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On Ruelle's property

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Abstract. In this paper we investigate the range of validity of Ruelle's property. First, we show that every finitely generated Fuchsian group has Ruelle's property. We also prove the existence of an infinitely generated Fuchsian group satisfying Ruelle's property. Concerning the negative results, we first generalize Astala and Zinsmeister's results [Mostow rigidity and Fuchsian groups. *C. R. Math. Acad. Sci. Paris* **311** (1990), 301–306; Teichmüller spaces and BMOA. *Math. Ann.* **289** (1991), 613–625] by proving that all convergence-type Fuchsian groups of the first kind fail to have Ruelle's property. Finally, we give some results about second-kind Fuchsian groups. [-3.2pc]

Key words: iterated function system, Markov map, Ruelle's property
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1. *Introduction*

In this paper we call a Fuchsian group a Möbius group acting on the unit disk Δ of the plane properly discontinuously and freely. Equivalently, a Fuchsian group is a discrete Möbius group acting on the circle without elliptic elements or else Fuchsian groups are exactly the groups coming from uniformization of hyperbolic Riemann surfaces. The limit set of a Fuchsian group G, denoted by $\Lambda(G)$, is the set of accumulation points of the G-orbit of any point $z \in \Delta$. Since the action of G is properly discontinuous, $\Lambda(G) \subset \partial\Delta$. A Fuchsian group G is said to be of the first kind if the limit set $\Lambda(G)$ is the entire circle. Otherwise, it is of the second kind. Points of the limit set Λ naturally correspond to geodesic rays with fixed base point $z_0 \in \Delta$. The limit set can be written as the disjoint union of two special subsets: the conical limit set $\Lambda_c(G)$, which corresponds to geodesics that return to some compact set infinitely often (the recurrent geodesics) and the escaping limit set, $\Lambda_e(G)$, which corresponds to geodesics escaping to infinity.

The critical exponent (or Poincaré exponent) of a Fuchsian group G is defined as

$$\delta(G) = \inf\left\{t : \sum_{g\in G} \exp(-t\rho(0, g(0))) < \infty\right\} \tag{1.1}$$

$$= \inf\left\{t : \sum_{g\in G}(1 - |g(0)|)^t < +\infty\right\}, \tag{1.2}$$

where ρ denotes the hyperbolic metric.

It is proved in [11] that, for any non-elementary group G, $\delta(G) = HD(\Lambda_c(G))$, the Hausdorff dimension of the conical limit set.

A Fuchsian group is said to be cocompact if the Riemann surface Δ/G is compact and cofinite if the quotient has finite hyperbolic area. A Fuchsian group G is said to be of divergence type if $\Sigma_{g\in G}(1 - |g(0)|) = \infty$. Otherwise, we say that it is of convergence type. It is well known that

$$\text{cocompact} \ \subset \ \text{cofinite} \ \subset \ \text{divergence-type} \ \subset \ \text{first} \quad \text{kind}.$$

All groups of the second kind are of convergence type but the converse is not true, as we shall see later.

We will call a Fuchsian group exceptional if it is the covering group of the sphere minus m disks and n points, where $1 \le m + n \le 3$, $(m, n) \ne (1, 0)$.

Let G be a Fuchsian group and let μ be a bounded measurable function on Δ such that $\|\mu\|_\infty < 1$ and

$$\mu(z) = \mu(g(z))\overline{g'(z)}/g'(z), \quad z \in \Delta, \ g \in G.$$

We say that μ is a G-compatible Beltrami coefficient (or complex dilatation). For a G-compatible Beltrami coefficient μ, there is a corresponding quasi-conformal mapping f_μ which is analytic outside Δ and such that

$$\mu(z) = \frac{\partial f_\mu}{\partial \bar{z}} \Big/ \frac{\partial f_\mu}{\partial z} \quad \text{almost every} \ z \in \Delta.$$

This map, f_μ, conjugates G to a quasi-Fuchsian group $G_\mu = f_\mu \circ G \circ f_\mu^{-1}$. We say that G_μ is a quasi-conformal deformation of G.

We can generalize to quasi-Fuchsian groups the notion of conical and escaping sets: we can also define the Poincaré exponent of such a group by replacing $(1 - |g(0)|)$ in (1.2) by $\mathrm{dist}(g(0), \partial f_\mu(\Delta))$, and the fact that the Poincaré exponent is equal to the dimension of the conical limit set remains true in this case (see [8, 11]).

A Fuchsian group G has Bowen's property if the limit set of any quasi-conformal deformation of G is either a circle or has Hausdorff dimension > 1. In 1979, Bowen [14] proved that if G is a cocompact Fuchsian group, then this dichotomy property holds. Soon after, Sullivan [26, 27] extended Bowen's property to all cofinite groups. In 1990, Astala and the second author [3] showed that Bowen's property fails for all convergence groups of the first kind. Then, in 2001, Bishop showed that Bowen's property holds for all divergence groups (see [8]).

We will say that a Fuchsian group G has Ruelle's property if, for any family of G-compatible Beltrami coefficients (μ_t) that is analytic in $t \in \Delta$, the map

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$t \mapsto HD(\Lambda(G_{\mu_t}))$ is real analytic in Δ. In 1982, Ruelle [24] showed that all cocompact groups have this property. In 1997, Anderson and Rocha [2] extended this result to finitely generated Fuchsian groups without parabolic elements. In [5, 6], Astala and the second author showed that, for Fuchsian groups corresponding to Denjoy–Carleson domains or infinite d-dimensional 'jungle gyms' with $d \geq 3$, Ruelle's property fails. In [9], Bishop gave a criterion for the failure of Ruelle's property that applies to many divergence-type examples including the d-dimensional 'jungle gym' with $d = 1, 2$, which thus implies that Ruelle's property is not equivalent to Bowen's.

In this paper, we continue to investigate the range of validity of Ruelle's property. First, by investigating the role of parabolic points and using Mauldin and Urbanski's [20] techniques, we prove the following theorem.

THEOREM 1.1. *Every finitely generated Fuchsian group has Ruelle's property.*

Using the same kinds of techniques, we also prove the existence of an infinitely generated Fuchsian group with Ruelle's property.

THEOREM 1.2. *There exists a sequence (s_n) of real numbers, increasing to infinity, such that the Fuchsian group uniformizing $S = \mathbb{C}\backslash\{s_n,\ n \geq 0\}$ has Ruelle's property.*

Remark Theorem 1.2 does not hold for any sequence (s_n). For example, $\mathbb{C} \setminus \mathbb{Z}$ is a $\mathbb{Z}$-covering of the twice-punctured sphere, as was noticed in [1] (we thank Mariusz Urbanski pointing this reference out to us), which implies, by the result of Bishop [9], that Ruelle's property fails in this case.

Concerning the negative results, we first generalize Astala and Zinsmeister's results in [3, 4] by proving the following theorem.

THEOREM 1.3. *All convergence-type Fuchsian groups of the first kind fail to have Ruelle's property.*

Concerning the second kind of Fuchsian groups (that is, convergence type) we prove the following theorem.

THEOREM 1.4. *Let S be an infinite area hyperbolic Riemann surface and let G be the universal covering group of S. Let γ be a closed geodesic in the surface S. Cutting S along γ, one obtains one or two bordered Riemann surfaces. We construct a new surface S' by gluing the one of infinite area with one or two funnels along γ. If G is of the first kind, then the corresponding second-kind covering group G' of S' fails to have Ruelle's property.*

2. *Proof of Theorem 1.3*

In order to prove this theorem, we will need the following lemma from [10], Lemma 2.1.

LEMMA 2.1. *Suppose that G is a Fuchsian group and μ is a G-compatible Beltrami coefficient. If $\{\mu_n\}$ is a family of G-compatible complex dilatations with L^∞ norms uniformly bounded by $k < 1$ that converges pointwise to μ, then*

$$\liminf_{n\to\infty} \delta(G_{\mu_n}) \geq \delta(G_\mu).$$

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For quasi-conformal deformations of Fuchsian groups, Bishop [9] gave the following result.

LEMMA 2.2. *If G is a torsion free non-exceptional type Fuchsian group, then G has a quasi-conformal deformation G_μ with $HD(\Lambda(G_\mu)) \geq \delta(G_\mu) > 1$.*

Let us recall some facts from BMO-Teichmuller theory, where the Bounded Mean Oscillation (BMO) space is defined as the measurable functions on the unit circle $\partial\Delta$ such that

$$\|f\|_{\mathrm{BMO}} = \sup_I M_I(|f - M_I(f)|) < \infty,$$

where the sup is over all subintervals I of the unit circle $\partial\Delta$ and

$$M_I(f) = \frac{1}{|I|}\int_I f(x)\,dx,$$

for more detail, see [16, Chs VI–VII]. A Carleson measure on the unit disk Δ is a positive measure ν such that there exists a constant C such that, for any $z \in \partial\Delta$ and any $r < 1$,

$$\nu(\Delta \cap D(z, r)) \leq Cr,$$

where $D(z, r)$ denotes the disk with center z and radius r. Garnett, Gehring and Jones have shown that if G is a convergence-type Fuchsian group, then

$$\sum (1 - |g(0)|)\delta_{g(0)}$$

is a Carleson measure on the unit disk Δ (δ_z stands for the Dirac mass at z); see [17] or [19, Lemma 2.2]. Now consider a function $\varphi \in L^\infty(\Delta)$ with a compact support included in some fundamental domain of G, with $\|\varphi\|_\infty < 1$. Then the function

$$\mu_\varphi(z) = \sum_{g \in G} \varphi(g(z))\frac{\overline{g'(z)}}{g'(z)}$$

is a G-compatible Beltrami coefficient. If μ is any G-compatible Beltrami coefficient, then μ does not descend via the canonical projection $\Delta \to \Delta/G$ to a function, but $|\mu|$ does: by a slight abuse of notation, we say that μ has compact support in Δ/G if $|\mu|$ has. Then the above-constructed Beltrami coefficient μ_φ has compact support and it follows easily from Garnett, Gehring and Jones' result [17] that if G is of convergence type, then

$$\frac{|\mu(z)|^2}{1 - |z|^2}\,dxdy$$

is a Carleson measure. From this, it is easy to deduce the following lemma (see [19, Lemma 2.2]).

LEMMA 2.3. *Suppose that G is a convergence-type Fuchsian group and μ is a G-compatible Beltrami coefficient. If μ is compactly supported on the surface Δ/G (we say that μ induces a compact deformation), then*

$$\frac{|\mu(z)|^2}{1 - |z|^2}\,dxdy \in CM(\Delta),$$

where $CM(\Delta)$ denotes the set of all Carleson measures of Δ.

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Now, let μ be a Beltrami coefficient such that

$$\frac{|\mu(z)|^2}{1-|z|^2}\,dxdy$$

is a Carleson measure. It is known that, in this case, $\log(f'_\mu)$ belongs to the space $BMOA(\Delta)$ with a norm controlled by the above Carleson measure norm. In particular, when the Carleson norm is small, then $\partial f_\mu(\Delta)$ is a rectifiable (chord-arc) curve [21, 25]. This is essential for the proof that convergence-type first-kind Fuchsian groups fail to have Bowen's property. For more details see [4, 5].

We can now prove the theorem. First, by Bishop's result, Lemma 2.2, there exists a G-compatible Beltrami coefficient μ such that $\delta(G_\mu) > 1$. Let (K_n) be an exhaustion of some fundamental domain by compact sets: if μ is a G-compatible Beltrami coefficient let

$$\mu_n(z) = \sum_{g\in G} \mu(g(z))\mathbf{1}_{K_n}(g(z))\frac{\overline{g'(z)}}{g'(z)}.$$

This is a compactly supported G-invariant Beltrami coefficient and the sequence $\{\mu_n\}$ converges pointwise to μ, so, for large n, μ_n satisfies $\delta(G_{\mu_n}) > 1$ by Bishop's result Lemma 2.1. But then, by Lemma 2.3,

$$\frac{|\mu_n(z)|^2}{1-|z|^2}\,dxdy \in CM(\Delta),$$

and if we consider the family $(t\mu_n)$, we see that $HD(\Lambda(G_{\mu_n}) > 1$, while, by the recalled results above, for small t, the curve $G_{t\mu}(\partial\Delta)$ is a chord-arc curve which is rectifiable, and hence $HD(\Lambda(G_{t\mu})) = 1$. This contradicts Ruelle's property.

3. *Proof of Theorem 1.4*

We begin with the following claim.

CLAIM. $HD(\Lambda_e(G')) = 1$.

Proof. To prove the claim, we need the following lemma, which is due to J.L. Fernandez and M. Melian (see [18, Theorem 1]).

LEMMA 3.1. *Suppose that G is a first-kind Fuchsian group such that the quotient Δ/G has infinite area. Then there are two possibilities.*

(i) *If G is of convergence type, then Λ_e has full measure.*

(ii) *If G is of divergence type, then Λ_e has measure zero, but its Hausdorff dimension is equal to* 1.

If G is of convergence type, we consider the lift of the closed geodesic γ in the unit disk. It consists of a nested set Σ of hyperbolic lines: the one intersecting the Dirichlet fundamental domain cuts it into two parts and we may assume that the origin belongs to a part that has infinite (hyperbolic) area. The hyperbolic lines in Σ of the first generation define a two-by-two disjoint family (I_j) of intervals of the unit circle. Let us call l_j the arc length of I_j: if

$$\sum l_j = 2\pi,$$

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then almost every geodesic issued from 0 would visit γ infinitely often, which would contradict (i) of Lemma 3.1. Thus $\Sigma l_j < 2\pi$ and the set of geodesics from 0 that never visit γ has positive measure. It follows that the escaping limit set of S' has positive measure and the claim follows.

Now, suppose that G is of divergence type and that the Riemann surface Δ/G has infinite area. A domain $D \subset S$ is called a geodesic domain if its relative boundary consists of finitely many non-intersecting closed simple geodesics and its area is finite. Fix a point $p \in S$. By [18, Theorem 4.1] we know that there exists a family $\{D_i\}_{i=0}^{\infty}$ of pairwise disjoint geodesic domains in S satisfying the following.

(i) The boundary of D_i and D_{i+1} have at least a simple closed geodesic in common.

(ii) $\lim_{i\to\infty} \operatorname{dist}(p, D_i) = \infty$.

Let Φ be the isometric embedding mapping from S' to S and let $D_i' = \Phi^{-1}(D_i)$ be the isometric embedding preimage of D_i. Without loss of generality, we may suppose that γ as stated in the theorem is part of the boundary of D_0. For the family $\{D_i\}_{i=0}^{\infty}$, the method used to prove [18, Theorem 1] by Fernandez and Melian is still valid. Modeled upon their method, we get that $HD(\Lambda_e(G')) = 1$. For the readers' convenience, we include some details taken from [18].

Let $\{D_i'\}_{i=0}^{+\infty}$ be the family of geodesic domains of S' constructed as above. For any i, let S_i' be the Riemann surface obtained from D_i' by pasting a funnel along each one of the simple closed geodesics of its boundary. For each i, we choose a simple closed geodesic γ_i from the common boundary $D_i' \cap D_{i+1}'$ and a point $P_i \in \gamma_i$. By [18, Theorem 4.1], and noticing that D_i' is the isometric image of D_i, we have $\delta_i \to 1$ when i tends to infinity, where δ_i is the Poincare exponent of S_i'.

For $\theta \in (0, \frac{1}{2}\pi)$, by [18, Theorem 5.1], we can choose a collection $\mathfrak{B}_i$ of geodesics in S_i' with initial and final endpoint P_i such that

$$L_i \leq \operatorname{length}(\gamma) \leq L_i + C(P_i), \quad \gamma \in \mathfrak{B}_i.$$

The number of geodesic arcs in $\mathfrak{B}_i$ is at least e^{σ_i}, and both the absolute value of the angles between γ and the closed geodesic γ_i are less than or equal to θ, where L_i is a constant such that $L_i \to \infty$ as $i \to \infty$, $C(P_i)$ is a constant depending only on the length of the geodesic γ_i, and $\sigma_i < \delta(S_i')$, $\sigma_i \to 1$ as $i \to \infty$. Note that for each i, D_i' is the convex core of S_i', which implies that every geodesic arc $\gamma \in \mathfrak{B}_i$ is contained in the convex core D_i'.

Furthermore, for each i, we may choose geodesic arcs γ_i^* with initial point P_i and final endpoint P_{i+1} such that

$$L_i \leq \operatorname{length}(\gamma_i^*) \leq L_i + C(P_{i+1}),$$

and both the absolute value of the angles between γ_i, γ_i^*, and γ_i^*, γ_{i+1} are less than or equal to θ.

Now we are going to construct a tree $\mathfrak{T}$ consisting of oriented geodesic arcs in the unit disk Δ.

First, lift γ_0^* to the unit disk starting at 0 (without loss of generality, we may suppose that 0 projects onto P_0). From the endpoint of the lifted γ_0^* (which project onto P_1), lift the family $\mathfrak{B}_1$; from each of the endpoints of these liftings (which still project onto P_1), lift again $\mathfrak{B}_1$. Keep lifting $\mathfrak{B}_1$ in this way M_1 times.

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Next, from each one of the endpoints obtained in the process above, we lift γ_1^*, and from each one of the endpoints of the lifting of γ_1^* (which project onto P_2), we lift the collection $\mathfrak{B}_2$ successively M_2 times, as above. By continuing this process indefinitely we obtain a tree $\mathfrak{T}$.

It is easy to see that $\mathfrak{T}$ contains uncountably many branches. The tips of the branches of $\mathfrak{T}$ are contained in the escaping limit set Λ_e of the covering group of S'.

By the proof of [18, Theorem 1.1], we know that, for a suitable sequence $\{M_i\}$ of repetitions, the dimension of the set of the tips of the branches of $\mathfrak{T}$ is one. By the construction of the tree $\mathfrak{T}$, we see that it is a unilaterally connected graph. Hence the geodesic corresponding to any branch of $\mathfrak{T}$ does not tend to the funnel with boundary γ. Hence the dimension of the escaping limit set Λ_e of the covering group G' is one.

We can now prove the theorem.

As in the proof of Theorem 1.3, by Lemma 2.1 and Lemma 2.2, we can choose a compactly supported G'-compatible Beltrami coefficient μ such that $HD(\Lambda(G'_\mu)) \geq \delta(G'_\mu) > 1$. Bishop [12] showed that the Hausdorff dimension of the escaping limit set is unchanged under any compact deformation. Hence, for the deformation group G'_μ, we have $HD(\Lambda_e(G'_\mu)) = HD(\Lambda_e(G'))$. By Lemma 2.3,

$$\frac{|\mu(z)|^2}{1-|z|^2}\,dxdy \in CM(\Delta)$$

and if we also consider the family $(t\mu)$, by [21, 25] we know that, for small t, the curve $\Lambda(G'_{t\mu})$ is a chord-arc curve that is rectifiable, and hence we have that $HD(\Lambda(G'_\mu)) > 1$ while $HD(f_{t\mu}(\partial\Delta)) = 1$ for small t. However, $HD(\Lambda_e(G'_\mu)) = 1$ for any $t \in [0, 1]$, and hence $HD(\Lambda(G'_{t\mu})) = 1$ for t small, which thus contradicts Ruelle's property. $\square$

4. *Proof of Theorem 1.1*

Before giving the proof of this theorem, we first recall some preliminaries.

Suppose that G is a finitely generated Fuchsian group of the first kind with a set of generators containing n parabolic elements. By the work of Bowen and Series [15], we know that there are countable partitions $\mathcal{P} = \{I_i\}_{i=1}^\infty$ of the unit circle S^1 into intervals I_i and a piecewise smooth map $f_G : S^1 \to S^1$ so that:

(1) the map f_G is strictly monotonic on each $I_i \in \mathcal{P}$ and extends to a C^2-function on $\overline{I}_i$ (in fact, $f_G|I_k = g_k|I_k$, for some fixed $g_k \in G$);

(2) if $f_G(I_k) \cap I_j \neq \emptyset$, then $f_G(I_k) \supset I_j$; and

(3) for all i, j, $\bigcup_{n=0}^\infty f^n(I_i) \supset I_j$.

The map f_G is called a Markov map for G. This Markov map defines an iterated function system (IFS). Let us recall the definition of an IFS (see [20]).

Let (X, ρ) be a non-empty compact metric space, let I be a countable index set with at least two elements and let

$$S = \{\phi_i : X \to X, i \in I\}$$

be a collection of injective contractions from X to X for which there exists $0 < s < 1$ such that

$$\rho(\phi_i(x), \phi_i(y)) \leq s\rho(x, y), \quad i \in I,\ (x, y) \in X.$$

Any such collection of contractions is called an IFS.

Let I^n denote the space of words of length n and let I^∞ denote the space of infinite sequences of symbols in I. Let $I^* = \bigcup_{n\geq 1} I^n$. For $\omega = \omega_1\omega_2\cdots\omega_n \in I^n$, $n \geq 1$, set

$$\phi_\omega = \phi_{\omega_1} \circ \phi_{\omega_2} \circ \cdots \circ \phi_{\omega_n}.$$

If $\omega \in I^* \cup I^\infty$ and $n \geq 1$ does not exceed the length of ω, we denote by $\omega|_n$ the word $\omega_1\omega_2\cdots\omega_n$. For $\omega \in I^\infty$, the set

$$\pi(\omega) = \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is a singleton and therefore we can define a map $\pi : I^\infty \to X$.

The set

$$J = \pi(I^\infty) = \bigcup_{\omega\in I^\infty} \bigcap_{n=1}^{\infty} \phi_{\omega|_n}(X)$$

is called the limit set associated to the system

$$S = \{\phi_i : X \to X, i \in I\}.$$

Let $\rho : I^\infty \to I^\infty$ be the left-shift map on I^∞, that is, $\rho(\omega) = \omega_2\omega_3\cdots$. Since $\phi_i(\pi(\omega)) = \pi(i\omega)$ for every $i \in I$, we get

$$\pi(\omega) = \phi_{\omega_1}(\pi(\rho(\omega)))$$

and

$$J = \bigcup_{i\in I} \phi_i(J).$$

For every $\sigma \geq 0$, we define

$$\psi(\sigma) = \sum_{i\in I} \|\phi_i'\|^\sigma \leq \infty,$$

where the norm $\|\cdot\|$ is the supremum norm taken over X. For $n \geq 1$, let

$$\psi_n(\sigma) = \sum_{\omega\in I^n} \|\phi_\omega'\|^\sigma.$$

By [20], we know that

$$\psi_n(\sigma) < \infty \Leftrightarrow \psi(\sigma) = \psi_1(\sigma) < \infty.$$

Let $\theta = \inf\{\sigma : \psi(\sigma) < \infty\}$. For $n \geq 1$, the function $\log(\psi_n)$ is convex on $(\theta, +\infty)$, and for these values of σ,

$$P(\sigma) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n(\sigma)$$

always exists and is finite if and only if $\psi(\sigma) < \infty$: the function P is called the topological pressure function. By [20], Lemma 3.2, we know that $P(\sigma)$ is strictly decreasing in the variable σ on the interval $(\theta, +\infty)$. The IFS is regular if and only if $P(\theta) = \infty$, which is equivalent to $\psi(\theta) = \infty$.

Let G be a finitely generated Fuchsian group of first kind with a set of generators containing finitely many parabolic elements. Let f_G be the associated Markov map. By

Bowen and Series' work [15], we know that there exists a subset $K \subset \partial\Delta$ which is the union of countably many open intervals $\bigcup I_i \subset \partial\Delta$ such that the first return map $f_K : K \to K$, $f_K(x) = f_G^{m(x)}(x)$, $m(x) = \inf\{m : f_G^m(x) \in K\}$ induced by f_G satisfies an additional expanding condition: there exists an integer $N > 0$ and a constant $\beta > 1$ such that $(f_K^N)'(x) \geq \beta$ for all $x \in K$. We will use the intervals in K as the index and denote the index set by I. Then we get an IFS by the map f_K as

$$S = \{\phi_i : \phi_i = f_K^{-1}|_i,\ i \in I\}.$$

These results remain valid for finitely generated second-kind Fuchsian groups as was proven by Anderson and Rocha [2] in the case of the set of generators containing no parabolic elements, but the Bowen–Series result goes through in this later case.

We can now prove the theorem.

Proof. Let (μ_t) be a family of G-compatible Beltrami coefficients that is analytic in $t \in \Delta$. By the self-similarity of the limit sets of quasi-Fuchsian groups G_t, in order to study the dimensions of the limit set of the group G_t, it is enough to study the dimensions of the images of K under quasi-conformal map f_{μ_t}.

In fact, we have viewed here the Bowen–Series construction as giving an IFS for K; we can look at it equivalently as giving a Markov partition for the whole circle, the latter one being the limit set of G as well as the limit set of the Markov map. If μ is a G-compatible Beltrami coefficient, we may transport the whole Markov partition (and the Markov map as well) on $\Lambda_\mu = f_\mu(\partial\Delta)$, which is also the limit set of G_μ. If we use the IFS point of view, it is clear that the Hausdorff dimension of $K_\mu = f_\mu(K)$ is the same as the Haudorff dimension of $\Lambda(G_\mu)$ by self-(almost) similarities. So, conjugating by f_{μ_t}, we get an IFS S_t induced by the IFS S as

$$S_t = \{\phi_i^t : \phi_i^t = f_{\mu_t} \circ \phi_i \circ f_{\mu_t}^{-1},\ i \in I,\ t \in \Delta\}.$$

Let

$$\psi_n^t(\sigma) = \sum_{\omega \in I_n^t} \|(\phi_\omega^t)'\|^\sigma, \quad n \geq 1$$

and let P be the topological pressure function

$$P(t, \sigma) = \lim_{n\to\infty} \frac{1}{n} \log \psi_n^t(\sigma), \quad t \in \Delta, \sigma \in (\theta_t, +\infty),$$

where $\theta_t = \inf_\sigma\{\sigma : \psi_1^t(\sigma) < \infty\}$.

If the set of generators of G_t contains no parabolic elements, then the index set I is finite. Thus $\theta_t = -\infty$ and the system is regular. When the set of generators of G_t contains some parabolic elements, we need the following lemma.

LEMMA 4.1. *For any $t \in \Delta$, $\theta_t = \frac{1}{2}$ and the IFS S_t is regular.* □

Proof. For fixed $t \in \Delta$, we need to show that

$$\psi_1^t(\sigma) = \sum_{i \in I} \|(\phi_i^t)'\|^\sigma < \infty,\ \sigma > \frac{1}{2},$$

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and

$$\psi_1^t\left(\frac{1}{2}\right) = \sum_{i\in I} \|(\phi_i^t)'\|^{\frac{1}{2}} = \infty.$$

Without loss of generality, we may suppose that the generators $\{\gamma_1, \ldots, \gamma_m, g\}$ of G contain only one parabolic element g. Now we divide I into two parts, $\mathcal{I}_h$ and $\mathcal{I}_p$, where

$$\mathcal{I}_h = \{i \in I : \phi_i \text{ is hyperbolic}\}$$

and

$$\mathcal{I}_p = I \setminus \mathcal{I}_h = \{i \in I : \phi_i \text{ is parabolic}\}$$

Then,

$$\psi_1^t(\sigma) = \sum_{i\in I} \|(\phi_i^t)'\|^{\sigma} = \sum_{i\in \mathcal{I}_h} \|(\phi_i^t)'\|^{\sigma} + \sum_{i\in \mathcal{I}_p} \|(\phi_i^t)'\|^{\sigma}.$$

By the property of the Markov map f_G and the definition of f_K, the index set $\mathcal{I}_h$ is a finite set (see [15, p. 160]). Hence

$$\sum_{i\in \mathcal{I}_h} \|(\phi_i^t)'\|^{\sigma} < \infty.$$

Since ∞ is an ordinary point of G_t, by [7] we know that

$$\sum_{i\in \mathcal{I}_p} \|(\phi^t)_i'\|^{\sigma} \asymp \sum \frac{1}{n^{2\sigma}},$$

where $A \asymp B$ means that $A/C < B < CB$ for some implicit constant C that depends only on the number of hyperbolic generators of G_t and the complex dilatation of f_{μ_t}. The lemma follows. □

Mauldin and Urbanski [20] showed that, for a regular system, the dimension of the limit set is the unique zero of the function $\sigma \mapsto P(t, \sigma)$. To finish the proof of the theorem, it remains only to prove that the zero varies real analytically with respect to t. This follows from the classical thermodynamic formalism (a generalization of the Perron–Frobenius theorem (see [13, 23])): $\exp P(t, \sigma)$ is an isolated eigenvalue of an transfer operator. The theorem follows from the implicit function theorem applied to $(t, \sigma) \mapsto \exp(P(t, \sigma))$. □

5. *Proof of Theorem 1.2*

Let $\mathbb{H}$ be the upper half plane $\{z : \mathrm{Im}(z) > 0\}$, let $\mathcal{D}_1^*$ be the closed disk with diameter $[0, 2]$ and let $\mathcal{D}_n^*$, $n \geq 2$ be the closed disk with diameter $[2^{n-1}, 2^n]$. We consider the domain

$$\Omega = \mathbb{H} \setminus \left(\left(\bigcup_{n\geq 1} \mathcal{D}_n^*\right) \cup \left(\bigcup_{n\geq 1} (-\mathcal{D}_n^*)\right)\right).$$

Let ϕ be the conformal mapping from Ω onto $\mathbb{H}$ fixing 0, 2 and ∞. We put $z_0 = 0$, and $z_n = \phi(2^n),\ n \geq 1$ and $z_n = \phi(-2^{-n}),\ n \leq -1$. Let σ_n be the reflection with respect to $\partial\mathcal{D}_n^*$ and $\tau(z) = -\bar{z}$. By Rubel and Ryff's construction [22] of the covering group of a Riemann surface $S = \mathbb{C} \setminus \{z_n\}$, the Fuchsian group Γ generated by $\{\tau \circ \sigma_n\}_{n=1}^{\infty}$ uniformizes the surface S in the sense that $S \simeq \mathbb{H}/\Gamma$.

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If we return to the disk model, we then have $S \simeq \Delta/G$, where $G = \eta \circ \Gamma \circ \eta^{-1}$ and $\eta : \mathbb{H} \to \Delta$ is an isomorphism which sends ∞ to i, 1 to 1 and 0 to $-i$. Denote $\mathcal{D}_n = \eta(\mathcal{D}_n^*)$. The disks $\mathcal{D}_n \bigcap \bar{\Delta}$ accumulate to i and the diameter of $\mathcal{D}_n$ is comparable to 2^{-n}.

Let F be the domain $\Delta \setminus \bigcup_{i \neq 0} \mathcal{D}_i$. The domain F is symmetric about the y-axis. Let E be the intersection of the closure of F with the unit circle $\partial\Delta$. By the construction of G, we know that E contains countably many points.

As in [15], we denote by $\mathcal{N}$ the set of images of the sides of F under G. Let us also denote by I_n, $n \in \mathbb{Z} \setminus \{0\}$, the intersection of $\mathcal{D}_n$ with $\partial\Delta$. For each point $e \in E$, we consider the set of all the elements of $\mathcal{N}$ passing through e that are not a side of F; we denote it by $\mathcal{N}_e$. We denote by $\mathcal{N}_E$ the set that contains all the elements in $\mathcal{N}$ meeting $\partial\Delta$ with only one endpoint in E. For each $n \in \mathbb{Z}$, the intervals formed by the intersection of the elements of $\mathcal{N}_E$ with $\partial\Delta$ then form a partition of each interval I_n. Let e_{n-1}, e_n (in anticlockwise order on $\partial\Delta$) be the endpoints of I_n. For $k \leq -1$, we denote by $I_{n,k}$ the subinterval of I_n with endpoints just as the $|k|$th and $(|k| + 1)$th points in clockwise order of the set of the intersection of elements of $\mathcal{N}_{e_{n-1}}$ with $\partial\Delta$. Similarly, for $k \geq 1$, we denote by $I_{n,k}$ the subinterval of I_n with endpoints just as the kth and $(k + 1)$th points in anticlockwise order of the set of the intersection of elements of $\mathcal{N}_{e_n}$ with $\partial\Delta$. In this case, $I_{n,0}$ is just the subinterval of I_n with endpoints just as the leftmost point in anticlockwise order of the set of the intersection of elements of $\mathcal{N}_{e_{n-1}}$ with $\partial\Delta$ and the rightmost point in the anticlockwise order of the set of the intersection of elements of $\mathcal{N}_{e_n}$ with $\partial\Delta$. Hence we have $I_n = \bigcup_{k \in \mathbb{Z}} I_{n,k}$. The set K in [15] is just $\bigcup_{n \in \mathbb{Z} \setminus \{0\}} I_{n,0}$.

On each interval I_n, $n \in \mathbb{Z} \setminus \{0\}$, the Markov map f is equal to $s_0 \circ s_n$, where s_0 is the reflection across the y-axis and s_n is the reflection across $\partial\mathcal{D}_n$. Then the induced map is equal to f on $I_{n,0}$ and to f^{n_k} on $I_{n,k}$, where n_k is the first integer such that $f^{n_k}(I_{n,k}) \subset K$.

Let $\mathcal{I} = \{(n, k), n \in \mathbb{Z} \setminus \{0\}, k \in \mathbb{Z}\}$. For $i \in \mathcal{I}$, let us put $\phi_i = f^{n_i}$ on I_i, where f is the Markov map. Then we get a dynamical system with a Markov partition $\{I_i, i \in \mathcal{I}, \phi_i\}$. The limit set of this dynamical system is now the unit circle minus the parabolic fixed points and the orbit of i under G. Let (μ_t) be a family of G-compatible Beltrami coefficients that is analytic in $t \in \Delta$ and G_t, the deformation group of G under the quasi-conformal mapping f_{μ_t}. Conjugating by f_{μ_t}, we get a dynamical system with a Markov partition on Λ_{μ_t} whose limit set is now Λ_{μ_t} minus the parabolic points and the orbit of $f_\mu(i)$ under G_t. Let us denote by I_i^t, $i \in \mathcal{I}$, the elements of this Markov partition and let d_i^t be their diameter.

In order to show that the Fuchsian group G has Ruelle's property, we need to show, as before, the regularity of this dynamical system. The difficulty here is that we have infinitely many parabolic elements in G, and we thus need uniform estimates.

LEMMA 5.1. *For any $t \in \Delta$, the IFS S_t is regular with the θ number equal to $\frac{1}{2}$.*

For fixed $t \in \Delta$, we need to show that

$$\psi_1^t(\sigma) := \sum_{i \in I} (d_i^t)^\sigma < +\infty, \ \sigma > \frac{1}{2},$$

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and

$$\psi_1^t(\tfrac{1}{2}) = +\infty.$$

To prove this, we will need two facts.

(1) If $|t| \leq k < 1$, then there exists $c(k) \in (0, 1)$ such that the maps f_{μ_t} are all $c(k)$-quasi-conformal and thus there exists $a(k) \in (0, 1)$ such that all these maps are $a(k)$-Hölder continuous with a uniform norm. As a consequence, for all $n \geq 1$ and $|t| \leq k$,

$$d_{n,\pm 1}^t \leq C2^{-na(k)},$$

with a uniform C.

(2) The maps f_{μ_t} conjugate all the parabolic elements to parabolic elements, and parabolic elements are conjugated to translations, by sending the parabolic fixed point to ∞. It follows that the endpoints of $I_{n,k}$, $k \geq 1$ (or $k \leq -1$) all lie on a same circle passing through the nth parabolic point. Combined with the first fact, we get

$$d_{n,\pm k}^t \leq \frac{C2^{-na(k)}}{k^2},$$

with a constant C independent of n, k.

This proves the first part of the lemma. To prove the second part, we notice that the last inequality may be reversed as

$$d_{n,\pm k}^t \geq \frac{d_{n,\pm 1}^t}{k^2},$$

from which the result follows.

As in the proof of Theorem 1.2, Mauldin and Urbanski [20] showed that, for a regular system, the dimension of the limit set is the unique zero of the function $\sigma \mapsto P(t, \sigma)$. By the classical thermodynamic formalism (a generalization of the Perron–Frobenius theorem, see [23]) we know that $\exp P(t, \sigma)$ is an isolated eigenvalue of a transfer operator. The theorem follows from the implicit function theorem applied to $(t, \sigma) \mapsto \exp(P(t, \sigma))$. $\square$

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