

Water entry of deformable spheres

Randy C. Hurd^{1,†}, Jesse Belden², Michael A. Jandron², D. Tate Fanning³,
Allan F. Bower⁴ and Tadd T. Truscott^{1,†}

¹Department of Mechanical and Aerospace Engineering, Utah State University, Logan, UT 84322, USA

²Naval Undersea Warfare Center Division Newport, Newport, RI 02841, USA

³Department of Mechanical Engineering, Brigham Young University, Provo, UT 84602, USA

⁴School of Engineering, Brown University, Providence, RI 02912, USA

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When a rigid body collides with a liquid surface with sufficient velocity, it creates a splash curtain above the surface and entrains air behind the sphere, creating a cavity below the surface. While cavity dynamics has been studied for over a century, this work focuses on the water entry characteristics of deformable elastomeric spheres, which has not been studied. Upon free surface impact, an elastomeric sphere deforms significantly, giving rise to large-scale material oscillations within the sphere resulting in unique nested cavities. We study these phenomena experimentally with high-speed imaging and image processing techniques. The water entry behaviour of deformable spheres differs from rigid spheres because of the pronounced deformation caused at impact as well as the subsequent material vibration. Our results show that this deformation and vibration can be predicted from material properties and impact conditions. Additionally, by accounting for the sphere deformation in an effective diameter term, we recover previously reported characteristics for time to cavity pinch off and hydrodynamic force coefficients for rigid spheres. Our results also show that velocity change over the first oscillation period scales with the dimensionless ratio of material shear modulus to impact hydrodynamic pressure. Therefore, we are able to describe the water entry characteristics of deformable spheres in terms of material properties and impact conditions.

Key words: aerodynamics, flow-structure interactions, interfacial flows (free surface)

1. Introduction

Water entry has been studied for over 100 years, with the earliest images taken by Worthington at the turn of the century (Worthington 1908), and much of the foundational work performed in the 1940s and 50s with military application in mind (May & Woodhull 1948; Richardson 1948; May 1952). The topic of water entry is still of interest today with several significant research papers published in the last 20 years investigating topics such as cavity physics, projectile dynamics and even ricochet off the water surface (Seddon & Moatamedi 2006; Duclaux *et al.* 2007;

[†] Email addresses for correspondence: randyhurd@gmail.com, taddtruscott@gmail.com

Duez *et al.* 2007; Aristoff & Bush 2009; Truscott, Epps & Belden 2014; Belden *et al.* 2016, respectively).

Cavity characteristics vary with Froude number, Bond number, Capillary number and by varying object geometry, rotation and wetting angle (Truscott *et al.* 2014). For example, low-speed impact events with sufficiently small capillary numbers will not form subsurface cavities (Duez *et al.* 2007). When a cavity forms it is often described by the manner in which the cavity collapses (or pinches off). Cavities are categorized according to the depth at which pinch off occurs, and these categories include: surface seal, deep seal, shallow seal and quasi-static seal (Aristoff *et al.* 2008; Aristoff & Bush 2009). The results herein occur within the high Bond number parameter space ($Bo > 300$), where surface tension is negligible and only deep seal type pinch-off events have been observed. Previous studies provide theoretical predictions for pinch-off time and depth which produce good agreement with experiments employing steel spheres (high solid–liquid density ratio) where deceleration can be neglected. Aristoff *et al.* (2010) revealed that a small mass ratio associated with a decelerating sphere can reduce the depth of pinch off, but does not alter pinch-off time.

Beyond revealing scaling for pinch-off depth and time, several studies have explored the effect of unique impact conditions on cavity physics and body dynamics. Simply changing the geometry of the projectile generates a cavity with a cross-section resembling the outer profile of the impacting body (Enriquez *et al.* 2012). For slender bodies it has been shown that even nose shape and entry angle can greatly alter cavity form and dynamics (May 1952; Bodily, Carlson & Truscott 2014). Spinning the projectile perpendicular to the free surface prior to impact creates asymmetrical cavities and generates unbalanced forces (Truscott & Techet 2009a). Similar findings have resulted from covering half of a hydrophilic sphere with a hydrophobic coating (Truscott & Techet 2009b). Both of these methods generate asymmetrical cavities and cause the impacting body to veer from the primary axis of travel. Some groups have extended the work to biological organisms, for instance, Chang *et al.* (2016) experimentally investigated plunge-diving birds using a simplified model. Their experiment involved an elastic beam attached to a rigid cone (representing the bird neck and head, respectively), and focused specifically on when buckling occurs as it relates to possible physical damage, as opposed to how significant deformations affect cavity shape and entry dynamics as discussed herein.

Recently, the authors investigated deformable spheres impacting a water surface at an oblique angle, primarily concerned with the effect of deformability on ricochet (Belden *et al.* 2016). It was shown that induced vibrations interact with the cavity in unique ways resulting in nested cavities, but also inefficient skipping. However, we are not aware of any research addressing the normal entry behaviour of deformable elastomeric spheres. Figure 1 presents two high resolution photographs which qualitatively display some of the differences between the water entry of rigid and highly deformable spheres, including differences in cavity shape and sphere deceleration. In this paper, we use an experimental approach to investigate the unique phenomena associated with water entry of highly deformable spheres.

2. Methods

We investigated the water entry characteristics of elastomeric spheres experimentally by varying sphere impact velocity U_0 , diameter D and material stiffness, as characterized by the neo-Hookean shear modulus G_∞ . Spheres were made from an incompressible platinum-cure silicone rubber called Dragon Skin[®], which is

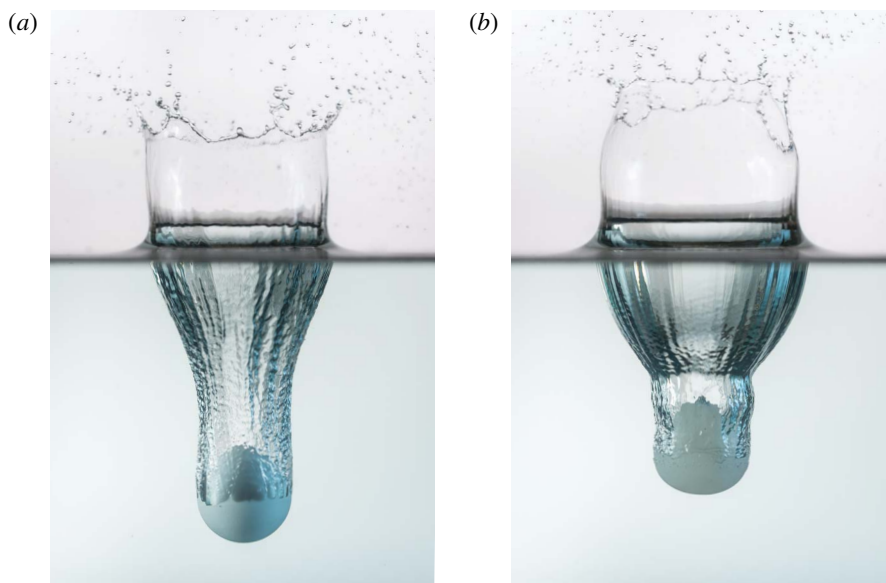


FIGURE 1. Two spheres with shear moduli G_∞ differing by four orders of magnitude experience very different water entry dynamics. (a) A rigid sphere ($G_\infty = 5.66 \times 10^5$ kPa), with a solid–liquid density ratio near unity, impacts the free surface forming a canonical subsurface cavity. (b) A deformable sphere ($G_\infty = 12.69$ kPa), with otherwise nearly identical properties and impact conditions as (a), forms an altered subsurface cavity due to relatively large deformations and material oscillation. Images were taken at the same time after impact. (Photo credit C. Mabey.)

produced by Smooth-On, Inc. Shear modulus was varied by adding a silicone thinner to the mixture to produce three discrete values ($G_\infty = 1.12, 6.70$ and 70.2 kPa), which were determined by sphere compression tests (see appendix A). The constituents of the silicone rubber were measured by mass ratio, mixed, then placed in a vacuum chamber to remove entrained air. Mixtures were poured into aluminium moulds to form spheres with two different diameters ($D = 51$ and 100 mm). Spheres had a density of $\rho_s = 1070$ kg m⁻³ and the density of water is represented by ρ_w . The water entry of rigid spheres with identical $\rho_s = 1070$ kg m⁻³ were also investigated for comparison.

The experimental set-up is summarized in figure 2(a). Spheres were dropped from three discrete heights (0.53, 1.53 and 2.27 m) into a 0.81×0.81 m² glass tank filled to ~ 1 m with water. The entry event was filmed using two Photron SA3 high-speed cameras at 2000 frames per second with diffuse back lighting. The scalar λ represents the deviation of the deformed sphere from the initial diameter. Before splash curtain dome over, the changing diameter λD of the sphere was measured by fitting a circle (cyan) to the top view of the sphere as shown in figure 2(b). After dome over, λD was measured below the free surface (side view figure 2c). The lowest point of the sphere y_b (red cross figure 2c) was also measured directly from the images. The separation line at the air–water–sphere interface is marked by a green horizontal line. An ellipse was then fitted to the edge of the sphere below the separation line (yellow outline). Because the sphere deformation is assumed to be symmetric about the y -axis, measurements of λ from the side and top camera views are assumed to represent the same quantity.

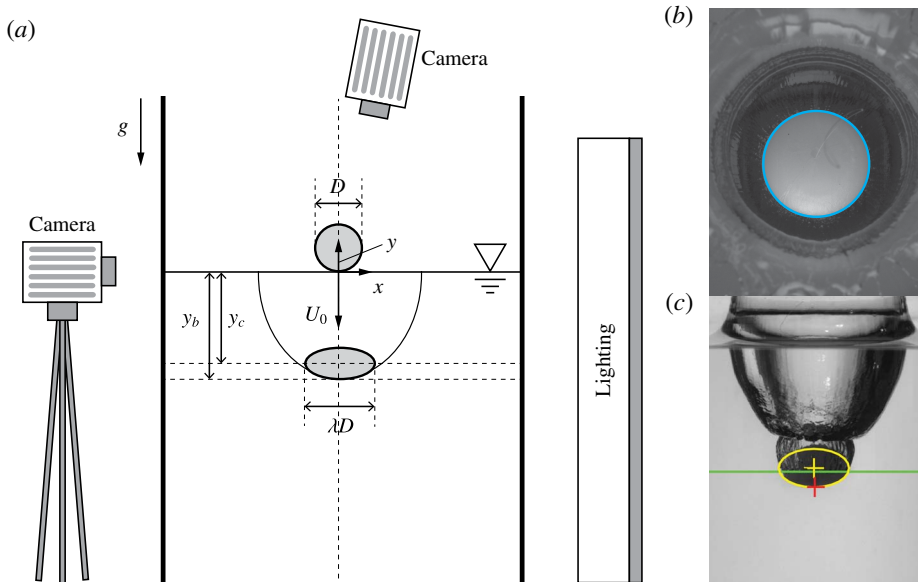


FIGURE 2. (a) Spheres falling into a water filled tank were filmed using high-speed cameras and diffuse back lighting. The event is described by the parameters: sphere diameter D , impact velocity U_0 , a scalar defining sphere deformation λ and the depth below the free surface of the sphere's lowest point y_b and the sphere's centre y_c . (b) Image processing was used to measure λD prior to dome over. (c) Image processing was used to locate y_b throughout the entry event (red cross). A curve was fitted to the edge of the sphere below the air–water–sphere interface (green line). A corresponding ellipsoid, with an assumed symmetry about the y -axis and a volume constrained by the undeformed sphere, was applied (yellow outline).

3. Results

Figure 1 displays high resolution images of two spheres with nearly identical impact conditions ($U_0 = 2.4 \text{ m s}^{-1}$, $D = 51 \text{ mm}$), except that the sphere in (a) has shear modulus $G_\infty = 5.66 \times 10^5 \text{ kPa}$ (rigid) while the sphere in (b) has shear modulus $G_\infty = 12.7 \text{ kPa}$. The cavity formed by the deformable projectile differs in the oscillatory profile of the cavity walls, in addition to being shallower and wider. Because the cavity physics and projectile dynamics are evidently different for a deformable sphere, characterizing the initial deformation and resulting material oscillation in the sphere is critical to understanding water entry physics for deformable objects.

Figure 3(a,b) lends additional insight into why a cavity produced by a deformable sphere deviates from that formed by a rigid sphere. At 12 ms after impact, the sphere has deformed significantly into an oblate spheroid, creating a wider cavity than a rigid sphere. Elastic forces cause the sphere to rebound from this initial deformation into a prolate spheroid with its major axis aligned with the vertical ($t = 29 \text{ ms}$). The continually oscillating sphere now proceeds to a second radial expansion that penetrates through the cavity wall, forming a smaller cavity within the first ($t = 43 \text{ ms}$), resulting in a so-called matryoshka cavity (Hurd *et al.* 2015; Belden *et al.* 2016). For this case pinch off occurs within the second cavity ($t = 96 \text{ ms}$).

For each experimental test, the position of the bottom of the sphere y_b was tracked through a series of high-speed images. In figure 3(c), y_b/D is plotted as a function

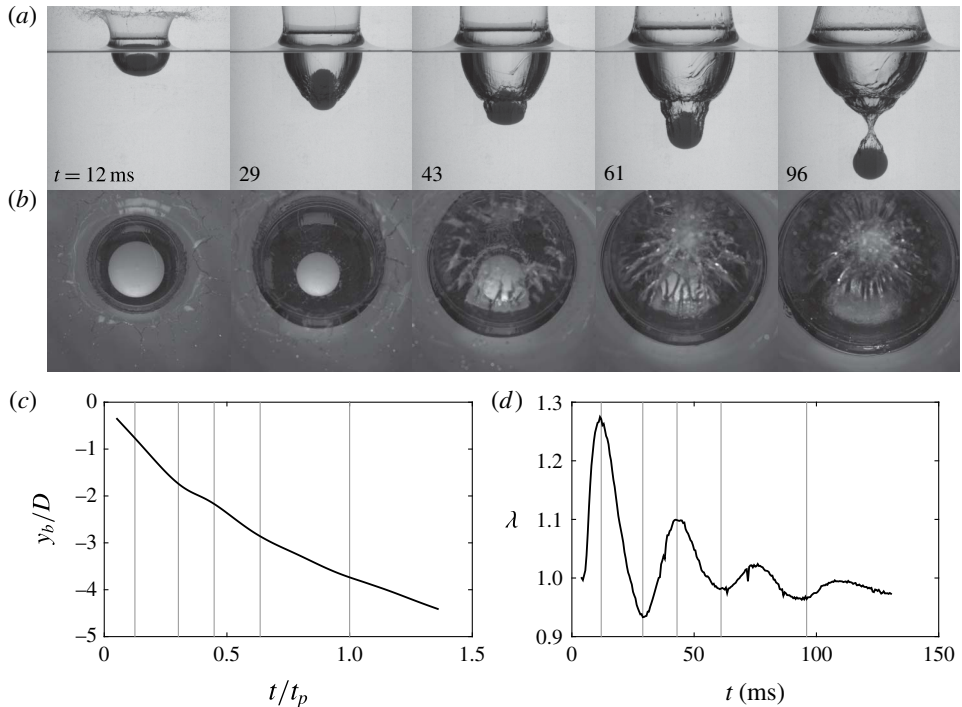


FIGURE 3. (a) A sphere deforms significantly as it impacts and enters the water ($G_\infty = 6.70$ kPa, $D = 51$ mm and $U_0 = 5.3$ m s $^{-1}$); after the initial deformation the sphere oscillates between oblate and prolate shapes, creating a second cavity within the first. Pinch off occurs within this second, smaller cavity ($t = 96$ ms, supplemental movie 1, available at <https://doi.org/10.1017/jfm.2017.365>). (b) The water entry event captured from a top view highlights the changing diameter and splash curtain dome over event. (c) The measured position of y_b is plotted against dimensionless time where vertical lines correspond to the images above. (d) Plotting the parameter λ as a function of time portrays a decaying sinusoid. Image sequences (a,b) are both shown in supplemental movie 1.

of dimensionless time t/t_p (t_p : time to pinch off), in which the sphere oscillation is evident. Figure 3(d) shows the measured value of λ , which reaches an initial large peak due to the impact event ($t = 12$ ms), and then decays throughout the water entry. This decay in λ is typical for all deformable sphere water entry events studied.

Figure 4(a) presents a simplified description of the sphere oscillation in which the sphere deforms into an oblate spheroid with symmetry about the y -axis. Here, λ represents the principal stretch in the x and z directions, and by conservation of volume the principal stretch in the y direction is $\lambda_y = 1/\lambda^2$. Defining λ in this way is based on the observation that the primary mode of deformation in the sphere during water entry is equi-biaxial tension, and λ is a measure of the principal stretch in the sphere. The parameter λ_{pN} represents the maximum stretch of the sphere in the x - z plane for the N th deformation period.

Based on the decaying behaviour of λ during water entry, we aim to address if the source of damping is in the sphere material, water or both. First, we isolate the response of the sphere by performing a series of tests in which the spheres are dropped onto horizontal rigid surfaces (appendix A, figure 12, supplemental movie 2). Impact with the rigid surface results in an initially large sphere deformation that decays in time. Based on these observations, we apply a viscoelastic model to the

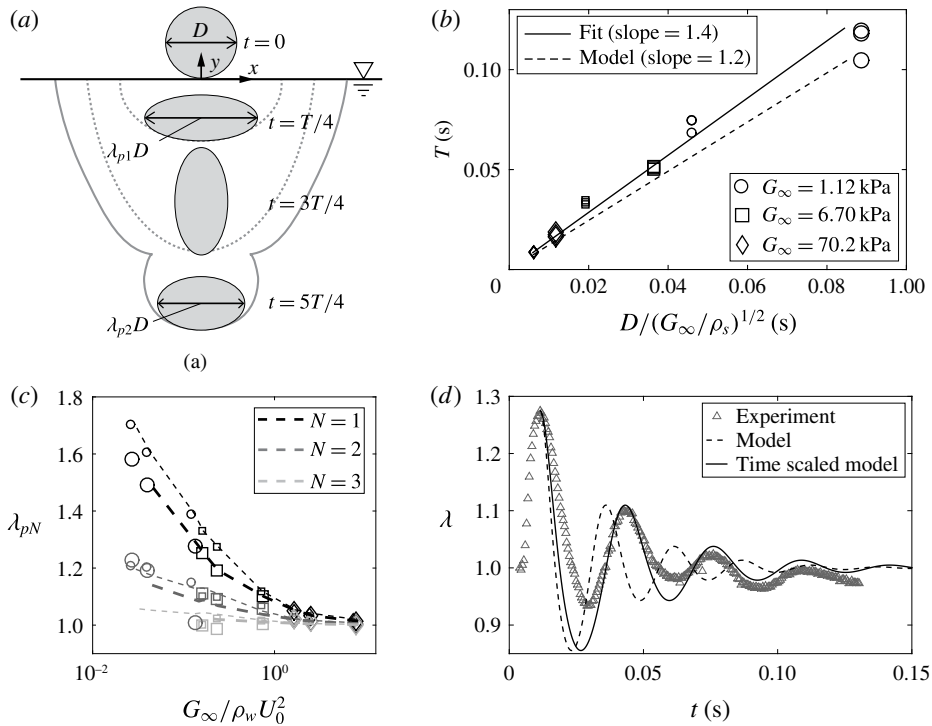


FIGURE 4. (a) When a deformable sphere impacts the water surface it flattens into an oblate spheroid with an increased cross-sectional diameter at $t = T/4$ ($\lambda_{p1}D$) before rebounding back into a sphere at $T/2$. The sphere then forms a prolate spheroid at $3T/4$, then returns to a spherical shape in a single period T (not shown). The principal stretch λ defines the deviation of the sphere from its un-stretched diameter D . The subscript pN refers to the maximum value of λ within the N th period. (b) The oscillation period of λ scales with $D/\sqrt{G_\infty/\rho_s}$ (the slope of the linear fit is 1.4). The analytical model predicts a slightly smaller slope (1.2). Large data points represent large spheres ($D = 100$ mm) and small data points represent small spheres ($D = 51$ mm). (c) The peak value of λ for a given period and given sphere radius appears to depend only on $G_\infty/\rho_w U_0^2$. Shading denotes period number (N) as indicated in the legend. Symbols represent experimental data and lines represent model prediction. Thin lines represent small spheres ($D = 51$ mm) and thick lines represent large spheres ($D = 100$ mm). (d) The measured values of λ in time, for the experimental case featured in figure 3, are represented by grey triangles. The behaviour predicted by the analytical model is represented by a dashed line. The solid line portrays a time-adjusted model with frequency shifted to correspond with the scaling in (b).

sphere material (Bergström & Boyce 1998) as summarized in appendix A. The model includes parameters to account for viscous damping, but the equilibrium stress is still governed by the hyperelastic neo-Hookean model (parameterized by shear modulus G_∞). The rigid surface impact data are used to calibrate the dynamic parameters of the viscoelastic model. Second, to see if the damping in the material model can explain the decay observed in the water entry events, we construct a simplified model of the sphere oscillation (derived in appendix A). The sphere is prescribed an initial stretch $\lambda_0 = \lambda_{p1}$ at $t = t_0$ and allowed to oscillate freely for $t > t_0$.

The analysis is performed for all experimental cases and the results are summarized in figure 4.

Figure 4(b) shows that the oscillation period predicted by the model is slightly less than that observed in the experiments. Because the viscoelastic model parameters were calibrated to an experimental test isolated from the water, we suggest the observed lower frequency (longer period) of the sphere in water is attributable to the added mass experienced by the sphere. Some mass of water has to be accelerated during portions of the sphere oscillation period (e.g. between $t = 3T/4$ and $t = 5T/4$ in figure 4a). However, if we scale the response in time by a ratio of experimental to modelled periods, T_{exp}/T_{model} , then the predicted oscillations show good agreement with the experiments. Despite the difference in period, the magnitude of the predicted peaks in λ are consistent with experiments (figure 4c,d), suggesting that the dominant source of damping is in fact in the sphere material. We note that the model agrees more accurately at the peaks than the valleys because an oscillating sphere complies with the idealization of the model (ellipsoidal assumption) more closely in its oblate shape than its prolate shape as is observed in figure 5(d).

Figure 5(a–d) shows the cavity growth and pinch off resulting from the impact of four spheres ($D = 51$ mm, $U_0 = 6.5$ m s⁻¹) with G_∞ decreasing from 5.66×10^5 kPa (a) to 1.12 kPa (d). The cavity in image sequence (b) is created by a sphere with a shear modulus of $G_\infty = 70.2$ kPa. The resulting cavity and pinch off strongly resembles those created by the rigid sphere in (a), except for the presence of small-scale undulations on the cavity walls due to sphere vibration. The sphere in (c) has a shear modulus an order of magnitude smaller than that in (b); it deforms significantly upon impact creating a much wider cavity and shallower pinch-off event. The smaller G_∞ results in higher magnitude but lower frequency oscillations, creating a second impact-like event within the first cavity. Deformations are even more pronounced in sequence (d). Pinch off occurs within the second cavity formed for image sequences (c) and (d). Spheres with lower values of G_∞ are often observed to decelerate so rapidly that they occupy the space where a deep seal would normally occur as seen in figure 5(e). In this instance the contact line of the second cavity recedes up the surface of the sphere and pinches off at the top.

Water entry events are often classified by cavity characteristics, with a common parameter being time to pinch off (t_p). The dimensionless time $t_p U_0/D$ is plotted against Froude number ($Fr = U_0/\sqrt{gD}$) in figure 6(a) for all tested cases, which is the same non-dimensionalization employed by Aristoff *et al.* (2010) for decelerating rigid spheres (dashed line). However, the scaling does not provide an effective data collapse for deformable elastomeric spheres. Instead we normalize using a new term $D_{eff} = \lambda_{pN}D$, which represents the maximum deformed diameter that the sphere assumes within the cavity in which pinch off occurs. For example, in the case seen in figure 5(d), pinch off occurred within the second cavity resulting in $D_{eff} = \lambda_{p2}D$. This adjustment provides a more convincing data collapse as can be seen in figure 6(b), where the solid line is a fit to the data (slope = 1.3).

We have described the effects of elastomeric sphere deformation on the global features of water entry, and now turn attention to sphere dynamics. Based on the description of the sphere as an ellipsoid (figure 4a), the position of the centre of mass is defined as

$$y_c = y_b + \frac{R}{\lambda^2}, \quad (3.1)$$

where, as already discussed, y_b is tracked from images (figure 3c). Any noise in measurements of λ would be amplified in (3.1); therefore, we use the time-scaled results of the model simulations to define λ . The velocity and acceleration of the

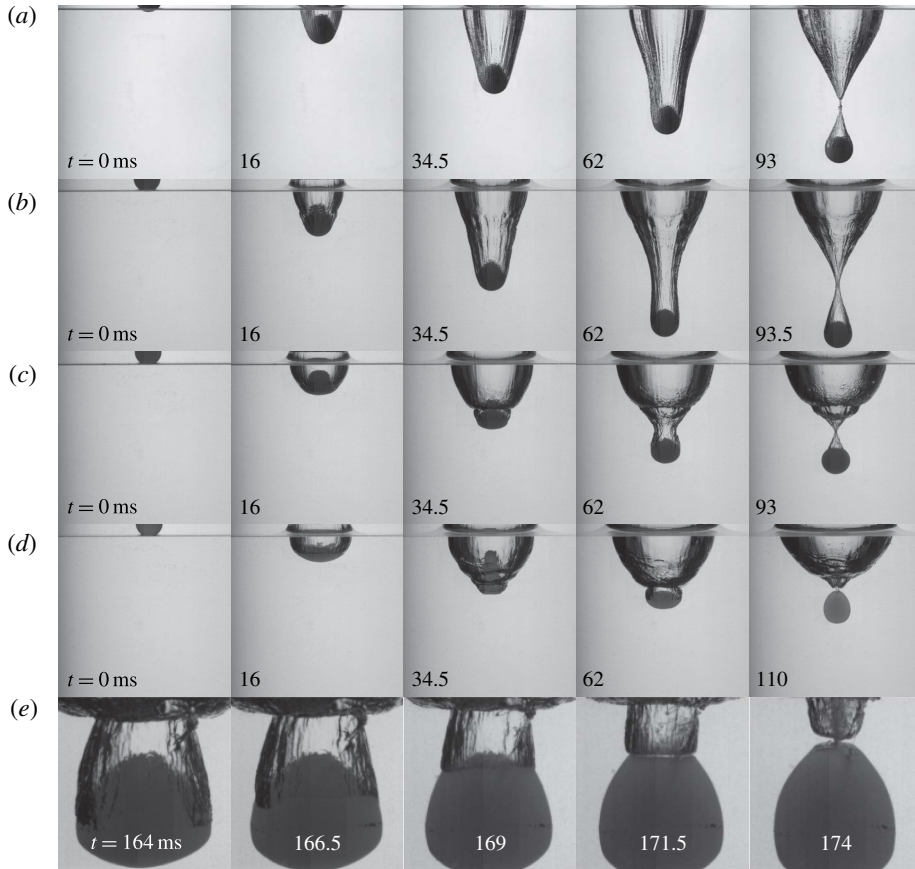


FIGURE 5. (a–d) The water entry of four spheres with identical diameter ($D = 51$ mm), density ($\rho_s = 1070$ kg m $^{-3}$) and impact velocity ($U_0 = 6.5$ m s $^{-1}$) but varying shear moduli: (a) $G_\infty = 5.66 \times 10^5$ kPa, (b) $G_\infty = 70.2$ kPa, (c) $G_\infty = 6.70$ kPa, (d) $G_\infty = 1.12$ kPa. (e) For the largest and most compliant spheres tested ($D = 100$ mm, $U_0 = 6.5$ m s $^{-1}$ and $G_\infty = 1.12$ kPa) spheres decelerate more rapidly, occupying the space where pinch off would occur. The attached cavity recedes upward along the sphere, pinching at the top of. Image sequences (a–e) correspond to supplemental movies 3–7 respectively.

centre of mass, \dot{y}_c and \ddot{y}_c , are computed from derivatives of smoothing splines fit to y_c , as was done in Truscott, Epps & Techet (2012). Figure 7(a–c) displays \dot{y}_c as a function of dimensionless time for all values of shear modulus G_∞ , where $D = 51$ mm and U_0 ranges from 3.0 to 6.5 m s $^{-1}$. The values of \dot{y}_c for rigid spheres ($G_\infty = 5.66 \times 10^5$) are plotted as blue curves. The vertical lines indicate the end of the first oscillation period for the elastomeric spheres (corresponding grey shades match the legend in (a)). Elastomeric spheres experience a greater deceleration than the rigid spheres as λD increases. However, the deformable spheres quickly transition to a deceleration rate similar to that of a rigid sphere as the magnitude of λ decreases (compare slopes of grey curves to blue curve after the first oscillation period marked by the vertical lines). Finally, after pinch off ($t/t_p > 1$), a steady state is reached and spheres fall at nearly constant velocity ($\lambda \rightarrow 1$). Notice that in (c) the softest

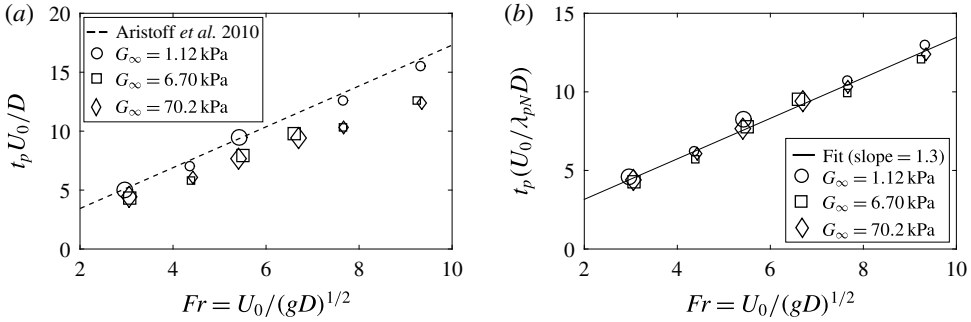


FIGURE 6. (a) Dimensionless pinch off plotted against Fr does not produce a convincing collapse as was the case with rigid spheres in the study by Aristoff *et al.* (2010). The dashed line represents the theoretical scaling proposed in the same study. (b) Rather a more accurate scaling is achieved by a dimensionless pinch-off time scaled by $D_{eff} = \lambda_{pN}$, which represents the maximum deformed diameter that the sphere assumes within the cavity in which pinch off occurs (slope of linear fit = 1.3).

spheres (lightest grey) lose nearly all of their velocity during the first deformation cycle, whereas more rigid spheres lose a significantly smaller portion.

We perform a scaling analysis of the water entry event to gain insight into the sphere deceleration over the first deformation period. For simplicity, added mass is neglected and thus the dominant forces include drag, gravity and buoyancy. Because the spheres are nearly neutrally buoyant, gravitational and buoyant terms cancel and a simple equation of motion for the impacting sphere can be expressed as

$$\rho_s \nabla \ddot{y}_c = \frac{1}{2} \rho_w A U^2 C_D, \tag{3.2}$$

where ∇ represents the volume of the sphere, A the cross-sectional area, U the velocity of the centre of mass and C_D the coefficient of drag. We simplify this expression by defining a characteristic acceleration $(U_1 - U_0)/T = \Delta U/T$, where U_1 denotes the velocity of the sphere centre of mass after the first deformation period. By noting that $\nabla \sim D^3$, $A \sim D^2$ and $C_D \sim 1$, we can approximate (3.2) as

$$\frac{\Delta U}{T} \approx \frac{\rho_w}{\rho_s} \frac{U_0^2}{D}. \tag{3.3}$$

We previously showed that $T \sim D/\sqrt{G_\infty/\rho_s}$ (figure 4b), and for tested spheres $\rho_w/\rho_s \sim 1$. This allows us to rearrange (3.3) to

$$\frac{\Delta U}{U_0} \approx \left(\frac{G_\infty}{\rho_w U_0^2} \right)^{-1/2}. \tag{3.4}$$

The velocity $\Delta U/U_0$ is plotted against $G_\infty/\rho_w U_0^2$ in figure 7(d). This dimensionless number, which is a ratio of material shear modulus to impact hydrodynamic pressure, collapses the data. For $G_\infty/\rho_w U_0^2 \gtrsim 0.2$, the data follow the scaling predicted by (3.4). However, in the limit of small G_∞ and large U_0 spheres deform significantly, and the argument $A \sim D^2$ no longer holds as there is a more complicated dependence of λ on the material properties and impact conditions. Furthermore, it is likely that

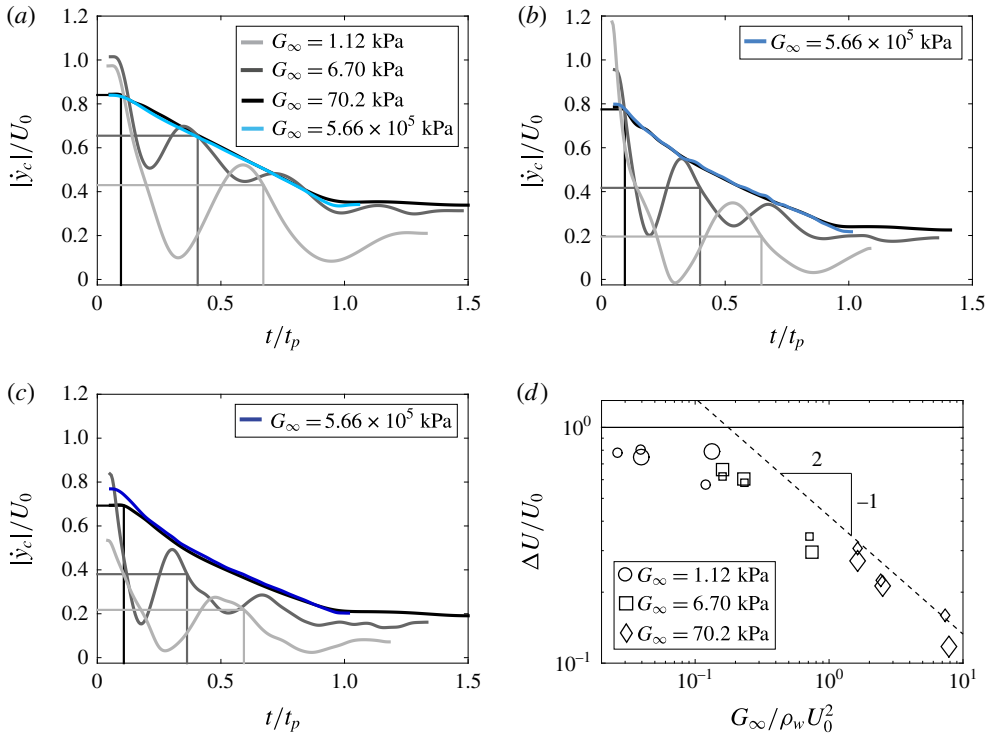


FIGURE 7. Dimensionless velocity ($|\dot{y}_c|/U_0$) is plotted against dimensionless time (t/t_p) for spheres impacting with three different velocities: (a) $U_0 = 3.0 \text{ m s}^{-1}$, (b) $U_0 = 5.3 \text{ m s}^{-1}$ and (c) $U_0 = 6.5 \text{ m s}^{-1}$. Compared to rigid spheres (blue curves), deformable spheres experience a larger deceleration rate after impact over the first cycle of sphere deformation. After the first oscillation period, deformable spheres follow a deceleration similar to rigid spheres, and then transition to a nearly constant velocity after pinch off. (d) The large initial change in velocity is investigated by plotting $\Delta U/U_0 = (U_1 - U_0)/U_0$, where U_1 is the velocity of the sphere centre of mass after one oscillation period, against $G_\infty/\rho_w U_0^2$. For $G_\infty/\rho_w U_0^2 \gtrsim 0.2$, $\Delta U/U_0$ scales with $(G_\infty/\rho_w U_0^2)^{-1/2}$, as predicted by a scaling analysis from the equation of motion for the sphere. For $G_\infty/\rho_w U_0^2 \lesssim 0.2$, $\Delta U/U_0$ asymptotes to the limit of 1, with nearly all of U_0 being lost over the first period of oscillation.

added mass plays a more significant role as $G_\infty/\rho_w U_0^2 \rightarrow 0$ (see appendix B). When $G_\infty/\rho_w U_0^2 < 0.2$, we find $\Delta U \rightarrow U_0$ within the first oscillation period. Nonetheless, the experimental data follow the proposed scaling well, and this allows us to predict how the impact dynamics of deformable spheres will differ from their rigid counterparts based on material properties and impact conditions.

At this point, it is worth commenting on the expected role of added mass during the water entry event. Prior research on rigid sphere water entry has shown that forces arising from added mass are significant in the early moments of impact, primarily at times before the entire sphere has passed the free surface (Faltinsen & Zhao 1998; Truscott *et al.* 2012, 2014). As discussed earlier, it is likely that the added fluid mass is responsible for the longer oscillation period of the spheres in water. This added mass would be expected to resist sphere acceleration in the direction of travel as the sphere oscillates. While added mass undoubtedly affects the physics of deformable

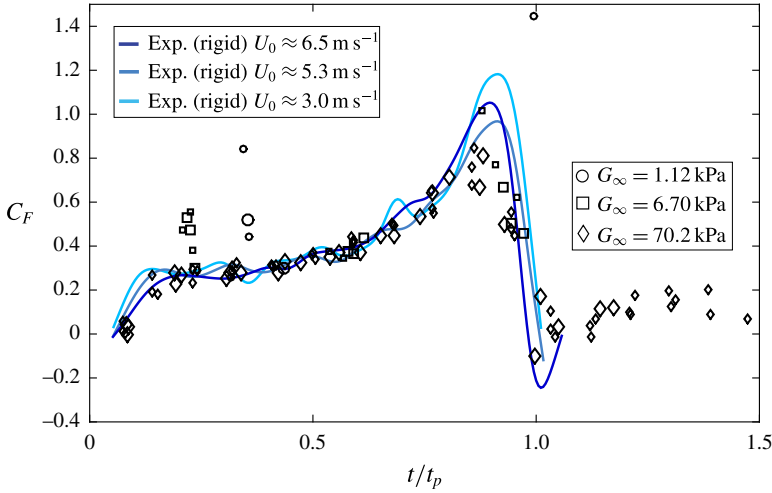


FIGURE 8. Force coefficients for deformable spheres (\bar{C}_F) calculated by averaging over each deformation period (y direction) are plotted as a function of dimensionless time (black symbols). The data encompass all sphere diameters, shear moduli and impact velocities tested. Force coefficients (C_F) for rigid spheres entering the free surface with the same specific gravity as the deformable spheres are plotted as a function of dimensionless time (blue curves). The period-averaged values for deformable spheres \bar{C}_F follow the instantaneous values for rigid spheres C_F , except during the first sphere deformation period in which deformable spheres experience larger drag from increased λD .

sphere water impact, we argue that it is unlikely to significantly affect the trends in $\Delta U/U_0$ for $G_\infty/\rho_w U_o^2 \gtrsim 0.2$ (see appendix B for more details). This is supported by the good agreement between the experimental data and the predicted trend in figure 7(d).

To further investigate how the water entry of a deformable elastomeric sphere differs from that of a rigid sphere, we calculate the total force coefficient acting on the sphere in the y direction as a function of time. The oscillating behaviour of the sphere results in a varying instantaneous force coefficient C_F . Therefore, we calculate a period-averaged force coefficient,

$$\bar{C}_F = \frac{\rho_s \nabla \bar{y}_c}{\frac{1}{2} \rho_w \bar{y}_b^2 \pi \left(\frac{\lambda_p N D}{2} \right)^2}, \tag{3.5}$$

where \bar{y}_c and \bar{y}_b are the acceleration and velocity of the centre of mass and sphere bottom averaged over a single oscillation period, respectively. Using (3.5), values for \bar{C}_F are plotted in figure 8 as a function of dimensionless time t/t_p for all experimental cases. The period-averaged values \bar{C}_F follow the instantaneous experimental values C_F for three cases of rigid sphere water entry (blue curves). This trend holds except for the first sphere deformation period ($t/t_p \approx 0.2$ to 0.4 depending on G_∞), for which the deformable spheres experience larger drag from increased λD . Over this period, the spheres deform into ellipsoids with a large aspect ratio and thus we expect the force coefficient to be larger. For example, for an ellipsoid with $\lambda = 1.3$, we expect the force coefficient to be between 3–7 times larger than that of a sphere, depending on Reynolds number (Daugherty & Franzini 1977).

4. Conclusion

We have shown that deformable elastomeric spheres form cavities that differ from those formed by rigid spheres by being shallower, wider and having undulatory cavity walls. These differences stem from the sphere flattening upon surface impact followed by material oscillation. We describe the deformation and oscillation in terms of both material properties and impact conditions. This allows us to define an effective diameter, which accounts for the deformation and provides effective scaling for time to pinch off and a period-averaged force coefficient. The large sphere deformation, particularly over the first period, is responsible for the increased loss in velocity as compared to rigid spheres. We have shown how this reduction in velocity scales with the ratio of material shear modulus to impact hydrodynamic pressure ($G_\infty/\rho_w U_0^2$). Surprisingly, we find that, except for the unique initial deceleration and altered cavity dynamics which we have quantified in terms of stiffness and impact velocity, the dynamics for the water entry of deformable elastomeric spheres mirrors that of rigid spheres.

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Supplementary movies

Supplementary movies are available at <https://doi.org/10.1017/jfm.2017.365>.

Appendix A. Viscoelastic model

A.1. Describing the sphere deformation

The model of sphere deformation is shown in figure 9. The deformation is described by assuming a volume preserving stretch that deforms the sphere into an ellipsoid, with semi-axes $(\lambda R, \lambda R, \lambda_3 R)$ aligned with the $\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ coordinate system. The incompressibility condition requires that $\lambda_3 = 1/\lambda^2$. The total deformation gradient can be expressed as

$$\mathbf{F} = \lambda(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda^2} \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (\text{A } 1)$$

where \otimes denotes the tensor product of two vectors.

We suppose that the solid can be idealized as a linear viscoelastic Bergstrom–Boyce material (Bergström & Boyce 1998). In this model, the total deformation gradient is decomposed into elastic and plastic parts $\mathbf{F} = \mathbf{F}^e \mathbf{F}^p$. For the simple deformation here, both \mathbf{F}^e and \mathbf{F}^p are volume preserving stretches parallel to the basis vectors, so we can write

$$\mathbf{F}^p = \lambda_p(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda_p^2} \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (\text{A } 2)$$

$$\mathbf{F}^e = \lambda_e(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda_e^2} \mathbf{e}_3 \otimes \mathbf{e}_3, \quad (\text{A } 3)$$

where

$$\lambda = \lambda_e \lambda_p. \quad (\text{A } 4)$$

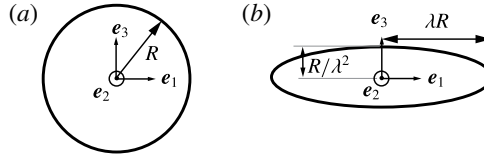


FIGURE 9. Nominal deformation of the sphere into an axisymmetric ellipsoid.

This allows us to calculate the left Cauchy–Green deformation tensor for the total and elastic deformation gradients

$$\mathbf{B} = \mathbf{F}\mathbf{F}^T = \lambda^2(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda^4}\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{A 5}$$

$$\mathbf{B}^e = \mathbf{F}^e \mathbf{F}^{eT} = \lambda_e^2(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda_e^4}\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{A 6}$$

The invariants of the tensors are

$$I_1 = \text{tr}(\mathbf{B}) = 2\lambda^2 + \frac{1}{\lambda^4}, \tag{A 7}$$

$$I_2 = \frac{1}{2}(I_1^2 - \mathbf{B} : \mathbf{B}) = \frac{1}{2} \left[\left(2\lambda^2 + \frac{1}{\lambda^4} \right)^2 - \left(2\lambda^4 + \frac{1}{\lambda^8} \right) \right], \tag{A 8}$$

$$I_1^e = \text{tr}(\mathbf{B}^e) = 2\lambda_e^2 + \frac{1}{\lambda_e^4}, \tag{A 9}$$

$$I_2^e = \frac{1}{2}(I_1^{e2} - \mathbf{B}^e : \mathbf{B}^e) = \frac{1}{2} \left[\left(2\lambda_e^2 + \frac{1}{\lambda_e^4} \right)^2 - \left(2\lambda_e^4 + \frac{1}{\lambda_e^8} \right) \right]. \tag{A 10}$$

We also need measures of total, elastic and plastic strain rates. We use the symmetric part of the velocity gradient as the strain rate measure

$$\mathbf{D} = \text{sym}(\dot{\mathbf{F}}\mathbf{F}^{-1}) = \mathbf{D}^e + \mathbf{D}^p, \tag{A 11}$$

$$\mathbf{D}^e = \text{sym}(\dot{\mathbf{F}}^e \mathbf{F}^{e-1}), \tag{A 12}$$

$$\mathbf{D}^p = \text{sym}(\mathbf{F}^e \dot{\mathbf{F}}^p \mathbf{F}^{p-1} \mathbf{F}^{e-1}). \tag{A 13}$$

For the simple stretch considered here, we obtain

$$\mathbf{D}^p = \frac{\dot{\lambda}_p}{\lambda_p}(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - 2\frac{\dot{\lambda}_p}{\lambda_p}\mathbf{e}_3 \otimes \mathbf{e}_3. \tag{A 14}$$

A.2. Material model theory

For the special case of an incompressible material, the Bergstrom–Boyce model assumes that the stress can be derived from an elastic strain energy of the form

$$U(I_1, I_2, I_1^e, I_2^e) = U_\infty(I_1, I_2) + U_T(I_1^e, I_2^e). \tag{A 15}$$

We can regard this as a nonlinear version of the 3-parameter Maxwell model (figure 10), in which U_T represents the energy in spring k_1 (this energy eventually relaxes to zero if a constant strain is applied to the material) and U_∞ represents the energy in spring k_2 .

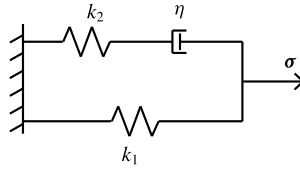


FIGURE 10. Nonlinear version of the 3-parameter Maxwell model that provides the framework for describing the stress in the Bergstrom–Boyce model.

The stresses are related to the derivatives of the strain energy in the usual way, giving

$$\boldsymbol{\sigma} = 2 \left[\left(\frac{\partial U_\infty}{\partial I_1} + I_1 \frac{\partial U_\infty}{\partial I_2} \right) \mathbf{B} - \frac{\partial U_\infty}{\partial I_2} \mathbf{B}^2 \right] + 2 \left[\left(\frac{\partial U_T}{\partial I_1^e} + I_1^e \frac{\partial U_T}{\partial I_2^e} \right) \mathbf{B}^e - \frac{\partial U_T}{\partial I_2^e} \mathbf{B}^{e^2} \right] + p \mathbf{1}. \tag{A 16}$$

To model the material used for the spheres presented in this paper, we choose U_∞ and U_T to be the incompressible neo-Hookean potential

$$U_\infty = \frac{G_\infty}{2} (I_1 - 3), \tag{A 17}$$

$$U_T = \frac{G_T}{2} (I_1^e - 3). \tag{A 18}$$

For this choice, we obtain

$$\boldsymbol{\sigma} = G_\infty \mathbf{B} + G_T \mathbf{B}^e + p \mathbf{1}. \tag{A 19}$$

We also need an evolution equation for the plastic part of the stretch \mathbf{F}^p . Bergstrom–Boyce suggest the following equation:

$$\mathbf{D}^p = \dot{\epsilon}_0 \left(\sqrt{I_1^p} - \sqrt{3} + \xi \right)^n \left(\frac{\tau_e}{\tau_0} \right)^m \frac{3}{2} \frac{\boldsymbol{\tau}}{\tau_e}, \tag{A 20}$$

where $\dot{\epsilon}_0$, m , n , τ_0 are material properties, ξ is a constant, $\boldsymbol{\tau} = G_T (\mathbf{B}^e - (1/3) \text{tr}(\mathbf{B}^e) \mathbf{1})$ is the deviatoric part of the ‘dynamic’ stress and $\tau_e = \sqrt{3 \boldsymbol{\tau} : \boldsymbol{\tau} / 2}$ is the Von Mises uniaxial equivalent dynamic stress. For the volume preserving stretching deformation considered here,

$$\begin{aligned} \frac{\boldsymbol{\tau}}{G_T} &= \lambda_e^2 (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) + \frac{1}{\lambda_e^4} \mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3} \left(2\lambda_e^2 + \frac{1}{\lambda_e^4} \right) (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) \\ &= \frac{1}{3} \left(\lambda_e^2 - \frac{1}{\lambda_e^4} \right) (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2 - 2\mathbf{e}_3 \otimes \mathbf{e}_3), \end{aligned} \tag{A 21}$$

and

$$\tau_e = G_T \left| \lambda_e^2 - \frac{1}{\lambda_e^4} \right|. \tag{A 22}$$

Invoking (A 14) in (A 20) we find

$$\frac{\dot{\lambda}_p}{\lambda_p} = \frac{1}{2} \dot{\epsilon}_0 \left(\sqrt{I_1^p} - \sqrt{3} + \xi \right)^n \left(\frac{\tau_e}{\tau_0} \right)^m \text{sign} \left(\lambda_e^2 - \frac{1}{\lambda_e^4} \right), \tag{A 23}$$

where

$$I_1^p = 2\lambda_p^2 + \frac{1}{\lambda_p^4}. \tag{A 24}$$

A.3. Dynamics

Finally, we need the equation of motion for λ , which will be obtained from the principle of virtual work (Bower 2009)

$$\int_V (\boldsymbol{\sigma} : \delta \mathbf{D}) dV + \int_V \rho_s (\mathbf{a} \cdot \delta \mathbf{v}) dV + \int_V \rho_s (\mathbf{b} \cdot \delta \mathbf{v}) dV - \int_A (\mathbf{t} \cdot \delta \mathbf{v}) dA = 0, \tag{A 25}$$

where $\delta \mathbf{v} = \delta \dot{\mathbf{V}} \mathbf{x}$ is a virtual velocity field, \mathbf{x} denote the coordinates of a material particle before deformation and

$$\delta \dot{\mathbf{V}} = \delta \dot{\lambda} (\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - \frac{2\delta \dot{\lambda}}{\lambda} \mathbf{e}_3 \otimes \mathbf{e}_3. \tag{A 26}$$

In this analysis, we neglect the effects of gravity and assume there are no external tractions; thus the third and fourth terms in (A 25) vanish. Invoking (A 5)–(A 6) and (A 19), the first term in (A 25) becomes

$$\int_V (\boldsymbol{\sigma} : \delta \mathbf{D}) dV = \frac{4}{3} \pi R^3 \left[2G_\infty \left(\lambda - \frac{1}{\lambda^5} \right) + 2G_T \frac{1}{\lambda} \left(\lambda_e^2 - \frac{1}{\lambda_e^4} \right) \right] \delta \lambda. \tag{A 27}$$

To evaluate the remaining terms, the following identities are useful

$$\left. \begin{aligned} \int_{V_0} dV_0 &= \frac{4\pi}{3} R^3, \\ \int_{V_0} x_i dV_0 &= 0, \\ \int_{V_0} x_i x_j dV_0 &= \frac{4\pi}{15} R^5 \delta_{ij}, \end{aligned} \right\} \tag{A 28}$$

where x_i denote the coordinates of a material particle with respect to the centre of the sphere and the integrals are evaluated over the undeformed sphere. The inertia term (second term in (A 25)) can be expressed as

$$\begin{aligned} \int_V \rho_s (\mathbf{a} \cdot \delta \mathbf{v}) dV &= \int_V \rho_s [\ddot{\mathbf{F}} \mathbf{x}] \cdot [\delta \dot{\mathbf{V}} \mathbf{x}] dV = \frac{4\pi}{15} \rho_s R^5 [\ddot{\mathbf{F}} \mathbf{F} : \delta \dot{\mathbf{V}} \mathbf{F}^{-T}] \\ &= 2 \left(\ddot{\lambda} - \frac{6\dot{\lambda}^2 - 2\lambda \ddot{\lambda}}{\lambda^7} \right) \delta \lambda. \end{aligned} \tag{A 29}$$

Collecting terms gives

$$\frac{4}{3} \pi R^3 \left[2G_\infty \left(\lambda - \frac{1}{\lambda^5} \right) + 2G_T \frac{1}{\lambda} \left(\lambda_e^2 - \frac{1}{\lambda_e^4} \right) \right] + \frac{8\pi}{15} \rho_s R^5 \left(\ddot{\lambda} - \frac{6\dot{\lambda}^2 - 2\lambda \ddot{\lambda}}{\lambda^7} \right) = 0. \tag{A 30}$$

Equations (A 4), (A 22)–(A 24) and (A 30) are solved in Matlab to resolve the stretch $\lambda(t)$ given initial conditions $\lambda(0) = \lambda_0$, $\lambda_p(0) = 1$ and $\dot{\lambda} = \dot{\lambda}_e = \dot{\lambda}_p = 0$.

A.4. Material model calibration

The material model is defined by 7 parameters: G_∞ , G_T , $\dot{\epsilon}_0$, m , n , τ_0 and ξ . Without access to the material testing facilities that would be required to fully characterize

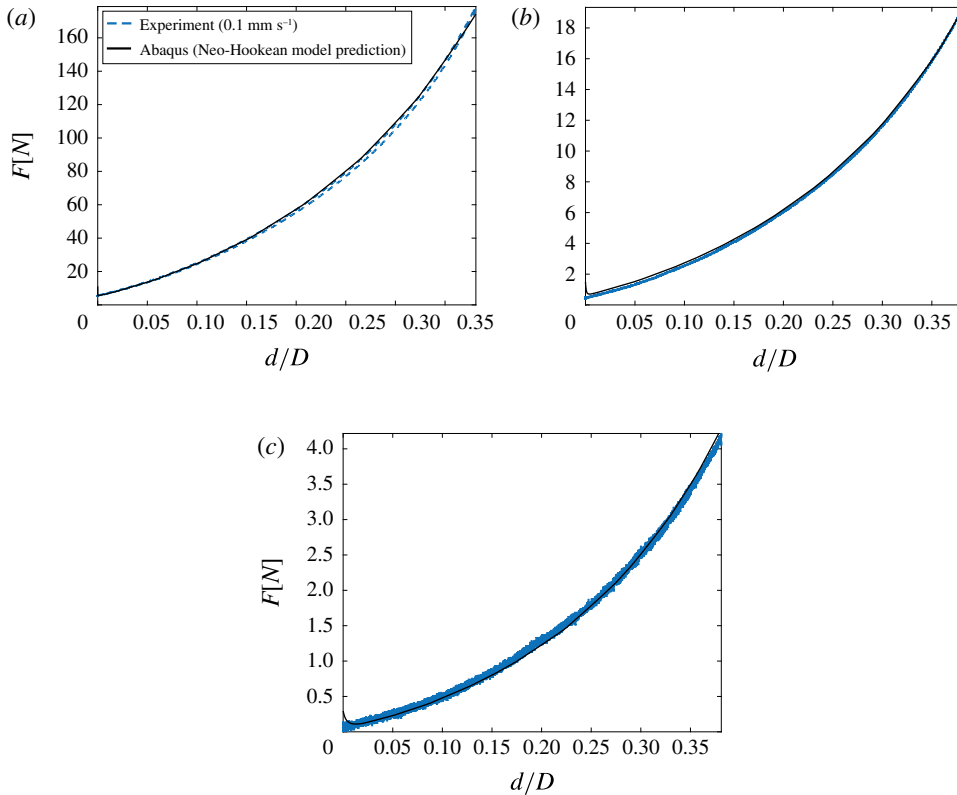


FIGURE 11. Results from quasi-static testing in which the actual spheres used in the water entry experiments are compressed on an Instron machine at a rate of 0.1 mm s^{-1} . The dashed blue lines show the experimental measurements of force as a function of normalized displacement d/D . The solid black lines are predictions from an Abaqus simulation using a hyperelastic neo-Hookean model. The shear modulus G_∞ was adjusted to find the best fit between the simulation and experiment. The three plots correspond to the three stiffness values: (a) $G_\infty = 70.2 \text{ kPa}$, (b) $G_\infty = 6.70 \text{ kPa}$ and (c) $G_\infty = 1.12 \text{ kPa}$.

the silicone materials used in this paper, we adopt a two-part approach to estimate parameters. First, the long-time modulus G_∞ is estimated from quasi-static testing in which the actual spheres used in the water entry experiments are compressed on an Instron machine.

This test set-up was then numerically modelled using the finite element software Abaqus where the sphere was modelled as an axisymmetric solid compressed between two rigid planes accounting for large deformation and frictionless contact. Commanding a displacement profile to match the experimental values, the resulting force is observed. Minimizing the difference in force between the numerical and experimental results is achieved by varying the neo-Hookean shear modulus, G_∞ . The assumption here is that the response is slow enough that the behaviour is quasi-static and all rate effects can be neglected, thus we only need to calibrate one parameter (Abaqus 2016). This is consistent with the strain energy defined in (A 17). We then varied G_∞ to find the value that produced the best fit between the numerically simulated and experimentally measured force–displacement curves. The results of these tests and numerical simulations are shown in figure 11.

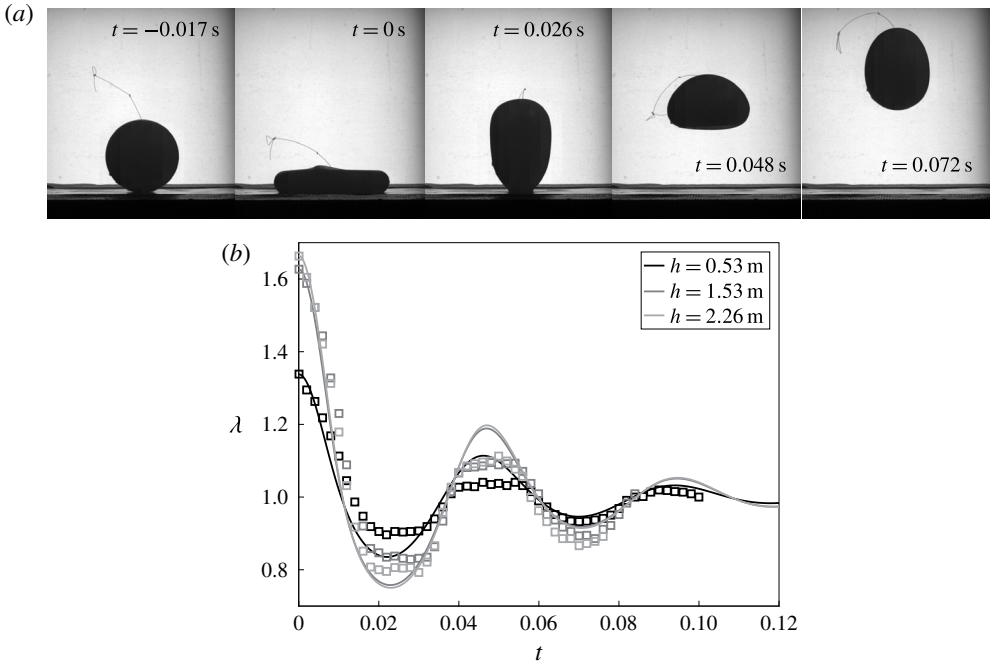


FIGURE 12. Experiments of the spheres impacting with a rigid horizontal surface were used to calibrate the viscoelastic material model. Results are shown for a sphere with diameter $D = 100$ mm and $G_\infty = 6.70$ kPa. (a) Sample high-speed images from the $h = 1.53$ m test. (b) The sphere is dropped from three heights above the table and the stretch λ is measured from high-speed images (squares). The viscoelastic parameters are found by simultaneously finding the best fit between simulations (lines) of $\lambda(t)$ and the experimental data for all drop heights for a given sphere. Shading indicates the height the spheres were dropped from.

| Sphere radius, R (m) | G_∞ (Pa) | G_T (Pa) | ξ | τ_0 | m | n | $\dot{\epsilon}_0$ |
|------------------------|-----------------|------------|--------|----------|-----|---------|--------------------|
| 0.025 | 74 690 | 74 690 | 0.0866 | 1.0 | 1.0 | -0.2481 | 0.0049 |
| 0.025 | 6 900 | 6 900 | 0.0866 | 1.0 | 1.0 | -0.1902 | 0.021 |
| 0.025 | 1 235 | 1 235 | 0.0866 | 1.0 | 1.0 | -1.0 | 0.0056 |
| 0.0487 | 74 690 | 74 690 | 0.0866 | 1.0 | 1.0 | -0.2402 | 0.0024 |
| 0.0487 | 6 900 | 6 900 | 0.0866 | 1.0 | 1.0 | -0.488 | 0.0041 |
| 0.0487 | 1 235 | 1 235 | 0.0866 | 1.0 | 1.0 | -0.50 | 0.0066 |

TABLE 1. Summary of material properties for the silicone spheres studied herein.

To estimate the ‘dynamic’ parameters of the material model, we perform an experiment in which all 6 spheres used in the water entry tests are dropped from 3 heights each onto a rigid horizontal surface. The maximum stretch in the plane of the image is measured, as shown in figure 12. The sphere response is then simulated using the dynamic model defined in (A 4), (A 22)–(A 24) and (A 30) with the initial stretch λ_0 set to the peak value measured in the experiment. We allow two parameters of the material model to be free – $\dot{\epsilon}_0$, n – and perform a nonlinear least-squares minimization to find the parameters that yield the best fit to the sphere stretch measurements. The material model parameters are summarized in table 1. The simulation results using

these material parameters to model the sphere response following impact with the rigid surface are shown in figure 12. In modelling the sphere response during water entry, these material parameters are used and the simulations is initialized with λ_0 measured from the experiments.

Appendix B. Added mass

The scaling analysis outlined in (3.2)–(3.4) neglected the effect of added mass. Here we include an added mass term in the equation of motion for y_c to evaluate the effect on the scaling arguments. Equation (3.2) becomes

$$\rho_s \forall \ddot{y}_c = -\rho_w \forall C_m \ddot{y}_c + \frac{1}{2} \rho_w U^2 C_D A, \quad (\text{B } 1)$$

where C_m is an added mass coefficient. Solving for \ddot{y}_c gives

$$\ddot{y}_c = \frac{\rho_w}{2\forall(\rho_s + C_m \rho_w)} U^2 C_D A. \quad (\text{B } 2)$$

Using the same scales as used in deriving (3.4) gives

$$\frac{\Delta U}{U_0} \approx \frac{\rho_w}{\rho_s + C_m \rho_w} \frac{U_0}{\sqrt{G_\infty / \rho_s}}, \quad (\text{B } 3)$$

and since for the spheres studied herein $\rho_s \approx \rho_w$, we find

$$\frac{\Delta U}{U_0} \approx \frac{1}{1 + C_m} \left(\frac{G_\infty}{\rho_w U_0^2} \right)^{-1/2}, \quad (\text{B } 4)$$

which differs from (3.4) only by the prefactor. Based on values of C_m for fully submerged ellipsoids (Newman 1977), we estimate a representative range of this prefactor as 0.35–0.86 corresponding to a range of $\lambda = 1.5$ –0.74. Therefore, as λ becomes larger, which occurs as $G_\infty / \rho_w U_0^2$ gets small, the added mass has a more profound effect on the relationship between $\Delta U / U_0$ and $G_\infty / \rho_w U_0^2$. However, for larger values of $G_\infty / \rho_w U_0^2$, equation (B 4) approaches (3.4). Indeed, for $G_\infty / \rho_w U_0^2 \gtrsim 0.2$ the data in figure 7(d) follow the trend predicted by the scaling analysis that excludes added mass.

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