

ON CANONICAL BASES AND INDUCTION OF W -GRAPHS

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Abstract. A canonical basis in the sense of Lusztig is a basis of a free module over a ring of Laurent polynomials that is invariant under a certain semilinear involution and is obtained from a fixed “standard basis” through a triangular base change matrix with polynomial entries whose constant terms equal the identity matrix. Among the better known examples of canonical bases are the Kazhdan–Lusztig basis of Iwahori–Hecke algebras (see Kazhdan and Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184), Lusztig’s canonical basis of quantum groups (see Lusztig, *Canonical bases arising from quantized enveloping algebras*, J. Amer. Math. Soc. **3**(2) (1990), 447–498) and the Howlett–Yin basis of induced W -graph modules (see Howlett and Yin, *Inducing W -graphs I*, Math. Z. **244**(2) (2003), 415–431; *Inducing W -graphs II*, Manuscripta Math. **115**(4) (2004), 495–511). This paper has two major theoretical goals: first to show that having bases is superfluous in the sense that canonicalization can be generalized to nonfree modules. This construction is functorial in the appropriate sense. The second goal is to show that Howlett–Yin induction of W -graphs is well-behaved a functor between module categories of W -graph algebras that satisfies various properties one hopes for when a functor is called “induction,” for example transitivity and a Mackey theorem.

§1. Introduction

The ring $\mathbb{Z}[v^{\pm 1}]$ of Laurent polynomials has an involutive automorphism $\bar{}$ defined by $\bar{v} := v^{-1}$.

If M is a free $\mathbb{Z}[v^{\pm 1}]$ -module equipped with an $\bar{}$ -semilinear involution ι and a “standard” basis $(t_x)_{x \in X}$ then a *canonical basis with respect to (t_x) and ι* in the sense of Lusztig is a basis (c_x) of M such that $\iota(c_x) = c_x$ and $c_x \in t_x + \sum_{y \in X} v\mathbb{Z}[v]t_y$ hold.

Kazhdan and Lusztig showed in [15] that the Iwahori–Hecke algebra of any Coxeter group (W, S) has a canonical basis with respect to the standard basis $(T_w)_{w \in W}$ and the involution $\iota(T_w) = T_{w^{-1}}^{-1}$, which is now known simply

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as the Kazhdan–Lusztig basis. (Note that Lusztig used a slightly different construction in [17] which essentially replaces $v\mathbb{Z}[v]$ with $v^{-1}\mathbb{Z}[v^{-1}]$ though this does not change results significantly.) The special features of the action of the Hecke algebra on itself with respect to this basis are captured in the definition of W -graphs in the same paper.

In [12] Howlett and Yin showed that given any parabolic subgroup $W_J \leq W$ and a W_J -graph (\mathfrak{C}, I, m) representing the H_J -module V , then the induced module $\text{Ind}_{H_J}^H(V) := H \otimes_{H_J} V$ is also represented by a W -graph. They constructed the W -graph explicitly in terms of a canonical basis of $\text{Ind}_{H_J}^H(V)$ and developed their ideas of inducing W -graphs further in [13].

In [9] Gyoja proved that given any *finite* Coxeter group (W, S) all complex representations of the Hecke algebra can in fact be realized by a W -graph. His proof was not constructive but introduced the W -graph algebra as an auxiliary object which I investigated further in my thesis [10] and in my previous paper [11]. The fundamental property of the W -graph algebra Ω is that the Hecke algebra is canonically embedded into $\mathbb{Z}[v^{\pm 1}]\Omega$ in such a way that a representation of H given by a W -graph canonically extends to a representation of Ω . And conversely given an Ω -module with a sufficiently nice basis, a W -graph can be obtained that realizes the associated representation of H (see 3.10 for details). In this sense W -graphs can (and I am advocating that they should) be understood not as combinatorial objects encoding certain matrices but as modules of an algebra.

This paper is organized as follows: Section 2 is about modules over (generalized) Laurent polynomial rings equipped with an $\bar{}$ -semilinear involution. It defines canonical modules and canonicalizations of modules. The main theorem in this section is Theorem 2.11 which proves a sufficient condition to recognize canonical modules and also shows that under the conditions present in the context of Hecke algebras (though no reference to Hecke algebras is made in this section) the canonicalization is unique and functorial with respect to positive maps.

Section 3 recalls the definition of Iwahori–Hecke algebras, W -graphs and W -graph algebras.

Section 4 proves that Howlett–Yin induction is well defined as a map $\Omega_J\text{-mod} \rightarrow \Omega\text{-mod}$. The proof is inspired by Lusztig’s elegant treatment of the μ -values in [17] instead of the more laborious proof in Howlett and Yin’s paper. The proof in the style of Lusztig has the additional bonus that it provides an algorithm to compute p -polynomials and μ -values without having to compute r -polynomials as an intermediate step. Specifically, this

is Algorithm 4.11. It is shown how this theorem recovers earlier results, including Howlett and Yin's. As an application it is proven that the W -graph algebra associated to a parabolic subgroup $W_J \leq W$ can be canonically identified with a subalgebra of the W -graph algebra of W .

Section 5 then proves that Howlett–Yin induction has many of the nice properties one expects: it is indeed a functor between the two module categories, it can be represented as tensoring with a certain bimodule, it satisfies a transitivity property and an analogue of the Mackey theorem.

Section 6 then applies these findings. An improved, more efficient algorithm to compute μ -values is given which generalizes ideas from Geck's PyCox software (see [6]). Additionally a very short proof of a result of Geck on induction of Kazhdan–Lusztig cells (from [5]) is given.

§2. Canonicalization of modules

Fix a commutative ring k and a totally ordered, abelian group $(\Gamma, +, \leq)$, i.e., \leq is a total order on Γ such that $x \leq y \implies x + z \leq y + z$ holds.

Consider the k -algebra $\mathcal{A} := k[\Gamma]$. As is common when considering group algebras of additively written groups, we will denote the group element $\gamma \in \Gamma$ as $v^\gamma \in \mathcal{A}$ and think of \mathcal{A} as the ring of “generalized Laurent polynomials in v ” with coefficients in k . This k -algebra has an involutive automorphism $\bar{}$ defined by $\overline{v^\gamma} := v^{-\gamma}$.

We also consider the smash product $\widehat{\mathcal{A}} := \mathcal{A} \rtimes \langle \iota \rangle$ where $\langle \iota \rangle$ is a cyclic group of order two acting as $\bar{}$ on \mathcal{A} .¹ An $\widehat{\mathcal{A}}$ -module is therefore the same as an \mathcal{A} -module M equipped with an $\bar{}$ -semilinear involution $\iota : M \rightarrow M$.

DEFINITION 2.1. Define $\mathcal{A}_{<0}$ as the k -submodule of \mathcal{A} spanned by all v^γ with $\gamma \geq 0$. Similarly define $\mathcal{A}_{\geq 0}$, $\mathcal{A}_{\leq 0}$ and $\mathcal{A}_{<0}$.

Note that $\mathcal{A}_{\geq 0}$ is a subalgebra of \mathcal{A} and $\mathcal{A}_{>0}$ an ideal inside it.

DEFINITION 2.2. Let M be an arbitrary k -module. The scalar extension $\mathcal{A} \otimes_k M$ is naturally an \mathcal{A} -module and via $\iota(a \otimes m) := \bar{a} \otimes m$ it is also an $\widehat{\mathcal{A}}$ -module which will be denoted by \widehat{M} . Any $\widehat{\mathcal{A}}$ -module V that is isomorphic to \widehat{M} for some $M \in k\text{-mod}$ is called a *canonical module* and any $\widehat{\mathcal{A}}$ -module isomorphism $c : \widehat{M} \rightarrow V$ is called a *canonicalization of V* .

¹Remember that given any k -algebra A , monoid G , and any monoid homomorphism $\phi : G \rightarrow (\text{End}(A), \circ)$ the algebra $A \rtimes_\phi G$ is defined as the k -algebra that has $A \otimes_k k[G]$ as its underlying k -module and extends the multiplication of A and $k[G]$ via $(a \otimes g) \cdot (b \otimes h) := a\phi(g)(b) \otimes gh$. It is also denoted by $A \rtimes G$ if the morphism ϕ is understood.

Saying that V is a $A \rtimes_\phi G$ module is equivalent to saying that V is an A -module and comes with an action of G on V such that $g(a \cdot v) = \phi(g)(a) \cdot gv$ holds.

If M is free and $(b_x)_{x \in X}$ is a basis of M , then the image of $(1 \otimes b_x)_{x \in X}$ under a canonicalization c is called the *canonical basis of V* associated to $(b_x)_{x \in X}$ and c .

REMARK 2.3. Note that $M \mapsto \widehat{M}$ and $f \mapsto \text{id}_A \otimes f$ is a faithful functor $k\text{-mod} \rightarrow \widehat{A}\text{-mod}$. It is left adjoint to the fixed point functor $V \mapsto \{x \in V \mid \iota(v) = v\}$.

EXAMPLE 2.4. As we will see, the Kazhdan–Lusztig basis $(C_w)_{w \in W}$ of an Iwahori–Hecke algebra $H = H(W, S)$ is a canonical basis of the $\widehat{\mathbb{Z}[v^{\pm 1}]}$ -module $V := H$ where $\iota : H \rightarrow H$ is defined by $\iota(T_w) = T_{w^{-1}}$.

The module M is the \mathbb{Z} -span of the “standard basis” $M := \bigoplus_{w \in W} \mathbb{Z}T_w$. Kazhdan and Lusztig’s classical result that an ι -invariant $\mathbb{Z}[v^{\pm 1}]$ -basis of H exists is now precisely the statement that $T_w \mapsto C_w$ defines a canonicalization map. In this sense $(C_w)_{w \in W}$ is the canonical basis of H associated to the standard basis.

EXAMPLE 2.5. Canonical bases of quantum group representations in the sense of Lusztig and Kashiwara (cf. [16] and [14]) are examples of canonical basis in the sense of the definition above.

In Lusztig’s notation $\mathbf{B} \subset \mathcal{L}$ is a canonical basis of the $\widehat{\mathbb{Z}[v^{\pm 1}]}$ -module $V := \mathcal{L} + \overline{\mathcal{L}}$ which is the $\mathbb{Z}[v^{\pm 1}]$ -span of any PBW-basis of \mathbf{U}^+ . The \mathbb{Z} -module M corresponds to $\mathcal{L}/v^{-1}\mathcal{L}$ (which is also isomorphic to $\mathcal{L} \cap \overline{\mathcal{L}}$ as well as $\overline{\mathcal{L}}/v\overline{\mathcal{L}}$) and the standard basis of M is the image of any PBW-basis of \mathbf{U}^+ in $\mathcal{L}/v^{-1}\mathcal{L}$.

EXAMPLE 2.6. Obviously most \widehat{A} -modules are not canonical. For example the only canonical \widehat{A} -module that is finitely generated over k is the zero module. Hence $V = k[i] = k[x]/(x^2 + 1)$ is not canonical where v operates as multiplication by i and ι operates as $i \mapsto -i$. Therefore the question arises how one can recognize if a given module is canonical and how one can find a canonicalization.

An obvious restatement of the definition is the following:

PROPOSITION 2.7. *Let V be an arbitrary \widehat{A} -module. Then V is canonical if and only if there exists k -submodule M of V such that*

- (a) $V = \bigoplus_{\gamma \in \Gamma} v^\gamma M$ as a k -module.
- (b) ι operates as 1 on M , i.e., $\iota \cdot m = m$ for all $m \in M$.

In this case $c : \widehat{M} \rightarrow V, a \otimes m \mapsto am$ is a canonicalization.

DEFINITION 2.8. Let V be an $\widehat{\mathcal{A}}$ -module and (X, \leq) a poset. A X -graded shadow of V is a collection $(M_x)_{x \in X}$ of k -submodules of V such that

- (a) $V = \bigoplus_{\substack{x \in X \\ \gamma \in \Gamma}} v^\gamma M_x$ as a k -module and
- (b) $\iota \cdot m_z \in m_z + \sum_{y < z} \mathcal{A}M_y$ for all $m_z \in M_z$.

REMARK 2.9. In light of the above proposition a shadow is something like a canonicalization “up to lower order error terms.” Theorem 2.11 shows that these error terms can be corrected by a “triangular base change” if the poset satisfies a finiteness condition. It therefore provides a sufficient criterion for the existence of a canonicalization which is inspired by the construction of the Kazhdan–Lusztig basis of Iwahori–Hecke algebras as well as similar constructions by Howlett and Yin, Deodhar, Geck and many more. Theorem 2.11 is precisely the common thread in all these constructions.

First we need a lemma.

LEMMA 2.10. Let V be an $\widehat{\mathcal{A}}$ -module, (X, \leq) a partially ordered set and $(M_x)_{x \in X}$ a X -graded shadow of V . Furthermore let $f \in V$ be an arbitrary element with $f = \sum_{x \in X_0} f_x$ for some finite $X_0 \subseteq X$ and $f_x \in \mathcal{A}_{>0}M_x$.

If f satisfies $\iota \cdot f = f$, then $f_x = 0$ for all $x \in X_0$.

Proof. Assume the contrary. Without loss of generality, we can also assume $f_x \neq 0$. Otherwise we could just shrink the set X_0 . Let $X_1 \subseteq X_0$ be the subset of all maximal elements of X_0 and $X_2 := X_0 \setminus X_1$. Thus

$$f \in \sum_{x \in X_1} \mathcal{A}_{>0}M_x + \sum_{x \in X_2} \mathcal{A}_{>0}M_x \subseteq \sum_{x \in X_1} \mathcal{A}_{>0}M_x + \sum_{x \in X_2} \sum_{y \leq x} \mathcal{A}M_y.$$

Because $f_x \in \mathcal{A}_{>0}M_x$, it is a $\mathcal{A}_{>0}$ -linear combination of elements of M_x , say $f_x = \sum_{i=1}^{n_x} a_{ix} m_{ix}$ for some $a_{ix} \in \mathcal{A}_{>0}$ and $m_{ix} \in M_x$.

Also note that ι maps every subspace of the form $\sum_{y \leq x} \mathcal{A}M_y$ into itself because $(M_x)_{x \in X}$ is an X -graded shadow. Thus

$$\begin{aligned} \iota f &\in \sum_{x \in X_1, i=1..n_x} \iota(a_{ix} m_{ix}) + \iota \left(\sum_{x \in X_2} \sum_{y \leq x} \mathcal{A}M_y \right) \\ &= \sum_{x \in X_1, i=1..n_x} \overline{a_{ix}} \iota(m_{ix}) + \sum_{x \in X_2} \sum_{y \leq x} \mathcal{A}M_y \\ &= \sum_{x \in X_1, i=1..n_x} \overline{a_{ix}} m_{ix} + \sum_{x \in X_2} \sum_{y \leq x} \mathcal{A}M_y \\ &\subseteq \sum_{x \in X_1} \mathcal{A}_{<0}M_x + \sum_{x \in X_2} \sum_{y \leq x} \mathcal{A}M_y. \end{aligned}$$

X_1 is a nonempty subset because X_0 is nonempty and finite. Comparing the x -components of f and ιf for $x \in X_1$ we find $f_x \in \mathcal{A}_{>0}M_x \cap \mathcal{A}_{<0}M_x = 0$ contrary to the assumption $f_x \neq 0$. \square

THEOREM 2.11. *Let (X, \leq) be a poset such that $(-\infty, y] := \{x \in X \mid x \leq y\}$ is finite for all $y \in X$.*

- (a) *If an $\widehat{\mathcal{A}}$ -module V has an X -graded shadow $(M_x)_{x \in X}$ then it is canonical and there exists a unique canonicalization $c : \widehat{M} \rightarrow V$ where $M := \bigoplus_{x \in X} M_x$ such that $c(1 \otimes m) \in m + \mathcal{A}_{>0}M$ for all $m \in M$. More precisely it satisfies $c(1 \otimes m_x) \in m_x + \sum_{y < x} \mathcal{A}_{>0}M_y$ for all $m_x \in M_x$.*
- (b) *The canonicalization above depends functorially on the shadow with respect to positive maps. More precisely let V_1, V_2 be two $\widehat{\mathcal{A}}$ -modules with X -graded shadows $(M_{i,x})_{x \in X}$ and canonicalizations $c_i : \widehat{M}_i \rightarrow V_i$ and let $\phi : V_1 \rightarrow V_2$ be an $\widehat{\mathcal{A}}$ -linear map with $\phi(\mathcal{A}_{\geq 0}M_1) \subseteq \mathcal{A}_{\geq 0}M_2$. There is an induced map $M_1 = \mathcal{A}_{\geq 0}M_1 / \mathcal{A}_{>0}M_1 \xrightarrow{\phi} \mathcal{A}_{\geq 0}M_2 / \mathcal{A}_{>0}M_2 = M_2$ and this induces an $\widehat{\mathcal{A}}$ -linear map $\widehat{\phi} : \widehat{M}_1 \rightarrow \widehat{M}_2$. This map satisfies $c_2 \circ \widehat{\phi} = \phi \circ c_1$ holds, i.e., the diagram in Figure 1 commutes.*

$$\begin{array}{ccc}
 \widehat{M}_1 & \xrightarrow{\widehat{\phi}} & \widehat{M}_2 \\
 c_1 \downarrow \cong & & c_2 \downarrow \cong \\
 V_1 & \xrightarrow{\phi} & V_2
 \end{array}$$

Figure 1. Functoriality of canonicalization of shadows.

Before we begin the proof observe that any \mathcal{A} -linear map $f : \mathcal{A}M_z \rightarrow \mathcal{A}M_y$ is uniquely determined by its restriction $M_z \rightarrow \mathcal{A}M_y$ which is a k -linear map and can be written as $f(m_z) = \sum_{\gamma \in \Gamma} v^\gamma f_\gamma(m_z)$ with uniquely determined k -linear maps $f_\gamma : M_z \rightarrow M_y$ that have the property that $\{\gamma \mid f_\gamma(m_z) \neq 0\}$ is finite for each $m_z \in M_z$ so that the sum is indeed well defined. Having this way of writing these maps in mind we can define $\bar{f} : \mathcal{A}M_z \rightarrow \mathcal{A}M_y$ to be the \mathcal{A} -linear map with $\bar{f}(m_z) := \sum_{\gamma \in \mathbb{Z}} v^{-\gamma} f_\gamma(m_z)$ for all $m_z \in M_z$. Note that $\bar{}$ is compatible with composition, i.e., $\overline{f \circ g} = \bar{f} \circ \bar{g}$.

We will use this notation for the proof to simplify the notation.

Proof. The uniqueness of c follows from the above lemma because if $c, c' : \widehat{M} \rightarrow V$ are two canonicalizations satisfying the stated property then $f := c(1 \otimes m) - c'(1 \otimes m)$ is an element of $\mathcal{A}_{>0}M$ with $\iota \cdot f = f$ so that $f = 0$ by Lemma 2.10.

Concerning the existence consider the \mathcal{A} -linear maps $\rho_{yz} : \mathcal{A}M_z \rightarrow \mathcal{A}M_y$ defined by

$$\forall m_z \in M_z : \iota \cdot m_z = \sum_y \rho_{yz}(m_z).$$

By assumption $\rho_{yz} = 0$ unless $y \leq z$ and $\rho_{zz}(m_z) = m_z$.

Following the usual construction of the Kazhdan–Lusztig polynomials and R -polynomials we will recursively construct \mathcal{A} -linear maps $\pi_{yz} : \mathcal{A}M_z \rightarrow \mathcal{A}_{\geq 0}M_y$ such that:

- $\pi_{yz} = 0$ unless $y \leq z$ and $\pi_{zz}(m_z) = m_z$
- $\pi_{xz} = \sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}}$.

The first step is to observe

$$(1) \quad \sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}} = \begin{cases} \text{id}_{\mathcal{A}M_x} & \text{if } x = z, \\ 0 & \text{otherwise.} \end{cases}$$

This follows from the fact that ι has order two:

$$\begin{aligned} m_z &= \iota \cdot \iota \cdot m_z \\ &= \sum_{y \leq z} \iota \cdot \underbrace{\rho_{yz}(m_z)}_{\in \mathcal{A}M_y} \\ &= \sum_{x \leq y \leq z} (\rho_{xy} \circ \overline{\pi_{yz}})(m_z). \end{aligned}$$

Fix $z \in X$. Define $\pi_{zz}(m_z) := m_z$ and $\pi_{xz} := 0$ for all $x \not\leq z$. If $x < z$ then assume inductively that π_{yz} is already known for all $x < y \leq z$. It is therefore possible to define

$$\alpha_{xz} := \sum_{x < y \leq z} \rho_{xy} \circ \overline{\pi_{yz}}.$$

This map satisfies

$$\begin{aligned} \alpha_{xz} &= \sum_{x < y \leq z} \rho_{xy} \circ \overline{\sum_{y \leq w \leq z} \rho_{yw} \circ \overline{\pi_{wz}}} \\ &= \sum_{x < y \leq w \leq z} \rho_{xy} \circ \overline{\rho_{yw}} \circ \pi_{wz} \end{aligned}$$

$$\begin{aligned}
 &= \sum_{x < w \leq z} \left(\sum_{x < y \leq w} \rho_{xy} \circ \overline{\rho_{yw}} \right) \circ \pi_{wz} \\
 &= \sum_{x < w \leq z} (0 - \rho_{xx} \circ \overline{\rho_{xw}}) \circ \pi_{wz} \\
 &= \sum_{x < w \leq z} -\overline{\rho_{xw}} \circ \pi_{wz} \\
 &= -\overline{\alpha_{xz}}.
 \end{aligned}$$

Therefore we obtain $\alpha_0 = 0$ in the decomposition $\alpha_{xz} = \sum_{\gamma \in \Gamma} v^\gamma \alpha_\gamma$. Now define $\pi_{xz} := \sum_{\gamma > 0} v^\gamma \alpha_\gamma$ so that $\alpha_{xz} = \pi_{xz} - \overline{\pi_{xz}}$ holds. This shows that $\pi_{xz}(m_z) \in \mathcal{A}_{>0}M_x$ as well as

$$\sum_{x \leq y \leq z} \rho_{xy} \circ \overline{\pi_{yz}} = \rho_{xx} \circ \overline{\pi_{xz}} + \alpha_{xz} = \overline{\pi_{xz}} + \alpha_{xz} = \pi_{xz}.$$

Thus the existence of all π_{xz} is established and we can define the \mathcal{A} -linear map $c : \widehat{M} \rightarrow V$ by

$$\forall m_z \in M_z : c(1 \otimes m_z) := \sum_{x \leq z} \pi_{xz}(m_z) = m_z + \sum_{x < z} \pi_{xz}(m_z).$$

It is bijective because it is “upper triangular with unit diagonal.” The map is also $\widehat{\mathcal{A}}$ -linear because

$$\begin{aligned}
 \iota \cdot c(1 \otimes m_z) &= \sum_{y \leq z} \iota \cdot \pi_{yz}(m_z) \\
 &= \sum_{x \leq y \leq z} \rho_{xy} \overline{\pi_{yz}}(m_z) \\
 &= \sum_{x \leq z} \pi_{xz}(m_z) \\
 &= c(1 \otimes m_z) \\
 &= c(\iota \cdot (1 \otimes m_z)).
 \end{aligned}$$

Finally we have to show that c is functorial. Let M_1, M_2, ϕ be as in the statement of the theorem and fix an arbitrary $m_1 \in M_1$. Then $c_1(m_1) \in m_1 + \mathcal{A}_{>0}M_1$ so that $\phi(c_1(m_1)) \in \phi(m_1) + \mathcal{A}_{>0}M_2$. Also $\widehat{\phi}(m_1) \in \phi(m_1) + \mathcal{A}_{>0}M_2$ by construction of $\widehat{\phi}$ so that $c_2(\widehat{\phi}(m_1)) \in \widehat{\phi}(m_1) + \mathcal{A}_{>0}M_2 =$

$\phi(m_1) + \mathcal{A}_{>0}M_2$. Therefore $f := \phi(c_1(m_1)) - c_2(\widehat{\phi}(m_1)) \in \mathcal{A}_{>0}M_2$. Additionally, since all four maps are $\widehat{\mathcal{A}}$ -linear and $m_1 \in \widehat{M}_1$ is ι -invariant f satisfies $\iota f = f$ so that Lemma 2.10 implies $f = 0$. This proves the commutativity of the diagram. \square

COROLLARY 2.12. *Let (X, \leq) be a poset such that $\{x \in X \mid x \leq y\}$ is finite for all $y \in X$. Furthermore let V be an $\widehat{\mathcal{A}}$ -module, U an $\widehat{\mathcal{A}}$ -submodule of V and $(M_x)_{x \in X}$ an X -graded shadow for V . Define $N_x := U \cap M_x$ for all $x \in X$.*

If U is generated as an \mathcal{A} -module by $\sum_{x \in X} N_x$, then (N_x) is an X -graded shadow for U , (M_x/N_x) is an X -graded shadow for V/U , the canonicalization $\widehat{M} \rightarrow V$ restricts to the canonicalization $\widehat{N} \rightarrow U$ and induces the canonicalization $\widehat{M}/\widehat{N} \rightarrow V/U$ on the quotients.

Proof. This follows immediately from functoriality of canonicalization applied to the embedding $U \hookrightarrow V$ and the quotient map $V \rightarrow V/U$, respectively. \square

REMARK 2.13. In terms of canonical bases this corollary recovers the theorem that if $(t_x)_{x \in X}$ is an \mathcal{A} -basis for V and U is spanned as an \mathcal{A} -module by a subset $(t_x)_{x \in Y}$ of that basis, then the canonical basis for U is the subset $(c_x)_{x \in Y}$ of the canonical basis $c_x := c(t_x)$ of V and the canonical basis of the quotient V/U is spanned by the vectors $(c_x)_{x \in X \setminus Y}$ (more precisely by their images under the quotient map $V \rightarrow V/U$).

REMARK 2.14. An important special case of this corollary is the case where U is of the form $U = \sum_{x \in I} \mathcal{A}M_x$ for some order ideal $I \trianglelefteq X$ (i.e., a subset with the property $x \in I \wedge y \leq x \implies y \in I$). Note that all such U are $\widehat{\mathcal{A}}$ -submodules by definition of X -graded shadows.

§3. Hecke algebras, W -graphs and W -graph algebras

For the rest of the paper fix a (not necessarily finite) Coxeter group (W, S) , a totally ordered, additive group Γ (which soon will be further restricted to be \mathbb{Z}) and a weight function $L : W \rightarrow \Gamma$, i.e., a function with $l(xy) = l(x) + l(y) \implies L(xy) = L(x) + L(y)$. We will use the shorthand $v_s := v^{L(s)} \in \mathbb{Z}[\Gamma]$ and the standard assumption $L(s) > 0$ for all $s \in S$.

DEFINITION 3.1. (Cf. [7]) The *Iwahori–Hecke algebra* $H = H(W, S, L)$ is the $\mathbb{Z}[\Gamma]$ -algebra which is freely generated by $(T_s)_{s \in S}$ subject only to the

relations

$$T_s^2 = 1 + (v_s - v_s^{-1})T_s \quad \text{and} \quad \underbrace{T_s T_t T_s \dots}_{m_{st} \text{ factors}} = \underbrace{T_t T_s T_t \dots}_{m_{st} \text{ factors}},$$

where m_{st} denotes the order of $st \in W$.

Because of the braid relations and Matsumoto’s theorem (cf. [18]), we can define the *standard basis elements* as

$$T_w := T_{s_1} T_{s_2} \cdots T_{s_l},$$

where $w = s_1 s_2 \dots s_l$ is any reduced expression of $w \in W$ in the generators. Note that $T_1 = 1$.

For each parabolic subgroup $W_J \leq W$ the Hecke algebra $H(W_J, J, L|_{W_J})$ will be identified with the *parabolic subalgebra* $H_J := \text{span}_{\mathbb{Z}[\Gamma]} \{T_w \mid w \in W_J\} \subseteq H$.

DEFINITION 3.2. (Cf. [15] and [7]) Let k be a commutative ring. A W -graph with edge weights in k is a triple (\mathfrak{C}, I, m) consisting of a finite set \mathfrak{C} of vertices, a vertex labeling map $I : \mathfrak{C} \rightarrow \{J \mid J \subseteq S\}$ and a family of edge weight matrices $m^s \in k^{\mathfrak{C} \times \mathfrak{C}}$ for $s \in S$ such that the following conditions hold:

- (a) $\forall x, y \in \mathfrak{C} : m_{xy}^s \neq 0 \implies s \in I(x) \setminus I(y)$.
- (b) The matrices

$$\omega(T_s)_{xy} := \begin{cases} -v_s^{-1} & \text{if } x = y, s \in I(x), \\ v_s & \text{if } x = y, s \notin I(x), \\ m_{xy}^s & \text{otherwise,} \end{cases}$$

induce a matrix representation $\omega : k[v^{\pm 1}]H \rightarrow k[v^{\pm 1}]^{\mathfrak{C} \times \mathfrak{C}}$.

The associated directed graph is defined as follows: The vertex set is \mathfrak{C} and there is a directed edge $x \leftarrow y$ if and only if $m_{xy}^s \neq 0$ for some $s \in S$. If this is the case, then the value m_{xy}^s is called a *weight* of the edge. The set $I(x)$ is called the *vertex label* of x .

REMARK 3.3. In the equal-parameter case (i.e., $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$) one can show $m_{xy}^s = m_{xy}^t$ for all $s, t \in I(x) \setminus I(y)$ so that it is well defined to speak of *the* weight of the edge $x \leftarrow y$.

This does no longer hold in the multiparameter case, so that one could say that the edges have a tuple of weights attached to them.

REMARK 3.4. Note that condition (a) and the definition of $\omega(T_s)$ already guarantee $\omega(T_s)^2 = 1 + (v_s - v_s^{-1})\omega(T_s)$ so that the only nontrivial requirement in condition (b) is the braid relation $\omega(T_s)\omega(T_t)\omega(T_s) \dots = \omega(T_t)\omega(T_s)\omega(T_t) \dots$.

Given a W -graph as above the matrix representation ω turns $k[\Gamma]^{\mathcal{C}}$ into a module for the Hecke algebra. It is natural to ask whether a converse is true. In many situations the answer is yes as shown by Gyoja.

THEOREM 3.5. *Let W be finite, $K \subseteq \mathbb{C}$ be a splitting field for W and assume $\Gamma = \mathbb{Z}$ and $L(s) = 1$ for all $s \in S$. Then every irreducible representation of $K(v)H$ can be realized as a W -graph module for some W -graph with edge weights in K .*

REMARK 3.6. The same is true in the multiparameter case if Lusztig’s conjecture **P15** or similar properties like Geck and Jacon’s (\spadesuit) and (\clubsuit) hold for (W, S, L) , see [7, 2.7.12] or [10, 4.3.5] for a proof.

REMARK 3.7. Gyoja also provides an example of a finite-dimensional representation of the affine Weyl group of type \tilde{A}_n that is not induced by a W -graph.

Convention. For the remainder of the paper we will assume $\Gamma = \mathbb{Z}$ (although we will still write Γ when referring to the group of exponents of the Laurent polynomials).

It is not strictly speaking necessary to do this since the results also hold in the general case, but the general definitions and proofs are much more technical because one has to work with infinite series of the form $\sum_{-L(s) < \gamma < L(s)} x_{s,\gamma} v^\gamma$ and must ensure their convergence in the appropriate sense in all proofs.

By restricting to $\Gamma = \mathbb{Z}$ all the relevant sums become finite sums and separate convergence arguments are unnecessary.

DEFINITION 3.8. (The W -graph algebra) Assume $\Gamma = \mathbb{Z}$ and consider the free algebra $\mathbb{Z}\langle e_s, x_{s,\gamma} \mid s \in S, -L(s) < \gamma < L(s) \rangle$. Define

$$j(T_s) := -v_s^{-1}e_s + v_s(1 - e_s) + \sum_{-L(s) < \gamma < L(s)} v^\gamma x_{s,\gamma} \in \mathbb{Z}[\Gamma] \otimes_{\mathbb{Z}} \mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$$

for all $s, t \in S$ and write

$$\sum_{\gamma \in \Gamma} v^\gamma \otimes y^\gamma(s, t) = \underbrace{j(T_s)j(T_t)j(T_s) \dots}_{m_{st} \text{ factors}} - \underbrace{j(T_t)j(T_s)j(T_t) \dots}_{m_{st} \text{ factors}}$$

for some $y^\gamma(s, t) \in \mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$.

Define Ω to be the quotient of $\mathbb{Z}\langle e_s, x_{s,\gamma} \rangle$ modulo the relations

- (a) $e_s^2 = e_s, e_s e_t = e_t e_s,$
- (b) $e_s x_{s,\gamma} = x_{s,\gamma}, x_{s,\gamma} e_s = 0,$
- (c) $x_{s,\gamma} = x_{s,-\gamma}$ and
- (d) $y^\gamma(s, t) = 0.$

for all $s, t \in S$ and all $\gamma \in \Gamma$.

Finally define the element

$$x_s := \sum_{\gamma \in \Gamma} v^\gamma x_{s,\gamma} \in \mathbb{Z}[\Gamma]\Omega.$$

REMARK 3.9. The definition immediately implies that $T_s \mapsto j(T_s)$ defines a homomorphism of $\mathbb{Z}[\Gamma]$ -algebras $j : H \rightarrow \mathbb{Z}[\Gamma]\Omega$. In fact this is an embedding as shown in [11, Corollary 10]. We will identify H with its image in $\mathbb{Z}[\Gamma]\Omega$ and suppress any mention of j from now on to simplify the notation.

Note that $C_s = T_s - v_s = -(v_s + v_s^{-1})e_s + x_s$.

REMARK 3.10. W -graph algebras have the distinguishing feature that each W -graph (\mathfrak{C}, I, m) with edge weights in k not only defines the structure of an H -module on $k[\Gamma]^\mathfrak{C}$ but that it induces a canonical $k\Omega$ -module structure on $k^\mathfrak{C}$ via

$$e_s \cdot \mathfrak{z} := \begin{cases} \mathfrak{z} & s \in I(\mathfrak{z}), \\ 0 & \text{otherwise,} \end{cases}$$

$$x_s \cdot \mathfrak{z} := \sum_{\mathfrak{r} \in \mathfrak{C}} m_{\mathfrak{r}\mathfrak{z}}^s \mathfrak{r}$$

for all $\mathfrak{z} \in \mathfrak{C}$. Then $k[\Gamma]^{\mathfrak{C} \times \mathfrak{C}}$ is a $k[\Gamma]\Omega$ -module and restriction to an H -module gives back the H -module in the definition.

Conversely if V is a $k\Omega$ -module that has a k -basis \mathfrak{C} with respect to which all e_s act as diagonal matrices, then V is obtained from a W -graph (\mathfrak{C}, I, m) in this way: m^s is the matrix representing the action of x_s and $I(\mathfrak{z}) = \{s \in S \mid e_s \mathfrak{z} = \mathfrak{z}\}$. In this way one can interpret Ω -modules as W -graphs up to choice of a basis. (See [11, Theorem 9] or [10, 4.2.18] for a detailed proof of these claims.)

Of course V does not need to have a basis at all if k is not a field so that $k\Omega$ -modules are indeed more general than W -graphs.

REMARK 3.11. Every $k\Omega$ -module V is also a k -module and gives us a canonical $\widehat{k[\Gamma]}$ -module $\widehat{V} = k[\Gamma] \otimes_k V$ which is also an $k[\Gamma]H$ -module by restriction along $k[\Gamma]H \hookrightarrow k[\Gamma]\Omega$.

The results about Howlett–Yin induction will all be proved via canonical-ization of $k[\Gamma]H$ -modules on which we will define the appropriate $k\Omega$ -module structure.

EXAMPLE 3.12. The trivial group is a Coxeter group $(1, \emptyset)$ and its associated W -graph algebra is just \mathbb{Z} .

A cyclic group of order 2 is a Coxeter group $(\{1, s\}, \{s\})$ of rank 1 and its associated W -graph algebra is as a free \mathbb{Z} -module with basis $\{e_s, 1 - e_s\} \cup \{x_{s,\gamma} \mid 0 \leq \gamma < L(s)\}$. The multiplication of the basis elements is completely determined by the relations because $x_{s,\gamma_1}x_{s,\gamma_2} = x_{s,\gamma_1}(e_sx_{s,\gamma_2}) = (x_{s,\gamma_1}e_s)x_{s,\gamma_2} = 0$.

§4. Howlett–Yin induction

Let M be any $k\Omega$ -module. Then $k[\Gamma]M$ is naturally a $k[\Gamma]k\Omega$ -module and by restriction of scalars it is also a $k[\Gamma]H$ -module which we will (somewhat abusing the notation) denote by $\text{Res}_H^\Omega M$.

Given any Ω_J -module M , its restriction to H_J can be induced to a H -module. The goal of this subsection is to prove that $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$ not only has the structure of an Ω -module but that this module structure can be chosen functorially in M . The specific construction of this functor is a generalization of such a construction Howlett and Yin gave in the equal-parameter case and for the special case that M is given by W -graph. We will prove it in the general case using ideas by Lusztig (see [17]).

4.1 Preparations

PROPOSITION 4.1. *Let M be a $k\Omega$ -module and $a \in \text{Res}_H^\Omega(M)$ be an arbitrary element. Then the following holds for all $s \in T$: $T_s a = -v_s^{-1}a \iff e_s a = a$.*

Proof. The forward implication can be seen as follows

$$\begin{aligned} -v_s^{-1}a &= (-v_s^{-1}e_s + v_s(1 - e_s) + x_s)a \\ \implies 0 &= ((v_s^{-1} + v_s)(1 - e_s) + x_s)a \\ \implies 0 &= ((v_s^{-1} + v_s)(1 - e_s)^2 + \underbrace{(1 - e_s)x_s}_{=0})a \end{aligned}$$

$$\begin{aligned}
 &= (v_s^{-1} + v_s)(1 - e_s)a \\
 \implies 0 &= (1 - e_s)a,
 \end{aligned}$$

where we used in the last step that $\text{Res}_H^\Omega(M) = k[\Gamma] \otimes_k M$ as $k[\Gamma]$ -modules to cancel $v_s^{-1} + v_s$. The backward implication is trivial: $T_s a = T_s e_s a = -v_s^{-1} e_s^2 a = -v_s^{-1} a$. \square

We will need the following well-known facts about cosets of parabolic subgroups:

LEMMA AND DEFINITION 4.2. *Let $J \subseteq S$ be any subset and W_J the associated parabolic subgroup. Then the following hold:*

- (a) $D_J := \{x \in W \mid \forall s \in J : l(xs) > l(x)\}$ is a set of representatives for the left cosets of W_J in W . Its elements are exactly the unique elements of minimal length in each coset. They have the property $l(xw) = l(x) + l(w)$ for all $w \in W_J$.
- (b) Deodhar’s lemma (cf. [2])
For all $w \in D_J$ and all $s \in S$ exactly one of the following cases happens:
 - (i) $sw > w$ and $sw \in D_J$
 - (ii) $sw > w$ and $sw \notin D_J$. In this case $s^w := w^{-1}sw \in J$.
 - (iii) $sw < w$. In this case $sw \in D_J$ holds automatically.

Thus for fixed $s \in S$ there is a partition $D_J = D_{J,s}^+ \sqcup D_{J,s}^0 \sqcup D_{J,s}^-$ and similarly for fixed $w \in D_J$ there is a partition $S = D_J^+(w) \sqcup D_J^0(w) \sqcup D_J^-(w)$ where

$$\begin{aligned}
 D_{J,s}^+ &:= \{w \mid sw > w, sw \in D_J\} & D_J^+(w) &:= \{s \mid sw > w, sw \in D_J\} \\
 D_{J,s}^0 &:= \{w \mid sw > w, sw \notin D_J\} & D_J^0(w) &:= \{s \mid sw > w, sw \notin D_J\} \\
 D_{J,s}^- &:= \{w \mid sw < w\} & D_J^-(w) &:= \{s \mid sw < w\}.
 \end{aligned}$$

If $J \subseteq K \subseteq S$ is another subset, then furthermore

- (c) $D_J^K := D_J \cap W_K$ is the set of distinguished left coset representatives for W_J in W_K and $D_K^S \times D_J^K \rightarrow D_J^S, (x, y) \mapsto xy$ is a length-preserving bijection.
- (d) If $x \in D_K^S$ and $y \in D_J^K$, then

$$\begin{aligned}
 D_J^+(xy) &= \{s \in D_K^0(x) \mid s^x \in D_J^+(y)\} \cup D_K^+(x) \\
 D_J^0(xy) &= \{s \in D_K^0(x) \mid s^x \in D_J^0(y)\} \\
 D_J^-(xy) &= \{s \in D_K^0(x) \mid s^x \in D_J^-(y)\} \cup D_K^-(x).
 \end{aligned}$$

Proof. See [8] for example. □

To apply the previous observations about canonical modules we need a semilinear map on our modules.

LEMMA AND DEFINITION 4.3. *If M is any $k\Omega$ -module, then we will fix the notation ι for the canonical $\bar{}$ -semilinear map $a \otimes m \mapsto \bar{a} \otimes m$ on $k[\Gamma] \otimes_k M$.*

Then the following hold:

- (a) *In the special case of $M = k\Omega$ itself, ι is ring automorphism of $k[\Gamma]\Omega$ with $\iota(T_s) = T_s^{-1} = T_s - (v_s - v_s^{-1})$. In particular ι restricts to a $\bar{}$ -semilinear involution of H .*
- (b) *For general M , furthermore $\iota(ax) = \iota(a)\iota(x)$ holds for all $a \in k[\Gamma]\Omega$, $x \in k[\Gamma]M$.*

Now let M be a $k\Omega_J$ -module and $V := \text{Res}_{H_J}^{\Omega_J}(M)$ its associated $k[\Gamma]H_J$ -module.

- (c) *$\iota(h \otimes x) := \iota(h) \otimes \iota(x)$ is a well-defined $\bar{}$ -semilinear involution on $\text{Ind}_{H_J}^H(V) = H \otimes_{H_J} V$.*
- (d) *The k -submodules $V_w := \{T_w \otimes m \mid m \in M\} \subseteq \text{Ind}_{H_J}^H(V)$ for $w \in D_J$ constitute a D_J -graded shadow on $\text{Ind}_{H_J}^H(V)$ where D_J is endowed with the Bruhat–Chevalley order.*
- (e) *The maps $\rho_{xz} : V_z \rightarrow k[\Gamma]V_x$ and $\pi_{xz} : V_z \rightarrow k[\Gamma]V_x$ in Theorem 2.11 are of the form*

$$\rho_{xz}(T_z \otimes m) = T_x \otimes r_{x,z}m \quad \text{and} \quad \pi_{xz}(T_z \otimes m) = T_x \otimes p_{x,z}m$$

for elements $r_{x,z} \in \mathbb{Z}[\Gamma]\Omega_J$, $p_{x,z} \in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J$ that are independent of M .

Proof. (a)+(b) The statements follow directly from the definition.

(c) For all $a \in H_J$ one has $\iota(ha) \otimes \iota(x) = \iota(h)\iota(a) \otimes \iota(x) = \iota(h) \otimes \iota(a)\iota(x) = \iota(h) \otimes \iota(ax)$ which proves the well-definedness of $h \otimes x \mapsto \iota(h) \otimes \iota(x)$.

(d) Note that

$$\iota(T_z \otimes m) = \sum_{w \in W} R_{w,z}T_w \otimes m = \sum_{x \in D_J} T_x \otimes \underbrace{\sum_{y \in W_J} R_{xy,z}T_y}_{=: r_{x,z}} m,$$

where $R_{w,z} \in \mathbb{Z}[\Gamma]$ are the Kazhdan–Lusztig R -polynomials, i.e., the polynomials defined by $\iota(T_z) = \sum_{w \in W} R_{w,z}T_w$.

Now note that $R_{xy,z} \neq 0$ implies $xy \leq z$ so that $x \leq xy \leq z$ and thus the summation only runs over x with $x \leq z$. If furthermore $x = z$ then $xy \leq z$ can only be true if $y = 1$. But we know $R_{z,z} = 1$. Therefore (V_w) really is a D_J -graded shadow of V .

(e) The claimed property for ρ_{xz} follows from the equation above. The analogous property for π_{xz} follows from the recursive construction of the π_{xz} in Theorem 2.11. □

LEMMA AND DEFINITION 4.4. *There is a unique family $\mu_{x,z}^s \in \mathbb{Z}[\Gamma]\Omega_J$ for $x, z \in D_J, s \in S$ such that the following properties hold*

- (a) $\mu_{x,z}^s = 0$ unless $x < z, z \in D_{J,s}^+ \cup D_{J,s}^0$, and $x \in D_{J,s}^0 \cup D_{J,s}^-$ hold.
- (b) $\overline{\mu_{x,z}^s} = \mu_{x,z}^s$
- (c) If $z \in D_{J,s}^+ \cup D_{J,s}^0$ and $x \in D_{J,s}^0 \cup D_{J,s}^-$, then

$$\mu_{x,z}^s + R + \sum_{x < y < z} p_{x,y} \mu_{y,z}^s \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$$

where

$$R = \begin{cases} -C_{s^x} p_{x,z} & z \in D_{J,s}^+, x \in D_{J,s}^0, \\ v_s^{-1} p_{x,z} & z \in D_{J,s}^+, x \in D_{J,s}^-, \\ p_{x,z} C_{s^z} - C_{s^x} p_{x,z} & z \in D_{J,s}^0, x \in D_{J,s}^0, \\ p_{x,z} C_{s^z} + v_s^{-1} p_{x,z} & z \in D_{J,s}^0, x \in D_{J,s}^-. \end{cases}$$

These elements satisfy:

- (d) $v_s \mu_{x,z}^s \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$.

Proof. Conditions (a)–(c) are precisely designed to give a recursive definition of $\mu_{x,z}^s$: the recursion happens along the poset $\{(x, z) \in D_J \times D_J \mid x \leq z\}$ with the order $(x, z) \sqsubset (x', z') : \iff z < z' \vee (z = z' \wedge x > x')$.

First note that this is a well-founded poset since intervals in the Bruhat–Chevalley order are finite so that no infinite descending chain can exist and recursive definitions really make sense.

Now if $\mu_{x,z}^s$ is known for all $(x, z) \sqsubset (x', z')$ then (c) determines the nonpositive part of $\mu_{x',z'}^s$ and by the symmetry condition (b) $\mu_{x',z'}^s$ is completely determined. This shows how to define $\mu_{x',z'}^s$ for $x' \leq z'$.

The last property of $\mu_{x,z}^s$ follows by induction from this recursive construction. Note that $v_s R \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ in all four cases because $p_{x,z} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ for all $x < z$. Assuming that $v_s \mu_{y,z}^s \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ already holds for

all $(y, z) \sqsubset (x, z)$ we find that

$$v_s \mu_{x,z}^s \equiv -v_s R - \sum_{x < y < z} p_{x,y} v_s \mu_{y,z}^s \equiv 0 \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J}$$

holds. □

4.2 Induction is well defined

THEOREM 4.5. *Let M be a $k\Omega_J$ -module and $\text{HY}_J^S(M)$ the k -module $\bigoplus_{w \in D_J} M$. Denote elements of the w -component of $\text{HY}_J^S(M)$ as $w|m$. Further write $\mu_{x,z}^s$ as $\sum_{-L(s) < \gamma < L(s)} \mu_{x,z}^{s,\gamma} \cdot v^\gamma$ with $\mu_{x,z}^{s,\gamma} \in \Omega_J$.*

With this notation $\text{HY}_J^S(M)$ becomes a $k\Omega$ -module via

$$e_s \cdot z|m := \begin{cases} 0 & z \in D_{J,s}^+, \\ z|e_{s^z} m & z \in D_{J,s}^0, \\ z|m & z \in D_{J,s}^-, \end{cases}$$

$$x_{s,\gamma} \cdot z|m := \begin{cases} \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m + sz|m & z \in D_{J,s}^+, \gamma = 0, \\ \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m & z \in D_{J,s}^+, \gamma \neq 0, \\ \sum_{x < z} x|\mu_{x,z}^{s,\gamma} m + z|x_{s^z,\gamma} m & z \in D_{J,s}^0, \\ 0 & z \in D_{J,s}^- \end{cases}$$

and the canonicalization $c_M : \text{Res}_H^\Omega \text{HY}_J^S(M) \rightarrow \text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J}(M)$, $z|m \mapsto \sum_y T_y \otimes p_{y,z} m$ is $k[\Gamma]H$ -linear.

REMARK 4.6. Comparing with [12, Theorem 5.3], this theorem gives a more general result. It includes [12, Theorem 5.3], as we will see in Proposition 4.10, but it also encompasses the multiparameter case (which Geck considered in [5] for the special case that M is a left cell module).

We will also see in Theorem 5.1 that the construction is functorial in the appropriate sense which is also not included in Howlett and Yin’s theorem. This more general, more abstract way of looking at induction will allow us to prove transitivity which was not included in Howlett and Yin’s paper. It also allows for simplification of many known results.

Proof of the main theorem. We want to define a representation $\omega : \Omega \rightarrow \text{End}(\text{HY}_J^S(M))$ and we already have a definition of $\omega(e_s)$ and $\omega(x_{s,y})$. Extend

these $k[\Gamma]$ -linearly to

$$\omega : \text{span}_{k[\Gamma]} \{1, e_s, x_{s,\gamma} \mid s \in S, -L(s) < \gamma < L(s)\} \rightarrow k[\Gamma] \text{End}(\text{HY}_J^S(M)).$$

We need to show that ω satisfies all the relations of Ω , that is

- (a) $\omega(e_s)^2 = \omega(e_s), \omega(e_s)\omega(e_t) = \omega(e_t)\omega(e_s),$
- (b) $\omega(e_s)\omega(x_{s,\gamma}) = \omega(x_{s,\gamma}), \omega(x_{s,\gamma})\omega(e_s) = 0,$
- (c) $\omega(x_{s,\gamma}) = \omega(x_{s,-\gamma})$ and
- (d) $\underbrace{\omega(T_s)\omega(T_t)\omega(T_s) \dots}_{m_{st} \text{ factors}} = \underbrace{\omega(T_t)\omega(T_s)\omega(T_t) \dots}_{m_{st} \text{ factors}}$ as an equation in $k[\Gamma] \text{End}(\text{HY}_J(M))$ where $m_{st} := \text{ord}(st)$.

The equations $\omega(e_s)^2 = \omega(e_s), \omega(e_s)\omega(e_t) = \omega(e_t)\omega(e_s)$ and $\omega(x_{s,\gamma}) = \omega(x_{s,-\gamma})$ follow directly from the definitions and the properties of μ .

To prove that $\omega(T_s)$ satisfies the braid relations, we use the $k[\Gamma]$ -linear bijection c and show $c(\omega(T_s)z|m) = T_s c(z|m)$ for all $z \in D_J$ and all $m \in M$. Since $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J}(M)$ is a $k[\Gamma]H$ -module, the braid relations hold on the right hand side and will therefore also hold on the left hand side. Because of the equality $C_s = T_s - v_s$ this is equivalent to showing $c(\omega(C_s)w|m) = C_s c(w|m)$. We compare these two elements of $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$:

On the left hand side we find:

$$\begin{aligned} c(\omega(C_s)z|m) &= c((-v_s + v_s^{-1})e_s + x_s) \cdot z|m) \\ &= \begin{cases} c\left(sz|m + \sum_{y < z} y|\mu_{y,z}^s m \right) & z \in D_{J,s}^+, \\ c\left(z|(-(v_s + v_s^{-1})e_{sz} + x_{sz})m + \sum_{y < z} y|\mu_{y,z}^s m \right) & z \in D_{J,s}^0, \\ -(v_s^{-1} + v_s)c(z|m) & z \in D_{J,s}^-, \end{cases} \\ &= \begin{cases} c\left(v_s z|m + sz|m + \sum_{y < z} y|\mu_{y,z}^s m \right) & z \in D_{J,s}^+, \\ c\left(z|C_{sz}m + \sum_{y < z} y|\mu_{y,z}^s m \right) & z \in D_{J,s}^0, \\ -(v_s + v_s^{-1})c(z|m) & z \in D_{J,s}^-, \end{cases} \end{aligned}$$

$$= \begin{cases} \sum_{x \in D_{J,s}} T_x \otimes \left(p_{x,sz}m + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s m \right) & z \in D_{J,s}^+, \\ \sum_{x \in D_{J,s}} T_x \otimes \left(p_{x,z} C_s z m + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s m \right) & z \in D_{J,s}^0, \\ \sum_{x \in D_{J,s}} T_x \otimes (-v_s - v_s^{-1}) p_{x,z} m & z \in D_{J,s}^-. \end{cases}$$

On the right hand side we find:

$$\begin{aligned} C_s c(z|m) &= \sum_{x \in D_{J,s}} T_s T_x \otimes p_{x,z} m + T_x \otimes (-v_s) p_{x,z} m \\ &= \sum_{x \in D_{J,s}^+} T_{sx} \otimes p_{x,z} m + T_x \otimes (-v_s) p_{x,z} m \\ &\quad + \sum_{x \in D_{J,s}^0} T_x \otimes T_{s^x} p_{x,z} m + T_x \otimes (-v_s) p_{x,z} m \\ &\quad + \sum_{x \in D_{J,s}^-} (T_{sx} + (v_s - v_s^{-1}) T_x) \otimes p_{x,z} m + T_x \otimes (-v_s) p_{x,z} m \\ &= \sum_{x \in D_{J,s}^-} T_x \otimes p_{sx,z} m + \sum_{x \in D_{J,s}^+} T_x \otimes (-v_s) p_{x,z} m \\ &\quad + \sum_{x \in D_{J,s}^0} T_x \otimes (T_{s^x} - v_s) p_{x,z} m \\ &\quad + \sum_{x \in D_{J,s}^+} T_x \otimes p_{sx,z} m + \sum_{x \in D_{J,s}^-} T_x \otimes (-v_s^{-1}) p_{x,z} m \\ &= \sum_{x \in D_{J,s}^+} T_x \otimes (p_{sx,z} - v_s p_{x,z}) m \\ &\quad + \sum_{x \in D_{J,s}^0} T_x \otimes C_{s^x} p_{x,z} m \\ &\quad + \sum_{x \in D_{J,s}^-} T_x \otimes (p_{sx,z} - v_s^{-1} p_{x,z}) m. \end{aligned}$$

Comparing the $T_x \otimes M$ components we find an equation of elements of $\mathbb{Z}[\Gamma]\Omega_J$ that needs to be satisfied. More specifically it is the equation in part (a) of Lemma 4.7 below.

Similarly the equations

$$\omega(e_s)\omega(x_s) = \omega(x_s) \quad \text{and} \quad \omega(x_s)\omega(e_s) = 0$$

translate into equations of elements of $\mathbb{Z}[\Gamma]\Omega_J$, the two equations in part (c) and part (d) of Lemma 4.7 below.

LEMMA 4.7. *The elements $p_{x,z}, \mu_{x,z}^s$ of $\mathbb{Z}[\Gamma]\Omega_J$ satisfy the following equations:*

(a) *For all $z \in D_J$ and all $x \in D_J$*

$$\left. \begin{array}{l} x \in D_{J,s}^+ \\ x \in D_{J,s}^0 \\ x \in D_{J,s}^- \end{array} \right\} \left. \begin{array}{l} p_{sx,z} - v_s p_{x,z} \\ C_{s^x} p_{x,z} \\ p_{sx,z} - v_s^{-1} p_{x,z} \end{array} \right\} = \begin{cases} p_{x,sz} + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s & z \in D_{J,s}^+, \\ p_{x,z} C_{s^z} + \sum_{x \leq y < z} p_{x,y} \mu_{y,z}^s & z \in D_{J,s}^0, \\ -(v_s + v_s^{-1}) p_{x,z} & z \in D_{J,s}^-. \end{cases}$$

(b) *For all $z \in D_{J,s}^0$ and all $x \in D_J$:*

$$p_{x,z} e_{s^z} = \begin{cases} -v_s p_{sx,z} e_{s^z} & x \in D_{J,s}^+, \\ e_{s^x} p_{x,z} e_{s^z} & x \in D_{J,s}^0, \\ -v_s^{-1} p_{sx,z} e_{s^z} & x \in D_{J,s}^-. \end{cases}$$

(c) *For all $z \in D_J$ and all $x \in D_{J,s}^0$: $e_{s^x} \mu_{x,z}^s = \mu_{x,z}^s$*

(d) *For all $z \in D_{J,s}^0$ and all $x \in D_J$: $\mu_{x,z}^s e_{s^z} = 0$*

The proof of this lemma can be found in the appendix of this paper. It is inspired by Lusztig’s proof of the analogous equations in [16], but is significantly longer. □

4.3 First applications

4.3.1 Recovering well-known examples of induced modules

We will start off by proving that our result encompasses several classic W -graph existence results, including Howlett and Yin’s [12, Theorem 5.1].

EXAMPLE 4.8. Starting with $J = \emptyset$ and the regular module $\Omega_\emptyset = \mathbb{Z}$ we obtain the special case $\text{HY}_\emptyset^S(\mathbb{Z}) =: KL^S$. As an H -module this is

isomorphic to $\text{Ind}_{H_\emptyset}^H(H_\emptyset) = H$ and the basis $\{z|1 \mid z \in W\}$ is identified with the Kazhdan–Lusztig basis $\{C_z \mid z \in W\}$ via the canonicalization map.

Thus we recover Kazhdan and Lusztig’s result (cf. [15, 1.3]) that the regular H -module is induced by a W -graph. The elements $\mu_{x,y}^s \in \mathbb{Z}[\Gamma]\Omega_\emptyset = \mathbb{Z}[\Gamma]$ equal the μ -values defined in [15] and [17] (in the case of unequal parameters) up to a sign. The elements $p_{x,y} \in \mathbb{Z}[\Gamma]$ are related to the Kazhdan–Lusztig polynomials via

$$p_{x,y} = (-1)^{l(x)+l(y)} \nu^{L(x)-L(y)} \overline{P_{x,y}}.$$

EXAMPLE 4.9. Starting with an arbitrary $J \subseteq S$ and arbitrary one-dimensional Ω_J -module² M , one obtains a W -graph structure on the induced module $\text{Ind}_{H_J}^H(M)$.

This module is called M^J by Doedhar in [3]. The elements $p_{x,y} \in \mathbb{Z}[\Gamma]\Omega_J$ act on M by multiplication with polynomials which are related to Doedhar’s and Couillens’s (cf. [1]) parabolic Kazhdan–Lusztig polynomials $P_{x,y}^J$ in a similar way as the polynomials in the previous example are related to the absolute Kazhdan–Lusztig polynomials.

We now show that Howlett–Yin induction is appropriately named, i.e., that it really recovers the construction in [12].

PROPOSITION 4.10. *Assume $L(s) = 1$ for all $s \in S$.*

Let $J \subseteq S$ be arbitrary and M a $k\Omega_J$ -module with a k -basis $\mathfrak{C} \subseteq M$ with respect to which e_s acts diagonally (i.e., a module given by a W -graph) for all $s \in J$. Let $c : \text{Res}_H^\Omega(\text{HY}_J^S(M)) \rightarrow \text{Ind}_{H_J}^H(\text{Res}_{H_J}^\Omega(M))$ be the canonicalization isomorphism.

Then $c(x|\mathfrak{r}) = C_{x,\mathfrak{r}}$ for all $x \in D_J, \mathfrak{r} \in \mathfrak{C}$ where $C_{x,\mathfrak{r}}$ denotes the canonical basis defined by Howlett and Yin in [12, Theorem 5.1]. The W -graphs given by the basis $\{x|\mathfrak{r} \mid x \in D_J, \mathfrak{r} \in \mathfrak{C}\}$ and by the basis $\{C_{x,\mathfrak{r}} \mid x \in D_J, \mathfrak{r} \in \mathfrak{C}\}$ are the same.

Proof. Our involution ι on $V := H \otimes_{H_J} k[\Gamma]M$ is the same as Howlett and Yin’s involution $\overline{}$ defined in the introductory paragraph of [12, Section 3].

Let $c : k[\Gamma] \text{HY}_J^S(M) \rightarrow V$ be the canonicalization isomorphism and $\tilde{C}_{x,\mathfrak{r}} := c(x|\mathfrak{r})$. Then $\{\tilde{C}_{x,\mathfrak{r}} \mid x \in D_J, \mathfrak{r} \in \mathfrak{C}\}$ is a $k[\Gamma]$ -basis of V that satisfies

²Remember that all one-dimensional H_J -modules are given by a unique W_J -graph so that there is absolutely no difference between H_J - and Ω_J -modules in this case

$\overline{\tilde{C}_{x,\mathfrak{r}}} = \tilde{C}_{x,\mathfrak{r}}$ as well as $\tilde{C}_{x,\mathfrak{r}} \in T_x \otimes \mathfrak{r} + \sum_{w < x} T_w \otimes k[\Gamma_{>0}]M$. Howlett and Yin's theorem (as well as Theorem 2.11) shows that there is a *unique* basis with this property. Therefore $\tilde{C}_{x,\mathfrak{r}} = C_{x,\mathfrak{r}}$ as claimed.

That the W -graphs are identical follows from the fact that the e_s act identical on both bases. In Theorem 4.5 we have chosen our definition such that

$$\begin{aligned} e_s \cdot x|\mathfrak{r} = x|\mathfrak{r} &\iff x \in D_{J,s}^- \vee (x \in D_{J,s}^0 \wedge e_{s^x}\mathfrak{r} = \mathfrak{r}) \\ &\iff (sx < x) \vee (sx > x \wedge sx \notin D_J \wedge s^x \in I(\mathfrak{r})). \end{aligned}$$

In [12, Theorem 5.3] the W -graph structure on V is defined in such a way that (using Howlett and Yin's notation Λ_s^-)

$$\begin{aligned} e_s \cdot C_{x,\mathfrak{r}} = C_{x,\mathfrak{r}} &\iff (x, \mathfrak{r}) \in \Lambda_s^- \\ &\iff (sx < x) \vee (sx > x \wedge sx \notin D_J \wedge s^x \in I(\mathfrak{r})). \end{aligned}$$

Because the canonicalization map c is H -linear, H acts identical on both bases too. x_s is a linear combination of e_s and T_s so that x_s acts identical on both bases too, i.e., the edge weight matrices are also identical which proves that the two W -graphs are identical. □

4.3.2 An algorithm to compute p and μ

Note that part (a) of Lemma 4.7 and the recursive definition of the μ lead to the following recursive algorithm to compute $p_{x,z}$ and $\mu_{x,z}^s$ for all $x, z \in D_J$ and all $s \in S$.

The recursion is again along the (well-founded!) order $(x, z) \sqsubset (x', z') : \iff z < z' \vee (z = z' \wedge x > x')$ on $\{(x, z) \in D_J \times D_J \mid x \leq z\}$.

Algorithm 4.11. Input: $J \subseteq S$ and $x, z \in W$.

Output: $p_{x,z} \in \mathbb{Z}[\Gamma]\Omega_J$ and $\mu_{x,z}^s \in \mathbb{Z}[\Gamma]\Omega_J$ for all $s \in S$.

1. If $x \not\leq z$, then $p_{x,z} = 0$ and $\mu_{x,z}^s = 0$.
2. If $x = z$, then $p_{x,z} = 1$ and $\mu_{x,z}^s = 0$.
3. If $x < z$, then choose any $t \in S$ with $tz < z$ and consider the following cases:

3.1 If $t \in D_J^+(x)$, then $p_{x,z} = -v_t p_{tx,tz}$.

3.2 If $t \in D_J^0(x)$, then $p_{x,z} = C_{t^x} p_{x,tz} - \sum_{y < tz} p_{x,y} \mu_{y,tz}^t$.

3.3 If $t \in D_J^-(x)$, then $p_{x,z} = p_{tx,tz} - v_t^{-1} p_{x,tz} - \sum_{y < tz} p_{x,y} \mu_{y,tz}^t$.

4. For all $s \in S$:

4.1 If $s \in D_J^+(x)$ or $s \in D_J^-(z)$, then $\mu_{x,z}^s = 0$.

4.2 Otherwise compute $\alpha := -R - \sum_{x < y < z} p_{x,y} \mu_{y,z}^s$, where R is defined as in 4.4. Write $\alpha = \alpha_- + \alpha_0 + \alpha_+$ where $\alpha_- \in \mathbb{Z}[\Gamma_{<0}] \Omega_J$, $\alpha_0 \in \Omega_J$, $\alpha_+ \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$. Then $\mu_{x,z}^s = \alpha_- + \alpha_0 + \overline{\alpha_-}$.

4.3.3 More about the algebraic structure of Ω

The fact that $m \mapsto 1|m$ is an injective map $M \rightarrow \text{HY}_J^S(M)$ provides a simple proof to [10, Conjecture 4.2.23] from the author’s thesis.

PROPOSITION 4.12. *Let k be a commutative ring. Then the parabolic morphism $j : k\Omega_J \rightarrow k\Omega$, $e_s \mapsto e_s$, $x_{s,\gamma} \mapsto x_{s,\gamma}$ is injective.*

Proof. Consider the Howlett–Yin induction $M := \text{HY}_J^S(\Omega_J)$ of the regular $k\Omega_J$ -module. It is a $k\Omega$ -module so that $f : k\Omega \rightarrow M$, $a \mapsto a \cdot 1|1$ is a morphism of $k\Omega$ -left-modules. For all $s \in J$ one finds $f(j(e_s)) = 1|e_s$ and $f(j(x_{s,\gamma})) = 1|x_{s,\gamma}$ so that $f(j(a)) = 1|a$ holds for all $a \in k\Omega_J$. In particular we find that $f \circ j$ is injective so that j is injective. \square

We will therefore suppress the embedding altogether and consider $k\Omega_J$ as a true subalgebra of $k\Omega$ from now on.

The Howlett–Yin induction also provides the correction to a small error in the proof of [10, Corollary 4.2.19] in the author’s thesis.

PROPOSITION 4.13. *Let k be a commutative ring. Define $E_J := \prod_{s \in J} e_s \prod_{s \in S \setminus J} (1 - e_s) \in k\Omega_J$. This element is nonzero in $k\Omega_J$.*

The fallacious argument in my thesis considered the Kazhdan–Lusztig W -Graph KL^S and assuming falsely that each $J \subseteq S$ occurs as a left descent set $D_L(w)$ for some $w \in W$. I concluded that E_J must act nontrivially on this W -graph. This only works for finite Coxeter groups because a subset $J \subseteq S$ in fact occurs as a left descent set if and only if W_J is finite (cf. [3, Lemma 3.6]). In particular S itself does not occur as a left descent set if W is infinite. Nevertheless S occurs in the W -graph of the sign representation and $E_S \in k\Omega$ is therefore nonzero. This is the idea of the following proof:

Proof. Consider the sign representation $M = k \cdot m_0$ of $k\Omega_J$, i.e., $e_s m_0 = m_0$, $x_s m_0 = 0$ for all $s \in J$. The element $1|m_0 \in \text{HY}_J^S(M)$ satisfies:

$$e_s 1|m_0 = \begin{cases} 1|m_0 & s \in J, \\ 0 & s \notin J, \end{cases}$$

for all $s \in S$ so that $E_J \cdot 1|m_0 = 1|m_0$ and therefore $E_J \neq 0$. \square

§5. Categorical properties of Howlett–Yin induction

We prove now that Howlett–Yin induction is very well-behaved. In particular it is a functor between module categories, given by tensoring with a certain bimodule, satisfies a transitivity property and a Mackey-type theorem.

5.1 Howlett–Yin induction as a functor between module categories

THEOREM 5.1. *Let M, M_1, M_2 be $k\Omega_J$ -modules, and $\phi : M_1 \rightarrow M_2$ a $k\Omega_J$ -linear map.*

(a) *Using the notation from Theorem 4.5, the map*

$$\text{HY}_J^S(\phi) : \text{HY}_J^S(M_1) \rightarrow \text{HY}_J^S(M_2), z|m_1 \mapsto z|\phi(m_1)$$

is Ω -linear. In particular HY_J^S is a functor $k\Omega_J\text{-Mod} \rightarrow k\Omega\text{-Mod}$.

(b) *$\text{HY}_J^S(\phi)$ commutes with the two canonicalizations, that is the diagram in Figure 2 commutes.*

$$\begin{array}{ccc} \text{Res}_H^\Omega \text{HY}_J^S(M_1) & \xrightarrow{\text{id} \otimes \text{HY}_J^S(\phi)} & \text{Res}_H^\Omega \text{HY}_J^S(M_2) \\ \downarrow c_{M_1} & & \downarrow c_{M_2} \\ \text{Ind}_{H,J}^H \text{Res}_{H,J}^{\Omega_J} M_1 & \xrightarrow{\text{Ind}_{H,J}^H \phi} & \text{Ind}_{H,J}^H \text{Res}_{H,J}^{\Omega_J} M_2 \end{array}$$

Figure 2.

Functoriality of Howlett–Yin induction.

In other words: the canonicalization c is a natural isomorphism $\text{Res}_H^\Omega \circ \text{HY}_J^S \rightarrow \text{Ind}_{H,J}^H \circ \text{Res}_{H,J}^{\Omega_J}$.

Proof. That $\text{HY}_J^S(\phi)$ is Ω -linear is readily verified with the definition of the Ω -action.

$\text{Ind}_{H,J}^H(\phi)$ is certainly H -linear. Therefore $\phi(T_x \otimes M_1) = T_x \otimes \phi(M_1) \subseteq T_x \otimes M_2$ holds for all $x \in D_J$. It also commutes with ι . By functoriality of canonicalization, the diagram in Figure 2 commutes. □

REMARK 5.2. The ordinary induction functor $\text{Ind}_{\Omega_J}^\Omega$ is given by a tensoring with the Ω - Ω_J -bimodule Ω . It is therefore natural to ask whether

HY_J^S can be described as a tensor functor and whether it satisfies a property similar to the Hom–tensor adjunction. The next proposition answers these questions in the affirmative.

PROPOSITION 5.3. *The functor HY_J^S is exact, commutes with direct sums and satisfies*

- (a) *There is a sub- Ω - Ω_J -bimodule $\mathfrak{I}_J^S \leq \Omega$ such that $\bar{a} \otimes m \mapsto a \cdot 1|m$ is a natural isomorphism $\Omega/\mathfrak{I}_J^S \otimes_{\Omega_J} M \rightarrow \text{HY}_J^S(M)$.*
- (b) *$\text{HY}_J^S(M)$ has the following universal mapping property in $k\Omega$ -mod:*

$$\text{Hom}(\text{HY}_J^S(M), X) \cong \{f : M \rightarrow \text{Res}_{\Omega_J}^\Omega(X) \mid \mathfrak{I}_J^S \cdot f(M) = 0\},$$

where the isomorphism is given by $F \mapsto (m \mapsto F(1|m))$.

Proof. It is clear from the definition that HY_J^S is exact and commutes with direct sums.

The Eilenberg–Watts theorem (which characterizes cocontinuous and right-exact functors between module categories, cf. [4, 19]) implies $\text{HY}_J^S \cong Q \otimes_{\Omega_J} -$ for some Ω - Ω_J -bimodule Q . In fact the proof is constructive. It shows that one can choose Q to be $\text{HY}_J^S(\Omega_J)$ and the isomorphism as $\text{HY}_J^S(\Omega_J) \otimes M \rightarrow \text{HY}_J^S(M)$, $z|a \otimes m \mapsto z|am$.

Furthermore $\text{HY}_J^S(\Omega_J)$ is generated by the element $1|1$: the Ω_J -submodule generated by $1|1$ is $1|\Omega_J$ and in general $1|M$ generates $\text{HY}_J^S(M)$ as an Ω -module. Therefore $\text{HY}_J^S(\Omega_J)$ is isomorphic to some quotient Ω/\mathfrak{I}_J^S via $a + \mathfrak{I}_J^S \mapsto a \cdot 1|1$.

The universal property follows from this presentation of the functor: Tensor the exact sequence $\mathfrak{I}_J^S \rightarrow \Omega \rightarrow \Omega/\mathfrak{I}_J^S \rightarrow 0$ with M . Right exactness of $- \otimes M$ implies that

$$\mathfrak{I}_J^S \otimes_{\Omega_J} M \rightarrow \Omega \otimes_{\Omega_J} M \rightarrow \text{HY}_J^S(M) \rightarrow 0$$

is exact. This provides a universal property of $\text{HY}_J^S(M)$ as the quotient of $\Omega \otimes M$ modulo the image of $\mathfrak{I}_J^S \otimes M \rightarrow \Omega \otimes M$. Combining this with Hom–Tensor-adjunction $\text{Hom}(\Omega \otimes M, X) \cong \text{Hom}(M, \text{Res}_{\Omega_J}^\Omega(X))$ we obtain the result. □

5.2 Transitivity

REMARK 5.4. Having a concept of “induction” directly leads to ask additional questions such as whether this is a transitive construction. Howlett and Yin did not address this question in their original papers. If I

were to guess I would say that the proliferation of indices and combinatorial formulas made such a proof infeasible. Here our more abstract approach to induction pays off by encapsulating all the work with recursive formulas involving p and μ .

LEMMA 5.5. *Let V_1, V_2 be two $k\Omega$ -modules and $f : V_1 \rightarrow V_2$ a k -linear map.*

Then f is $k\Omega$ -linear if and only if the induced map $k[\Gamma] \otimes_k V_1 \rightarrow k[\Gamma] \otimes_k V_2$ is H -linear and $f(e_s m) = e_s f(m)$ holds for all $m \in V_1$.

Proof. Because $T_s = -v_s^{-1}e_s + v_s e_s + x_s$ the assumptions imply $f(x_s m) = x_s f(m)$ as elements of $k[\Gamma] \otimes V_2 = \bigoplus_{\gamma} v^{\gamma} V_2$. Now by definition $x_s = \sum_{\gamma} x_{s,\gamma} v^{\gamma}$ so that $\sum_{\gamma} f(x_{s,\gamma} m) v^{\gamma} = \sum_{\gamma} x_{s,\gamma} f(m) v^{\gamma}$. Comparing coefficients gives Ω -linearity. The reverse implication is clear. \square

THEOREM 5.6. *Howlett–Yin induction is transitive. More precisely, if $J \subseteq K \subseteq S$, then*

$$\tau_M : \text{HY}_K^S(\text{HY}_J^K(M)) \rightarrow \text{HY}_J^S(M), w|z|m \mapsto wz|m$$

is a natural $k\Omega$ -module isomorphism.

Proof. Consider the diagram in Figure 3. Here $t : \text{Ind}_{H_K}^{H_S} \circ \text{Ind}_{H_J}^{H_K} \rightarrow \text{Ind}_{H_J}^{H_S}$ is the natural isomorphism mapping $h_1 \otimes (h_2 \otimes m) \mapsto h_1 h_2 \otimes m$.

$$\begin{array}{ccc}
 \text{Res}_{H_S}^{\Omega_S} \text{HY}_K^S \text{HY}_J^K M & \xrightarrow{\text{id}_{k[\Gamma]} \otimes \tau_M} & \text{Res}_{H_S}^{\Omega_S} \text{HY}_J^S M \\
 \downarrow c_{\text{HY}_J^K(M)} & & \downarrow c_M \\
 \text{Ind}_{H_K}^{H_S} \text{Res}_{H_K}^{\Omega_K} \text{HY}_J^K M & & \\
 \downarrow \text{Ind}_{H_K}^{H_S}(c_M) & & \downarrow \\
 \text{Ind}_{H_K}^{H_S} \text{Ind}_{H_J}^{H_K} \text{Res}_{H_J}^{\Omega_J} M & \xrightarrow{t_M} & \text{Ind}_{H_J}^{H_S} \text{Res}_{H_J}^{\Omega_J} M
 \end{array}$$

Figure 3.
Transitivity of Howlett–Yin induction.

We will show that this diagram commutes. Note that t and c are natural H -linear isomorphisms. In particular this expresses $\text{id}_{k[\Gamma]} \otimes \tau_M$ as

a composition of H_S -linear natural isomorphisms. We will easily verify that τ_M is in fact Ω_S -linear so that τ_M really is a natural isomorphism between Ω_S -modules.

To prove the diagram commutes we will show that the counter-clockwise composition of arrows from $\text{Res}_{H_S}^{\Omega_S} \text{HY}_J^S M$ to $\text{Ind}_{H_J}^{H_S} \text{Res}_{H_J}^{\Omega_J} M$ equals the canonicalization c_M .

First note that all maps in the diagram are in fact $\widehat{k[\Gamma]}$ -linear: $\text{id}_{k[\Gamma]} \otimes \tau_M$ is trivially $\widehat{k[\Gamma]}$ -linear. $c_{\text{HY}_J^K(M)}$ is $\widehat{k[\Gamma]}$ -linear because it is a canonicalization. $\text{Ind}_{H_K}^{H_S}(c_M)$ is $\widehat{k[\Gamma]}$ -linear because c_M is and $\text{Ind}_{H_K}^{H_S}$ maps $\widehat{k[\Gamma]}$ -linear maps to $\widehat{k[\Gamma]}$ -linear maps. That $t_{\text{Res}_{H_K}^{\Omega_K}(M)}$ is also $\widehat{k[\Gamma]}$ -linear can readily be verified.

Next we prove that the counter-clockwise composition maps $xy|m$ into $T_{xy} \otimes m + \sum_{w \in D_J^S} T_w \otimes k[\Gamma]_{>0} M$ for all $(x, y) \in D_K^S \times D_J^K$:

$$\begin{aligned} 1 \otimes xy|m &\xrightarrow{\text{id} \otimes \tau_M^{-1}} 1 \otimes x|y|m \\ &\xleftarrow{c_{\text{HY}_J^K(M)}} T_x \otimes y|m + \sum_{u \in D_K^S} T_u \otimes k[\Gamma]_{>0} \text{HY}_J^K(M) \\ &\xleftarrow{\text{Ind}_{H_K}^{H_S}(c_M)} T_x \otimes T_y \otimes m + \sum_{(u,v) \in D_K^S \times D_J^K} T_u \otimes T_v \otimes k[\Gamma]_{>0} M \\ &\xleftarrow{t_{\text{Res}_{H_J}^{\Omega_J}(M)}} T_{xy} \otimes m + \sum_{uw \in D_J^S} T_{uw} \otimes k[\Gamma]_{>0} M. \end{aligned}$$

Theorem 2.11 shows that c_M is the only $\widehat{k[\Gamma]}$ -linear map that maps $1 \otimes xy|m$ into $T_{xy} \otimes m + \sum_{w \in D_J^S} T_w \otimes k[\Gamma]_{>0} M$. This completes our proof that $\text{id}_{k[\Gamma]} \otimes \tau_M$ is H -linear and natural in M .

Furthermore $\tau_M(e_s x|y|m) = e_s \tau_M(x|y|m)$ follows directly from Lemma 4.2 and the definition of the Ω -action so that τ_M is Ω -linear by Proposition 5.5. □

5.3 The Mackey theorem for Howlett–Yin induction

REMARK 5.7. The next natural question is whether there exists a Mackey decomposition for the Howlett–Yin induction. Recall that the Mackey formula for group representations says

$$\text{Res}_{W_K}^W \text{Ind}_{W_J}^W(V) = \bigoplus_d \text{Ind}_{W_K \cap {}^d W_J}^{W_K} \text{Res}_{W_K \cap {}^d W_J}^{d W_J}(dV)$$

for all $k[W_J]$ -modules V . Here the sum runs over a set of representatives d for W_K - W_J -double cosets and ${}^d(-)$ denotes conjugation by d . The conjugated representation dV is isomorphic as a $k[W_K \cap {}^dW_J]$ -module to $d \otimes V \subseteq \text{Ind}_{W_J}^W(V)$.

One can show that parabolic subgroups of Coxeter groups are well-behaved in that $W_K \cap {}^dW_J = W_{K \cap {}^dJ}$ if one chooses d of minimal length in its double coset.

A similar formula also holds at the level of Hecke algebras:

$$\text{Res}_{H_K}^H \text{Ind}_{H_J}^H(V) = \bigoplus_d \text{Ind}_{H_{K \cap {}^dJ}}^{H_K}({}^dV)$$

for all H_J -modules V . Here d runs over the set of representatives of W_K - W_J -double cosets of shortest length and dV is the $H_{K \cap {}^dJ}$ -module $T_d \otimes V \subseteq \text{Ind}_{H_J}^H(V)$. The reason for both of these formulas is that W decomposes as disjoint union of double cosets and H decomposes as a direct sum of H_K - H_J -bimodules accordingly.

Unfortunately there is no reason to expect that $\text{HY}_J^S(M)$ decomposes into a direct sum over double cosets because Ω , unlike $k[W]$ and H , does not have such a direct sum decomposition (and in fact $\text{HY}_J^S(M)$ can be an indecomposable Ω -module). Instead we will find a *filtration* indexed by the double cosets whose layers play the role of the direct summands in the Mackey decomposition.

LEMMA 5.8. *Let $J, K \subseteq S$. Define $D_{KJ} := D_K^{-1} \cap D_J$. Then*

- (a) D_{KJ} is a system representative of W_K - W_J -double cosets in W . More precisely $d \in D_{KJ}$ if and only if it is the unique element of minimal length within its double coset.
- (b) $D_J^S = \coprod_{d \in D_{KJ}} D_{K \cap {}^dJ}^K \cdot d$.
- (c) For $d \in D_{KJ}$ and $x \in D_{K \cap {}^dJ}^K$:

$$D_J^*(xd) \cap K = D_{K \cap {}^dJ}^*(x) \cap K,$$

where $* \in \{+, 0, -\}$.

Proof. See [8, 2.1.6–2.1.9]. □

REMARK 5.9. Fix some $d \in D_{KJ}$. For any Ω_J -module M one can define an $\Omega_{K \cap {}^dJ}$ -module dM by

$$e_s \cdot {}^d m := {}^d(e_s m) \quad \text{and} \quad x_s \cdot {}^d m := {}^d(x_s m).$$

Similarly for any H_J -module V one can define a $H_{K \cap dJ}$ -module dV by

$$T_s \cdot {}^d v := {}^d(T_s v).$$

Note that dV is isomorphic to the $H_{K \cap dJ}$ -submodule $T_d \otimes V \subseteq \text{Ind}_{H_J}^{H_S} V$.

THEOREM 5.10. *Let $J, K \subseteq S$. Furthermore let M be a $k\Omega_J$ -module and $V := \text{Res}_{H_J}^{\Omega_J} M$ its associated kH_J -module. For all $d \in D_{KJ}$ define the following k -submodules of $\text{HY}_J^S(M)$ and $\text{Ind}_{H_J}^{H_S}(V)$, respectively:*

$$F^{\leq d} \text{HY}_J^S(M) := \sum_{\substack{a \in D_{KJ}, a \leq d \\ w \in D_{K \cap dJ}^K}} wa | M$$

$$F^{\leq d} \text{Ind}_{H_J}^{H_S}(V) := \sum_{\substack{a \in D_{KJ}, a \leq d \\ w \in D_{K \cap dJ}^K}} T_{wa} \otimes V.$$

The following hold for all $d \in D_{KJ}$:

- (a) $F^{\leq d} \text{HY}_J^S(M)$ is a Ω_K -submodule, $F^{\leq d} \text{Ind}_{H_J}^{H_S}(V)$ is a H_K - and $\widehat{k[\Gamma]}$ -submodule and the canonicalization map c_M identifies these with each other.
- (b) The map $\Psi_M^d : \text{HY}_{K \cap dJ}^K({}^dM) \rightarrow F^{\leq d} \text{HY}_J^S(M) / F^{< d} \text{HY}_J^S(M)$ which is defined by $w | {}^d m \mapsto wd | m$ for all $w \in D_{K \cap dJ}^K$ is a natural isomorphism of Ω_K -modules.

REMARK 5.11. In [13] Howlett and Yin also proved a Mackey-style theorem, which is a bit weaker than what is claimed here. Howlett and Yin only identify sub- W_K -graphs of the induced W -graph and prove that they are the same as the W_K -graphs for summands appearing in the Mackey formula.

Speaking in terms of modules this proves that $\text{HY}_{K \cap dJ}^K({}^dM)$ appears as some subquotient of $\text{Res}_{\Omega_K}^{\Omega_S}(\text{HY}_J^S(M))$. This is similar to describing a module by listing its composition factors. The new theorem states not only that these composition factors arise somewhere in the module but also identifies an explicit filtration in which they arise. (And also generalizes the result to the multiparameter case and nonfree modules.)

Proof. That $F^{\leq d} \text{HY}_J^S(M)$ is a Ω_K -submodule follows directly from the definition of the Ω -action on $\text{HY}_J^S(M)$ and the observation

$w'd' \leq wd \implies d' \leq d$ for all $w, w' \in W_K, d, d' \in D_{KJ}$. That $F^{\leq d} \text{Ind}_{H_J}^{H_S}(V)$ is a H_K -submodule follows from the fact that it is equal to $\sum_{a \leq d} \text{span} \{T_w \mid w \in W_K a W_J\} \otimes V$.

To prove the second claim, observe that the given map is certainly a k -linear bijection. Next we will show that it makes the diagram in Figure 4 commute.

$$\begin{array}{ccc}
 F^{\leq d} \text{HY}_J^S(M) / F^{< d} \text{HY}_J^S(M) & \xleftarrow{\Psi_M^d} & \text{HY}_{K \cap dJ}^K(dM) \\
 \downarrow \overline{c_M} & & \downarrow c_{dM} \\
 F^{\leq d} \text{Ind}_{H_J}^{H_S}(V) / F^{< d} \text{Ind}_{H_J}^{H_S}(V) & \xleftarrow{T_{xd} \otimes m \leftarrow T_x \otimes {}^d m} & \text{Ind}_{H_{K \cap dJ}}^{H_K}(dV)
 \end{array}$$

Figure 4.
Mackey isomorphism for Howlett–Yin induction.

This will again be done by utilizing the uniqueness of the canonicalization map. Observe that all maps involved in the diagram are $k[\widehat{\Gamma}]$ -linear bijections. It is also readily verified that both the counter-clockwise and the clockwise compositions map $w|{}^d m$ into $T_{wd} \otimes m + \sum_{x \in D_{K \cap dJ}^K} T_{xd} \otimes k[\Gamma]_{>0} M$ so that the diagram indeed commutes by uniqueness of the canonicalization map.

Because canonicalization is H_K -linear, we conclude that Ψ_M^d is also H_K -linear. It follows directly from the above lemma that $\Psi_M^d(e_s \cdot w|{}^d m) = e_s \cdot \Psi_M^d(w|{}^d m)$ for $s \in K$. Proposition 5.5 implies again that Ψ_M^d is indeed Ω_K -linear. □

§6. More applications

6.1 An improved algorithm to compute p and μ

Denote with $p_{x,z}^J$ and $\mu_{x,z}^{s,J}$ the elements in $\mathbb{Z}[\Gamma]\Omega_J$ from Definitions 4.3 and 4.4, respectively, to make the dependence from $J \subseteq S$ explicit. Note that these elements do not depend on S in the sense that for any parabolic subgroup $W_J \subseteq W_K \subseteq W$ with $x, z \in W_K$ the elements computed with respect to the inclusion $J \subseteq S$ are the same as when computed with respect to the inclusion $J \subseteq K$.

PROPOSITION 6.1. *Suppose $J \subseteq K \subseteq S$. Let $u, x \in D_K^S$ and $v, y \in D_J^K$. Furthermore let $\omega_J^K : \Omega_K \rightarrow \Omega_J^{D_J^K \times D_J^K}$ be the matrix representation induced by the action of Ω_K on $\text{HY}_J^K(\Omega_J) = \bigoplus_{v \in D_J^K} v|\Omega_J$.*

- (a) *If $u \not\leq x$, then $\mu_{uv,xy}^{s,J}$ and $p_{uv,xy}^J = 0$.*
- (b) *If $u = x$, then*

$$\mu_{uv,xy}^{s,J} = \begin{cases} \mu_{v,y}^{s^x,J} & s \in D_K^0(x) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad p_{uv,xy}^J = p_{v,y}^J.$$

- (c) *If $u < x$, then*

$$\mu_{uv,xy}^{s,J} = \omega_J^K(\mu_{u,x}^{s,K})_{v,y} \quad \text{and} \quad p_{uv,xy}^J = \sum_{\substack{t \in D_J^K \\ v \leq t}} p_{v,t}^J \cdot \omega_J^K(p_{u,x}^K)_{ty}.$$

Proof. Considering that the diagram in Figure 3 is commutative, one finds that $x|y|m \in \text{HY}_K^S(\text{HY}_J^K(M))$ is mapped both to

$$\sum_{u \leq x} \sum_{v \leq t} T_u T_v \otimes p_{v,t}^J \omega_J^K(p_{u,x}^K)_{ty} \cdot m \quad \text{and to} \quad \sum_{u,v} T_{uv} \otimes p_{uv,xy}^J \cdot m.$$

Setting $M = \Omega_J$ and $m = 1$, we obtain the equations for $p_{uv,xy}^J$.

We consider the identification $\text{HY}_K^S \text{HY}_J^K(M) \cong \text{HY}_J^S(M)$ from Theorem 5.6 and the action of x_s on both modules:

$$x_s \cdot xy|m = \begin{cases} sxy|m + \sum_{uv} uv \left| \mu_{uv,xy}^{s,J} m \right. & s \in D_J^+(xy), \\ xy|x_s xy|m + \sum_{uv} uv \left| \mu_{uv,xy}^{s,J} m \right. & s \in D_J^0(xy), \\ 0 & s \in D_J^-(xy), \end{cases}$$

$$x_s \cdot x|y|m = \begin{cases} sx|y|m + \sum_{u < x} u \left| \mu_{u,x}^{s,K} \cdot y|m \right. & s \in D_K^+(x), \\ x|x_s x \cdot y|m + \sum_{u < x} u \left| \mu_{u,x}^{s,K} \cdot y|m \right. & s \in D_K^0(x), \\ 0 & s \in D_K^-(x), \end{cases}$$

$$= \begin{cases} sx|y|m + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^+(x), \\ x|s^x y|m + \sum_{v < y} x|v|\mu_{v,y}^{s^x,J} m \\ \quad + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^+(y), \\ x|y|x_{s^x y}m + \sum_{v < y} x|v|\mu_{v,y}^{s^x,J} m \\ \quad + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^0(y), \\ 0 & s \in D_K^0(x), s^x \in D_J^-(y), \\ 0 & s \in D_K^-(x), \end{cases}$$

Therefore

$$\sum_{uv} u|v|\mu_{uv,xy}^{s,J} m = \begin{cases} \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^+(x), \\ \sum_{v < y} x|v|\mu_{v,y}^{s^x,J} m \\ \quad + \sum_{u < x} u|\mu_{u,x}^{s,K} \cdot y|m & s \in D_K^0(x), s^x \in D_J^+(y) \cup D_J^0(y), \\ 0 & \text{otherwise.} \end{cases}$$

Comparing the component $u | \text{HY}_J^K(M)$, we find

$$\sum_v v|\mu_{uv,xy}^{s,J} m = \begin{cases} \mu_{u,x}^{s,K} \cdot y|m & u < x, \\ \sum_{v < y} v|\mu_{v,y}^{s^x,J} m & u = x, s \in D_K^0(x), \\ 0 & \text{otherwise.} \end{cases}$$

Now set $M := \Omega_J$ and $m := 1$. □

This suggests the following algorithm for computing $p_{w,z}^J$ and $\mu_{w,z}^{s,J}$ for all $w, z \in D_J^S$ and all $s \in S$:

Algorithm 6.2. Input: $J \subseteq S$.

Output: $p_{w,x}, \mu_{w,x}^s \in \mathbb{Z}[\Gamma]\Omega_J$ for all $w, x \in D_J$ and all $s \in S$.

1. Choose a flag $J = K_0 \subsetneq K_1 \subsetneq \dots \subsetneq K_n = S$.
2. For all $i = 0, \dots, n - 1$, all $u, x \in D_{K_i}^{K_{i+1}}$, and all $s \in K_{i+1}$ compute $p_{u,x}^{K_i}$ and $\mu_{u,x}^{s,K_i} \in \Omega_{K_i}$ with Algorithm 4.11.

3. For $i = 0, \dots, n - 1$ compute $p_{w,x}^J$ and $\mu_{w,z}^{s,J}$ for all $w, z \in D_J^{K_{i+1}}$ and all $s \in K_{i+1}$ as follows:
 - 3.1 Write $w = uv, z = xy$ with $u, x \in D_{K_i}^{K_{i+1}}$ and $v, y \in D_J^{K_i}$.
 - 3.2 If $u \not\leq x$, then $\mu_{w,z}^{s,J} = 0$ and $p_{w,z}^J = 0$.
 - 3.3 If $u = x$, then $\mu_{w,z}^{s,J} = \begin{cases} \mu_{v,y}^{s^x,J} & s \in D_{K_i}^0(x) \\ 0 & \text{otherwise} \end{cases}$ and $p_{w,z}^J = p_{v,y}^{K_i}$.
 - 3.4 If $u < x$, then compute $\omega_J^{K_i}(\mu_{v,y}^{s,K_i})$ and $\omega_J^{K_i}(p_{u,x}^{K_i})$. Assemble the $p_{v',y'}^J$ with $v', y' \in D_J^{K_i}$ into the matrix $P \in (\mathbb{Z}[\Gamma_{\geq 0}]\Omega_J)^{D_J^{K_i} \times D_J^{K_i}}$. Then $\mu_{w,z}^{s,J} = \omega_J^{K_i}(\mu_{u,x}^{s,K_i})_{v,y}$ and $p_{w,z}^J = (P \cdot \omega_J^{K_i}(p_{u,x}^{K_i}))_{v,y}$.

REMARK 6.3. Note that the action of Ω_{K_i} on $\text{HY}_J^{K_i}(\Omega_J)$ only involves values of $\mu_{w',z'}^{s',J}$ where $w', z' \in D_J^{K_i}$ and $s' \in K_i$ which are already known by the previous iteration of the loop.

The big advantage of this algorithm over a direct computation of all $\mu_{w,z}^{s,J}$ with Algorithm 4.11 is that the expensive recursion over D_J^S is replaced by n collectively cheaper recursions over $D_{K_0}^{K_1}, D_{K_1}^{K_2}, \dots, D_{K_{n-1}}^{K_n}$ so that fewer polynomials $p_{w,z}$ need to be computed and the computed elements are less complex (measured for example by the maximal length of occurring words in the generators e_s, x_s of Ω) and therefore need less memory.

Additionally the n calls to Algorithm 4.11 in step 2 are independent of each other and can be executed in parallel which can lead to a sizeable speed-up.

REMARK 6.4. The Mackey isomorphism Ψ^d from Theorem 5.10 translates into the equation

$$\mu_{yd,wd}^{s,J} = \kappa_d(\mu_{y,w}^{s,K \cap dJ})$$

for all $d \in D_{KJ}$, all $y, w \in D_{K \cap dJ}^K$, and all $s \in K$ where $\kappa_d : \Omega_{K \cap dJ} \rightarrow \Omega_{K^d \cap J}$ is the isomorphism $e_s \mapsto e_{s^d}, x_s \mapsto x_{s^d}$.

Provided one knows all $\mu_{u,v}^{s,T}$ for all $T \subseteq K$, all $u, v \in D_T^K$, and all $s \in K$, one can use that knowledge to partially calculate $\mu_{y,w}^{s,J}$. This in turn might be used to give the recursion from Algorithm 4.11 a head start and reduce the necessary recursion depth.

6.2 Induction of left cells

Transitivity of Howlett–Yin induction also enables us to effortlessly improve a result of Geck regarding the induction of cells (see [5]).

REMARK 6.5. Recall the definition of (left) Kazhdan–Lusztig cells: Define a preorder $\preceq_{\mathcal{L}}$ on W by defining $\{y \in W \mid y \preceq_{\mathcal{L}} z\}$ to be the smallest subset $\mathfrak{C} \subseteq W$ such that the subspace $\text{span}_{\mathbb{Z}[\Gamma]} \{C_y \mid y \in \mathfrak{C}\}$ is an H -submodule of H . The preorder then defines an equivalence relation $\sim_{\mathcal{L}}$ as usual by $x \sim_{\mathcal{L}} y : \iff x \preceq_{\mathcal{L}} y \preceq_{\mathcal{L}} x$. The equivalence classes of this relation are called *left cells*.

PROPOSITION 6.6.

- (a) $\mathfrak{C} \subseteq W$ is $\preceq_{\mathcal{L}}$ -downward closed if and only if $\text{span}_{\mathbb{Z}} \{x|1 \mid x \in \mathfrak{C}\}$ is a Ω -submodule of $\text{HY}_{\emptyset}^S(\Omega_{\emptyset})$.
- (b) If $\mathfrak{C} \subseteq W_J$ is a union of left cells, then $D_J^S \cdot \mathfrak{C} \subseteq W$ is also a union of left cells.

Proof. Consider the Kazhdan–Lusztig W -graph $\text{HY}_{\emptyset}^S(\Omega_{\emptyset})$. Then $\{x|1 \mid x \in W\}$ constitute a \mathbb{Z} -basis of this module which (under the canonicalization map) corresponds to the basis $\{C_x \mid x \in W\}$. That \mathfrak{C} is a $\preceq_{\mathcal{L}}$ -downward closed means that $\text{span}_{\mathbb{Z}[\Gamma]} \{C_x \mid x \in \mathfrak{C}\}$ is an H -submodule of H . Because $e_s \cdot x|1 \in \{0, x|1\}$ for all x and s , every subset of the form $\text{span}_{\mathbb{Z}} \{x|1 \mid x \in \mathfrak{C}\}$ is closed under multiplication with e_s . Since e_s and T_s together generate Ω this proves the first statement.

Now let $\mathfrak{C} \subseteq W_J$ be $\preceq_{\mathcal{L}}$ -downward closed and $M := \text{span}_{\mathbb{Z}} \{x|1 \mid x \in \mathfrak{C}\}$ be the corresponding submodule of $\text{HY}_{\emptyset}^J(\Omega_{\emptyset})$. Then $\text{HY}_J^S(M) = \text{span}_{\mathbb{Z}} \{w|x|1 \mid w \in D_J^S, x \in \mathfrak{C}\}$ is a submodule of $\text{HY}_J^S \text{HY}_{\emptyset}^J(\Omega_{\emptyset}) \cong \text{HY}_{\emptyset}^S(\Omega_{\emptyset})$. In other word $D_J^S \cdot \mathfrak{C}$ is a $\preceq_{\mathcal{L}}$ -downward closed set of W . Because every union of cells can be written as a set difference $\mathfrak{C}_1 \setminus \mathfrak{C}_2$ for some downward closed sets $\mathfrak{C}_2 \subseteq \mathfrak{C}_1 \subseteq W$, this proves the second statement. □

REMARK 6.7. Modifying Algorithm 6.2 such that only μ -values for elements in $D_J^S \cdot \mathfrak{C}$ are computed, we recover Geck’s PyCox algorithm for the decomposition into left cells.

Appendix A. Proof of Lemma 4.7

We will prove the four equations in Lemma 4.7 simultaneously with a double induction. We will induct over $l(z)$ and assume that all four equations hold for all pairs (x', z') with $l(z') < l(z)$. For any fixed z we will induct over $l(z) - l(x)$. Observe that all equations are trivially true if $l(x) > l(z) + 1$ because all occurring p and μ are zero. We will therefore assume that the equations also hold for all pairs (x', z) with $l(x') > l(x)$.

Proof of part (a) of Lemma 4.7. We denote with f_{xz} the difference between the right hand side and the left hand side of the equation. Then by the above considerations:

$$c(\omega(C_s)z|m) - C_s c(z|m) = \sum_x T_x \otimes f_{xz} m.$$

We will show $f_{xz} \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$ and conclude $f_{xz} = 0$ using Lemma 2.10. Note that both $c(\omega(C_s)z|m)$ as well as $C_s c(z|m)$ are ι -invariant elements of $\text{Ind}_{H_J}^H \text{Res}_{H_J}^{\Omega_J} M$ because c is $\widehat{k[\Gamma]}$ -linear and $\iota(C_s) = C_s$.

Case 1: $z \in D_{J,s}^+$.

Case 1.1.+1.2: $x \in D_{J,s}^0 \cup x \in D_{J,s}^-$

In both cases $f_{xz} \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$ by definition of μ .

Case 1.3.: $x \in D_{J,s}^+$.

Observe that $p_{sx,z} = 1 \iff sx = z \iff x = sz \iff p_{x,sz} = 1$ so that $p_{sx,z} - p_{x,sz}$ is always an element of $\mathbb{Z}[\Gamma_{>0}] \Omega_J$. Thus

$$\begin{aligned} f_{xz} &= \underbrace{p_{z,sx} - p_{sx,z}}_{\in \mathbb{Z}[\Gamma_{>0}] \Omega_J} + \underbrace{v_s p_{x,z}}_{\in \mathbb{Z}[\Gamma_{>0}]} + \sum_y p_{x,y} \mu_{y,z}^s \\ &\equiv \sum_y p_{x,y} \mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J} \\ &= \sum_{y \in D_{J,s}^+} p_{x,y} \underbrace{\mu_{y,z}^s}_{=0} + \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} p_{x,y} \mu_{y,z}^s + \sum_{\substack{y \in D_{J,s}^- \\ x < y < z}} p_{x,y} \mu_{y,z}^s. \end{aligned}$$

Now since $x \in D_{J,s}^+$ we cannot have $x = y$ in both the second and the third sum so that we can use part (a) of Lemma 4.7 for (x, y) and part (c) of Lemma 4.7 for (y, z) in the second sum as well as part (a) of Lemma 4.7 for (x, y) in the third sum so that we obtain

$$\begin{aligned} f_{xz} &= \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (-v_s^{-1})(p_{x,y} C_s^y - p_{sx,y} + \sum_{x < y' < y} p_{x,y'} \mu_{y',y}^s) (e_{s^y} \mu_{y,z}^s) \\ &\quad + \sum_{y \in D_{J,s}^-} \underbrace{(p_{sx,y} (-v_s))}_{\in \mathbb{Z}[\Gamma_{\geq 0}]} \underbrace{\mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{>0}]} \\ &\equiv \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (-v_s^{-1})(p_{x,y} C_s^y - p_{sx,y} + \sum_{y'} p_{x,y'} \mu_{y',y}^s) e_{s^y} \mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J}. \end{aligned}$$

Now $y < z$ so that we can use part (d) of Lemma 4.7 for (y', y) and obtain

$$\begin{aligned} f_{xz} &= \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (-v_s^{-1}) (p_{x,y} C_{s^y} - p_{sx,y} + \sum_{y'} p_{x,y'} \underbrace{\mu_{y',y}^s}_{=0} e_{s^y} \mu_{y,z}^s) \\ &= \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (-v_s^{-1}) (p_{x,y} C_{s^y} e_{s^y} - p_{sx,y} e_{s^y}) \mu_{y,z}^s \\ &= \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (-v_s^{-1}) (p_{x,y} (-v_s - v_s^{-1}) e_{s^y} - p_{sx,y} e_{s^y}) \mu_{y,z}^s. \end{aligned}$$

Applying part (b) of Lemma 4.7 to (x, y) we find

$$\begin{aligned} f_{xz} &= \sum_{y \in D_{J,s}^0} (-v_s^{-1}) (-v_s p_{sx,y} e_{s^y} (-v_s - v_s^{-1}) - p_{sx,y} e_{s^y}) \mu_{y,z}^s \\ &= \sum_{y \in D_{J,s}^0} (-v_s^{-1}) (v_s^2 p_{sx,y} e_{s^y}) \mu_{y,z}^s \\ &= \sum_{y \in D_{J,s}^0} \underbrace{-p_{sx,y}}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} e_{s^y} \underbrace{v_s \mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} \\ &\equiv 0 \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}. \end{aligned}$$

Case 2: $z \in D_{J,s}^0$.

Case 2.1+2.2: $x \in D_{J,s}^0 \cup x \in D_{J,s}^-$

In both cases $f_{xz} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$ by definition of μ .

Case 2.3 $x \in D_{J,s}^+$.

In this case $sx \neq z$ so that $p_{sx,z} \in \mathbb{Z}[\Gamma_{>0}]\Omega_J$. Therefore:

$$\begin{aligned} f_{xz} &= p_{x,z} C_{s^z} + \sum_{y < z} p_{x,y} \mu_{y,z}^s - p_{sx,z} + v_s p_{x,z} \\ &\equiv p_{x,z} T_{s^z} + \sum_{y < z} p_{x,y} \mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}. \end{aligned}$$

Because $x' := sx$ satisfies $x' > x$ and $x' \in D_{J,s}^-$ we can use the induction hypothesis for (x', z) and (x', y) so that

$$\begin{aligned}
 f_{xz} &= p_{sx',z}T_{sz} + \sum_{y < z} p_{sx',y}\mu_{y,z}^s \\
 &= \left(\underbrace{p_{x',z}C_{sz} + v_s^{-1}p_{x',z}}_{=p_{x',z}T_{sz}^{-1}} + \sum_{y'} p_{x',y'}\mu_{y',z}^s \right) T_{sz} + \sum_{y \in D_{J,s}^+} p_{x,y} \underbrace{\mu_{y,z}^s}_{=0} \\
 &\quad + \sum_{y \in D_{J,s}^0} \left(\underbrace{p_{x',y}C_{sy} + v_s^{-1}p_{x',y}}_{=p_{x',y}T_{sy}^{-1}} + \sum_{y'} p_{x',y'}\mu_{y',y}^s \right) \mu_{y,z}^s \\
 &\quad + \sum_{y \in D_{J,s}^-} (-v_s + v_s^{-1})p_{x',y} + v_s^{-1}p_{x',y} \mu_{y,z}^s \\
 &= \underbrace{p_{x',z}}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} + \sum_{x' \leq y' < z} p_{x',y'}\mu_{y',z}^s T_{sz} \\
 &\quad + \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} \left(p_{x',y}T_{sy}^{-1} + \sum_{x' \leq y' < y} p_{x',y'}\mu_{y',y}^s \right) \mu_{y,z}^s \\
 &\quad + \sum_{y \in D_{J,s}^-} - \underbrace{p_{x',y}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J} \cdot \underbrace{v_s\mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} \\
 &\equiv \sum_{x' \leq y' < z} p_{x',y'}\mu_{y',z}^s (-v_s^{-1}e_{sz} + v_s(1 - e_{sz}) + x_{sz}) \\
 &\quad \times \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} p_{x',y}T_{sy}^{-1}\mu_{y,z}^s + \sum_{\substack{y \in D_{J,s}^0, y' \in D_J \\ x' \leq y' < y < z}} p_{x',y'}\mu_{y',y}^s\mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}]\Omega_J}.
 \end{aligned}$$

Because $x < x'$ we can use part (d) of Lemma 4.7 for (y', z) in the first sum, part (c) of Lemma 4.7 for (y, z) in the second and the third sum as well as part (d) of Lemma 4.7 for (y', y) in the third sum to obtain

$$\begin{aligned}
 f_{xz} &= \sum_{x' \leq y' < z} \underbrace{p_{x',y'}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]\Omega_J} \underbrace{\mu_{y',z}^s v_s}_{\in \mathbb{Z}[\Gamma_{>0}]\Omega_J} (1 - e_{sz}) \\
 &\quad + \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} p_{x',y} \underbrace{T_{sy}^{-1}e_{sy}}_{=-v_s e_{sy}} \mu_{y,z}^s + \sum_{\substack{y \in D_{J,s}^0, y \in D_J \\ x' \leq y' < y < z}} p_{x',y'} \underbrace{\mu_{y',y}^s e_{sy}}_{=0} \mu_{y,z}^s
 \end{aligned}$$

$$\begin{aligned} &\equiv \sum_{\substack{y \in D_{J,s}^0 \\ x' \leq y}} \underbrace{p_{x',y}}_{\in \mathbb{Z}[\Gamma_{\geq 0}]} (-e_{sy}) \underbrace{v_s \mu_{y,z}^s}_{\in \mathbb{Z}[\Gamma_{> 0}]\Omega_J} \pmod{\mathbb{Z}[\Gamma_{> 0}]\Omega_J} \\ &\equiv 0 \pmod{\mathbb{Z}[\Gamma_{> 0}]\Omega_J} \end{aligned}$$

which is what we wanted to prove.

Case 3: $z \in D_{J,s}^-$.

Case 3.1: $x \in D_{J,s}^-$.

In this case $sx \in D_{J,s}^+$ so that $sx \neq z$ and thus $p_{sx,z} \in \mathbb{Z}[\Gamma_{> 0}]\Omega_J$. We infer

$$f_{xz} = p_{sx,z} - v_s^{-1} p_{x,z} + v_s p_{x,z} + v_s^{-1} p_{x,z} = p_{sx,z} + v_s p_{x,z} \in \mathbb{Z}[\Gamma_{> 0}]\Omega_J.$$

Case 3.2: $x \in D_{J,s}^0$.

In this case the equation is equivalent to $(1 - e_{sx})p_{x,z} = 0$ by Proposition 4.1. If $x \not\leq z$ then this is vacuously true because $p_{x,z} = 0$.

Since $z \in D_{J,s}^-$ we can write $z = sz'$ for some $z' \in D_{J,s}^+$ with $z' < z$. We can apply part (a) of Lemma 4.7 to (x, z') and find:

$$\begin{aligned} (1 - e_{sx})p_{x,z} &= (1 - e_{sx})p_{x,sz'} \\ &= \underbrace{(1 - e_{sx})}_{=C_{s^x}} (C_{s^x} p_{x,z'} - \sum_y p_{x,y} \mu_{y,z'}^s) \\ &= - \sum_{y \in D_{J,s}^+} (1 - e_{sx}) p_{x,y} \underbrace{\mu_{y,z'}^s}_{=0} \\ &\quad - \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z'}} (1 - e_{sx}) p_{x,y} \mu_{y,z'}^s - \sum_{\substack{y \in D_{J,s}^- \\ x \leq y < z'}} (1 - e_{sx}) p_{x,y} \mu_{y,z'}^s. \end{aligned}$$

Because $z' < z$ we can apply part (c) of Lemma 4.7 to (y, z') in the second sum and part (a) of Lemma 4.7 to (x, y) (in the equivalent formulation $(1 - e_{sx})p_{x,y} = 0$) in the third sum so that we obtain

$$(1 - e_{sx})p_{x,z} = - \sum_{\substack{y \in D_{J,s}^0 \\ x \leq y < z'}} (1 - e_{sx}) p_{x,y} e_{sy} \mu_{y,z'}^s.$$

Using part (b) of Lemma 4.7 for (x, y) all the summands vanish.

Case 3.3: $x \in D_{J,s}^+$.

In this case $x' := sx$ satisfies $x' > x$ and $x' \in D_{J,s}^-$ so that we find

$$\begin{aligned} f_{xz} &= p_{sx,z} - v_s p_{x,z} + v_s p_{x,z} + v_s^{-1} p_{x,z} \\ &= p_{x',z} + v_s^{-1} p_{sx',z} \end{aligned}$$

which—using the induction hypothesis for (x', z) —equals

$$\begin{aligned} &= p_{x',z} + v_s^{-1}(-v_s p_{x',z}) \\ &= 0. \end{aligned}$$

□

Since we have now proven part (a) of Lemma 4.7 we can use it to prove the other equations:

Proof of part (b) and part (d) of Lemma 4.7. If $x \in D_{J,s}^\pm$ we can multiply the equation in part (a) of Lemma 4.7 with e_{sz} from the right:

$$\begin{aligned} (p_{sx,z} - v_s^{\pm 1} p_{x,z})e_{sz} &= p_{x,z} C_{sz} e_{sz} + \mu_{x,z}^s e_{sz} + \sum_{x < y < z} p_{x,y} \mu_{y,z}^s e_{sz} \\ &= p_{x,z}(-v_s - v_s^{-1})e_{sz} + \mu_{x,z}^s e_{sz} + \sum_{x < y < z} p_{x,y} \mu_{y,z}^s e_{sz}. \end{aligned}$$

Now we can apply part (d) of Lemma 4.7 to (y, z) in the sum and obtain:

$$\begin{aligned} &= p_{x,z}(-v_s - v_s^{-1})e_{sz} + \mu_{x,z}^s e_{sz} \\ \implies (p_{sx,z} + v_s^{\mp 1} p_{x,z})e_{sz} &= \mu_{x,z}^s e_{sz}. \end{aligned}$$

If $x \in D_{J,s}^+$, then $\mu_{x,z}^s = 0$ so that $p_{sx,z} = -v_s^{-1} p_{x,z}$ and $\mu_{x,z}^s e_{sz} = 0$. If $x \in D_{J,s}^-$, then the left hand side is contained in $\mathbb{Z}[\Gamma_{>0}]\Omega_J$ because $sx, x \neq z$ while the right hand side is $\bar{}$ -invariant so that both sides equal to zero.

Now consider the case $x \in D_{J,s}^0$. Then the following holds by part (a) of Lemma 4.7 for (x, z) :

$$\begin{aligned} (v_s + v_s^{-1})p_{x,z}e_{sz} &= -p_{x,z} C_{sz} e_{sz} \\ &= -C_{sx} p_{x,z} e_{sz} + \mu_{x,z} e_{sz} + \sum_{x < y < z} p_{x,y} \mu_{y,z}^s e_{sz}. \end{aligned}$$

In the sum we can use part (d) of Lemma 4.7 for (y, z) and find:

$$\begin{aligned} (v_s + v_s^{-1})p_{x,z}e_{sz} &= -C_{sx} p_{x,z} e_{sz} + \mu_{x,z} e_{sz} \\ \implies \mu_{x,z}^s e_{sz} &= (C_{sx} + v_s + v_s^{-1})p_{x,z} e_{sz} \end{aligned}$$

$$\begin{aligned} &= ((v_s + v_s^{-1})(1 - e_s) + x_s)p_{x,z}e_{sz} \\ &= ((v_s + v_s^{-1}) + x_s)(1 - e_s)p_{x,z}e_{sz}. \end{aligned}$$

Now consider the exponents that occur at both sides of the equation: On the left hand side all exponents are $< L(s)$. On the right hand side this means $v_s(1 - e_s)p_{x,z}e_{sz} = 0$ because $p_{x,z} \in \mathbb{Z}[\Gamma_{\geq 0}] \Omega_J$. Therefore $(1 - e_s)p_{x,z}e_{sz} = 0$ and $\mu_{y,z}^s e_{sz} = 0$. □

Proof of lemma part (c) of Lemma 4.7. Without loss of generality, we assume $x < z$ and $z \in D_{J,s}^+ \cup D_{J,s}^0$ since otherwise $\mu_{x,z}^s = 0$. Because $\overline{\mu_{x,z}^s} = \mu_{x,z}^s$ we only need to prove $(1 - e_{s^x})\mu_{x,z}^s \equiv 0 \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J}$.

Lemma part (a) of Lemma 4.7 for (x, z) implies

$$\begin{aligned} 0 &= (1 - e_{s^x})C_{s^x}p_{x,z} \\ &= (1 - e_{s^x})p_{x,x}\mu_{x,z}^s + \left\{ \begin{array}{ll} (1 - e_{s^x})p_{x,sz} & z \in D_{J,s}^+ \\ (1 - e_{s^x})p_{x,z}C_{s^z} & z \in D_{J,s}^0 \end{array} \right\} \\ &\quad + \sum_{x < y < z} (1 - e_{s^x})p_{x,y}\mu_{y,z}^s. \end{aligned}$$

Because $x \in D_{J,s}^0$, $x = sz$ cannot hold in the first case so that $p_{x,sz} \in \mathbb{Z}[\Gamma_{>0}] \Omega_J$. In the second case we use $C_{s^z} = e_{sz}C_{s^z}$ and the now proven part (b) of Lemma 4.7. We obtain

$$\begin{aligned} 0 &\equiv (1 - e_{s^x})\mu_{x,z}^s + \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} (1 - e_{s^x})p_{x,y}\mu_{y,z}^s \\ &\quad + \sum_{\substack{y \in D_{J,s}^- \\ x < y < z}} (1 - e_{s^x})p_{x,y}\mu_{y,z}^s \pmod{\mathbb{Z}[\Gamma_{>0}] \Omega_J}. \end{aligned}$$

Note that $(1 - e_{s^x})p_{x,y} = 0$ for $y \in D_{J,s}^-$ by part (a) of Lemma 4.7 applied to (x, y) and $\mu_{y,z}^s = e_{sy}\mu_{y,z}^s$ for $y \in D_{J,s}^0$ by part (c) of Lemma 4.7. Using part (b) of Lemma 4.7 we find

$$\begin{aligned} 0 &\equiv (1 - e_{s^x})\mu_{x,z}^s + \sum_{\substack{y \in D_{J,s}^0 \\ x < y < z}} \underbrace{(1 - e_{s^x})p_{x,y}e_{sy}}_{=0} \mu_{y,z}^s \\ &= (1 - e_{s^x})\mu_{x,z}^s. \end{aligned} \quad \square$$

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