_i - b_j| \geq \frac{C}{n^2},$$

where $C > 0$ depends on the condenser $(\partial\mathbb{D}, \overline{\phi(\mathbb{D})})$. We obtain an explicit formula for a finite rank operator to estimate the approximation numbers of C_ϕ.

In the following section, we state several known results from function theory and potential theory that will be used in the proofs of our main results. In particular, the approach to condenser capacity via discrete energies will be described and the rational functions corresponding to the extremal points for the discrete energies will be used in our proof to approximate analytic functions on compact subsets of $\mathbb{D}$. In § 3, we will state and prove the result that the equality (1.1) holds for weighted Besov spaces B^p_w and the validity of the equality (1.2) will be proved in § 4.

2. Background material

In this section, we collect some results from function theory and potential theory.

2.1. Equilibrium measure and potential

The logarithmic potential of a positive Borel measure μ with compact support in $\mathbb{C}$ is the function

$$U_\mu(z) := \int \log \frac{1}{|z-w|} d\mu(w), \quad z \in \mathbb{C}.$$

We note that the potential U_μ is a harmonic function outside the support of μ. The logarithmic capacity of a compact set $K \subset \mathbb{C}$ is defined by

$$c(K) = \exp\left(-\inf_\mu \iint \log \frac{1}{|z-w|} d\mu(z) d\mu(w)\right),$$

where the above infimum is taken over all Borel probability measures μ supported on K.

Let (E, F) be a condenser and let $\sigma = \sigma_E - \sigma_F \in S(E, F)$. The potential of σ is defined by

$$U_\sigma(z) := \int \log \frac{1}{|z-w|} d\sigma(w) = U_{\sigma_E}(z) - U_{\sigma_F}(z), \qquad z \in \mathbb{C}.$$

Since U_{σ_E} is harmonic on $\mathbb{C} \setminus E$ and U_{σ_F} is harmonic on $\mathbb{C} \setminus F$, the potential U_σ is well defined for every $z \in \mathbb{C}$, although it may take the values $\pm\infty$.

Let (E, F) be a condenser with finite equilibrium energy. Then, there exists a unique measure $\tau \in S(E, F)$ such that $I(E, F) = I(\tau)$ and τ is called the equilibrium measure of (E, F). Also, according to the fundamental theorem of potential theory for condensers, there exist real numbers $V_E \geq 0$, $V_F \leq 0$ and Borel sets $Z_E \subset \partial E$, $Z_F \subset \partial F$ (possibly

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empty) having zero logarithmic capacity, such that the following equalities hold for the equilibrium energy and the equilibrium potential U_τ:

$$V_F \le U_\tau(z) \le V_E, \text{ for every } z \in \mathbb{C}, \tag{2.1}$$

$$U_\tau(z) = V_E, \text{ for every } z \in E \setminus Z_E, \tag{2.2}$$

$$U_\tau(z) = V_F, \text{ for every } z \in F \setminus Z_F, \tag{2.3}$$

$$I(E, F) = V_E - V_F. \tag{2.4}$$

When the open set $\mathbb{C} \setminus (E \cup F)$ is regular for the Dirichlet problem, the equilibrium potential is continuous on $\mathbb{C}$ and $Z_E = Z_F = \emptyset$. In particular, when the compact sets E and F are connected, the equilibrium potential satisfies a Hölder continuity property described in the following theorem proved by J. Siciak [20, pp. 205, 210].

Theorem B (Siciak [20]). *Let $(E,\, F)$ be a condenser, where E and F are non-degenerate continua. Let τ be the equilibrium measure of $(E,\, F)$. Then, there exist constants $C_1 = C_1(E,\, F) > 0$ and $\alpha = \alpha(E,\, F) \in (0,\, 1)$, such that*

$$|U_\tau(z) - V_E| \le C_1 \operatorname{dist}(z, E)^\alpha \tag{2.5}$$

and

$$|U_\tau(z) - V_F| \le C_1 \operatorname{dist}(z, F)^\alpha, \tag{2.6}$$

for every $z \in \mathbb{C}$.

2.2. Discrete energies

Let $(E,\, F)$ be a condenser and suppose that both sets E and F contain infinitely many points. For any integer $n \ge 2$, let

$$L_n(E, F) := \{(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in E^n \times F^n : \alpha_i \ne \alpha_j \text{ and } \beta_i \ne \beta_j, i \ne j\}.$$

The n-th discrete energy of $(E,\, F)$ is defined by

$$W_n(E, F) = \frac{1}{n(n-1)} \inf \left\{ \sum_{1 \le i < j \le n} \log \left(\frac{|\alpha_i - \beta_j||\alpha_j - \beta_i|}{|\alpha_i - \alpha_j||\beta_i - \beta_j|} \right) \right\},$$

where the infimum is taken over all point configurations $(\alpha_1,\, \ldots,\, \alpha_n,\, \beta_1,\, \ldots,\, \beta_n) \in L_n(E,\, F)$. Each configuration $(a_1,\, \ldots,\, a_n,\, b_1,\, \ldots,\, b_n) \in L_n(E,\, F)$ for which the above infimum is attained will be called an extremal configuration for $(E,\, F)$ and the points $a_1,\, \ldots,\, a_n,\, b_1,\, \ldots,\, b_n$ are called n-th Bagby points. From the compactness of E and F it follows that, for every integer $n \ge 2$, there exists an extremal configuration in $L_n(E,\, F)$.

Although every discrete signed measure in $S(E,\, F)$ has infinite energy, the above sum may be considered as a discrete version of the energy of a discrete measure having point masses at the points a_i and b_i, $i = 1,\, \ldots,\, n$. Bagby [2] proved the following theorem relating the equilibrium energy with the discrete energies $W_n(E,\, F)$ of a condenser.

Theorem C (Bagby [2]). *Let $(E,\, F)$ be a condenser and suppose that both sets E and F contain infinitely many points. Then the sequence $(W_n(E,\, F))$ is increasing and*

$$I(E,F) = \lim_{n\to\infty} W_n(E,F).$$

Moreover, assuming that $(a_1,\, \ldots,\, a_n,\, b_1,\, \ldots,\, b_n) \in L_n(E,\, F)$ is an extremal configuration and letting

$$\sigma_n = \frac{1}{n}\left(\sum_{i=1}^{n}\delta_{a_i} - \sum_{i=1}^{n}\delta_{b_i}\right) \in S(E,F),$$

it is true (see [2]) that the sequence of the measures σ_n converges in the weak-star sense to the equilibrium measure of $(E,\, F)$ and the potentials U_{σ_n} converge locally uniformly to the equilibrium potential of $(E,\, F)$ in $\mathbb{C}\setminus(E\cup F)$. We will need the following result concerning the rate of convergence of the potentials U_{σ_n} proved by Kloke [12, Theorem 2.7, p. 194] (see also [10] for condensers in higher-dimensional Euclidean spaces).

Theorem D (Kloke [12]). *Let $(E,\, F)$ be a condenser such that both E and F are unions of a finite number of mutually disjoint and non-degenerate continua. Let τ be the equilibrium measure of $(E,\, F)$. Also, for every integer $n \geq 2$, let*

$$(a_1,\ldots,a_n,b_1,\ldots,b_n) \in L_n(E,F)$$

be an extremal configuration for $(E,\, F)$ and let

$$\sigma_n = \frac{1}{n}\left(\sum_{i=1}^{n}\delta_{a_i} - \sum_{i=1}^{n}\delta_{b_i}\right) \in S(E,F).$$

Then, there exists a constant $C_2 = C_2(E,\, F) > 1$ such that

$$|U_\tau(z) - U_{\sigma_n}(z)| \leq \frac{32\log(C_2 n)}{n},$$

for every

$$z \in \left\{w\in\mathbb{C} : \operatorname{dist}(w,\partial E) \geq \frac{1}{n^2} \text{ and } \operatorname{dist}(w,\partial F) \geq \frac{1}{n^2}\right\}.$$

2.3. Diameters in the space of continuous functions

Let K be a compact subset of $\mathbb{D}$ and let $C(K)$ be the Banach space of continuous functions on K, equipped with the norm

$$\|f\|_K = \sup_{z\in K}|f(z)|, \qquad f\in C(K).$$

Let

$$B := \{f\in H^\infty : \|f\|_\infty \leq 1\}$$

be the unit ball in H^∞. Taking restrictions on K, we may consider B as a subset of $C(K)$. For every $n\in\mathbb{N}$, let X_n denote the family of n-dimensional linear subspaces of $C(K)$.

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The n-dimensional diameter of B in $C(K)$ is defined by

$$d_n(B, C(K)) := \inf_{E \in X_n} \left[\sup_{f \in B} \left(\inf_{g \in E} \|f - g\|_K \right) \right].$$

We will need the following result about the n-dimensional diameter of B in $C(K)$; see [22, Theorem 7, p. 353] or [9, p. 249].

Theorem E (Widom [22]). *Let K be a compact subset of $\mathbb{D}$. There exists a constant $C_3 := C_3(K) > 0$ such that*

$$d_n(B, C(K)) \geq C_3 \exp\left(-2\pi n / \mathrm{Cap}(\partial\mathbb{D}, K)\right), \quad n \in \mathbb{N}.$$

3. Symbols with compact image in the unit disc

In this section, we will prove that the asymptotic formula (1.1) holds for composition operators on weighted Besov spaces. Before stating and proving our first main result, we will prove some helpful lemmas. The first one is standard, it concerns the norms of the evaluation functionals and it follows from the subharmonicity property of the modulus of an analytic function. We include its proof for the convenience of the reader. We will denote by $D(z, r)$ the open disc centred at $z \in \mathbb{C}$ with radius $r > 0$.

Lemma 3.1. *Let $w : \mathbb{D} \mapsto (0, +\infty]$ be a lower semicontinuous function on $L^1(\mathbb{D})$ and let $p > 1$. For every $a \in \mathbb{D}$, the linear functionals $L_a(f) = f(a)$ and $T_a(f) = f'(a)$ are bounded on B^p_w. Also, for every compact subset K of $\mathbb{D}$,*

$$\sup_{a \in K} \|L_a\| < +\infty \text{ and } \sup_{a \in K} \|T_a\| < +\infty.$$

Proof. Let K be a compact subset of $\mathbb{D}$ and let $d = \mathrm{dist}(K, \partial\mathbb{D})/2$. Since w is lower semicontinuous on $\mathbb{D}$, it attains its lower bound on any compact subset of $\mathbb{D}$. In particular,

$$M := \min\left\{ w(z) : z \in \overline{D(0, 1 - d)} \right\} > 0.$$

Let $f \in B^p_w$ and $a \in K$. From the subharmonicity of $|f'|^p$,

$$\begin{aligned} |T_a(f)|^p = |f'(a)|^p &\leq \frac{1}{\pi d^2} \int_{D(a,d)} |f'(z)|^p dA(z) \\ &\leq \frac{1}{\pi d^2 M} \int_{D(a,d)} |f'(z)|^p w(z) dA(z) \\ &\leq \frac{1}{\pi d^2 M} \int_{\mathbb{D}} |f'(z)|^p w(z) dA(z) \end{aligned}$$

and

$$|T_a(f)| \leq \left(\frac{1}{\pi d^2 M} \right)^{1/p} \left(\int_{\mathbb{D}} |f'(z)|^p w(z) dA(z) \right)^{1/p} \leq \left(\frac{1}{\pi d^2 M} \right)^{1/p} \|f\|_{B^p_w}.$$

Since $a \in K \subset \mathbb{D}$ were arbitrary, we conclude that T_a is bounded on B^p_w for every $a \in \mathbb{D}$ and

$$\sup_{a\in K} \|T_a\| \leq \left(\frac{1}{\pi d^2 M}\right)^{1/p} < +\infty. \tag{3.1}$$

The corresponding results for L_a follow from (3.1) and the inequality

$$\begin{aligned}|L_a(f)| = |f(a)| &= \left| f(0) + \int_0^a f'(z)dz \right| \\ &\leq |f(0)| + |a| \Big(\sup_{b\in[0,a]} \|T_b\| \Big) \|f\|_{B^p_w} \\ &\leq \Big(1 + \sup_{b\in[0,a]} \|T_b\|\Big) \|f\|_{B^p_w}.\end{aligned}$$

□

In the following lemma, we will describe the symbols ϕ, with $\|\phi\|_\infty < 1$, for which $C_\phi : B^p_w \mapsto B^p_w$ is bounded.

Lemma 3.2. *Let $w : \mathbb{D} \mapsto (0, +\infty]$ be a lower semicontinuous function on $L^1(\mathbb{D})$, let $p > 1$ and let $\phi \in H^\infty$ satisfying $\|\phi\|_\infty < 1$. Then $C_\phi : B^p_w \mapsto B^p_w$ is bounded if and only if $\phi \in B^p_w$.*

Proof. Suppose that $\phi \in B^p_w$. Then, since $\overline{\phi(\mathbb{D})}$ is a compact subset of $\mathbb{D}$, we have

$$\begin{aligned}\int_{\mathbb{D}} |C_\phi(f)'(z)|^p w(z) dA(z) &= \int_{\mathbb{D}} |f'(\phi(z))|^p |\phi'(z)|^p w(z) dA(z) \\ &\leq \left(\sup_{z\in\overline{\phi(\mathbb{D})}} |f'(z)|^p \right) \|\phi\|^p_{B^p_w} < +\infty,\end{aligned}$$

and $C_\phi(f) \in B^p_w$, for every $f \in B^p_w$. By Lemma 3.1, we get that convergence in B^p_w implies uniform convergence on compact subsets of $\mathbb{D}$. From the closed graph theorem, it follows that $C_\phi : B^p_w \mapsto B^p_w$ is bounded.

Conversely, we have $\phi = C_\phi(I) \in B^p_w$, where $I \in B^p_w$ is the identity function. □

The main step in the proof of the first main result will be the approximation of functions in B^p_w by rational functions on the closure of the image of the symbol of the composition operator. The success of this approach on getting a lower bound for the approximation numbers is based on the following result.

Let X and Y be two Banach spaces and let $T : X \mapsto Y$ be a bounded linear operator. For every integer $n \geq 1$, we define

$$\tilde{a}_n(T) := \inf \{\|T - R\| : R : X \mapsto Y \text{ is linear, rank}(R) < n \text{ and } R(X) \subset T(X)\}.$$

Clearly, $a_n(T) \leq \tilde{a}_n(T)$. In the other direction, we have the following result.

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Proposition 3.3. *Let X and Y be two Banach spaces and let $T : X \mapsto Y$ be a bounded linear operator. For every integer $n \geq 1$, $\tilde{a}_n(T) \leq n a_n(T)$.*

Proof. Let $\epsilon > 0$. Let $R : X \mapsto Y$ be a linear operator satisfying $\operatorname{rank}(R) < n$ and $\|T - R\| < a_n(T) + \epsilon$. Set $m := \operatorname{rank}(R)$. Then $X/\ker(R)$ is an m-dimensional normed space. By Auerbach's lemma (see e.g. [23, II.E.11, p. 75]), there exist $\xi_1, \ldots, \xi_m \in (X/\ker(R))$ and $\psi_1, \ldots, \psi_m \in (X/\ker(R))^*$ such that, for all $j, k = 1, \ldots, m$,

$$\|\xi_j\| = 1, \quad \|\psi_k\| = 1 \text{ and } \psi_k(\xi_j) = \delta_{jk},$$

where

$$\delta_{jk} = \begin{cases} 0, & \text{if } j \neq k, \\ 1, & \text{if } j = k. \end{cases}$$

Writing $\pi : X \mapsto X/\ker(R)$ for the quotient map, for each j, we may pick $x_j \in X$ such that $\pi(x_j) = \xi_j$ and $\|x_j\| < 1 + \epsilon$. Also, for each k, define $\varphi_k \in X^*$ by $\varphi_k := \psi_k \circ \pi$. Clearly we have $\|\varphi_k\| \leq 1$ for each k and $\varphi_k(x_j) = \delta_{jk}$, for all j, k. Thus

$$R = \sum_{k=1}^{m} R x_k \otimes \varphi_k, \tag{3.2}$$

because every vector in X can be written as a linear combination of $x_1, \ldots, x_m$ and a vector in $\ker(R)$, and the two sides of (3.2) agree on all such vectors. Define

$$\tilde{R} := \sum_{k=1}^{m} T x_k \otimes \varphi_k.$$

Clearly $\tilde{R} : X \mapsto Y$ is linear with $\operatorname{rank}(\tilde{R}) \leq m$ and $\tilde{R}(X) \subset T(X)$. Hence

$$\begin{aligned} \tilde{a}_n(T) &\leq \|T - \tilde{R}\| \\ &\leq \|T - R\| + \|R - \tilde{R}\| \\ &= \|T - R\| + \left\| \sum_{k=1}^{m} (R x_k - T x_k) \otimes \varphi_k \right\| \\ &\leq \|T - R\| + \sum_{k=1}^{m} \|R - T\| \|x_k\| \|\varphi_k\| \\ &\leq \|T - R\| (1 + m(1 + \epsilon)) \\ &\leq (a_n(T) + \epsilon) n (1 + \epsilon). \end{aligned}$$

Letting $\epsilon \to 0$ we obtain the result. □

We now proceed to state and prove our first main result.

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Theorem 3.4. *Let $w : \mathbb{D} \mapsto (0, +\infty]$ be a lower semicontinuous function on $L^1(\mathbb{D})$, let $p > 1$ and let $\phi \in B_w^p$ satisfying $\|\phi\|_\infty < 1$. Then the formula*

$$\lim_{n\to\infty} (a_n(C_\phi))^{1/n} = \exp(-2\pi/\mathrm{Cap}(\partial\mathbb{D}, \overline{\phi(\mathbb{D})})) \tag{3.3}$$

holds, for the approximation numbers of $C_\phi : B_w^p \mapsto B_w^p$.

Proof. Let $\partial\mathbb{D} := \mathbb{T}$, $K := \overline{\phi(\mathbb{D})}$ and note that both $\mathbb{T}$ and K are non-degenerate continua. Therefore, the condenser $(\mathbb{T}, K)$ has a positive capacity. Let τ be the equilibrium measure of $(\mathbb{T}, K)$. For every integer $n \geq 2$, let $(a_1, \ldots, a_n, b_1, \ldots, b_n) \in L_n(\mathbb{T}, K)$ be an extremal configuration and let

$$\sigma_n = \frac{1}{n}\left(\sum_{i=1}^n \delta_{a_i} - \sum_{i=1}^n \delta_{b_i}\right) \in S(\mathbb{T}, K).$$

The potential of σ_n is

$$\begin{aligned} U_{\sigma_n}(z) &= \frac{1}{n}(\sum_{i=1}^n \log\frac{1}{|z-a_i|} - \sum_{i=1}^n \log\frac{1}{|z-b_i|}) \\ &= \frac{1}{n}\log\left|\frac{(z-b_1)\cdots(z-b_n)}{(z-a_1)\cdots(z-a_n)}\right|. \end{aligned}$$

Consider the rational function

$$R_n(z) = \frac{(z-b_1)\cdots(z-b_n)}{(z-a_1)\cdots(z-a_n)}$$

and note that R_n is analytic on $\mathbb{D}$ and $U_{\sigma_n}(z) = \frac{1}{n}\log|R_n(z)|$. First, we will use the rational functions R_n to obtain finite rank linear operators that will approximate C_ϕ in order to prove that

$$\limsup_{n\to\infty} (a_n(C_\phi))^{1/n} \leq \exp(-2\pi/\mathrm{Cap}(\mathbb{T}, K)). \tag{3.4}$$

We start by establishing upper and lower bounds for $|R_n|$.

Fix $\epsilon \in (0, \mathrm{dist}(\mathbb{T}, K)/2)$ and $n \in \mathbb{N}$ satisfying $1/n^2 < \mathrm{dist}(\mathbb{T}, K)/2$. Let $\gamma_\epsilon := \partial D(0, 1-\epsilon)$ (positively oriented circle) and $A_n := \{z \in \mathbb{D} : \mathrm{dist}(z, K) = 1/n^2\}$. From Theorem B and Theorem D, it follows that

$$U_{\sigma_n}(z) \leq U_\tau(z) + \frac{32\log(C_2 n)}{n} \leq V_K + \frac{C_1}{n^{2\alpha}} + \frac{32\log(C_2 n)}{n}, \tag{3.5}$$

for every $z \in A_n$ and

$$U_{\sigma_n}(z) \geq U_\tau(z) - \frac{32\log(C_2 n)}{n} \geq V_\mathbb{T} - C_1\epsilon^\alpha - \frac{32\log(C_2 n)}{n}, \tag{3.6}$$

for every $z \in \gamma_\epsilon$, where the constants $C_1 > 0$, $C_2 > 1$ and $\alpha \in (0, 1)$ depend only on the condenser $(\mathbb{T}, K)$. From (3.5) and (3.6), we obtain the inequalities

$$|R_n(z)| \leq \exp(nV_K + C_1 n^{1-2\alpha} + 32\log(C_2 n)), \tag{3.7}$$

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for every $z \in A_n$ and

$$|R_n(z)| \geq \exp(nV_{\mathbb{T}} - nC_1\epsilon^\alpha - 32\log(C_2 n)), \tag{3.8}$$

for every $z \in \gamma_\epsilon$. Also, from the maximum principle, it follows that the inequality (3.7) holds for every z in the component of $\mathbb{C} \setminus A_n$ containing K.

For $f \in B^p_w$, let

$$F_n(z,f) := \frac{R_n(z)}{2\pi i}\int_{\gamma_\epsilon}\frac{f'(\zeta)}{R_n(\zeta)(\zeta - z)}d\zeta, \qquad z \in K.$$

From the residue theorem, we get that

$$F_n(z,f) = f'(z) - H_n(z,f), \tag{3.9}$$

where

$$H_n(z,f) := R_n(z)\sum_{i=1}^n \frac{f'(b_i)}{R_n'(b_i)(b_i - z)}.$$

Let

$$I_n(z,f) := f(\phi(0)) + \sum_{i=1}^n f'(b_i)\int_0^z \frac{R_n(\zeta)}{R_n'(b_i)(b_i-\zeta)}d\zeta$$

be a primitive of $H_n(\cdot,\, f)$ on $\mathbb{D}$. We note that

$$\begin{aligned}\int_{\mathbb{D}} |(I_n(\phi(z),f))'(z)|^p w(z)dA(z) &= \int_{\mathbb{D}} |I_n'(\phi(z),f)|^p|\phi'(z)|^p w(z)dA(z)\\ &\leq \left(\sup_{z\in K}|I_n'(z,f)|^p\right)\|\phi\|^p_{B^p_w} < +\infty\end{aligned}$$

and $I_n(\cdot,\, f)\circ\phi \in B^p_w$, for every $f \in B^p_w$. We consider the operator $J_n : B^p_w \mapsto B^p_w$, defined by $J_n(f) := I_n(\cdot,\, f)\circ\phi$. Then J_n is a bounded linear operator and $J_n(B^p_w)$ is contained in the linear span of the functions

$$z \mapsto \int_0^{\phi(z)} \frac{R_n(\zeta)}{R_n'(b_i)(b_i-\zeta)}d\zeta \in B^p_w, \quad i = 1,\ldots,n.$$

We obtain that $\operatorname{rank}(J_n) \leq n$. Therefore, for every $f \in B^p_w$,

$$\begin{aligned} a_{n+1}(C_\phi) &\leq \|C_\phi - J_n\|\\ &= \sup_{\|f\|_{B^p_w}=1}\left(\int_{\mathbb{D}}|(C_\phi(f) - J_n(f))'(z)|^p w(z)dA(z)\right)^{1/p}\\ &= \sup_{\|f\|_{B^p_w}=1}\left(\int_{\mathbb{D}}|f'(\phi(z)) - H_n(\phi(z),f)|^p|\phi'(z)|^p w(z)dA(z)\right)^{1/p}\\ &= \sup_{\|f\|_{B^p_w}=1}\left(\int_{\mathbb{D}}|F_n(\phi(z),f)|^p|\phi'(z)|^p w(z)dA(z)\right)^{1/p}, \end{aligned} \tag{3.10}$$

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where in (3.10) the equality (3.9) has been used. From the inequalities (3.7) and (3.8), we get that, for every $z \in K$,

$$\begin{aligned} |F_n(z,f)| &\le \frac{|R_n(z)|}{2\pi} \int_{\gamma_\epsilon} \frac{|f'(\zeta)|}{|R_n(\zeta)||\zeta - z|} |d\zeta| \\ &\le \frac{\exp(n(V_K - V_{\mathbb{T}}) + C_1 n^{1-2\alpha} + nC_1\epsilon^\alpha + 64\log(C_2 n))}{(\mathrm{dist}(\mathbb{T},K)/2)} \cdot \\ &\quad \cdot (1-\epsilon) \sup_{\zeta \in \gamma_\epsilon} |f'(\zeta)|. \end{aligned} \tag{3.11}$$

We note that (3.11) holds for every analytic function f on $\mathbb{D}$. From (2.4), (3.10) and (3.11), it follows that

$$\begin{aligned} &a_{n+1}(C_\phi) \\ &\le \frac{\exp(-nI(\mathbb{T},K) + C_1 n^{1-2\alpha} + nC_1\epsilon^\alpha + 64\log(C_2 n))}{(\mathrm{dist}(\mathbb{T},K)/2)} \cdot \\ &\cdot \left(\int_{\mathbb{D}} |\phi'(z)|^p w(z) dA(z)\right)^{1/p} (1-\epsilon) \sup_{\zeta \in \gamma_\epsilon} |f'(\zeta)|. \end{aligned} \tag{3.12}$$

Raising (3.12) to the power $1/(n+1)$ and letting $n \to +\infty$ we get

$$\limsup_{n \to +\infty} (a_n(C_\phi))^{1/n} \le \exp\left(-I(\mathbb{T},K) + C_1\epsilon^\alpha\right).$$

Letting $\epsilon \to 0$, we obtain (3.4).

Next, we will use Proposition 3.3 to get a lower bound for the approximation numbers of C_ϕ. Let $\epsilon > 0$, let $n \in \mathbb{N}$ and let $P_m : B^p_w \mapsto B^p_w$ be a linear operator with $\mathrm{rank}(P_m) = m < n$, satisfying $P_m(B^p_w) \subset C_\phi(B^p_w)$. Let $E := \{h' : \mathbb{D} \mapsto \mathbb{C} : h \circ \phi \in P_m(B^p_w)\}$ and note that, taking restriction on the set K, E is a linear subspace of $C(K)$ with $\dim(E) = m$. Therefore,

$$d_m(B, C(K)) \le \sup_{f \in B} (\inf_{g \in E} \|f - g\|_K), \tag{3.13}$$

where B is the unit ball in H^∞. Let $f_0 \in B$ such that

$$\sup_{f \in B} (\inf_{g \in E} \|f - g\|_K) \le (1+\epsilon) \inf_{g \in E} \|f_0 - g\|_K. \tag{3.14}$$

From Theorem E, there exists $C_3 > 0$ such that

$$C_3 \exp(-nI(\mathbb{T},K)) \le C_3 \exp(-mI(\mathbb{T},K)) \le d_m(B, C(K)). \tag{3.15}$$

From (3.13), (3.14) and (3.15), we obtain that

$$C_3 \exp(-nI(\mathbb{T},K)) \le (1+\epsilon) \inf_{g \in E} \|f_0 - g\|_K. \tag{3.16}$$

We will now estimate $\|C_\phi - P_m\|$. Let I_0 be a primitive of f_0 on $\mathbb{D}$, satisfying $I_0(\phi(0)) = 0$. We have

$$\|I_0\|_{B^p_w} = \left(\int_{\mathbb{D}} |f_0(z)|^p w(z) dA(z)\right)^{1/p} \le \left(\int_{\mathbb{D}} w(z) dA(z)\right)^{1/p} := C_w.$$

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Therefore, I_0/C_w lies in the unit sphere of B_w^p. Let $P_m(I_0/C_w) = h_0 \circ \phi$ and note that $C_w h_0' \in E$. From (3.16) we obtain that

$$\begin{aligned}\|C_\phi - P_m\| &\geq \|C_\phi(I_0/C_w) - P_m(I_0/C_w)\|_{B_w^p} \\ &= \frac{1}{C_w}\left(\int_{\mathbb{D}} |f_0(\phi(z)) - C_w h_0'(\phi(z))|^p |\phi'(z)|^p w(z) dA(z)\right)^{1/p} \\ &\geq \frac{1}{C_w} \inf_{g\in E} \|f_0 - g\|_K \|\phi\|_{B_w^p} \\ &\geq \frac{C_3\|\phi\|_{B_w^p}}{C_w(1+\epsilon)} \exp(-nI(\mathbb{T}, K)). \end{aligned} \tag{3.17}$$

Taking infimum over all linear operators $P_m : B_w^p \mapsto B_w^p$ with $\operatorname{rank}(P_m) = m < n$ and $P_m(B_w^p) \subset C_\phi(B_w^p)$, we obtain that

$$\tilde{a}_n(C_\phi) \geq \frac{C_3\|\phi\|_{B_w^p}}{C_w(1+\epsilon)} \exp(-nI(\mathbb{T}, K)). \tag{3.18}$$

From Proposition 3.3 and (3.18), it follows that

$$a_n(C_\phi) \geq \frac{C_3\|\phi\|_{B_w^p}}{C_w(1+\epsilon)n} \exp(-nI(\mathbb{T}, K)). \tag{3.19}$$

Raising (3.19) to the power $1/n$ and letting $n \to +\infty$, we get

$$\liminf_{n\to+\infty} (a_n(C_\phi))^{1/n} \geq \exp(-I(\mathbb{T}, K)). \tag{3.20}$$

Finally, the equality (3.3) follows from (3.4) and (3.20). The proof is complete. □

4. Symbols with non-compact image in the unit disc

In this section, we will prove that the asymptotic formula (1.2) holds for composition operators on weighted Besov spaces. We will see that formula (1.2) follows from Theorem 3.4 and the following well-known properties.

The first is a property of approximation numbers (see e.g. [23, III.G.2, p. 237]).

Theorem F (Wojtaszczyk [23]). *Let X be a Banach space and let T, $S : X \mapsto X$ be two bounded linear operators. For all integers n, $m \geq 1$,*

$$a_{n+m-1}(T \circ S) \leq a_n(T) a_m(S).$$

The second property is a lower bound for the capacity of a condenser involving the diameters of its plates and the distance between them; see e.g. [21, Lemma 7.38, p. 95].

Theorem G (Vuorinen [21]). *There exists a constant $C_4 > 0$ such that, for every condenser (E, F), where both E and F are non-degenerate continua, we have*

$$\operatorname{Cap}(E, F) \geq C_4 \log\left(1 + \frac{\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}}{\operatorname{dist}(E, F)}\right).$$

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We will now state and prove our second main result.

Theorem 4.1. *Let $w : \mathbb{D} \mapsto (0, +\infty]$ be a lower semicontinuous function on $L^1(\mathbb{D})$, let $p > 1$ and let $\phi \in B^p_w$ satisfying $\|\phi\|_\infty = 1$. Then the formula*

$$\lim_{n\to\infty} (a_n(C_\phi))^{1/n} = 1 \tag{4.1}$$

holds, for the approximation numbers of $C_\phi : B^p_w \mapsto B^p_w$.

Proof. Let $r \in (0, 1)$ and consider the bounded linear operator $L_r : B^p_w \mapsto B^p_w$, defined by $L_r f(z) = f(rz)$, $f \in B^p_w$. Also, note that $C_\phi \circ L_r = C_{r\phi}$, $\|r\phi\|_\infty = r$ and $\overline{r\phi(\mathbb{D})} = \overline{\{rz : z \in \phi(\mathbb{D})\}}$ is a compact subset of $\mathbb{D}$. From Theorem F, it follows that

$$a_n(C_{r\phi}) = a_n(C_\phi \circ L_r) = a_{n+1-1}(C_\phi \circ L_r) \leq a_n(C_\phi)a_1(L_r) = a_n(C_\phi)\|L_r\|,$$

for every $n \in \mathbb{N}$. Therefore,

$$\frac{a_n(C_{r\phi})^{1/n}}{\|L_r\|^{1/n}} \leq a_n(C_\phi)^{1/n}, \quad n \in \mathbb{N}.$$

Letting $n \to +\infty$, from Theorem 3.4, we obtain that

$$\exp(-2\pi/\mathrm{Cap}(\partial\mathbb{D}, \overline{r\phi(\mathbb{D})})) \leq \liminf_{n\to+\infty}(a_n(C_\phi))^{1/n}. \tag{4.2}$$

From Theorem G, it follows that

$$\begin{aligned}\mathrm{Cap}(\partial\mathbb{D}, \overline{r\phi(\mathbb{D})}) &\geq C_4 \log(1 + \frac{\min\{\mathrm{diam}(\partial\mathbb{D}), \mathrm{diam}(\overline{r\phi(\mathbb{D})})\}}{\mathrm{dist}(\partial\mathbb{D}, \overline{r\phi(\mathbb{D})})}) \\ &= C_4 \log(1 + \frac{\mathrm{diam}(\overline{r\phi(\mathbb{D})})}{1-r}). \end{aligned}\tag{4.3}$$

Letting $r \to 1$, from (4.2) and (4.3), we get

$$1 \leq \liminf_{n\to+\infty}(a_n(C_\phi))^{1/n}. \tag{4.4}$$

On the other hand,

$$\limsup_{n\to+\infty}(a_n(C_\phi))^{1/n} \leq \limsup_{n\to+\infty}(a_1(C_\phi))^{1/n} = 1. \tag{4.5}$$

The conclusion follows from (4.4) and (4.5). □

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References

1. A. Aleman, *The multiplication operator on Hilbert spaces of analytic functions*, (Habilitation, FernUniversität Hagen, 1993).
2. T. Bagby, The modulus of a plane condenser, *J. Math. Mech.* **17** (1967), 315–329.
3. G. Bao, N. G. Göğüş and S. Pouliasis, On Dirichlet spaces with a class of superharmonic weights, *Canad. J. Math.* **70** (2018), 721–741.
4. C. Costara and T. Ransford, Which de Branges-Rovnyak spaces are Dirichlet spaces (and vice versa)?, *J. Funct. Anal.* **265** (2013), 3204–3218.
5. C. C. Cowen and B. D. MacCluer, *Composition operators on spaces of analytic functions*, Studies in Advanced Mathematics (CRC Press, 1995).
6. V. N. Dubinin, *Condenser capacities and symmetrization in geometric function theory*, Translated from the Russian by Nikolai G. Kruzhilin (Springer, Basel, 2014).
7. O. El-Fallah, K. Kellay, J. Mashreghi and T. Ransford, *A primer on the Dirichlet space*, Cambridge Tracts in Mathematics, Volume 203 (Cambridge University Press, 2014).
8. O. El-Fallah, K. Kellay, H. Klaja, J. Mashreghi and T. Ransford, Dirichlet spaces with superharmonic weights and de Branges-Rovnyak spaces, *Complex Anal. Operator Theory* **10** (2016), 97–107.
9. S. D. Fisher, *Function theory on planar domains. A second course in complex analysis*, Pure and Applied Mathematics (John Wiley and Sons, 1983).
10. M. Götz, Approximating the condenser equilibrium distribution, *Math. Z.* **236**(4) (2001), 699–715.
11. H. Kloke, Punktsysteme mit extremalen Eigenschaften für ebene Kondensatoren, Dissertation Universität Dortmund, 1984.
12. H. Kloke, On the capacity of a plane condenser and conformal mapping, *J. Reine Angew. Math.* **358** (1985), 179–201.
13. N. S. Landkof, *Foundations of Modern Potential Theory* (Springer-Verlag, 1972).
14. D. Li, H. Queffélec and L. Rodríguez-Piazza, A spectral radius type formula for approximation numbers of composition operators, *J. Funct. Anal.* **267**(12) (2014), 4753–4774.
15. S. Richter, A representation theorem for cyclic analytic two-isometries, *Trans. Amer. Math. Soc.* **328** (1991), 325–349.
16. S. Richter and C. Sundberg, A formula for the local Dirichlet integral, *Michigan Math. J.* **38** (1991), 355–379.
17. D. Sarason, Local Dirichlet spaces as de Branges-Rovnyak spaces, *Proc. Amer. Math. Soc.* **125** (1997), 2133–2139.
18. J. H. Shapiro, *Composition operators and classical function theory*, Universitext, Tracts in Mathematics (Springer-Verlag, 1993).
19. S. M. Shimorin, Complete Nevanlinna-Pick property of Dirichlet-type spaces, *J. Funct. Anal.* **191** (2002), 276–296.
20. J. Siciak, Wiener's type regularity criteria on the complex plane, *Ann. Polon. Math.* **66** (1997), 203–221.
21. M. Vuorinen, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math. Volume 1319 (Springler-Verlag, 1988).
22. H. Widom, Rational approximation and n-dimensional diameter, *J. Approx. Theory* **5** (1972), 343–361.
23. P. Wojtaszczyk, *Banach spaces for analysts*, Cambridge Studies in Advanced Mathematics, Volume 25 (Cambridge University Press, 1991).
24. K. Zhu, *Operator theory in function spaces*, 2nd edn, Mathematical Surveys and Monographs, Volume 138 (American Mathematical Society, 2007).