# KUROSH RADICALS OF RINGS WITH OPERATORS

N. DIVINSKY AND A. SULIŃSKI

If $A$ is an algebra over a commutative ring with unity, $\Phi$, then the Jacobson radical of the algebra $A$ is equal to the Jacobson radical of $A$, thought of as a ring (**1**, p. 18, Theorem 1). The present note extends this result to all radical properties (in the sense of Kurosh **2**) and allows $\Phi$ to be any set of operators on $A$.

If $A$ is a ring and $\Phi$ is an arbitrary set, we say that $\Phi$ is a *set of operators for* $A$ if for any $\alpha$ in $\Phi$ and any $x$ in $A$, the composition $\alpha x$ is defined and is an element of $A$, and if this composition satisfies the following two conditions:

$$\alpha(x + y) = \alpha x + \alpha y,$$

$$\alpha(xy) = (\alpha x)y = x(\alpha y),$$

for any $\alpha$ in $\Phi$ and any $x$ and $y$ in $A$.

We shall say that an ideal $I$ of $A$ is $\Phi$-*admissible* if $I$ is an ordinary ring ideal of $A$ and if $\alpha I \leqslant I$ for every $\alpha$ in $\Phi$. If $A$ has a unity element, then every ideal $I$ of $A$ is $\Phi$-admissible because $\alpha x = \alpha(1.x) = (\alpha.1)x$. Since $\alpha.1$ is some element in $A$, and $x$ is in $I$, $(\alpha.1)x = \alpha x$ is also in $I$ and thus $\alpha I \leqslant I$.

However, when $A$ does not have a unity element, there may exist ring ideals which are not $\Phi$-admissible. For example, if $A$ is the additive group of rational numbers, with zero multiplication, and if $\Phi$ is the ring of rational numbers, with $\alpha x$, for $\alpha$ in $\Phi$ and $x$ in $A$, defined to be the ordinary rational number product, then $\Phi$ is a set of operators for $A$. The additive group of integers, $I$, is then an ideal of $A$, but it is not $\Phi$-admissible.

If $P$ is a property of rings, we shall say that a ring $R$ is a *P-ring* if it has property $P$. We shall say that an ideal $I$ of a ring $S$ is a *P-ideal* if $I$, thought of as a ring, has property $P$.

A property $P$ of rings is said to be a *radical property* (**2**) if it satisfies the following three conditions:

(1) every homomorphic image of a $P$-ring is again a $P$-ring;

(2) every ring $R$ contains a $P$-ideal which contains all the $P$-ideals of $R$;

(3) if $P$ is the maximal $P$-ideal, or $P$-radical, of a ring $R$, then $R/P$ is $P$-semi-simple, i.e. $R/P$ has no non-zero $P$-ideals.

This general definition of radical property includes all the well-known radicals: Jacobson, Baer, Levitzki, Brown–McCoy, etc.

If $A$ is any ring, with operator domain $\Phi$, and $P$ is a radical property, we shall consider both the $P$-radical of $A$ and the $\Phi$-admissible $P$-ideal of $A$ which

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contains all the $\Phi$-admissible $P$-ideals of $A$. This maximal $\Phi$-admissible $P$-ideal of $A$ is the $\Phi$-$P$-radical of $A$, or the $P$-radical of $A$ when $A$ is thought of as a ring with operators. Of course the $P$-radical of $A$, thought of merely as a ring, contains the $P$-radical of $A$, thought of as a ring with operators. It not only seems possible for these two radicals to be different, but it is not even clear that the $P$-radical of $A$, thought of as a ring with operators, must exist.

We shall now prove the following

THEOREM. *If $A$ is any ring with operator domain $\Phi$ and $P$ is any radical property, then the $P$-radical of $A$, thought of as a ring with operators, exists and is equal to the $P$-radical of $A$, thought of merely as a ring.*

We require the following

LEMMA. *Let $A$ be a ring with operator domain $\Phi$. Let $P$ be a radical property and $I$ a $P$-ideal of $A$, $I$ not necessarily $\Phi$-admissible. Then for every $\alpha$ in $\Phi$, $I_\alpha \equiv I + \alpha I$ is also a $P$-ideal of $A$, though $I_\alpha$ may not be $\Phi$-admissible either.*

*Proof.* It is clear that $I_\alpha$ is an ideal of $A$, for if $x$ is in $A$ and $y$ is in $I$, then $x.\alpha y = \alpha.xy$. Since $xy$ is in $I$, $\alpha.xy$ is in $\alpha I$ and thus $A.\alpha I \leqslant \alpha I$. Similarly $\alpha I.A \leqslant \alpha.IA \leqslant \alpha I$.

To see that $I_\alpha$ is a $P$-ideal, consider $h$, the natural homomorphism from $I_\alpha$ to $I_\alpha/I$. Then let $g$ be a mapping from $I$ to $I_\alpha/I$, defined as follows:

$$\text{for any } y \text{ in } I,\ g(y) \equiv h(\alpha y).$$

To see that $g$ is a homomorphism, consider

$$\begin{aligned} g(y_1 + y_2) &= h(\alpha[y_1 + y_2]) \\ &= h(\alpha y_1 + \alpha y_2) \\ &= h(\alpha y_1) + h(\alpha y_2) \\ &= g(y_1) + g(y_2), \end{aligned}$$

and

$$\begin{aligned} g(y_1 y_2) &= h(\alpha[y_1 y_2]) \\ &= h(\alpha y_1 \cdot y_2) \\ &= 0 \text{ in } I_\alpha/I \end{aligned}$$

because $\alpha y_1 \cdot y_2$ is in $I$ and $h$ maps $I$ into the 0 coset of $I_\alpha/I$. However,

$$\begin{aligned} g(y_1)\cdot g(y_2) &= h(\alpha y_1)\cdot h(\alpha y_2) \\ &= h(\alpha y_1 \cdot \alpha y_2) \\ &= h(y_1 \cdot \alpha(\alpha y_2)) \\ &= 0 \text{ in } I_\alpha/I \end{aligned}$$

because $y_1 \cdot \alpha(\alpha y_2)$ is in $I$. Therefore $g(y_1 \cdot y_2) = g(y_1)\cdot g(y_2)$, and $g$ is a homomorphism.

To see that $g$ is a homomorphism onto $I_\alpha/I$, take any element

$$y_1 + \alpha y_2 + I = \alpha y_2 + I$$

in $I_\alpha/I$. Consider $g(y_2) = h(\alpha y_2) = \alpha y_2 + I$. This proves that $g$ is a homomorphism of the $P$-ring $I$ onto the ring $I_\alpha/I$. Since $P$ is a radical property, by (1), we can conclude that $I_\alpha/I$ is also a $P$-ring.

We can then conclude that $I_\alpha$ itself must be a $P$-ring, for if $I_\alpha$ is not a $P$-ring, then $I_\alpha$ has a $P$-radical $J \neq I_\alpha$ (Property (2)) which contains all the $P$-ideals of $I_\alpha$. In particular $J$ contains $I$. Furthermore, $I_\alpha/J$ is $P$-semi-simple by Property (3). However,

$$I_\alpha/J = (I_\alpha/I)/(J/I)$$

and this is a homomorphic image of the $P$-ring $I_\alpha/I$. Thus $I_\alpha/J$ must also be a $P$-ring (Property (1)). Then $I_\alpha/J$ is both $P$-semi-simple and a $P$-ring and this can only happen if $I_\alpha/J = 0$. This is a contradiction unless $J = I_\alpha$, and $I_\alpha$ is itself a $P$-ring.

This concludes the proof of the lemma and we can now prove the theorem.

*Proof.* Let $R$ be the $P$-radical of $A$, thought of merely as a ring. Then $R$ is a $P$-ideal of $A$, $R$ contains all the $P$-ideals of $A$ and in particular all the $\Phi$-admissible $P$-ideals of $A$.

If $R$ is $\Phi$-admissible itself, then it is also the $P$-radical of $A$, thought of as a ring with operators, and in this case the theorem is true.

However, since $R$ is a $P$-ideal of $A$, $R + \alpha R$ is also a $P$-ideal of $A$, for any $\alpha$ in $\Phi$, by the lemma. But $R$ contains all the $P$-ideals of $A$. Thus $R + \alpha R \leqslant R$ and in particular $\alpha R \leqslant R$. This is true for every $\alpha$ in $\Phi$ and thus $R$ is $\Phi$-admissible and the theorem is established.

We observe that since the associative law has not been used in establishing this result, the theorem holds true for rings which are not necessarily associative.

## References

1. N. Jacobson, *Structure of rings*, Amer. Math. Soc. Coll. Publ., vol. *37* (1956).
2. A. Kurosh, *Radicals of rings and algebras*, Mat. Sbornik, *33* (75) (1953), 13–26.

*University of British Columbia and*
*Institute of Mathematics, Polish Academy of Sciences*

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