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NONDIVISIBILITY AMONG IRREDUCIBLE CHARACTER CO-DEGREES

NEDA AHANJIDEH

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Abstract

For a character χ of a finite group G, the number $\chi^c(1) = [G : \ker\chi]/\chi(1)$ is called the co-degree of χ. A finite group G is an NDAC-group (no divisibility among co-degrees) when $\chi^c(1) \nmid \phi^c(1)$ for all irreducible characters χ and ϕ of G with $1 < \chi^c(1) < \phi^c(1)$. We study finite groups admitting an irreducible character whose co-degree is a given prime p and finite nonsolvable NDAC-groups. Then we show that the finite simple groups ${}^2B_2(2^{2f+1})$, where $f \geq 1$, $\mathrm{PSL}_3(4)$, Alt_7 and J_1 are determined uniquely by the set of their irreducible character co-degrees.

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1. Introduction and preliminaries

In this paper, $G$is a finite groupand p is a prime number. Let $\mathrm{Irr}(G)$ denote the set of (complex) irreducible characters of G. For a normal subgroup N of G and $\theta \in \mathrm{Irr}(N)$, let $I_G(\theta)$ denote the inertia group of θ in G and let $\mathrm{Irr}(G|\theta)$ be the set of the irreducible constituents of the induced character θ^G. Also, $n_p(G)$ denotes the number of Sylow p-subgroups of G. If m is a positive integer, m_p denotes the p-part of m. For a character χ of G, the number $\chi^c(1) = [G : \ker\chi]/\chi(1)$ is called the co-degree of χ (see [11]). Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$ and $\mathrm{cd}(G) = \{\chi(1) : \chi \in \mathrm{Irr}(G)\}$. In [1, 3, 4, 5, 8, 11], various properties of the co-degrees of irreducible characters of finite groups are studied. By [11, Theorem A], if $p \mid |G|$, then p divides some element of $\mathrm{Codeg}(G)$. In [1, 3, 4], it is shown that the p-parts of the co-degrees of irreducible characters of finite groups can control the structure of groups. Our first result is the following theorem of this type.

THEOREM 1.1. *Let G be a finite group and p a prime. Then $p \in \mathrm{Codeg}(G)$ if and only if either p divides $|G/G'|$ or G has a normal subgroup K such that G/K is a Frobenius group whose Frobenius kernel has order p and whose Frobenius complement is cyclic.*

A finite group G is an NDAD-group (no divisibility among degrees) when $\alpha \nmid \beta$ for all $\alpha, \beta \in \mathrm{cd}(G)$ with $1 < \alpha < \beta$. In [10], the finite nonsolvable NDAD-groups were classified. Similarly, a finite group G is an NDAC-group (no divisibility among co-degrees) when $\alpha \nmid \beta$ for all $\alpha, \beta \in \mathrm{Codeg}(G)$ with $1 < \alpha < \beta$. In [2], we showed that if $\gcd(\alpha, \beta)$ is either 1 or a prime for all $\alpha, \beta \in \mathrm{Codeg}(G)$ with $1 < \alpha < \beta$, then G is solvable. Our second result is the following classification theorem.

THEOREM 1.2. *Let G be a nonsolvable NDAC-group and let U be the maximal normal solvable subgroup of G. Then G is perfect and G/U is isomorphic to one of the simple groups $J_1, {}^2B_2(2^{2f+1})$, where $f \geq 1$, Alt_7, $\mathrm{PSL}_3(4)$ or $\mathrm{PSL}_2(2^f)$, where $f > 1$.*

Let P be an elementary abelian 2-group of order 16. Then there is an action of Alt_5 on P such that for the extension G of P by Alt_5, we have $\mathrm{Codeg}(G) = \{1, 12, 15, 20, 64\}$. So, G is an NDAC-group. This example shows that in Theorem 1.2, U is not necessarily a central subgroup of G.

In 1990, Huppert conjectured that if S is a nonabelian simple group such that $\mathrm{cd}(G) = \mathrm{cd}(S)$, then $G \cong S \times A$, where A is abelian. Let $q > 3$ be a prime power. In [5], the analogue of Huppert's conjecture for character co-degrees has been verified for the simple group $\mathrm{PSL}_2(q)$, that is, if $\mathrm{Codeg}(G) = \mathrm{Codeg}\mathrm{PSL}_2(q))$, then $G \cong \mathrm{PSL}_2(q)$. We continue this investigation and prove the following result as a corollary of Theorem 1.2.

COROLLARY 1.3. *Let S be one of the simple groups $J_1, {}^2B_2(2^{2f+1})$, where $f \geq 1$, Alt_7 or $\mathrm{PSL}_3(4)$. If $\mathrm{Codeg}(G) = \mathrm{Codeg}(S)$, then G is isomorphic to S.*

2. Proofs of the main results

LEMMA 2.1 [11, Lemma 2.1]. *Let N be a normal subgroup of G. Then $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. Also, if $\psi \in \mathrm{Irr}(N)$, then $\psi^c(1) \mid \chi^c(1)$ for every $\chi \in \mathrm{Irr}(G|\psi)$.*

LEMMA 2.2. *Let $N = S_1 \times \cdots \times S_t$ be a minimal normal subgroup of G, where $S_i \cong S$, a nonabelian simple group. Then there exists $\phi \in \mathrm{Irr}(N)$ that extends to G with $\ker \phi = \{1\}$.*

PROOF. This follows immediately from [13, 12] and [6, Theorems 3–4 and Lemma 5]. □

LEMMA 2.3. *Let N be a minimal normal subgroup of G. If N is abelian and $\mathrm{Codeg}(G) = \mathrm{Codeg}(G/N)$, then $|N|$ divides $|G/N|$.*

PROOF. By [4, Proposition 6(i)], if $\chi \in \mathrm{Irr}(G)$ is such that $N \nleq \ker \chi$, then $|N| \mid \chi^c(1)$. However, $\mathrm{Codeg}(G) = \mathrm{Codeg}(G/N)$. Hence, $\chi^c(1) \mid |G/N|$. So, the lemma follows. □

2.1. Proof of Theorem 1.1. If p divides $|G/G'|$, then [8, Lemma 2.2] completes the proof. Next let K be a normal subgroup of G such that G/K is a Frobenius group whose Frobenius kernel is F/K and $|F/K| = p$. Let $\alpha \in \mathrm{Irr}(F/K) - \{1_{F/K}\}$ and $\chi \in \mathrm{Irr}(G/K|\alpha)$. Then $I_{G/K}(\alpha) = F/K$ and, hence, $\chi^c(1) = |F/K| = p$. By Lemma 2.1, $p \in \mathrm{Codeg}(G)$,

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as required. Conversely, let $\chi^c(1) = p$ for some $\chi \in \mathrm{Irr}(G)$. If $\chi(1) = 1$, then $\chi \in \mathrm{Irr}(G/G')$. So, $\chi^c(1) = p \mid |G/G'|$, as required. Now let $\chi(1) \neq 1$. Fix $K = \ker\chi$ and set $\bar{G} = G/K$. Then $\bar{G}$ is nonabelian, $p \mid |\bar{G}|$ and $\chi(1) = |\bar{G}|/p$. Let $\bar{P}$ be a p-subgroup of $\bar{G}$ of order p. If $C_{\bar{G}}(\bar{P}) \neq \bar{P}$, then there exists an abelian subgroup $\bar{A}$ of $\bar{G}$ such that $\bar{P} < \bar{A}$. By [9, Problem 2.9(b)], $\chi(1) = |\bar{G}|/p \leq [\bar{G} : \bar{A}]$. Consequently, $|\bar{A}| \leq p$, which is a contradiction. This forces $C_{\bar{G}}(\bar{P}) = \bar{P}$. Also, $\chi^2(1) < |\bar{G}|$. Therefore, $|\bar{G}|^2/p^2 < |\bar{G}|$. Hence, $|\bar{G}| < p^2$. This shows that $\bar{P}$ is a Sylow p-subgroup of $\bar{G}$ and $n_p(\bar{G}) < p$. It follows that $n_p(\bar{G}) = 1$, so $\bar{P}$ is normal in $\bar{G}$. Since $\bar{G}/\bar{P} = N_{\bar{G}}(\bar{P})/C_{\bar{G}}(\bar{P}) \lesssim \mathrm{Aut}(\bar{P})$, $C_{\bar{G}}(\bar{P}) = \bar{P}$ and $\mathrm{Aut}(\bar{P})$ is a cyclic group of order $p - 1$, it follows that $\bar{G}$ is a Frobenius group whose Frobenius kernel is $\bar{P}$ and whose Frobenius complement is cyclic. Now the theorem follows. □

2.2. Proof of Theorem 1.2. In order to prove Theorem 1.2, we need to prove the following propositions.

PROPOSITION 2.4. *Every nonsolvable* NDAC-*group is perfect.*

PROOF. Let G be a minimal counterexample. Assume that M is a minimal normal subgroup of G such that G/M is nonsolvable. Since the hypothesis is inherited by quotients, G/M is perfect. Hence, $G'M = G$ and $M \not\leq G'$. Thus, $M \neq M \cap G' \trianglelefteq G$. As M is a minimal normal subgroup of G, $M \cap G' = \{1\}$ and so $G = G' \times M$. Consequently, M is an elementary abelian p-group for some prime p. Therefore, M admits an irreducible character ϕ whose co-degree is p. Since G is nonsolvable, G' is nonsolvable. So, there is a prime divisor q of $|G'|$ such that $p \neq q$. By [11, Theorem A], there is a $\chi \in \mathrm{Irr}(G')$ such that $q \mid \chi^c(1)$. Set $\chi_1 = 1_{G'} \times \phi$ and $\chi_2 = \chi \times \phi$. Then $\chi_1, \chi_2 \in \mathrm{Irr}(G)$, $\chi_1^c(1) = p$ and $pq \mid \chi_2^c(1)$, which is a contradiction. Now suppose that for every normal subgroup L of G, G/L is solvable. This forces G to have M as its unique minimal normal subgroup, because G is nonsolvable. We observe that M is nonabelian. As $G' \neq G$ and G/M is solvable, $M \neq G$ and $(G/M)' \neq G/M$. By Lemma 2.2, there is a $\phi \in \mathrm{Irr}(M)$ that extends to $\chi \in \mathrm{Irr}(G)$ and $\ker\phi = \{1\}$. Since M is the unique minimal normal subgroup of G and $\ker\phi = \{1\}$, we have $\ker\chi = \{1\}$. Hence, $|G/M| < \chi^c(1)$ and $|G/M|$ divides $\chi^c(1)$. On the other hand, $G/M \neq (G/M)'$. Let p be a prime divisor of $[G/M : (G/M)']$. By Theorem 1.1, there exists a $\psi \in \mathrm{Irr}(G/M)$ whose co-degree is p. However, $\psi \in \mathrm{Irr}(G)$, $\psi^c(1) \mid \chi^c(1)$ and $\psi^c(1) \neq \chi^c(1)$. This is a contradiction. Therefore, G is perfect, as required. □

PROPOSITION 2.5. *Let G be a nonsolvable* NDAC-*group. If U is the maximal normal solvable subgroup of G, then G/U is a nonabelian simple group.*

PROOF. We work by induction on $|G|$. By Lemma 2.1, for every proper normal subgroup L of G, G/L is an NDAC-group. First, suppose that every normal subgroup of G is nonsolvable. Then $U = \{1\}$. Let M_1 and M_2 be two distinct minimal normal subgroups of G. Then M_1 and M_2 are nonsolvable. Also, $M_1M_2/M_1 \trianglelefteq G/M_1$. So, G/M_1 is a nonsolvable NDAC-group. By induction, $G/M_1 = M_1M_2/M_1 \times U_0/M_1$, where U_0/M_1 is solvable. By Proposition 2.4, $G' = G$. Hence, $U_0/M_1 = \{M_1\}$. Consequently,

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$G = M_1 \times M_2$. Let $\psi_1 \in \mathrm{Irr}(M_1) - \{1_{M_1}\}$ and $\psi_2 \in \mathrm{Irr}(M_2) - \{1_{M_2}\}$. Set $\chi_1 = \psi_1 \times 1_{M_2}$ and $\chi_2 = \psi_1 \times \psi_2$. Then $\chi_1, \chi_2 \in \mathrm{Irr}(G) - \{1_G\}$, $\chi_1^c(1) = |M_1|/\psi_1(1)$ divides $\chi_2^c(1) = |M_1||M_2|/(\psi_1(1)\psi_2(1))$ and $\chi_1^c(1) \neq \chi_2^c(2)$, which is a contradiction.

Next let M be the unique minimal normal subgroup of G. If $M = G$, then G is nonabelian and simple, as desired. If $G \neq M$, then there is a $\psi \in \mathrm{Irr}(M)$ that extends to $\chi \in \mathrm{Irr}(G)$ and $\ker\psi = \{1\}$, by Lemma 2.2. So, $\ker\chi = \{1\}$. Thus, $|G/M| \mid \chi^c(1)$ and $|G/M| \neq \chi^c(1)$. For every $\phi \in \mathrm{Irr}(G/M) - \{1_{G/M}\}$, we have $1 \neq \phi^c(1) \mid |G/M|$. Consequently, $\phi^c(1) \neq \chi^c(1)$ and $\phi^c(1) \mid \chi^c(1)$, which is a contradiction. Finally, suppose that M is a minimal normal subgroup of G, which is abelian. Then G/M is a nonsolvable NDAC-group. By induction, G/M is an extension of a solvable group by a simple group and so is G. Now the proof is complete. □

Proof of Theorem 1.2. By Propositions 2.4 and 2.5, G is perfect and G/U is nonabelian and simple. On the other hand, G/U is an NDAC-group and, for every $\psi \in \mathrm{Irr}(G/U) - \{1_{G/U}\}$, we have $\psi(1) > 1$ and $\ker\psi = 1$. So, $\alpha \nmid \beta$ for all $\alpha, \beta \in \mathrm{cd}(G/U)$ with $1 < \alpha < \beta$. Hence, G/U is a nonabelian simple NDAD-group. Therefore, [10, Theorem A] completes the proof. □

Proof of Corollary 1.3. By Theorem 1.1, Codeg(S) does not contain any prime number and neither does Codeg(G). So, Theorem 1.1 forces G to be perfect. This implies that G is a nonsolvable NDAC-group, because Codeg(G) = Codeg(S) and S is an NDAC-group. If U is the maximal normal solvable subgroup of G, then Theorem 1.2 shows that G/U is isomorphic to one of the simple groups $\mathrm{PSL}_2(2^f)$, where $f > 1$, ${}^2B_2(2^{2f+1})$, where $f \geq 1$, Alt_7, $\mathrm{PSL}_3(4)$ and J_1. We note that

$$\mathrm{Codeg}(\mathrm{PSL}_2(2^f)) = \{1, 2^f(2^f-1), 2^f(2^f+1), (2^{2f}-1)\}, \tag{2.1}$$

$$\begin{aligned}\mathrm{Codeg}({}^2B_2(2^{2f+1})) = \{&1, (2^{2(2f+1)}+1)(2^{2f+1}-1), 2^{2(2f+1)}(2^{2f+1}-1),\\ &2^{f+1}(2^{2(2f+1)}+1), 2^{2(2f+1)}(2^{2f+1}-2^{f+1}+1),\\ &2^{2(2f+1)}(2^{2f+1}+2^{f+1}+1)\},\end{aligned} \tag{2.2}$$

$$\mathrm{Codeg}(\mathrm{PSL}_3(4)) = \{1, 2^4.3^2.7, 2^6.3^2, 2^6.7, 2^6.5, 3^2.5.7\}, \tag{2.3}$$

$$\mathrm{Codeg}(\mathrm{Alt}_7) = \{1, 2^2.3^2.5, 2^2.3^2.7, 2^3.3.7, 2^2.3.5.7, 2^3.3.5, 2^3.3^2\}, \tag{2.4}$$

$$\begin{aligned}\mathrm{Codeg}(J_1) = \{&1, 3.5.11.19, 2.3.5.7.11, 2^3.3.5.19, 7.11.19,\\ &2^3.3.5.11, 2^3.3.5.7\}.\end{aligned} \tag{2.5}$$

Thus, Codeg(G/U) contains two different elements whose 2-parts are $|G/U|_2$. Since Codeg(G/U) ⊆ Codeg(G) = Codeg(S), considering the elements of Codeg(S),

$$|S|_2 = |G/U|_2. \tag{2.6}$$

This shows that if $S \cong \mathrm{Alt}_7$ or J_1, then $|G/U|_2 = 2^3$ and so $G/U \cong \mathrm{Alt}_7$, $\mathrm{PSL}_2(2^3)$ or J_1. However, $2.3.5.7.11 \in \mathrm{Codeg}(J_1)$, $11 \nmid |\mathrm{Alt}_7|$, $2^3.3^2 \in \mathrm{Codeg}(\mathrm{Alt}_7)$ and $3^2 \nmid |J_1|$. Also, $2^3.7 \in \mathrm{Codeg}(\mathrm{PS}L_2(2^3))$ and $2^3.7 \notin \mathrm{Codeg}(\mathrm{Alt}_7), \mathrm{Codeg}(J_1)$. This shows that if

$S \cong \mathrm{Alt}_7$ or J_1, then $G/U \cong S$. If $S \cong {}^2B_2(q)$ or $\mathrm{PSL}_3(4)$, where $q = 2^{2f+1} \geq 8$, then

$$\mathrm{Codeg}(S) - \{1\} \text{ contains exactly one odd number which is } |S|/|S|_2, \tag{2.7}$$

by (2.2) and (2.3). As $|\mathrm{Alt}_7|_2, |J_1|_2 < |S|_2$, it follows that $G/U \not\cong \mathrm{Alt}_7, J_1$, by (2.6). In the remaining possibilities, (2.1)–(2.3) show that $|G/U|/|G/U|_2 \in \mathrm{Codeg}(G/U)$. Since $\mathrm{Codeg}(G/U) \subseteq \mathrm{Codeg}(G)$ and $|G/U|/|G/U|_2$ is odd, we see from (2.7) that $|G/U|/|G/U|_2 = |S|/|S|_2$. So, (2.6) forces $|G/U| = |S|$. Now we can check easily that $G/U \cong S$. So, in all cases,

$$G/U \cong S. \tag{2.8}$$

Next, we claim that $U = \{1\}$. Working towards a contradiction, let $U \neq \{1\}$ and let M be a maximal normal subgroup of G such that $M \lneqq U$. Then U/M is a minimal normal subgroup of G/M. As U is solvable, U/M is an elementary abelian r-group for some prime divisor r of $|G|$. Let $1_{U/M} = \lambda_1, \ldots, \lambda_t$ be the representatives of the action of G/M on $\mathrm{Irr}(U/M)$. If O_i is the G/M-orbit of λ_i, then $1 + \Sigma_{i=2}^t |O_i|\lambda_i(1)^2 = \Sigma_{\lambda \in \mathrm{Irr}(U/M)} \lambda(1)^2 = |U/M| \equiv_r 0$. Hence, there exists an $i > 1$ such that $r \nmid |O_i|$. Since $|O_i| = [G/M : I_{G/M}\lambda_i)]$, it follows that

$$|G/M|_r \mid |I_{G/M}(\lambda_i)|. \tag{2.9}$$

Also,

$$[G/M : I_{G/M}(\lambda_i)] = |O_i| < |U/M|. \tag{2.10}$$

Let $\chi \in \mathrm{Irr}(G/M|\lambda_i)$. If $I_{G/M}(\lambda_i) = G/M$, then $\chi_{U/M} = e\lambda_i$ for some positive integer e. So, $\ker\chi \cap U/M = \ker\lambda_i$. However, $\ker\chi \cap U/M \trianglelefteq G/M$ and U/M is a minimal normal subgroup of G/M. Thus, either $\ker\chi \cap U/M = U/M$ or $\{M\}$. In the former case, $U/M \leq \ker\chi$. Hence, $\chi_{U/M} = e1_{U/M}$, which is a contradiction. In the latter case, $\ker\lambda_i = \ker\chi \cap U/M = \{M\}$. Hence, U/M is a cyclic group of order r. Since $(G/M)/C_{G/M}(U/M) = N_{G/M}(U/M)/C_{G/M}(U/M) \lesssim \mathrm{Aut}(U/M)$, $\mathrm{Aut}(U/M)$ is a cyclic group of order $r - 1$ and $G = G'$, we see that $C_{G/M}(U/M) = G/M$. Hence, $U/M \leq Z(G/M)$. On the other hand, $(G/M)' = G/M$. Thus, $|U/M| = r$ and G/M is a Schur cover of G/U. Therefore, $S \cong \mathrm{Alt}_7$ and $r \in \{2, 3\}$, $S \cong {}^2B_2(8)$ and $r = 2$ or $S \cong PSL_3(4)$ and $r \in \{2, 3\}$. By [7], we can check that $\mathrm{Codeg}(G/M) \nsubseteq \mathrm{Codeg}(S) = \mathrm{Codeg}(G)$, which is a contradiction with Lemma 2.1. Hence, $I_{G/M}(\lambda_i) \neq G/M$. This yields $C_{G/M}(U/M) = U/M$. We note that $U/M \leq I_{G/M}(\lambda_i)$ and $(G/M)/(U/M) \cong G/U \cong S$. Consequently,

$$\frac{I_{G/M}(\lambda_i)}{U/M} \text{ is isomorphic to a proper subgroup of } S. \tag{2.11}$$

By (2.8) and Lemma 2.1,

$$\mathrm{Codeg}(S) = \mathrm{Codeg}(G/U) = \mathrm{Codeg}\left(\frac{G/M}{U/M}\right) \subseteq \mathrm{Codeg}(G/M) \subseteq \mathrm{Codeg}(G) = \mathrm{Codeg}(S).$$

Hence, $\mathrm{Codeg}(G/M) = \mathrm{Codeg}(G/U) = \mathrm{Codeg}((G/M)/(U/M))$. It follows from Lemma 2.3 that $|U/M| \leq |(G/M)/(U/M)|_r = |G/U|_r$. So, (2.8) and (2.10) force

$$[G/M : I_{G/M}(\lambda_i)] < |S|_r. \tag{2.12}$$

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We continue the proof by examining each of the four possibilities for the simple group in Corollary 1.3 in turn.

Case 1: $S \cong {}^2B_2(q)$, *where* $q = 2^{2f+1} \geq 8$. By (2.8), $G/U \cong {}^2B_2(q)$. As stated in [10, Lemma 3.8], any proper subgroup of $S = {}^2B_2(q)$ is isomorphic to a subgroup of ${}^2B_2(q_0)$, where $q_0 = 2^{2e+1} \mid q$, or to a subgroup of some subgroup of order $q^2(q-1), 2(q-1), 4(q-2^{f+1}+1)$ or $4(q+2^{f+1}+1)$. If $|I_{G/M}(\lambda_i)|$ divides $|{}^2B_2(q_0)|$, then $q \nmid \chi^c(1)$, $q-1 \nmid \chi^c(1)$ and $(q \pm 2^{f+1}+1) \nmid \chi^c(1)$ for every $\chi \in \mathrm{Irr}(G|\lambda_i)$, because $\chi^c(1) \mid |I_{G/M}(\lambda_i)|$. This leads to a contradiction by considering the elements of $\mathrm{Codeg}(S)$ in (2.2). It follows from (2.9), (2.11) and the above statement that

$$\left|\frac{I_{G/M}(\lambda_i)}{U/M}\right| \text{ divides } \begin{cases} q^2(q-1) & \text{if } r=2, \\ q^2(q-1) & \text{if } r \mid q-1, \\ 4(q-2^{f+1}+1) \text{ or } 4(q+2^{f+1}+1) & \text{if } r \mid q^2+1. \end{cases}$$

Thus, $|S|_r < [G/M : I_{G/M}(\lambda_i)]$, contradicting (2.12).

Case 2: $S \cong \mathrm{PSL}_3(4)$. By (2.8), $G/U \cong \mathrm{PSL}_3(4)$. By [7], any proper subgroup of $\mathrm{PSL}_3(4)$ is isomorphic to a subgroup of some subgroup of order 72, 360, 168 or 960. It follows from (2.9), (2.11) and the above statement that

$$\frac{I_{G/M}(\lambda_i)}{U/M} \text{ divides } 960 \text{ if } r=2,\ 360 \text{ if } r=3,\ 360 \text{ or } 960 \text{ if } r=5,\ 168 \text{ if } r=7.$$

Thus, if $r \neq 2$, then $|S|_r < [G/M : I_{G/M}(\lambda_i)]$, contradicting (2.12). Now let $r=2$. Then $2 \nmid [G/M : I_{G/M}(\lambda_i)]$ and $21 \leq [G/M : I_{G/M}(\lambda_i)] < |U/M| \leq |S|_2 = 64$, by (2.9), (2.10) and Lemma 2.3. Since $|I_{G/M}(\lambda_i)/(U/M)| \mid 960$, either $|I_{G/M}(\lambda_i)/(U/M)| = 320$ and $|U/M| = 64$ or $|I_{G/M}(\lambda_i)/(U/M)| = 960$ and $|U/M| \in \{32, 64\}$. Let $\chi \in \mathrm{Irr}(G/M|\lambda_i)$. Then χ is faithful. Hence, $\chi^c(1) = |I_{G/M}(\lambda_i)|/e_\chi$, where $e_\chi = \langle \chi_{U/M}, \lambda_i \rangle$. Since $\chi^c(1) \in \mathrm{Codeg}(G/M) = \mathrm{Codeg}(G)$ and $3^2, 7 \nmid \chi^c(1)$, we have $\chi^c(1) = 2^6.5$, by (2.3). Consequently, if $|I_{G/M}(\lambda_i)/(U/M)| = 320$ and $|U/M| = 64$, then $e_\chi = 2^6$ and, if $|I_{G/M}(\lambda_i)/(U/M)| = 960$ and $|U/M| \in \{32, 64\}$, then $2^5.3 \mid e_\chi$. Therefore, $2^{12} \leq e_\chi^2 \leq [I_{G/M}(\lambda_i) : U/M] \leq 2^6.3.5$, which is impossible.

Case 3: $S \cong \mathrm{Alt}_7$. By (2.8), $G/U \cong \mathrm{Alt}_7$. By [7], any proper subgroup of $S = \mathrm{Alt}_7$ is isomorphic to a subgroup of some subgroup of order $2^3.3^2.5, 2^3.3.7, 2^3.3.5$ or $2^3.3^2$. From (2.9), (2.11) and the above statement,

$$\left|\frac{I_{G/M}(\lambda_i)}{U/M}\right| \text{ divides } \begin{cases} 2^3.3^2.5 \text{ or } 2^3.3.7 & \text{if } r=2, \quad 2^3.3^2.5 \quad \text{if } r=3, \\ 2^3.3^2.5 & \text{if } r=5, \quad 2^3.3.7 \quad \text{if } r=7. \end{cases}$$

Thus, if either $r \neq 2, 3$ or $r \in \{2, 3\}$ and $|I_{G/M}(\lambda_i)/(U/M)| \neq 2^3.3^2.5$, it follows that $|S|_r < [G/M : I_{G/M}(\lambda_i)]$, contradicting (2.12). Next assume that $r \in \{2, 3\}$ and $|I_{G/M}(\lambda_i)/(U/M)| = 2^3.3^2.5$. Thus, $|U/M| \geq 7$, by (2.10). If $r=2$, then $|U/M| \mid |G/U|_2 = 8$ and, if $r=3$, then $|U/M| \mid |G/U|_3 = 9$, by Lemma 2.3. So, $|U/M| \in \{8, 9\}$. We note that U/M is an elementary abelian r-group. Thus, $G/U \lesssim \mathrm{Aut}(U/M)$, which is isomorphic to $GL_3(2)$ or $GL_2(3)$. Hence, $3^2 \nmid |G/U|$, which is a contradiction.

Case 4: $S \cong J_1$. By (2.8), $G/U \cong J_1$. By [7], any proper subgroup of $S = J_1$ is isomorphic to a subgroup of some subgroup of order $2^2.3.5.11, 2^3.3.7, 2^3.3.5, 2.3.19, 2.5.11,$

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$2^2.3.5$ or 2.3.7. By (2.11), $|I_{G/M}(\lambda_i)/(U/M)|$ divides one of the above numbers. Thus, $|S|_r < [G/M : I_{G/M}(\lambda_i)]$, contradicting (2.12).

Therefore, $U = \{1\}$ and $G \cong S$, as required. □

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NEDA AHANJIDEH, Department of Pure Mathematics,
Faculty of Mathematical Sciences, Shahrekord University,
P.O. Box 115, Shahrekord, Iran
e-mail: ahanjideh.neda@sci.sku.ac.ir, ahanjidn@gmail.com

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