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INHOMOGENEOUS INCOMPRESSIBLE VISCOUS FLOWS WITH SLOWLY VARYING INITIAL DATA

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Abstract The purpose of this paper is to provide a large class of initial data which generates global smooth solution of the 3D inhomogeneous incompressible Navier–Stokes system in the whole space $\mathbb{R}^3$. This class of data is based on functions which vary slowly in one direction. The idea is that 2D inhomogeneous Navier–Stokes system with large data is globally well-posed and we construct the 3D approximate solutions by the 2D solutions with a parameter. One of the key point of this study is the investigation of the time decay properties of the solutions to the 2D inhomogeneous Navier–Stokes system. We obtained the same optimal decay estimates as the solutions of 2D homogeneous Navier–Stokes system.

Keywords: inhomogeneous incompressible Navier–Stokes equations; slow variable; decay estimate; anisotropic Littlewood–Paley theory

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1. Introduction

In this paper, we investigate the global well-posedness of three-dimensional (3D) incompressible inhomogeneous Navier–Stokes system with large initial data slowly varying in one space variable. In general, inhomogeneous Navier–Stokes system in $\mathbb{R}^d$ reads

$$\text{(INSdD)}\quad \begin{cases} \partial_t\rho + u\cdot\nabla\rho = 0,\\ \rho\partial_t u + \rho u\cdot\nabla u - \Delta u + \nabla\Pi = 0,\\ \operatorname{div} u = 0,\\ (\rho, u)|_{t=0} = (\rho_0, u_0). \end{cases}$$

Here the unknown ρ is a function from $[0,T]\times\mathbb{R}^d$ into the interval $]0,\infty[$ which represents the density of fluid at time t and point x,[1] the unknown $u = (u^1,\dots,u^d)$

[1]We do want to avoid vacuum.

is a time dependent vector field on $\mathbb{R}^d$ which represents the velocity of fluid locating at position x and time t and Π is a function from $[0, T] \times \mathbb{R}^d$ to $\mathbb{R}$ which represents the pressure at point x and time t, which ensures the incompressibility of the fluid. The choice of $\mathbb{R}^d$ as a domain is a real simplification because as we shall see later on the pressure is uniquely determined by the divergence free condition on the vector field u (the case of periodic boundary condition i.e., the flat torus $\mathbb{T}^d$ as a domain also works).

Let us notice that in the case when $\rho_0 \equiv 1$, the system (INSdD) turns out to be the homogeneous incompressible Navier–Stokes system. We have to keep in mind that the system of (INSdD) is more complex than this one.

This system (INSdD) can be used as a model to describe a fluid that is incompressible but has non-constant density. Basic examples are mixture of incompressible and non-reactant flows, flows with complex structure (e.g. blood flow or model of rivers), fluids containing a melted substance, etc.

First of all, this equation satisfies some *a priori* estimates. Let us first study the *a priori* estimate on the density. It is classical to consider the density ρ as a perturbation of the homogeneous density arbitrarily chosen to be equal to 1. Let us introduce the notation

$$\varrho \stackrel{\text{def}}{=} \rho - 1$$

which will be used all along this text.

This system has three major basic features. First of all, the incompressibility expressed by the fact that the vector field u is divergence free gives

$$\forall p \in [1, \infty], \quad \|\varrho(t)\|_{L^p} = \|\varrho_0\|_{L^p} \quad \text{and} \quad \|\rho(t)\|_{L^\infty} = \|\rho_0\|_{L^\infty}. \tag{1.1}$$

Moreover, the second equation of (INSdD), called the momentum equation, implies a control of the total kinetic energy which is formally expressed by

$$\frac{1}{2}\int_{\mathbb{R}^d} \rho(t,x)|u(t,x)|^2\,dx + \int_0^t \|\nabla u(t')\|_{L^2}^2\,dt' = \frac{1}{2}\int_{\mathbb{R}^d} \rho_0(x)|u_0(x)|^2\,dx. \tag{1.2}$$

This third basic feature is the scaling invariance. Indeed, if (ρ, u, Π) is a solution of (INSdD) on $[0, T] \times \mathbb{R}^d$, then $(\rho, u, \Pi)_\lambda$ defined by

$$(\rho, u, \Pi)_\lambda(t,x) \stackrel{\text{def}}{=} \big(\rho(\lambda^2 t, \lambda x), \lambda u(\lambda^2 t, \lambda x), \lambda^2 \Pi(\lambda^2 t, \lambda x)\big) \tag{1.3}$$

is also a solution of (INSdD) on $[0, T/\lambda^2] \times \mathbb{R}^d$. This leads to the notion of critical regularity.

Based on the energy estimate (1.2), Simon (see also Kazhikov [22]) constructed in [32] global weak solutions of (INSdD) with finite energy (see the book by Lions [24] for the variable viscosity case).

In the case of smooth data with no vacuum, Ladyvzenskaja and Solonnikov first addressed in [23] the question of unique solvability of (INSdD). More precisely, they considered the system (INSdD) in a bounded domain Ω with homogeneous Dirichlet boundary condition for u. Under the assumptions that $u_0 \in W^{2-\frac{2}{p},p}(\Omega)$ $(p > d)$ is divergence free and vanishes on $\partial\Omega$ and that $\rho_0 \in C^1(\Omega)$ is bounded away from zero, then they proved in [23] that

- the system (INS2D) is globally well-posed;
- the system (INS3D) is Local well-posed. If in addition u_0 is small in $W^{2-\frac{2}{p},p}(\Omega)$, then global well-posedness holds true.

More recently, Paicu, Zhang and the second author proved in [28] the following well-posedness result for (INS3D) with small data.

Theorem 1.1. *Let us consider an initial data (ρ_0, u_0) in $L^\infty(\mathbb{R}^3)\times H^1(\mathbb{R}^3)$. Let us assume that for some positive constant C_0,*

$$C_0^{-1}\leqslant \rho_0(x)\leqslant C_0.$$

Then there exists a constant $\varepsilon_0>0$ depending only on C_0 such that if $\|u_0\|_{L^2}\|\nabla u_0\|_{L^2}\leqslant \varepsilon_0$, then the system (INS3D) has a unique global solution (ρ, u).

Let us notice that the smallness condition in Theorem 1.1 is scaling invariant. Moreover, the fact that in dimension two, the system (INS2D) is globally well-posed is related to the fact that in dimension two, the quantity

$$\frac{1}{2}\int_{\mathbb{R}^2}\rho(t,x)|u(t,x)|^2\,dx+\int_0^\infty \|\nabla u(t')\|_{L^2}^2\,dt'$$

is scaling invariant under the transformation (1.3).

In this text, we shall consider slowly varying initial data i.e., a family of initial data of the form

$$(\rho_{0,\varepsilon,\eta}, u_{0,\varepsilon,\eta})\stackrel{\mathrm{def}}{=}\big(1+\eta[\varsigma_0]_\varepsilon, ([v_0^{\mathrm{h}}]_\varepsilon, 0)\big), \tag{1.4}$$

where ε and η are two positive real parameters, ς_0 is a smooth function, and v_0^{h} is a smooth divergence free two-dimensional (2D) vector field which depends on a real parameter z. All along this text, we use the notation, for a function f on $\mathbb{R}^3$,

$$[f]_\varepsilon(x_{\mathrm{h}}, x_3)\stackrel{\mathrm{def}}{=} f(x_{\mathrm{h}}, \varepsilon x_3).$$

Here we are interested in the size of the initial data. We do not intent to solve (INS3D) for rough initial data, instead we want to exhibit a large class of initial data which are 'large' in the sense that they do not satisfy any previous smallness hypothesis which ensures global existence of regular solutions. The main theorem of this text is the following.

Theorem 1.2. *Let us consider initial profiles ς_0 and v_0^{h} which are functions and vector fields in $\mathcal{S}(\mathbb{R}^3)$ such that $\mathrm{div}_{\mathrm{h}} v_0^{\mathrm{h}}=0$ and such that for any z in $\mathbb{R}$ and any j in $\{1,2\}$*

$$\int_{\mathbb{R}^2}\varsigma_0(x_{\mathrm{h}}, z)v_0^{\mathrm{h}}(x_{\mathrm{h}}, z)\,dx_{\mathrm{h}}=0 \quad \textit{and} \quad \int_{\mathbb{R}^2}x_j\varsigma_0(x_{\mathrm{h}}, z)v_0^{\mathrm{h}}(x_{\mathrm{h}}, z)\,dx_{\mathrm{h}}=0. \tag{1.5}$$

Then there exists two positive constants η_0 and ε_0 which depend on norms of ς_0 and v_0^{h} such that if $\eta\leqslant\eta_0$ and $\varepsilon\leqslant\varepsilon_0$, the initial date defined by (1.4) generate a unique global smooth solution of (INS3D).

Let us make some comments about this theorem. Slowly varying data has been introduced by Gallagher and the first author in [7] in the case of homogeneous incompressible Navier–Stokes equations, i.e., the case when $\rho \equiv 1$. The above theorem is proved in [7] in this case. The motivation of this work was to provide a large class of examples of initial data which are large (in the homogeneous incompressible Navier–Stokes, it means essentially that the $\dot{B}^{-1}_{\infty,\infty}$ norm of the initial data defined by

$$\|a\|_{\dot{B}^{-1}_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{t>0} t^{\frac{1}{2}} \|e^{t\Delta} a\|_{L^\infty} \tag{1.6}$$

is large), which is the case here, because $\|[v_0^{\mathrm{h}}]_\varepsilon\|_{\dot{B}^{-1}_{\infty,\infty}}$ has the same size as $\|v_0^{\mathrm{h}}\|_{\dot{B}^{-1}_{\infty,\infty}}$. The idea of the proof in [7] was to use that homogeneous 2D incompressible Navier–Stokes equation with initial data $v_0^{\mathrm{h}}(\cdot, z)$ is globally well-posed and then to prove the real solution was close (in some appropriated way) to $[v^{\mathrm{h}}]_\varepsilon$.

Slowly varying turns out to be a useful tool to study the set $\mathcal{G}$ of initial data in the space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ which generates unique global smooth solutions to 3D homogeneous Navier–Stokes system. Since the work in [21] by Gallagher, Iftimie and Planchon, it is known that this set is open and connected. In [9], Gallagher and the two authors used slowing varying initial data to prove that through each point of $\mathcal{G}$ passes an uncountable number of arbitrarily long segments which are included in $\mathcal{G}$.

The study of initial data in the homogeneous case as presented above can be qualified as 'well-prepared' using the language of singular perturbation theory. The 'ill-prepared' case has been studied by Gallagher, Paicu and the first author in [8]; they proved that the initial data

$$\left([w_0^{\mathrm{h}}]_\varepsilon, \frac{1}{\varepsilon}[w_0^3]_\varepsilon\right)$$

generates a unique global smooth solution of the homogeneous incompressible Navier–Stokes equation when the profile w is a divergence free vector field and which is small in a Banach space of analytic function with respect to the vertical variable.

Let us see why the result of Theorem 1.2 is in some sense a 'ill-prepared' result. In order to explain this, we recall the precise definition of the Besov norms from [5] for instance.

Definition 1.1. Let us consider a smooth function φ on $\mathbb{R}$, the support of which is included in $[3/4, 8/3]$ such that

$$\forall \tau > 0, \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{and} \quad \chi(\tau) \stackrel{\text{def}}{=} 1 - \sum_{j\geqslant 0} \varphi(2^{-j}\tau) \in \mathcal{D}([0, 4/3]).$$

Let us define

$$\Delta_j a = \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), \quad \text{and} \quad S_j a = \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}).$$

Let (p, r) be in $[1, +\infty]^2$ and s in $\mathbb{R}$. We define the Besov norm by

$$\|a\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \|(2^{js}\|\Delta_j a\|_{L^p})_j\|_{\ell^r(\mathbb{Z})}.$$

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We remark that in the particular case when $p = r = 2$, the Besov spaces $\dot{B}^s_{p,r}$ coincides with the classical homogeneous Sobolev spaces $\dot{H}^s$.

All the well-posedness results of (INS3D) for small data requires that for $p \in]1, 6[$

$$\|[v_0]_\varepsilon\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)} \ll 1 \quad \text{with or without } \eta\|[\varsigma_0]_\varepsilon\|_{\dot{B}^{\frac{3}{p}}_{p,1}} \ll 1.$$

One may check the references [1–4, 15–18, 20, 27–29] for details.

Let us also mention that Paicu and the two authors proved this theorem in [10] in the case when $\eta \leqslant \varepsilon^\sigma$ with $\sigma > \frac{1}{4}$.

Note that a divergence free vector field with the components of which are integrable is mean free. Thus, Hypothesis (1.5) implies in particular that

$$\forall z \in \mathbb{R}, \quad \int_{\mathbb{R}^2} (1 + \eta\varsigma_0(x_\mathrm{h}, z)) v_0^\mathrm{h}(x_\mathrm{h}, z)\, dx_\mathrm{h} = 0. \tag{1.7}$$

Let us notice that the hypothesis about the momentum of $\varsigma_0 v_0^\mathrm{h}$ ensures in particular that $\varsigma_0 v_0^\mathrm{h}$ belongs to the anisotropic space $\mathcal{B}_2^{-1,\frac{3}{2}}$ (see forthcoming Definition 2.1). Following observations of the first author and Gallagher in [7], it is easy to prove that

$$\|[\varsigma_0]_\varepsilon\|_{\dot{B}^{\frac{3}{p}}_{p,1}(\mathbb{R}^3)} \gtrsim \varepsilon^{-\frac{1}{p}} \quad \text{and} \quad \|[v_0]_\varepsilon\|_{\dot{B}^{-1+\frac{3}{p}}_{p,1}(\mathbb{R}^3)} \gtrsim \varepsilon^{-\frac{1}{p}}. \tag{1.8}$$

Therefore, the result of Theorem 1.2 is of 'ill-prepared' type because of inequality (1.8), where the norm coincides with the one given by (1.6) in the case when $(s, p, r) = (-1, \infty, \infty)$. Yet we do not require any analytic assumption on the initial data.

Let us complete this section by the notations of the paper:

For $a \lesssim b$, we mean that there is a uniform constant C, which may be different on different lines, such that $a \leqslant Cb$. We denote by $(a|b)_{L^2}$ the $L^2(\mathbb{R}^d)$ inner product of a and b. For X, X_1 Banach spaces, T a positive real number and q in $[1, +\infty]$, we denote the norm $\|\cdot\|_{X\cap X_1} \stackrel{\text{def}}{=} \|\cdot\|_X + \|\cdot\|_{X_1}$ and $L^q_T(X)$ for the set of measurable functions on $[0, T]$ with values in X, such that $t \longmapsto \|f(t)\|_X$ belongs to $L^q([0, T])$. We denote

$$L^p_T(L^q_\mathrm{h}(L^r_\mathrm{v})) = L^p([0, T]; L^q(\mathbb{R}_{x_\mathrm{h}}; L^r(\mathbb{R}_z)))$$

with $x_\mathrm{h} = (x_1, x_2)$, and $\nabla_\mathrm{h} = (\partial_{x_1}, \partial_{x_2})$, $\Delta_\mathrm{h} = \partial^2_{x_1} + \partial^2_{x_2}$. Finally Δ_ε stands for $\Delta_\mathrm{h} + \varepsilon^2\partial_z^2$, ∇_ε for $(\nabla_\mathrm{h}, \varepsilon\partial_z)$, and $\|f\|_{X_\mathrm{h}}$ for the X norm of f in the horizontal variable x_h.

2. Structure and main ideas of the proof

Because we shall seemingly consider the density functions as perturbations of the reference density 1, it is natural to set

$$a \stackrel{\text{def}}{=} \frac{1}{\rho} - 1 = -\frac{\varrho}{1+\varrho}$$

so that System (INSdD) translates into

$$\text{(INSdD)} \quad \begin{cases} \partial_t a + u\cdot\nabla a = 0, \\ \partial_t u + u\cdot\nabla u - (1+a)(\Delta u - \nabla\Pi) = 0, \\ \operatorname{div} u = 0, \\ (a, u)|_{t=0} = (a_0, u_0). \end{cases}$$

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Even if our main motivation comes from dimension 3, we shall consider this system in both $\mathbb{R}^2$ and $\mathbb{R}^3$. In dimension 3, we use systematically the notation $x = (x_{\rm h}, x_3)$ and $x = (x_{\rm h}, z)$ in the case when z represents εx_3.

For proving Theorem 1.2, we follow the idea of [7], namely, using the fact that the 2D incompressible inhomogeneous Navier–Stokes system is globally well-posed. We shall construct the approximate solutions of (INS3D) with initial data slowly varying in one space variable in the following way. Let us denote by $(a^{\rm h}, v^{\rm h}, \Pi^{\rm h})$ the (global) solution of INS2D with initial data $(a_0(\cdot, z), v_0^{\rm h}(\cdot, z))$, that is

$$\begin{cases} \partial_t a^{\rm h} + v^{\rm h} \cdot \nabla_{\rm h} a^{\rm h} = 0, \\ \partial_t v^{\rm h} + v^{\rm h} \cdot \nabla_{\rm h} v^{\rm h} - (1 + a^{\rm h})(\Delta_{\rm h} v^{\rm h} - \nabla_{\rm h} \Pi^{\rm h}) = 0, \\ \operatorname{div}_{\rm h} v^{\rm h} = 0, \\ a^{\rm h}|_{t=0} = a_0(x_{\rm h}, z), \quad v^{\rm h}|_{t=0} = v_0^{\rm h}(x_{\rm h}, z), \end{cases} \tag{2.1}$$

which can also be equivalently written as

$$\begin{cases} \partial_t \rho^{\rm h} + v^{\rm h} \cdot \nabla_{\rm h} \rho^{\rm h} = 0, \\ \partial_t(\rho^{\rm h} v^{\rm h}) + \operatorname{div}_{\rm h}(\rho^{\rm h} v^{\rm h} \otimes v^{\rm h}) - \Delta_{\rm h} v^{\rm h} + \nabla_{\rm h} \Pi^{\rm h} = 0, \\ \operatorname{div}_{\rm h} v^{\rm h} = 0, \\ \rho^{\rm h}|_{t=0} = 1 + \eta\varsigma_0(x_{\rm h}, z), \quad v^{\rm h}|_{t=0} = v_0^{\rm h}(x_{\rm h}, z), \end{cases} \tag{2.2}$$

where $\rho^{\rm h} \stackrel{\rm def}{=} \frac{1}{1+a^{\rm h}}$ and $a_0 = -\frac{\varsigma_0}{1+\eta\varsigma_0}\eta$. As in [7], we consider $([a^{\rm h}]_\varepsilon, [v^{\rm h}]_\varepsilon, [\Pi^{\rm h}]_\varepsilon)$ as the first order approximation of the solution to (INS3D) and let us write the solution $(a_\varepsilon, u_\varepsilon, \widetilde{\Pi}_\varepsilon)$ as

$$(a_\varepsilon, u_\varepsilon, \widetilde{\Pi}_\varepsilon) = ([a^{\rm h}]_\varepsilon, ([v^{\rm h}]_\varepsilon, 0), [\Pi^{\rm h}]_\varepsilon) + \varepsilon(b_\varepsilon, \overline{R}_\varepsilon, \overline{\Pi}_\varepsilon). \tag{2.3}$$

It is easy to observe that

$$\partial_t \overline{R}_\varepsilon + [v^{\rm h}]_\varepsilon \cdot \nabla_{\rm h} \overline{R}_\varepsilon + \overline{R}_\varepsilon \cdot \nabla([v^{\rm h}]_\varepsilon, 0) + \varepsilon \overline{R}_\varepsilon \cdot \nabla \overline{R}_\varepsilon - (1 + a_\varepsilon)\big(\Delta \overline{R}_\varepsilon - \nabla \overline{\Pi}_\varepsilon\big) = -\overline{E}_\varepsilon,$$

where the error term $\overline{E}_\varepsilon$ is given by

$$\begin{aligned} \overline{E}_\varepsilon \stackrel{\rm def}{=} & \frac{1}{\varepsilon}([\partial_t v^{\rm h} + v^{\rm h} \cdot \nabla_{\rm h} v^{\rm h} - (1 + a^{\rm h})(\Delta_{\rm h} v^{\rm h} - \nabla_{\rm h} \Pi^{\rm h})]_\varepsilon, 0) \\ & - \varepsilon(1 + a_\varepsilon)([\partial_z^2 v^{\rm h}]_\varepsilon, 0) - (1 + a_\varepsilon)(0, [\partial_z \Pi^{\rm h}]_\varepsilon) - b_\varepsilon\big([\Delta_{\rm h} v^{\rm h} - \nabla_{\rm h} \Pi^{\rm h}]_\varepsilon, 0\big). \end{aligned} \tag{2.4}$$

At this stage, we need to define precisely the norms we shall use to measure the size of all the terms above. As already commented in the introduction, this point is crucial. Let us define the anisotropic Besov norms.

Definition 2.1. Let us consider two functions φ and χ given by Definition 1.1 and let us define the operators of localization in horizontal and vertical frequencies by

$$\begin{aligned} \Delta_k^{\rm h} a = \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_{\rm h}|)\widehat{a}), \quad \Delta_\ell^{\rm v} a = \mathcal{F}^{-1}(\varphi(2^{-\ell}|\zeta|)\widehat{a}), \\ S_k^{\rm h} a = \mathcal{F}^{-1}(\chi(2^{-k}|\xi_{\rm h}|)\widehat{a}), \quad S_\ell^{\rm v} a = \mathcal{F}^{-1}(\chi(2^{-\ell}|\zeta|)\widehat{a}). \end{aligned}$$

Now let us define the norm we are going to use in this text. For p in $[1,\infty]$, and (s,s') in $\mathbb{R}^2$, we define

$$\|a\|_{\mathcal{B}_p^{s,s'}} \stackrel{\text{def}}{=} \sum_{(k,\ell)\in\mathbb{Z}^2} 2^{ks+\ell s'} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} a\|_{L^p}.$$

We shall also use the following norm on force f which involves the action of the heat flow. Let p be in $]3,4[$, and T a positive time, we define

$$\|f\|_{\mathcal{F}_p(T)} \stackrel{\text{def}}{=} \left\| \int_0^t e^{(t-t')\Delta} f(t')\,dt' \right\|_{X(T)} \quad \text{with}$$

$$\|f\|_{X(T)} \stackrel{\text{def}}{=} \|f\|_{L^4_T(\mathcal{B}_p^{-\frac12+\frac2p,\frac1p})} + \|f\|_{L^2_T(\mathcal{B}_p^{\frac2p,\frac1p})} + \|\partial_3 f\|_{L^{\frac43}_T(\mathcal{B}_p^{-\frac12+\frac2p,\frac1p})} + \|\nabla f\|_{L^1_T(\mathcal{B}_p^{\frac2p,\frac1p})} + \|\partial_3^2 f\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac3p,-\delta})} \quad \text{for } \delta\in\,]0,1-3/p[.$$

Let us make some comments about this definition. We first point out that the norms of $\mathcal{B}_p^{s,s'}$ are homogeneous with respect to the vertical variable. More precisely, we have

$$\|[a]_\varepsilon\|_{\mathcal{B}_p^{s,s'}} \sim \varepsilon^{s'-\frac1p}\|a\|_{\mathcal{B}_p^{s,s'}}. \tag{2.5}$$

In particular, the norm $\|\cdot\|_{\mathcal{B}_p^{s,\frac1p}}$ is invariant under the vertical dilation.

We now investigate the relation between L^1 norm in time with value in some anisotropic Besov spaces and the norm $\mathcal{F}_p(T)$. For any p in $[1,\infty]$, and for any (α,β) in $\mathbb{R}^2$ such that

$$\alpha+\beta = -1+\frac3p, \quad \alpha \leqslant -1+\delta+\frac3p \quad\text{and}\quad \beta\leqslant\frac1p,$$

then we have

$$\|f\|_{\mathcal{F}_p(T)} \leqslant C_{\alpha,\beta}\|f\|_{L^1([0,T];\mathcal{B}_p^{\alpha,\beta})}. \tag{2.6}$$

We postpone its proof in the Appendix A.

We also use frequently some law of product in particular (see [10, Lemma 2.3])

$$\|ab\|_{\mathcal{B}_p^{s_1+s_1'-\frac2p,s_2+s_2'-\frac1p}} \lesssim \|a\|_{\mathcal{B}_p^{s_1,s_2}}\|b\|_{\mathcal{B}_p^{s_1',s_2'}} \tag{2.7}$$

where the two sums s_1+s_1' and s_2+s_2' are positive and s_1 and s_1' (respectively s_2 and s_2') are less than or equal to $2/p$ (respectively $1/p$).

Now let us analyze the constraints we have for the choice of norms for the different terms in the external force given by (2.4). For those which are purely of the form $[f]_\varepsilon$, there is in fact no choice. Indeed, since no positive power of ε appears, the choice of norm to the space $\mathcal{B}_p^{\sigma,\frac1p}$ is mandatory. This space must be L^1 in time because we want R_ε to be in $L^1_t(Lip)$ due to the control of the transport equation. Then the parabolic scaling determines the index σ of the horizontal regularity. The space must be

$$L^1_t(\mathcal{B}_p^{-1+\frac2p,\frac1p}).$$

Let us see whether the term $[\partial_z\Pi^{\mathrm{h}}]_\varepsilon$ which appears in (2.4) belongs to this space or not. Let us compute this horizontal pressure. Applying the horizontal divergence to the

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momentum equation of (2.2), we write that

$$-\Delta_{\mathrm{h}}\Pi^{\mathrm{h}} = \mathrm{div}_{\mathrm{h}}(\partial_t(\rho^{\mathrm{h}} v^{\mathrm{h}})) + \mathrm{div}_{\mathrm{h}}\,\mathrm{div}_{\mathrm{h}}(\rho^{\mathrm{h}} v^{\mathrm{h}} \otimes v^{\mathrm{h}}).$$

Using the fact that v^{h} is divergence free, we infer that for $\varrho^{\mathrm{h}} \stackrel{\mathrm{def}}{=} \rho^{\mathrm{h}} - 1$,

$$\begin{gathered}\Pi^{\mathrm{h}} = \Pi^{\mathrm{h}}_L + \Pi^{\mathrm{h}}_Q \quad \text{with} \\ \Pi^{\mathrm{h}}_L \stackrel{\mathrm{def}}{=} -\Delta_{\mathrm{h}}^{-1}\,\mathrm{div}_{\mathrm{h}}(\partial_t(\varrho^{\mathrm{h}} v^{\mathrm{h}})) \quad \text{and} \quad \Pi^{\mathrm{h}}_Q \stackrel{\mathrm{def}}{=} -\Delta_{\mathrm{h}}^{-1}\,\mathrm{div}_{\mathrm{h}}\,\mathrm{div}_{\mathrm{h}}(\rho^{\mathrm{h}} v^{\mathrm{h}} \otimes v^{\mathrm{h}}).\end{gathered} \tag{2.8}$$

It will be possible to prove that the term $[\partial_z \Pi^{\mathrm{h}}_Q]_\varepsilon$ belongs to $L^1(\mathbb{R}^+; \mathcal{B}_2^{0,\frac{1}{2}})$; this will be a consequence of Theorem 4.1 concerning the large time decay estimates of v^{h}. On the other hand, it is not possible to prove that $\partial_z \Pi_L$ belongs to the Besov space $\dot{B}^0_{2,1}$ horizontally. Indeed, it is equivalent to the fact that a homogeneous Fourier multiplier of order -1 applied to a product belongs to $\dot{B}^0_{2,1}$ in the horizontal variable. The lowest possible regularity of a product is L^1. But the space L^1 is included in $\dot{B}^{-1}_{2,\infty}$ in dimension two and even not in the homogeneous Sobolev space $\dot{H}^{-\frac{1}{2}}$. In order to bypass this difficulty, we introduce a correction term. In order to define it, let us consider the vector field $w_\varepsilon(t, x_{\mathrm{h}}, z)$ the solution of

$$\begin{cases} \partial_t w^{\mathrm{h}}_\varepsilon - \Delta_\varepsilon w^{\mathrm{h}}_\varepsilon = -\nabla_{\mathrm{h}} \Pi^1_\varepsilon, \\ \partial_t w^3_\varepsilon - \Delta_\varepsilon w^3_\varepsilon = -\varepsilon^2 \partial_z \Pi^1_\varepsilon + \partial_z \Pi^{\mathrm{h}}_L, \\ \mathrm{div} w_\varepsilon = 0 \quad \text{and} \quad w_{\varepsilon|t=0} = 0. \end{cases} \tag{2.9}$$

Let us introduce the following Ansatz. We search the solution $(a_\varepsilon, u_\varepsilon, \widetilde{\Pi}_\varepsilon)$ of (INS3D) of the form

$$\begin{gathered}(a_\varepsilon, u_\varepsilon, \widetilde{\Pi}_\varepsilon) = ([a^{\mathrm{h}}]_\varepsilon, u_{\varepsilon,\mathrm{app}}, \Pi_{\varepsilon,\mathrm{app}}) + \varepsilon(b_\varepsilon, R_\varepsilon, \Pi_\varepsilon) \quad \text{with} \\ (u_{\varepsilon,\mathrm{app}}, \Pi_{\varepsilon,\mathrm{app}}) \stackrel{\mathrm{def}}{=} \big(([v^{\mathrm{h}}]_\varepsilon, 0), [\Pi^{\mathrm{h}}]_\varepsilon) + \varepsilon((\varepsilon[w^{\mathrm{h}}_\varepsilon]_\varepsilon, [w^3_\varepsilon]_\varepsilon), \varepsilon[\Pi^1_\varepsilon]_\varepsilon).\end{gathered} \tag{2.10}$$

Then $(R_\varepsilon, \nabla\Pi_\varepsilon)$ solves the system

$$(\mathrm{INS3D})_\varepsilon \begin{cases} \partial_t R_\varepsilon + u_{\varepsilon,\mathrm{app}} \cdot \nabla R_\varepsilon + R_\varepsilon \cdot \nabla u_{\varepsilon,\mathrm{app}} + \varepsilon R_\varepsilon \cdot \nabla R_\varepsilon - (1 + a_\varepsilon)(\Delta R_\varepsilon - \nabla \Pi_\varepsilon) = -E_\varepsilon, \\ \mathrm{div} R_\varepsilon = 0 \quad \text{and} \quad R_\varepsilon|_{t=0} = 0. \end{cases}$$

where the error term E_ε is given by

$$E_\varepsilon \stackrel{\mathrm{def}}{=} \frac{1}{\varepsilon}(\partial_t u_{\varepsilon,\mathrm{app}} + u_{\varepsilon,\mathrm{app}} \cdot \nabla u_{\varepsilon,\mathrm{app}} - (1 + a_\varepsilon)(\Delta u_{\varepsilon,\mathrm{app}} - \nabla \Pi_{\varepsilon,\mathrm{app}})). \tag{2.11}$$

Of course the key point then is the estimate of the error term. Let us analyze it. First we write that

$$\begin{aligned} E_\varepsilon = {} & \frac{1}{\varepsilon}([\partial_t v^{\mathrm{h}} + v^{\mathrm{h}} \cdot \nabla^{\mathrm{h}} v^{\mathrm{h}} - (1 + a^{\mathrm{h}})(\Delta_{\mathrm{h}} v^{\mathrm{h}} - \nabla_{\mathrm{h}} \Pi^{\mathrm{h}})]_\varepsilon, 0) \\ & + [v^{\mathrm{h}} \cdot \nabla_{\mathrm{h}}(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon) + \varepsilon w_\varepsilon \cdot \nabla(v^{\mathrm{h}}, 0) + \varepsilon^2 w_\varepsilon \cdot \nabla(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)]_\varepsilon \\ & - \varepsilon(1 + a_\varepsilon)([\partial_z^2 v^{\mathrm{h}}]_\varepsilon, 0) - b_\varepsilon([\Delta_{\mathrm{h}} v^{\mathrm{h}} - \nabla_{\mathrm{h}} \Pi^{\mathrm{h}}]_\varepsilon, 0) \\ & + [\partial_t(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon) - \Delta_\varepsilon(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon) - \varepsilon \nabla_\varepsilon \Pi^1_\varepsilon - (0, \partial_z \Pi^{\mathrm{h}}_L)]_\varepsilon \\ & - a_\varepsilon[\Delta_\varepsilon(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon) - (0, \partial_z \Pi^{\mathrm{h}}) - \varepsilon \nabla_\varepsilon \Pi^1_\varepsilon]_\varepsilon - (0, [\partial_z \Pi^{\mathrm{h}}_Q]_\varepsilon). \end{aligned}$$

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By definition of $(a^{\rm h}, v^{\rm h}, \Pi^{\rm h})$ and $(w_\varepsilon, \Pi^1_\varepsilon)$, we can write

$$
\begin{aligned}
&E_\varepsilon = \sum_{\ell=1}^{4} E_\varepsilon^\ell \quad \text{with}\\
&E_\varepsilon^1 \stackrel{\rm def}{=} [v^{\rm h}\cdot\nabla_{\rm h}(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3) - (0, \partial_z \Pi_Q^{\rm h}) + \varepsilon w_\varepsilon\cdot\nabla(v^{\rm h}, 0) + \varepsilon^2 w_\varepsilon\cdot\nabla(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3)]_\varepsilon,\\
&E_\varepsilon^2 \stackrel{\rm def}{=} -\varepsilon(1+a_\varepsilon)([\partial_z^2 v^{\rm h}]_\varepsilon, 0),\\
&E_\varepsilon^3 \stackrel{\rm def}{=} -a_\varepsilon[\Delta_\varepsilon(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3) - (0, \partial_z\Pi^{\rm h}) - \varepsilon\nabla_\varepsilon \Pi_\varepsilon^1]_\varepsilon \quad \text{and}\\
&E_\varepsilon^4 \stackrel{\rm def}{=} -b_\varepsilon([\Delta_{\rm h} v^{\rm h} - \nabla_{\rm h}\Pi^{\rm h}]_\varepsilon, 0).
\end{aligned}
\tag{2.12}
$$

Let us remark that the term $v^{\rm h}$ is ubiquitous in the error term E_ε, even in w_ε because $\Pi_L^{\rm h}$ depends on this vector field $v^{\rm h}$. Thus the property of $v^{\rm h}$ are crucial for the understanding of the error term E_ε.

Section 3 is devoted to the systematic study of the time decay of $v^{\rm h}$. This section is devoted to the 2D case and can be of independent interest. We generalize the decay in time estimates obtained in the case of homogeneous Navier–Stokes equation by Wiegner in [33] (see also the works [6, 19, 30, 31] and see [13] for the application of this method to a singular perturbed 2D Navier–Stokes system). We remark that to obtain this optimal time decay estimate for $v^{\rm h}$, we need to use a completely new formulation (see (3.22) below) of the inhomogeneous Navier–Stokes system.

As it can be observed in the term E_ε^2, we need L^1 in time estimate of term that involves second derivative of $v^{\rm h}$ with respect to the vertical variable z. This is the purpose of Theorem 4.1. A first consequence of this study is the following proposition, the proof of which will be presented in § 4.

Proposition 2.1. *Under the hypothesis of Theorem 1.2, we have*

$$
\begin{aligned}
\|v^{\rm h}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})} + \|\partial_z v^{\rm h}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})} + \|\nabla v^{\rm h}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}&\\
+ \|\partial_z v^{\rm h}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac34,\frac34})} + \|\partial_z \Pi^{\rm h}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})} \leqslant \mathcal{C}_0.&
\end{aligned}
$$

Here and in all that follows, we always denote $\mathcal{C}_0$ to be a positive constant which depends on norms of the profile ς_0 and $v_0^{\rm h}$ of the initial data and which may be changed from line to line.

The decay estimates of $v^{\rm h}$ obtained in Theorem 4.1 allow to prove the following proposition.

Proposition 2.2. *Let $(w_\varepsilon, \Pi_\varepsilon^1)$ be the solution of the System (2.9), then we have*

$$
\begin{aligned}
\varepsilon\|(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3)\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})} + \|(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}&\\
+ \|\nabla_\varepsilon(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})} + \|\nabla_\varepsilon(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})} \leqslant \mathcal{C}_0.&
\end{aligned}
$$

Moreover, for any α in $]0, 1[$, *we have*

$$
\|\Delta_\varepsilon(\varepsilon w_\varepsilon^{\rm h}, w_\varepsilon^3) - \varepsilon\nabla_\varepsilon\Pi_\varepsilon^1\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\alpha,\frac12})} \leqslant C_\alpha\mathcal{C}_0.
$$

The proof of this proposition is the purpose of Section 5.

Using law of product (2.7), the above two propositions imply that

$$\|E_\varepsilon^1\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac{1}{2}})} \leqslant \mathcal{C}_0, \tag{2.13}$$

which is the content of Corollary 5.1.

The two terms E_ε^2 and E_ε^3 are of a different nature. They contain of course terms which are rescaled functions of $(a^{\mathrm{h}}, v^{\mathrm{h}}, \Pi^{\mathrm{h}})$ and w_ε multiplied by the function a_ε. Their control demands estimates on the function a_ε. This requires the following induction hypothesis.

Let p be in $]3,4[$ and $\mathcal{R}_0$ be a positive real number which will be chosen large enough later on, we define $\overline{T}_\varepsilon$ as

$$\overline{T}_\varepsilon \overset{\text{def}}{=} \sup\left\{t < T_\varepsilon^\star / \|R_\varepsilon\|_{L_t^4(\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}})} + \|\nabla R_\varepsilon\|_{L_t^1(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{R}_0\right\} \tag{2.14}$$

where $T_\varepsilon^\star$ denotes the life span of the regular solution of (INS3D) associated with the initial data $(1+\eta[\varsigma_0]_\varepsilon, ([v_0^{\mathrm{h}}]_\varepsilon, 0))$. Under the above induction hypothesis, the regularity of a_ε is controlled thanks to the following proposition.

Proposition 2.3. *Let $(u_\varepsilon)_\varepsilon$ be a family of divergence free vector fields and a_0 a function in L^p with derivatives also in L^p. Let us consider the family $(a_\varepsilon)_\varepsilon$ of the solutions to*

$$\begin{cases} \partial_t a_\varepsilon + u_\varepsilon \cdot \nabla a_\varepsilon = 0, \\ a_{\varepsilon|t=0} = [a_0]_\varepsilon. \end{cases} \tag{2.15}$$

Then for any s in $]0, 1-1/p[$, we have

$$\|a_\varepsilon(t)\|_{\mathcal{B}_p^{s,\frac{1}{p}}} \lesssim \|a_0\|_{L^p}^{1-s-\frac{1}{p}} \|\nabla a_0\|_{L^p}^{s+\frac{1}{p}} \exp\left(C\int_0^t U_\varepsilon(t')\,dt'\right) \quad \textit{with}$$

$$U_\varepsilon(t') \overset{\text{def}}{=} \|\nabla_{\mathrm{h}} u_\varepsilon^{\mathrm{h}}(t')\|_{L^\infty} + \frac{1}{\varepsilon}\|\partial_3 u_\varepsilon^{\mathrm{h}}(t')\|_{L^\infty} + \varepsilon\|\nabla_{\mathrm{h}} u_\varepsilon^3(t')\|_{L^\infty}.$$

Proof. Let us change the variable by defining

$$\widetilde{a}_\varepsilon(t, x_{\mathrm{h}}, z) \overset{\text{def}}{=} a\left(t, x_{\mathrm{h}}, \frac{z}{\varepsilon}\right) \quad \text{and} \quad \widetilde{u}_\varepsilon(t, x_{\mathrm{h}}, z) \overset{\text{def}}{=} \left(u_\varepsilon^{\mathrm{h}}\left(t, x_{\mathrm{h}}, \frac{z}{\varepsilon}\right), \varepsilon u_\varepsilon^3\left(t, x_{\mathrm{h}}, \frac{z}{\varepsilon}\right)\right).$$

The transport equation (2.15) becomes

$$\begin{cases} \partial_t \widetilde{a}_\varepsilon + \widetilde{u}_\varepsilon \cdot \nabla \widetilde{a}_\varepsilon = 0, \\ \widetilde{a}|_{t=0} = a_0. \end{cases}$$

Let us remark that, because u_ε is divergence free, we have $\|\nabla \widetilde{u}_\varepsilon\|_{L^\infty} \sim U_\varepsilon(t)$. It is well known that *isotropic* Besov norms with regularity index less than 1 are propagated by the Lipschitz norm of the convection velocity. More precisely (see for instance Theorem 3.14 of [5]), we have for s in $]0, 1-1/p[$,

$$\begin{aligned} \|\widetilde{a}_\varepsilon(t)\|_{\dot{B}_{p,1}^{s+\frac{1}{p}}(\mathbb{R}^3)} &\leqslant \|a_0\|_{\dot{B}_{p,1}^{s+\frac{1}{p}}(\mathbb{R}^3)} \exp\left(C\int_0^t \|\nabla \widetilde{u}_\varepsilon(t')\|_{L^\infty}\,dt'\right) \\ &\leqslant \|a_0\|_{\dot{B}_{p,1}^{s+\frac{1}{p}}(\mathbb{R}^3)} \exp\left(C\int_0^t U_\varepsilon(t')\,dt'\right). \end{aligned}$$

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As we have (see [12, Lemma 4.3] for instance)

$$\|a\|_{\mathcal{B}_p^{s,\frac{1}{p}}} \lesssim \|a\|_{\dot{B}^{s+\frac{1}{p}}_{p,1}(\mathbb{R}^3)} \quad \text{and} \quad \|a_0\|_{\dot{B}^{s+\frac{1}{p}}_{p,1}(\mathbb{R}^3)} \lesssim \|a_0\|_{L^p}^{1-s-\frac{1}{p}} \|\nabla a_0\|_{L^p}^{s+\frac{1}{p}},$$

the proposition is proved because $\|a(t)\|_{\mathcal{B}_p^{s,\frac{1}{p}}} \sim \|\widetilde{a}(t)\|_{\mathcal{B}_p^{s,\frac{1}{p}}}$. □

Under the induction hypothesis (2.14), we have the following corollary.

Corollary 2.1. *Let $(a_\varepsilon, u_\varepsilon, \widetilde{\Pi}_\varepsilon)$ be a smooth enough solution of (INS3D) on $[0, \overline{T}_\varepsilon[$. Then for any s in $]0, 1-1/p[$, a constant C exists such that, for any time t less than $\overline{T}_\varepsilon$, we have*

$$\|a_\varepsilon(t)\|_{\mathcal{B}_p^{s,\frac{1}{p}}} \leqslant \mathcal{C}_0 \eta \exp(C\mathcal{R}_0).$$

Proof. In view of (2.10), we have

$$\int_0^t U_\varepsilon(t')\,dt' \lesssim \int_0^t (\|\nabla v^{\mathrm{h}}(t')\|_{L^\infty} + \varepsilon^2 \|\nabla w_\varepsilon(t')\|_{L^\infty} + \|\nabla R_\varepsilon(t')\|_{L^\infty})\,dt',$$

which together with Propositions 2.1 and 2.2 ensures that

$$\int_0^t U_\varepsilon(t')\,dt' \lesssim \mathcal{C}_0 + \mathcal{R}_0.$$

Then applying Proposition 2.3, we conclude the proof of the corollary. □

With Corollary 2.1, we can establish the estimates of the terms E_ε^2 and E_ε^3. Indeed for any p in $]3, 4[$, $q \in]p/(p-2), 2p/(p-1)[$ and δ in $]1/q - 1/p, 1-3/p[$, so that $-1 + \delta + 1/p + 2/q, 1 + 1/q - 1/p - \delta \in]0, 1[$. Then it follows from (2.5) and Lemma 4.1 that

$$\begin{aligned}\varepsilon \|[\partial_z^2 v^{\mathrm{h}}]_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})} &\leqslant C\varepsilon^{1-\delta-\frac{1}{p}} \|\partial_z v^{\mathrm{h}}\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},1-\delta})} \\ &\leqslant C\varepsilon^{1-\delta-\frac{1}{p}} \|\partial_z v^{\mathrm{h}}\|_{L^1_T(\mathcal{B}_q^{-1+\delta+\frac{1}{p}+\frac{2}{q},1+\frac{1}{q}-\frac{1}{p}-\delta})}.\end{aligned}$$

Applying the law of product (2.7), gives

$$\varepsilon \|a_\varepsilon [\partial_z^2 v^{\mathrm{h}}]_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})} \leqslant C\varepsilon^{1-\delta-\frac{1}{p}} \|a_\varepsilon\|_{L^\infty_T(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \|\partial_z v^{\mathrm{h}}\|_{L^1_T(\mathcal{B}_q^{-1+\delta+\frac{1}{p}+\frac{2}{q},1+\frac{1}{q}-\frac{1}{p}-\delta})}.$$

Therefore, thanks to (4.14) of Lemma 4.3 and Corollary 2.1, we conclude

$$\|E_\varepsilon^2\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})} \leqslant \mathcal{C}_0 \varepsilon^{1-\delta-\frac{1}{p}} \exp(C\mathcal{R}_0). \tag{2.16}$$

Similarly we deduce from the law of product (2.7) that

$$\begin{aligned}&\|a_\varepsilon[\Delta_\varepsilon(\varepsilon w_\varepsilon^{\mathrm{h}}, w_\varepsilon^3) - (0, \partial_z \Pi^{\mathrm{h}}) - \varepsilon \nabla_\varepsilon \Pi_\varepsilon^1]_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \\ &\leqslant \|a_\varepsilon\|_{L^\infty_T(\mathcal{B}_p^{\frac{1}{p},\frac{1}{p}})} \left(\|\Delta_\varepsilon(\varepsilon w_\varepsilon^{\mathrm{h}}, w_\varepsilon^3) - \varepsilon \nabla_\varepsilon \Pi_\varepsilon^1\|_{L^1_T(\mathcal{B}_p^{-1+\frac{3}{p},\frac{1}{p}})} + \|\partial_z \Pi^{\mathrm{h}}\|_{L^1_T(\mathcal{B}_p^{-1+\frac{3}{p},\frac{1}{p}})} \right),\end{aligned}$$

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which together with Propositions 2.2 and 2.1, and (4.8) ensures that

$$\|E^3_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{C}_0\eta\exp(C\mathcal{R}_0). \tag{2.17}$$

The term E^4_ε given by (2.12) is much more difficult to be treated. It is indeed here that we encounter the difficulty of our method (which originates from the framework of parabolic system) due to the transport equation. Let us investigate the equation on b_ε, which is given by (2.3),

$$\partial_t b_\varepsilon + u_\varepsilon\cdot\nabla b_\varepsilon + R_\varepsilon\cdot\nabla[a^{\mathrm{h}}]_\varepsilon + \varepsilon[w_\varepsilon\cdot\nabla a^{\mathrm{h}}]_\varepsilon = 0 \quad \text{with } b_{\varepsilon|t=0} = 0. \tag{2.18}$$

The control of b_ε is given by the following proposition.

Proposition 2.4. *Under the induction hypothesis (2.14), we can decompose $b_\varepsilon = \overline{b}_\varepsilon + \widetilde{b}_\varepsilon$ such that, for any p in $]3,4[$, there holds, for any t less than $\overline{T}_\varepsilon$,*

$$\|\overline{b}_\varepsilon(t)\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}} \leqslant \mathcal{C}_0\eta(1+\mathcal{R}_0)\langle t\rangle^{\frac{1}{2}} \quad \textit{and} \tag{2.19}$$

$$\|\widetilde{b}_\varepsilon(t)\|_{L^p} \leqslant \mathcal{C}_0\varepsilon^{1-\frac{1}{p}}(1+\mathcal{R}_0)^2\langle t\rangle. \tag{2.20}$$

Let us notice that the norms of b_ε grows in time. As we need L^1 in time control on the remainder term R_ε, it seems a disaster. In fact, it is compensated by the time decay of v^{h} established in Theorem 4.1 below. The proof of this proposition is the purpose of § 6. Then we can obtain the following estimates for $E^4_\varepsilon = E^{4,1}_\varepsilon + E^{4,2}_\varepsilon$,

$$\|E^{4,1}_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{C}_0\eta(1+\mathcal{R}_0) \quad \text{and} \quad \|E^{4,2}_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})} \leqslant \mathcal{C}_0\varepsilon^{1-\frac{1}{p}}(1+\mathcal{R}_0)^2, \tag{2.21}$$

which is the content of Corollary 6.1.

Now we are in position to solve globally the coupled system $(\mathrm{INS3D})_\varepsilon$ with (2.18). Let us think this system as a perturbation of a semi-linear parabolic system. We first compute $\nabla\Pi_\varepsilon$. The point is that the resolution operator of the elliptic system

$$\operatorname{div}((1+a)\nabla\Pi - f) = 0$$

can be written as

$$\Delta\Pi + \operatorname{div}(a\nabla\Pi) = \operatorname{div} f$$

and then

$$(\mathrm{Id} - \mathcal{M}_a)\nabla\Pi = \nabla\Delta^{-1}\operatorname{div} f \quad \text{with } \mathcal{M}_a g \stackrel{\text{def}}{=} -\nabla\Delta^{-1}\operatorname{div}(ag) \tag{2.22}$$

It is obvious that if a is a bounded function, the operator $\mathcal{M}_a$ is a bounded linear operator from $(L^2(\mathbb{R}^d))^d$ into itself and that

$$\|\mathcal{M}_a g\|_{L^2} \leqslant \|a\|_{L^\infty}\|g\|_{L^2}.$$

Thus, if $\|a\|_{L^\infty}$ is less than 1, the operator $\mathrm{Id} - \mathcal{M}_a$ is invertible on $(L^2(\mathbb{R}^d))^d$, and

$$\nabla\Pi = (\mathrm{Id} - \mathcal{M}_a)^{-1}\nabla\Delta^{-1}\operatorname{div} f. \tag{2.23}$$

This leads to the following definition of the modified Leray projection operator on divergence free vector field.

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Definition 2.2. Let a be a bounded function with the L^∞ norm of which is less than 1. We can define the modified Leray projection operator on divergence free vector fields associated with a (denoted $\mathbb{P}_a$) by

$$\mathbb{P}_a f \stackrel{\text{def}}{=} f - (1+a)(\mathrm{Id} - \mathcal{M}_a)^{-1}(\nabla \Delta^{-1} \operatorname{div} f).$$

Let us remark that it is a bounded operator on $(L^2(\mathbb{R}^d))^d$ and that if the function a is identically equal to 0, then the operator $\mathbb{P}_a$ is the classical Leray projection operator on divergence free vector fields.

Moreover, in the case when the L^∞ norm of a_ε is less than 1, the system $(\text{INS3D})_\varepsilon$ can be equivalently reformulated as

$$\begin{cases} \partial_t R_\varepsilon - \Delta R_\varepsilon = \mathbb{P}_{a_\varepsilon}(a_\varepsilon \Delta R_\varepsilon - \operatorname{div}(u_{\varepsilon,\text{app}} \otimes R_\varepsilon + R_\varepsilon \otimes u_{\varepsilon,\text{app}} + \varepsilon R_\varepsilon \otimes R_\varepsilon) - E_\varepsilon), \\ \operatorname{div} R_\varepsilon = 0 \quad \text{and} \quad R_\varepsilon|_{t=0} = 0. \end{cases} \tag{2.24}$$

We shall conclude the proof of Theorem 1.2 in § 7 by proving that the solution of the coupled system of equations (2.18) and (2.24) is global provided that η and ε are small enough.

3. Decay estimates for 2D flows

In this section, we investigate the decay properties of the global regular solution $(\rho, u, \nabla\Pi)$ of 2D incompressible inhomogeneous Navier–Stokes system (INS2D).

In this section, we use the following notations:

$$E_0(t) \stackrel{\text{def}}{=} \|\sqrt{\rho}u(t)\|_{L^2}^2, \quad E_1(t) \stackrel{\text{def}}{=} \|\nabla u(t)\|_{L^2}^2, \quad E_2(t) \stackrel{\text{def}}{=} \|\sqrt{\rho}\partial_t u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^4,$$
$$E_3(t) \stackrel{\text{def}}{=} \|\nabla \partial_t u(t)\|_{L^2}^2 + E_2^{\frac{3}{2}}(t) + E_2(t)\|\nabla\rho(t)\|_{L^\infty}^2, \quad E_i \stackrel{\text{def}}{=} E_i(0), \quad i = 0, 1, 2, 3,$$
$$\mathcal{C}(E_0) \quad \text{is an increasing function of } E_0,$$
$$a \lesssim b \Longrightarrow a \leqslant \mathcal{C}(E_0) b.$$

Moreover, in this section, we denote by x a generic point of $\mathbb{R}^2$. The main result of this section is the following theorem.

Theorem 3.1. *Let us consider the smooth solution $(\rho, u, \nabla\Pi)$ of (INS2D) associated with the initial data (ρ_0, u_0). In addition we assume that*

$$U_0 \stackrel{\text{def}}{=} \int_{\mathbb{R}^2} |x|\,|u_0(x)|\,dx < \infty, \quad \int_{\mathbb{R}^2} \rho_0 u_0(x)\,dx = 0, \quad \textit{and} \quad \frac{3}{4} \leqslant \rho_0(x) \leqslant \frac{5}{4}. \tag{3.1}$$

For any T greater than or equal $T_0(\rho_0, u_0)$ with

$$T_0(\rho_0, u_0) \stackrel{\text{def}}{=} \max\left\{\frac{U_0}{E_0}, \|\varrho_0\|_{L^2}^2\right\},$$

we have the following decay property for the total kinetic energy

$$\|u(t)\|_{L^2}^2 \lesssim E_0 \langle t\rangle_T^{-2}. \tag{3.2}$$

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For higher order derivatives of u, *we get for* T *large enough,*

$$\|\nabla u(t)\|_{L^2} + \|u(t)\|_{L^\infty} \lesssim E_1^{\frac{1}{2}} \langle t\rangle_T^{-(\frac{3}{2})}, \tag{3.3}$$

$$\|\partial_t u(t)\|_{L^2} + \|\nabla^2 u(t)\|_{L^2} + \|\nabla \Pi(t)\|_{L^2} \lesssim E_2^{\frac{1}{2}} \langle t\rangle_T^{-2}, \tag{3.4}$$

$$\|\nabla\partial_t u(t)\|_{L^2} \lesssim E_3^{\frac{1}{2}} \langle t\rangle_T^{-(\frac{5}{2})} \quad \text{and} \quad \|\nabla^3 u(t)\|_{L^2} + \|\nabla^2 \Pi(t)\|_{L^2} \lesssim E_3^{\frac{1}{2}} \langle t\rangle_T^{-(\frac{5}{2})} \log\langle t\rangle_T, \tag{3.5}$$

$$\|\nabla u(t)\|_{L^\infty} \lesssim E_1^{\frac{1}{4}} E_3^{\frac{1}{4}} \langle t\rangle_T^{-2} \log^{\frac{1}{2}} \langle t\rangle_T. \tag{3.6}$$

Here and in the rest of this section, we always denote $\langle \tau\rangle \overset{\text{def}}{=} (e+\tau)$ *and* $h_T(t) \overset{\text{def}}{=} h(t/T)$.

We remark that the decay rate of $\|u(t)\|_{L^2}$ given by (3.2) is optimal even in the case of classical Navier–Stokes system (see [25, Theorem A]).

3.1. Global energy estimates for the linearized system

Let $(\rho, u, \nabla\Pi)$ be a global classical solution of (INS2D). We first consider some basic energy estimate for linearized equations of 2D inhomogeneous incompressible Navier–Stokes system

$$\text{(LINS2D)} \begin{cases} \partial_t \rho + u\cdot\nabla\rho = 0, \\ \rho\partial_t v + \rho u\cdot\nabla v - \Delta v + \nabla\Pi_v = f + L(t)v, \\ \operatorname{div} u = \operatorname{div} v = 0, \\ \rho|_{t=0} = \rho_0, \quad v|_{t=0} = v_0. \end{cases}$$

• L^2 *energy estimate*

If the operator L is such that $\|L(t)\|_{\mathcal{L}(L^2)}$ belongs to $L^1(\mathbb{R}^+)$, then by multiplying (LINS2D) by the quantity

$$\exp\left(-\int_0^t \|L(t')\|_{\mathcal{L}(L^2)}\, dt'\right)$$

reduces to the case when $L(t)$ is a non-positive operator in the sense that $(L(t)v|v)_{L^2}$ is non-positive. Then the operator L can be ignored in the energy estimates. We assume this from now on.

We shall assume that all the vector fields and functions are smooth in time with value in any Sobolev space.

First of all, let us notice that the energy estimate, obtained by taking into account the fact that the vector field u is divergence free, writes

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^2}^2 \leqslant (f|v)_{L^2}. \tag{3.7}$$

By integration this gives

$$\begin{aligned} \frac{1}{2}\|\sqrt{\rho}v(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla v(t')\|_{L^2}^2\, dt' &\leqslant \frac{1}{2}\|\sqrt{\rho}v(t_0)\|_{L^2}^2 + \int_{t_0}^t (f(t')|v(t'))_{L^2} \\ &\leqslant \frac{1}{2}\|\sqrt{\rho}v(t_0)\|_{L^2}^2 + \int_{t_0}^t \left\|\frac{f}{\sqrt{\rho}}(t')\right\|_{L^2} \|\sqrt{\rho}v(t')\|_{L^2}\, dt'. \end{aligned}$$

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From this, we deduce that for any non-negative t_0 and any t greater than t_0

$$\frac{1}{2}\|\sqrt{\rho}v\|^2_{L^\infty([t_0,t];L^2)} + \int_{t_0}^t \|\nabla v(t')\|^2_{L^2}\,dt' \leqslant \|\sqrt{\rho}v(t_0)\|^2_{L^2} + 2\left(\int_{t_0}^t \left\|\frac{f}{\sqrt{\rho}}(t')\right\|_{L^2} dt'\right)^2. \tag{3.8}$$

In particular, since $(\rho, u, \nabla\Pi)$ is a classical solution of (INS2D), the above argument leads to

$$\frac{1}{2}\|\sqrt{\rho}u(t)\|^2_{L^2} + \int_{t_0}^t \|\nabla u(t')\|^2_{L^2}\,dt' = \frac{1}{2}\|\sqrt{\rho}u(t_0)\|^2_{L^2} \quad \text{for } 0 \leqslant t_0 \leqslant t, \tag{3.9}$$

which implies in particular that

$$\int_{t_0}^t \|u(t')\|^4_{L^4}\,dt' \lesssim \|u\|^2_{L^\infty([t_0,t];L^2)}\|\nabla u\|^2_{L^2([t_0,t];L^2)} \lesssim \|u(t_0)\|^4_{L^2}. \tag{3.10}$$

Moreover, the fact that the vector field u is divergence free implies that

$$\min_{x\in\mathbb{R}^2} \rho(t,x) = \min_{x\in\mathbb{R}^2} \rho_0(x) \quad \text{and} \quad \forall p \in [1,\infty], \quad \|\varrho(t)\|_{L^p} = \|\varrho_0\|_{L^p}. \tag{3.11}$$

- *The estimates for the first order derivatives*

The basic result is the following lemma.

Lemma 3.1. *Let ρ, u, v and f satisfy (LINS2D). Then for any non-negative t_0 and t with t greater than or equal to t_0, we have,*

$$\begin{aligned}&\|\nabla v(t)\|^2_{L^2} + \int_{t_0}^t (\|\sqrt{\rho}\partial_t v(t')\|^2_{L^2} + \|\nabla^2 v(t')\|^2_{L^2} + \|\nabla\Pi_v(t')\|^2_{L^2})\,dt' \\ &\leqslant C\left(\|\nabla v(t_0)\|^2_{L^2} + \int_{t_0}^t \|f(t')\|^2_{L^2}\,dt'\right)\exp(C\|u(t_0)\|^4_{L^2}).\end{aligned} \tag{3.12}$$

Moreover, some decay on ∇v can be obtained through the following inequalities. If s is a positive real number, we have.

$$\begin{aligned}&\langle t\rangle_T^s\|\nabla v(t)\|^2_{L^2} + \int_{t_0}^t \langle t'\rangle_T^s(\|\sqrt{\rho}\partial_t v(t')\|^2_{L^2} + \|\nabla^2 v(t')\|^2_{L^2} + \|\nabla\Pi_v(t')\|^2_{L^2})\,dt' \\ &\leqslant C\left(\langle t_0\rangle_T^s\|\nabla v(t_0)\|^2_{L^2} + \int_{t_0}^t \langle t'\rangle_T^{s-1}\|\nabla v(t')\|^2_{L^2}\frac{dt'}{T} + \int_{t_0}^t \langle t'\rangle_T^s\|f(t')\|^2_{L^2}\,dt'\right) \\ &\times\exp(C\|u(t_0)\|^4_{L^2}),\end{aligned} \tag{3.13}$$

and

$$t\|\nabla v(t)\|^2_{L^2} \leqslant C\left(\|v(t/2)\|^2_{L^2} + t\int_{t/2}^t \|f(t')\|^2_{L^2}\,dt'\right)\exp(C\|u(t/2)\|^4_{L^2}). \tag{3.14}$$

Proof. Multiplying the momentum part of (LINS2D) by $\partial_t v$ and integrating the resulting equation over $\mathbb{R}^2$, we obtain

$$\|\sqrt{\rho}\partial_t v\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|^2_{L^2} = -(\rho u\cdot\nabla v|\partial_t v)_{L^2} + (f|\partial_t v)_{L^2}.$$

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As ρ lies between 1/2 and 2, we get

$$\frac{3}{4}\|\sqrt{\rho}\partial_t v\|_{L^2}^2 + \frac{1}{2}\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 \leqslant 4\|(u\cdot\nabla v)(t)\|_{L^2}^2 + 4\|f(t)\|_{L^2}^2.$$

It is important now to make precise the idea in the framework of the time evolutionary Stokes problem, one time derivative of v is equivalent to two space derivatives of v. In the case of the system (LINS2D), this equivalent inequality is described by the following lemma.

Lemma 3.2. *Let $(\rho, v, \nabla\Pi_v)$ be a solution of (LINS2D). Then we have, for any p in the interval $]1,\infty[$,*

$$\|\nabla^2 v\|_{L^p} + \|\nabla\Pi_v\|_{L^p} \leqslant C(\|\sqrt{\rho}\partial_t v\|_{L^p} + \|u\|_{L^{2p}}^2\|\nabla v\|_{L^2} + \|f\|_{L^p}).$$

In the case when p equals to 2, we have the opposite inequality

$$\|\sqrt{\rho}\partial_t v\|_{L^2}^2 \leqslant C(\|\nabla^2 v\|_{L^2}^2 + \|u\|_{L^4}^4\|\nabla v\|_{L^2}^2 + \|f\|_{L^2}^2).$$

Proof. Observing that

$$\text{(SSE)}\qquad \begin{cases} -\Delta v + \nabla\Pi_v = f - \rho\partial_t v - \rho u\cdot\nabla v,\\ \operatorname{div} v = 0,\end{cases}$$

we deduce from the classical estimate on Stokes operator that for any p in $]1,\infty[$

$$\begin{aligned}\|\nabla^2 v\|_{L^p} + \|\nabla\Pi_v\|_{L^p} &\leqslant C(\|f\|_{L^p} + \|\rho\partial_t v\|_{L^p} + \|\rho u\cdot\nabla v\|_{L^p})\\ &\leqslant C\left(\|f\|_{L^p} + \|\sqrt{\rho}\partial_t v\|_{L^p} + \|u\|_{L^{2p}}\|\nabla v\|_{L^{2p}}\right).\end{aligned}\tag{3.15}$$

By using the 2D interpolation inequality that

$$\|a\|_{L^{2p}(\mathbb{R}^2)} \leqslant C_p\|a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla a\|_{L^p(\mathbb{R}^2)}^{\frac{1}{2}},\tag{3.16}$$

we get

$$\|\nabla^2 v\|_{L^p} + \|\nabla\Pi_v\|_{L^p} \leqslant \tfrac{1}{2}\|\nabla^2 v\|_{L^p} + C(\|f\|_{L^p} + \|\sqrt{\rho}\partial_t v\|_{L^p} + \|u\|_{L^{2p}}^2\|\nabla v\|_{L^2}).$$

This proves the first inequality. Because v_t is divergence free, we get, by taking the L^2 scalar product of $\partial_t v$ with (SSE), that

$$\begin{aligned}\|\sqrt{\rho}\partial_t v\|_{L^2}^2 &= (f|\partial_t v)_{L^2} + (\Delta v|\partial_t v)_{L^2} - (\sqrt{\rho}u\cdot\nabla v|\sqrt{\rho}\partial_t v)_{L^2}\\ &\leqslant \left(\sqrt{2}(\|f\|_{L^2} + \|\Delta v\|_{L^2}) + C\|u\|_{L^4}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}\right)\|\sqrt{\rho}\partial_t v\|_{L^2}.\end{aligned}$$

Hölder inequalities imply that

$$\|\sqrt{\rho}v_t\|_{L^2}^2 \leqslant \frac{1}{2}\|\sqrt{\rho}\partial_t v\|_{L^2}^2 + C(\|f\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2 + C\|u\|_{L^4}^2\|\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2}).$$

This leads to the second inequality and the lemma is proved. □

Continuation of the proof of Lemma 3.1. Applying the above lemma in the case when p equals to 2, we obtain that

$$\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 + \|\sqrt{\rho}\partial_t v\|_{L^2}^2 + \frac{1}{C}(\|\nabla^2 v\|_{L^2}^2 + \|\nabla\Pi_v\|_{L^2}^2) \leqslant C(\|f\|_{L^2}^2 + \|u\|_{L^4}^4\|\nabla v\|_{L^2}^2). \tag{3.17}$$

Gronwall lemma implies that, for any non-negative t_0 and t such that t is greater than or equal to t_0,

$$\begin{aligned}
&\|\nabla v(t)\|_{L^2}^2 + \int_{t_0}^t \left(\|\sqrt{\rho}\partial_t v(t')\|_{L^2}^2 + \frac{1}{C}\|\nabla^2 v(t')\|_{L^2}^2 + \frac{1}{C}\|\nabla\Pi_v(t')\|_{L^2}^2\right) dt' \\
&\quad \leqslant C\left(\|\nabla v(t_0)\|_{L^2}^2 + \int_{t_0}^t \|f(t')\|_{L^2}^2\, dt'\right)\exp\left(C\int_{t_0}^t \|u(t')\|_{L^4}^4\, dt'\right)
\end{aligned}$$

which together with (3.10) implies (3.12).

Let us prove (3.13). For any $s>0$, by multiplying $\langle t\rangle_T^s$ to (3.17), we get

$$\begin{aligned}
&\frac{d}{dt}(\langle t\rangle_T^s\|\nabla v(t)\|_{L^2}^2) + \langle t\rangle_T^s(\|\sqrt{\rho}\partial_t v\|_{L^2}^2 + \|\nabla^2 v\|_{L^2}^2 + \|\nabla\Pi_v\|_{L^2}^2) \\
&\quad \leqslant C(\langle t\rangle_T^s\|f\|_{L^2}^2 + \|u\|_{L^4}^4\langle t\rangle_T^s\|\nabla v\|_{L^2}^2 + \frac{1}{T}\langle t\rangle_T^{s-1}\|\nabla v\|_{L^2}^2).
\end{aligned}$$

Applying Gronwall's lemma and using (3.10) leads to (3.13).

Similarly from inequality (3.17), we deduce that for any non-negative t_0 and t such that t is greater than or equal to t_0,

$$\begin{aligned}
\frac{d}{dt}((t-t_0)\|\nabla v(t)\|_{L^2}^2) &= \|\nabla v(t)\|_{L^2}^2 + (t-t_0)\frac{d}{dt}\|\nabla v(t)\|_{L^2}^2 \\
&\lesssim \|\nabla v(t)\|_{L^2}^2 + (t-t_0)\|f(t)\|_{L^2}^2 + \|u(t)\|_{L^4}^4(t-t_0)\|\nabla v(t)\|_{L^2}^2.
\end{aligned}$$

Applying Gronwall's inequality yields

$$(t-t_0)\|\nabla v(t)\|_{L^2}^2 \lesssim \int_{t_0}^t (\|\nabla v(t')\|_{L^2}^2 + (t'-t_0)\|f(t')\|_{L^2}^2)\, dt' \exp\left(C\int_{t_0}^t \|u(t')\|_{L^4}^4\, dt'\right),$$

which together with the energy estimate on v, namely inequality (3.8), ensures

$$\begin{aligned}
(t-t_0)\|\nabla v(t)\|_{L^2}^2 &\lesssim \left(\|v(t_0)\|_{L^2}^2 + \left(\int_{t_0}^t \|f(t')\|_{L^2}\, dt'\right)^2 + \int_{t_0}^t (t'-t_0)\|f(t')\|_{L^2}^2\, dt'\right) \\
&\qquad \times \exp\left(C\int_{t_0}^t \|u(t')\|_{L^4}^4\, dt'\right).
\end{aligned}$$

Taking t_0 equal to $t/2$ in the above inequality gives

$$t\|\nabla v(t)\|_{L^2}^2 \lesssim \left(\|v(t/2)\|_{L^2}^2 + t\int_{t/2}^t \|f(t')\|_{L^2}^2\, dt'\right)\exp\left(C\int_{t/2}^t \|u(t')\|_{L^4}^4\, dt'\right).$$

This together with (3.10) concludes the proof of (3.14). □

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3.2. Sharp L^2 decay estimates

The purpose of this subsection is to prove the following proposition.

Proposition 3.1. *Let $T_0(\rho_0, u_0)$ be given by Theorem 3.1 and $T_1(\rho_0, u_0) \stackrel{\text{def}}{=} \max\{T_0, E_0/E_1\}$. Then under the assumptions of Theorem 3.1, there holds (3.2) for $T \geqslant T_0$. And for $T \geqslant T_1$, we have following decay estimate*

$$\|\nabla u(t)\|_{L^2}^2 \lesssim E_1 \langle t \rangle_T^{-3}. \tag{3.18}$$

Proof. It is based on the method introduced by Wiegner in [33] in order to study the decay of the energy of the classical Navier–Stokes system in two space dimension (see also [30, 31]). The idea is to use a cutoff in the frequency space adapted to time. More precisely, let us consider a positive constant T (which can be understood as a scaling parameter which has the dimension of time), and g any positive real function defined on $\mathbb{R}^+$ such that

$$g^2(\tau) \leqslant 3\langle \tau \rangle^{-1} \quad \text{with } \langle \tau \rangle \stackrel{\text{def}}{=} (e + \tau). \tag{3.19}$$

Let us define

$$S_T(t) \stackrel{\text{def}}{=} \{\xi \in \mathbb{R}^2, \sqrt{T}|\xi| \leqslant \sqrt{2} g_T(t)\} \quad \text{and} \quad v_\flat(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{S_T(t)} \widehat{v}(t)). \tag{3.20}$$

Here we adapt this method to the inhomogeneous case through the following lemma.

Lemma 3.3. *Let (ρ, u, v) solve $(LINS2D)$. Then we have*

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho} v(t)\|_{L^2}^2 + \frac{1}{T} g_T^2(t) \|\sqrt{\rho} v(t)\|_{L^2}^2 \leqslant \frac{2}{T} g_T^2(t) \|v_\flat(t)\|_{L^2}^2 + (f(t)|v(t))_{L^2}.$$

Proof. As $v(t) - v_\flat(t)$ and $v_\flat(t)$ are orthogonal in all Sobolev spaces, we get in particular that

$$\|\nabla v(t)\|_{L^2}^2 = \|\nabla v_\flat(t)\|_{L^2}^2 + \|\nabla(v(t) - v_\flat(t))\|_{L^2}^2.$$

By definition of $S_T(t)$ and again by the orthogonality between $v(t) - v_\flat(t)$ and $v_\flat(t)$, we get

$$\begin{aligned} \|\nabla(v(t) - v_\flat(t))\|_{L^2}^2 &\geqslant \frac{2}{T} g_T^2(t) \|v(t) - v_\flat(t)\|_{L^2}^2 \\ &\geqslant \frac{2}{T} g_T^2(t) \|v(t)\|_{L^2} - \frac{2}{T} g_T^2(t) \|v_\flat(t)\|_{L^2}^2. \end{aligned}$$

As $\rho(t,x)$ is less than or equal to 2, the energy estimate (3.7) implies the lemma. □

The interest of this lemma is that the term on the left is typically a term that creates decay. Of course, the control of the term $v_\flat$ associated with (very) low frequencies is the term that tends to prevent the decay. It must be estimated in a careful way. Writing a general theory with external force seems too ambitious. We are going to restrict ourselves to two cases: the case when $u = v$ and $f \equiv 0$, namely, the case of solution of (INS2D), and later on the case of a family of solution $v^{\mathrm{h}}(\cdot, z)$ of (INS2D) where z is a real parameter and then v represents derivatives of $v^{\mathrm{h}}(\cdot, z)$ with respect to the parameter z (see [14, §§ 4 and 5] for details).

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Lemma 3.4. *Under the hypothesis of Proposition 3.1, we have, for any T greater than or equal to T_0,*

$$\|u_\flat(t)\|_{L^2}^2 \leqslant \frac{1}{4}\|\sqrt{\rho}u(t)\|_{L^2}^2 + CE_0^2\langle t\rangle_T^{-2} + Cg_T^6(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}\frac{dt'}{T}\right)^2 + Cg_T^4(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}^2\frac{dt'}{T}\right)^2.$$

Proof. It relies on the rewriting of the momentum equation of (INS2D) as

$$\partial_t u - \Delta u + \nabla\Pi = -\partial_t(\varrho u) - \operatorname{div}(\rho u\otimes u). \tag{3.21}$$

If $\mathbb{P}$ denotes the Leray projection on divergence free vector fields on $\mathbb{R}^2$, the above relation writes in term of Fourier transform

$$\widehat{u}(t,\xi) = e^{-t|\xi|^2}\widehat{u}_0(\xi) - \int_0^t e^{-(t-t')|\xi|^2}\partial_t\mathcal{F}\mathbb{P}(\varrho u)(t',\xi)\,dt' - \int_0^t e^{-(t-t')|\xi|^2}\mathcal{F}\mathbb{P}(\operatorname{div}(\rho u\otimes u))(t',\xi)\,dt'.$$

By integration by parts in time, we get that

$$\int_0^t e^{-(t-t')|\xi|^2}\partial_t\mathcal{F}\mathbb{P}(\varrho u)(t',\xi)\,dt' = \mathcal{F}\mathbb{P}(\varrho u)(t,\xi) - e^{-t|\xi|^2}\mathcal{F}\mathbb{P}(\varrho_0 u_0)(\xi) - \int_0^t e^{-(t-t')|\xi|^2}|\xi|^2\mathcal{F}\mathbb{P}(\varrho u)(t',\xi)\,dt'.$$

Let us notice that in the integral term, we exchange one time derivative for two space derivatives. This gives the following key formula

$$\widehat{u}(t,\xi) = e^{-t|\xi|^2}\mathcal{F}\mathbb{P}(\rho_0 u_0)(\xi) - \mathcal{F}\mathbb{P}(\varrho u)(t,\xi) + \int_0^t e^{-(t-t')|\xi|^2}|\xi|^2\mathcal{F}\mathbb{P}(\varrho u)(t',\xi)\,dt' - \int_0^t e^{-(t-t')|\xi|^2}\mathcal{F}\mathbb{P}(\operatorname{div}(\rho u\otimes u))(t',\xi)\,dt'. \tag{3.22}$$

Because $\mathbb{P}$ decreases the modulus of the Fourier transform, we get for any t and ξ,

$$|\widehat{u}(t,\xi)|^2 \leqslant 2e^{-2t|\xi|^2}|\mathcal{F}(\rho_0 u_0)(\xi)|^2 + 2|\mathcal{F}(\varrho u)(t,\xi)|^2 + 2|\xi|^4\left(\int_0^t |\mathcal{F}(\varrho u)(t',\xi)|\,dt'\right)^2 + 2|\xi|^2\left(\int_0^t |\mathcal{F}(\rho u\otimes u)(t',\xi)|\,dt'\right)^2. \tag{3.23}$$

To estimate $\|u_\flat(t)\|_{L^2}$, we have to integrate the above inequality over $S_T(t)$. In order to do it, we make pointwise estimates in the Fourier variable.

First, let us observe that u_0 and thus $\rho_0 u_0$ belongs to $L^1(\mathbb{R}^2, |x|\,dx)$. Because of the fact that $\rho_0 u_0$ is mean free, we infer that

$$|\mathcal{F}(\rho_0 u_0)(\xi)| \leqslant |\xi|\,\|D_\xi\mathcal{F}(\rho_0 u_0)\|_{L^\infty} \leqslant |\xi|\int_{\mathbb{R}^2}\rho_0(x)|u_0(x)|\,|x|\,dx.$$

By integration on $S_T(t)$, this gives, because T is greater than T_0 and $g^2(\tau)\leqslant 3\langle\tau\rangle^{-1}$,

$$\int_{S_T(t)} e^{-2t|\xi|^2}|\mathcal{F}(\rho_0 u_0)(\xi)|^2\,d\xi \lesssim E_0^2\langle t\rangle_T^{-2}. \tag{3.24}$$

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Let us observe that, thanks to (3.11), we get, for T greater than or equal to T_0

$$\begin{aligned}|\mathcal{F}(\varrho u)(t',\xi)| &\leqslant \|\varrho(t')\|_{L^2}\|u(t')\|_{L^2}\\ &\leqslant \|\varrho_0\|_{L^2}\|u(t')\|_{L^2} \leqslant T_0^{\frac{1}{2}}\|u(t')\|_{L^2}.\end{aligned}$$

From this, we infer that,

$$\int_{S_T(t)} |\xi|^4 \left(\int_0^t |\mathcal{F}(\varrho u)(t',\xi)|\,dt'\right)^2 d\xi \lesssim g_T^6(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}\frac{dt'}{T}\right)^2. \tag{3.25}$$

Along the same lines, we get that

$$|\mathcal{F}(\rho u\otimes u)(t',\xi)| \lesssim \|\sqrt{\rho}u(t')\|_{L^2}^2.$$

Thus we get

$$\int_{S_T(t)} |\xi|^2 \left(\int_0^t |\mathcal{F}(\rho u\otimes u)(t',\xi)|\,dt'\right)^2 d\xi \lesssim g_T^4(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}^2\frac{dt'}{T}\right)^2. \tag{3.26}$$

Because of the hypothesis on ρ_0, we obviously have

$$\begin{aligned}\int_{S_T(t)} 2|\mathcal{F}(\varrho u)(t,\xi)|^2\,d\xi &\leqslant 4(2\pi)^2\|\varrho\|_{L^\infty}^2\|\sqrt{\rho}u(t)\|_{L^2}^2\\ &\leqslant 4(2\pi)^2\|\varrho_0\|_{L^\infty}^2\|\sqrt{\rho}u(t)\|_{L^2}^2 \leqslant \frac{1}{4}(2\pi)^2\|\sqrt{\rho}u(t)\|_{L^2}^2.\end{aligned}$$

Together with estimates (3.24)–(3.26), we achieve the proof of the lemma. □

Continuation of the proof of Proposition 3.1. The above lemmas give immediately that

$$\begin{aligned}\frac{d}{dt}\|\sqrt{\rho}u(t)\|_{L^2}^2 + \frac{1}{T}g_T^2(t)\|\sqrt{\rho}u(t)\|_{L^2}^2 &\lesssim \frac{1}{T}E_0^2\langle t\rangle_T^{-3} + \frac{1}{T}g_T^8(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}\frac{dt'}{T}\right)^2\\ &\quad + \frac{1}{T}g_T^6(t)\left(\int_0^t \|\sqrt{\rho}u(t')\|_{L^2}^2\frac{dt'}{T}\right)^2. \end{aligned}\tag{3.27}$$

Let us define

$$G(\tau) \stackrel{\text{def}}{=} \exp\left(\int_0^\tau g^2(\tau')\,d\tau'\right). \tag{3.28}$$

The above formula writes after integration

$$\begin{aligned}\|\sqrt{\rho}u(t)\|_{L^2}^2 G_T(t) - E_0 &\lesssim E_0^2\int_0^t \langle t'\rangle_T^{-3}G_T(t')\frac{dt'}{T}\\ &\quad + \int_0^t g_T^8(t')G_T(t')\left(\int_0^{t'} \|\sqrt{\rho}u(t'')\|_{L^2}\frac{dt''}{T}\right)^2\frac{dt'}{T}\\ &\quad + \int_0^t g_T^6(t')G_T(t')\left(\int_0^{t'} \|\sqrt{\rho}u(t'')\|_{L^2}^2\frac{dt''}{T}\right)^2\frac{dt'}{T}. \end{aligned}\tag{3.29}$$

Now we iterate this inequality several times to get the final decay estimates of u given by Proposition 3.1. Let us first choose the function g as

$$g^2(\tau) = 3(\langle\tau\rangle \log\langle\tau\rangle)^{-1} \quad \text{which gives } G(\tau) = \log^3\langle\tau\rangle.$$

Using that $\|\sqrt{\rho}u(t)\|_{L^2}^2$ is less than or equal to the initial energy E_0, inequality (3.29) writes

$$\begin{aligned}\|\sqrt{\rho}u(t)\|_{L^2}^2 \log^3\langle t\rangle_T - E_0 &\lesssim E_0(1+E_0)\int_0^t \langle t'\rangle_T^{-1}\frac{dt'}{T}\\ &\lesssim E_0(1+E_0)\log\langle t\rangle_T.\end{aligned}$$

We deduce that

$$\|\sqrt{\rho}u(t)\|_{L^2}^2 \log^2\langle t\rangle_T \lesssim E_0(1+E_0). \tag{3.30}$$

Now let us plug this estimate into inequality (3.29) with the choice $g^2(\tau) = \langle\tau\rangle^{-1}$, which gives $G(\tau) = \langle\tau\rangle$. This leads to

$$\begin{aligned}\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle_T - E_0 \lesssim\ & E_0^2\int_0^t \langle t'\rangle_T^{-2}\frac{dt'}{T} + \int_0^t \langle t'\rangle_T^{-3}\left(\int_0^{t'} \|\sqrt{\rho}u(t'')\|_{L^2}\frac{dt''}{T}\right)^2\frac{dt'}{T}\\ &+ \int_0^t \langle t'\rangle_T^{-2}\left(\int_0^{t'} \|\sqrt{\rho}u(t'')\|_{L^2}^2\frac{dt''}{T}\right)^2\frac{dt'}{T}.\end{aligned}$$

Let us define $V(t) \stackrel{\text{def}}{=} \sup_{t'\leqslant t}(\|\sqrt{\rho}u(t')\|_{L^2}^2\langle t'\rangle_T)$. We get

$$\begin{aligned}V(t) - E_0 &\lesssim E_0^2\int_0^t \langle t'\rangle_T^{-2}\frac{dt'}{T} + (1+E_0)^2\int_0^t \langle t'\rangle_T^{-2}H_T(t')V(t')\frac{dt'}{T} \quad \text{with}\\ H(\tau) &\stackrel{\text{def}}{=} 1 + \left(\int_0^\tau \langle\tau'\rangle^{-\frac{1}{2}}\log^{-1}\langle\tau'\rangle\, d\tau'\right)^2.\end{aligned}$$

As we have that $H(\tau) \lesssim 1 + \langle\tau\rangle_T\log^{-2}\langle\tau\rangle_T$, the function $\langle t'\rangle_T^{-2}H_T(t')$ is integrable and then Gronwall lemma gives

$$\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle_T \leqslant \mathcal{C}(E_0)E_0.$$

Let us plug this estimate into (3.29) and choose $g^2(\tau) = 2\langle\tau\rangle^{-1}$ which gives $G(\tau) = \langle\tau\rangle^2$. We infer that

$$\begin{aligned}\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle_T^2 - E_0 &\leqslant E_0^2\int_0^t \langle t'\rangle_T^{-1}\frac{dt'}{T} + \mathcal{C}(E_0)E_0\int_0^t \langle t'\rangle_T^{-1}\log^2\langle t'\rangle_T\frac{dt'}{T}\\ &\leqslant \mathcal{C}(E_0)E_0\log^3\langle t\rangle_T.\end{aligned}$$

Finally resuming the above estimate into (3.29) once again with the choice $g^2(\tau) = \alpha\langle\tau\rangle^{-1}$, for $\alpha \in]2,3[$ gives $G(\tau) = \langle\tau\rangle^\alpha$ and

$$\begin{aligned}\|\sqrt{\rho}u(t)\|_{L^2}^2\langle t\rangle_T^\alpha - E_0 \lesssim\ & E_0^2\int_0^t \langle t'\rangle_T^{\alpha-3}\frac{dt'}{T} + \mathcal{C}(E_0)E_0\int_0^t \langle t'\rangle_T^{\alpha-4}\log^5\langle t'\rangle_T\frac{dt'}{T}\\ &+ \mathcal{C}(E_0)E_0\int_0^t \langle t'\rangle_T^{\alpha-5}\log^6\langle t'\rangle_T\frac{dt'}{T},\end{aligned}$$

which implies the estimate (3.2).

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Let us prove (3.18). Applying (3.14) with $v = u$ and $f \equiv 0$, and inequality (3.2), we get

$$t\|\nabla u(t)\|_{L^2}^2 \lesssim \|u(t/2)\|_{L^2}^2 \lesssim E_0\langle t\rangle_T^{-2},$$

which write, in the case when T is greater than T_1,

$$\|\nabla u(t)\|_{L^2}^2 \lesssim \frac{E_0}{T}\frac{T}{t}\langle t\rangle_T^{-2} \lesssim E_1\frac{T}{t}\langle t\rangle_T^{-2}.$$

Moreover, inequality (3.12) implies that $\|\nabla u(t)\|_{L^2} \lesssim \|\nabla u_0\|_{L^2}^2$ which proves (3.18). □

3.3. Decay estimates for the second and third derivatives of u

The main idea which seems to be the simplest one at the first glance consists in the differentiation of the momentum equation of (INS2D) with respect to the space variables and then trying to apply result of the previous subsections. However, for this particular system, this quite natural idea fails. The reason is due to the fact that term of the type $\nabla_x\rho$ will appear in this process. Their control demands a control of the norm which is L^1 in time with value in Lip in space for the vector field u. This control cannot be assumed and has to be proved. The main idea to overcome this difficulty consists in differentiating the momentum equation of (INS2D) with respect to the time variable. As shown by Lemma 3.2, this represents the estimate of the second space derivatives of u.

All the results of this subsection relies on the following lemma.

Lemma 3.5. *Let (ρ, u, v) solve (LINS2D). Then we have, for any positive constant $\mathcal{T}$,*

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}v_t(t)\|_{L^2}^2 + \frac{3}{4}\|\nabla v_t\|_{L^2}^2 \leqslant (f_t|v_t)_{L^2} + CF_{1,\mathcal{T}}(u(t))\|\sqrt{\rho}v_t(t)\|_{L^2}^2 + CF_{2,\mathcal{T}}(u(t),v(t))$$
$$+ C\|\nabla v_t(t)\|_{L^2}^{\frac12}\|\nabla u_t(t)\|_{L^2}^{\frac12}\|\nabla v(t)\|_{L^2}\|\sqrt{\rho}v_t(t)\|_{L^2}^{\frac12}\|\sqrt{\rho}u_t(t)\|_{L^2}^{\frac12}$$

with

$$F_{1,\mathcal{T}}(u) \overset{\text{def}}{=} \|u\|_{L^4}^4 + \frac{1}{\mathcal{T}}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 \quad \textit{and}$$

$$F_{2,\mathcal{T}}(u,v) \overset{\text{def}}{=} \|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2} + C\|u\|_{L^2}\|\nabla v\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2 + \mathcal{T}\|\nabla^2 v\|_{L^2}^2.$$

Before applying and then proving this lemma, let us make some comments about it. First of all, the parameter $\mathcal{T}$ is a scaling parameter, the role of which will appear in a while. Inequalities (3.10) and (3.12) imply that, for any positive t,

$$\int_t^\infty F_{1,\mathcal{T}}(u(t))\,dt \leqslant \|u(t)\|_{L^2}^4 + \frac{\|\nabla u(t)\|_{L^2}^2}{\mathcal{T}}\mathcal{C}(E_0). \tag{3.31}$$

In the same spirit, it can easily be inferred that, for any positive t,

$$\begin{aligned}\int_t^\infty F_{2,\mathcal{T}}(u(t'),v(t'))\,dt' \leqslant\ & \|\nabla u\|_{L^\infty([t,\infty[;L^2)}\|\nabla^2 u\|_{L^2([t,\infty[\times\mathbb{R}^2)}\\ & \times\|\nabla v\|_{L^\infty([t,\infty[;L^2)}\|\nabla^2 v\|_{L^2([t,\infty[\times\mathbb{R}^2)}\\ & + C\|u\|_{L^\infty([t,\infty[;L^2)}\|\nabla v\|_{L^\infty([t,\infty[;L^2)}^2\|\nabla^2 u\|_{L^2([t,\infty[\times\mathbb{R}^2)}^2\\ & + \mathcal{T}\|\nabla^2 v\|_{L^2([t,\infty[\times\mathbb{R}^2)}^2.\end{aligned}$$

Inequality (3.12) implies that for any positive t,

$$\int_t^\infty F_{2,\mathcal{T}}(u(t'),v(t'))\,dt' \lesssim \left(\|\nabla v(t)\|_{L^2}^2 + \int_t^\infty \|f(t')\|_{L^2}^2\,dt'\right) \times (\|\nabla u(t)\|_{L^2}^2 + \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^2 + \mathcal{T}). \tag{3.32}$$

Let us establish now the following corollary.

Corollary 3.1. *Let $(\rho, u, \nabla\Pi)$ be a smooth enough solution of (INS2D), then we have for any positive t_0 and any t greater than or equal to t_0,*

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla^2 u(t)\|_{L^2}^2 + \|\nabla\Pi(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla u_t(t')\|_{L^2}^2\,dt' \lesssim E_2(t_0).$$

Proof. Let us apply Lemma 3.5 with $u = v$ and $f \equiv 0$. This gives

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \frac{3}{4}\|\nabla u_t(t)\|_{L^2}^2 \leqslant CF_{1,\mathcal{T}}(u(t))\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + CF_{2,\mathcal{T}}(u(t),u(t)) + C\|\nabla u_t(t)\|_{L^2}\|\nabla u(t)\|_{L^2}\|\sqrt{\rho}u_t(t)\|_{L^2}.$$

Using the convexity inequality, this gives

$$\frac{d}{dt}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 \lesssim \widetilde{F}_{1,\mathcal{T}}(u(t))\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + F_{2,\mathcal{T}}(u(t),u(t)) \tag{3.33}$$

with $\widetilde{F}_{1,\mathcal{T}}(w) \stackrel{\text{def}}{=} F_{1,\mathcal{T}}(w) + \|\nabla w\|_{L^2}^2$. Then Gronwall lemma implies that for any positive t_0 and any t greater than or equal to t_0,

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla u_t(t')\|_{L^2}^2\,dt' \lesssim \left(\|\sqrt{\rho}u_t(t_0)\|_{L^2}^2 + \int_{t_0}^\infty F_{2,\mathcal{T}}(u(t'),u(t'))\,dt'\right) \times \exp\left(\int_{t_0}^\infty \widetilde{F}_{1,\mathcal{T}}(u(t'))\,dt'\right).$$

Inequalities (3.9), (3.31) and (3.32) applied with $u = v$ gives

$$\begin{aligned}\|\sqrt{\rho}u_t(t)\|_{L^2}^2 &+ \int_{t_0}^t \|\nabla u_t(t')\|_{L^2}^2\,dt'\\ &\lesssim (\|\sqrt{\rho}u_t(t_0)\|_{L^2}^2 + (1+\|u(t_0)\|_{L^2})\|\nabla u(t_0)\|_{L^2}^2(\|\nabla u(t_0)\|_{L^2}^2 + \mathcal{T}))\\ &\quad\times \exp\left(C\left(\|\sqrt{\rho}u(t_0)\|_{L^2}^2 + \|u(t_0)\|_{L^2}^4 + \frac{\|\nabla u(t_0)\|_{L^2}^2}{\mathcal{T}}\mathcal{C}(E_0)\right)\right).\end{aligned}$$

Choosing $\mathcal{T} = \|\nabla u(t_0)\|_{L^2}^2$ ensures

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \int_{t_0}^t \|\nabla u_t(t')\|_{L^2}^2\,dt' \lesssim E_2(t_0),$$

which together with the first inequality of Lemma 3.2 implies

$$\|\nabla^2 u(t)\|_{L^2} + \|\nabla\Pi(t)\|_{L^2} \lesssim \|u_t(t)\|_{L^2} + \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^2 \lesssim E_2^{\frac{1}{2}}(t_0).$$

This finishes the proof of the corollary. □

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Proof of Lemma 3.5. By applying ∂_t to the momentum part of (LINS2D), we obtain

$$\rho\partial_t v_t + \rho u\cdot\nabla v_t - \Delta v_t + \nabla\partial_t\Pi_v = \widetilde{f}_t \quad \text{with } \widetilde{f}_t \stackrel{\text{def}}{=} -\rho_t v_t - \rho_t u\cdot\nabla v - \rho u_t\cdot\nabla v + f_t. \tag{3.34}$$

Now let us observe that as ρ is transported by the flow of u, we have $\rho_t = -\mathrm{div}(\rho u)$. Thus the new external force becomes

$$\widetilde{f}_t = \mathrm{div}(\rho u)v_t + \mathrm{div}(\rho u)u\cdot\nabla v - \rho u_t\cdot\nabla v + f_t.$$

Applying the basic energy estimate (3.7), we get

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}v_t\|_{L^2}^2 + \|\nabla v_t\|_{L^2}^2 = (\widetilde{f}_t|v_t)_{L^2}.$$

The key point consists in estimating the term $(\widetilde{f}_t|v_t)_{L^2}$. It follows by integration by parts that

$$\begin{aligned}
&(\widetilde{f}_t|v_t)_{L^2} = (f_t|v_t)_{L^2} + \textstyle\sum_{i=1}^5 \mathcal{E}_i(t) \quad \text{with}\\
&\mathcal{E}_1(t) \stackrel{\text{def}}{=} -2(\rho u\cdot\nabla v_t|v_t)_{L^2}\\
&\mathcal{E}_2(t) \stackrel{\text{def}}{=} -(\rho(u\cdot\nabla u)\cdot\nabla v|v_t)_{L^2}\\
&\mathcal{E}_3(t) \stackrel{\text{def}}{=} -(\rho(u\otimes u):\nabla^2 v|v_t)_{L^2}\\
&\mathcal{E}_4(t) \stackrel{\text{def}}{=} -(\rho u\cdot\nabla v|u\cdot\nabla v_t)_{L^2} \quad \text{and}\\
&\mathcal{E}_5(t) \stackrel{\text{def}}{=} -(\rho u_t\cdot\nabla v|v_t)_{L^2}.
\end{aligned} \tag{3.35}$$

By using the 2D interpolation inequality (3.16), we get

$$\begin{aligned}
|\mathcal{E}_1(t)| &\lesssim \|u\|_{L^4}\|\nabla v_t\|_{L^2}\|v_t\|_{L^4}\\
&\lesssim \|\nabla v_t\|_{L^2}^{\frac{3}{2}}\|u\|_{L^4}\|v_t\|_{L^2}^{\frac{1}{2}} \leqslant \epsilon\|\nabla v_t\|_{L^2}^2 + C_\epsilon\|u\|_{L^4}^4\|v_t\|_{L^2}^2.
\end{aligned} \tag{3.36}$$

Using again 2D interpolation inequality (3.16) yields

$$\begin{aligned}
|\mathcal{E}_2(t)| &\leqslant \|u\|_{L^4}\|\nabla u\|_{L^4}\|\nabla v\|_{L^4}\|v_t\|_{L^4}\\
&\lesssim \|\nabla u\|_{L^2}^{\frac{1}{2}}\|\nabla^2 u\|_{L^2}^{\frac{1}{2}}\|\nabla v\|_{L^2}^{\frac{1}{2}}\|\nabla^2 v\|_{L^2}^{\frac{1}{2}}\|u\|_{L^4}\|v_t\|_{L^2}^{\frac{1}{2}}\|\nabla v_t\|_{L^2}^{\frac{1}{2}}.
\end{aligned}$$

Hölder inequality implies that

$$\mathcal{E}_2(t) \leqslant \epsilon\|\nabla v_t\|_{L^2}^2 + C_\epsilon\|u\|_{L^4}^4\|v_t\|_{L^2}^2 + C_\epsilon\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla v\|_{L^2}\|\nabla^2 v\|_{L^2}. \tag{3.37}$$

Using the 2D interpolation inequality

$$\|a\|_{L^\infty(\mathbb{R}^2)} \lesssim \|a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}\|\nabla^2 a\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}, \tag{3.38}$$

we get

$$\begin{aligned}
|\mathcal{E}_3(t)| &\leqslant \|u\|_{L^\infty}^2\|\nabla^2 v\|_{L^2}\|v_t\|_{L^2}\\
&\lesssim \|\nabla^2 v\|_{L^2}\|u\|_{L^2}\|\nabla^2 u\|_{L^2}\|v_t\|_{L^2} \lesssim \mathcal{T}\|\nabla^2 v\|_{L^2}^2 + \frac{1}{\mathcal{T}}\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\|v_t\|_{L^2}^2.
\end{aligned} \tag{3.39}$$

Similarly using again 2D interpolation inequality (3.38), we get

$$\begin{aligned}|\mathcal{E}_4(t)| &\leqslant \|u\|_{L^\infty}^2\|\nabla v\|_{L^2}\|\nabla v_t\|_{L^2}\\ &\leqslant \epsilon\|\nabla v_t\|_{L^2}^2 + C_\epsilon\|u\|_{L^2}^2\|\nabla^2 u\|_{L^2}^2\|\nabla v\|_{L^2}^2.\end{aligned} \tag{3.40}$$

In order to estimate $\mathcal{E}_5$, we use (3.16) which gives

$$|\mathcal{E}_5(t)| \lesssim \|\nabla v_t(t)\|_{L^2}^{\frac{1}{2}}\|\nabla u_t(t)\|_{L^2}^{\frac{1}{2}}\|\nabla v(t)\|_{L^2}\|\sqrt{\rho}v_t(t)\|_{L^2}^{\frac{1}{2}}\|\sqrt{\rho}u_t(t)\|_{L^2}^{\frac{1}{2}}.$$

Together with inequalities (3.36)–(3.40), this yields Lemma 3.5. □

Now let us investigate the decay properties of the second order space derivatives or of one time derivative of u. They are described by the following proposition.

Proposition 3.2. *For any T greater than or equal to $T_2(\rho_0, u_0) \overset{\text{def}}{=} \max\{T_1, E_2/E_1\}$ and under the assumptions of Theorem 3.1, inequality (3.4) holds.*

Proof. We follow the same lines as the proof of inequality (3.14) with the following computations. Using relation (3.33), we get that for any positive t_0 and t such that t is greater than or equal to t_0,

$$\begin{aligned}&\frac{d}{dt}((t-t_0)\|\sqrt{\rho}u_t(t)\|_{L^2}^2) + (t-t_0)\|\nabla u_t\|_{L^2}^2\\ &\quad\lesssim \|\sqrt{\rho}u_t(t)\|_{L^2}^2 + \widetilde{F}_{1,\mathcal{T}}(u(t))(t-t_0)\|\sqrt{\rho}u_t(t)\|_{L^2}^2 + (t-t_0)F_{2,\mathcal{T}}(u(t),u(t)).\end{aligned}$$

Gronwall lemma along with (3.9) and (3.31) implies that

$$\begin{aligned}(t-t_0)\|\sqrt{\rho}u_t(t)\|_{L^2}^2 &\lesssim \left(\int_{t_0}^t \|\sqrt{\rho}u_t(t')\|_{L^2}^2\,dt' + \int_{t_0}^t (t'-t_0)F_{2,\mathcal{T}}(u(t'),u(t'))\,dt'\right)\\ &\quad\times\exp\left(\|\sqrt{\rho}u(t_0)\|_{L^2}^2 + \|u(t_0)\|_{L^2}^4 + \frac{\|\nabla u(t_0)\|_{L^2}^2}{\mathcal{T}}\right).\end{aligned} \tag{3.41}$$

It follows from (3.32) that

$$\begin{aligned}\int_{t_0}^t (t'-t_0)F_{2,\mathcal{T}}(u(t'),u(t'))\,dt' &\leqslant t\int_{t_0}^t F_{2,\mathcal{T}}(u(t'),u(t'))\,dt'\\ &\lesssim t(\|\nabla u(t_0)\|_{L^2}^2 + \mathcal{T})\|\nabla u(t_0)\|_{L^2}^2.\end{aligned}$$

Resuming the above estimates into (3.41) and choosing $\mathcal{T} = \|\nabla u(t_0)\|_{L^2}^2$ yields

$$(t-t_0)\|\sqrt{\rho}u_t(t)\|_{L^2}^2 \lesssim \|\nabla u(t_0)\|_{L^2}^2 + t\|\nabla u(t_0)\|_{L^2}^4.$$

Taking t_0 equals to $\frac{t}{2}$ in the above inequality, then inequality (3.18) of Proposition 3.1 ensures that

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2 \lesssim E_1\frac{T}{t}\frac{1}{T}\langle t\rangle_T^{-3} + E_1^2\langle t\rangle_T^{-6}.$$

Using Corollary 3.1, we infer that for $t \geqslant T_2$

$$\|\sqrt{\rho}u_t(t)\|_{L^2}^2 \lesssim (E_2 + E_1^2 + E_1/T)\langle t\rangle_T^{-4} \lesssim E_2\langle t\rangle_T^{-4}. \tag{3.42}$$

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While it follows from Lemma 3.2 that

$$\|\nabla^2 u(t)\|_{L^2} + \|\nabla\Pi(t)\|_{L^2} \lesssim \|u_t(t)\|_{L^2} + \|u(t)\|_{L^2}\|\nabla u(t)\|_{L^2}^2,$$

which together with inequalities (3.2), (3.18) and (3.42) leads to inequality (3.4). The proposition is proved. □

Corollary 3.2. *Under the assumptions of Theorem 3.1, we have*

$$\|u(t)\|_{L^\infty} \lesssim E_0^{\frac14} E_2^{\frac14} \langle t\rangle_T^{-(\frac32)} \quad \text{and} \tag{3.43}$$

$$\int_t^\infty \|\nabla u(t')\|_{L^\infty}\,dt' \lesssim (\sqrt{E_2}T)^{\frac34}\langle t\rangle_T^{-1} + (\sqrt{E_2}T)^{\frac14}\langle t\rangle_T^{-2}. \tag{3.44}$$

Moreover, we have the following estimates on the density. For any p in $[2,\infty]$, we have

$$\|\nabla\rho(t)\|_{L^p} \lesssim \|\nabla\rho_0\|_{L^p}, \tag{3.45}$$

$$\|\rho_t(t)\|_{L^p} \lesssim \|\nabla\rho_0\|_{L^p} E_0^{\frac14} E_2^{\frac14}\langle t\rangle_T^{-(\frac32)}, \tag{3.46}$$

$$\|\nabla^2\rho(t)\|_{L^2} \lesssim \|\nabla^2\rho_0\|_{L^2} + \|\nabla\rho_0\|_{L^\infty} E_2^{\frac12} T \quad \text{and} \tag{3.47}$$

$$\|\nabla\rho_t(t)\|_{L^2} \lesssim (E_2^{\frac14}\|\nabla^2\rho_0\|_{L^2} + \|\nabla\rho_0\|_{L^\infty}(E_2^{\frac34}T + E_1^{\frac12}))\langle t\rangle_T^{-(\frac32)}. \tag{3.48}$$

Proof. Inequality (3.43) follows directly from inequalities (3.2) and (3.4) and from the interpolation inequality (3.38). By using 2D interpolation inequality, we get, by applying Lemma 3.2 with p equal to 4, that

$$\begin{aligned}\|\nabla u(t)\|_{L^\infty} &\leqslant C\|\nabla u(t)\|_{L^4}^{\frac12}\|\nabla^2 u(t)\|_{L^4}^{\frac12}\\ &\leqslant C\|\nabla u(t)\|_{L^4}^{\frac12}(\|u_t\|_{L^4}^{\frac12} + C\|u\|_{L^8}\|\nabla u(t)\|_{L^2}^{\frac12}).\end{aligned}$$

Using that $\|u(t)\|_{L^8} \leqslant C\|u(t)\|_{L^2}^{\frac14}\|\nabla u(t)\|_{L^2}^{\frac34}$, we infer that

$$\|\nabla u(t)\|_{L^\infty} \leqslant C\|\nabla u(t)\|_{L^2}^{\frac14}\|\nabla^2 u(t)\|_{L^2}^{\frac14}\|\sqrt{\rho}u_t\|_{L^2}^{\frac14}\|\nabla u_t\|_{L^2}^{\frac14} + C\|\nabla u(t)\|_{L^2}^{\frac32}\|u(t)\|_{L^2}^{\frac14}\|\nabla^2 u(t)\|_{L^2}^{\frac14}.$$

Applying Hölder inequality with respectively $(\frac18, \frac34, \frac18)$ and $(\frac34, \frac14)$ gives

$$\begin{aligned}\int_t^\infty \|\nabla u(t')\|_{L^\infty}\,dt' &\lesssim \Big(\int_t^\infty \|\nabla u(t')\|_{L^2}^2\,dt'\Big)^{\frac34}\Big(\int_t^\infty \|u(t')\|_{L^2}\|\nabla^2 u(t')\|_{L^2}\,dt'\Big)^{\frac14}\\ &+ \Big(\int_t^\infty \|\nabla u(t')\|_{L^2}^2\,dt'\Big)^{\frac18}\Big(\int_t^\infty \|\nabla^2 u(t')\|_{L^2}^{\frac13}\|\sqrt{\rho}u_t(t')\|_{L^2}^{\frac13}\,dt'\Big)^{\frac34}\Big(\int_t^\infty \|\nabla u_t(t')\|_{L^2}^2\,dt'\Big)^{\frac18},\end{aligned}$$

which together with (3.9), (3.2)–(3.4) and Corollary 3.1 ensures (3.44).

Inequality (3.45) comes simply from the density equation after differentiation which is

$$\partial_t\nabla\rho + u\cdot\nabla\nabla\rho = -\nabla u\cdot\nabla\rho. \tag{3.49}$$

Gronwall lemma and (3.44) allows to conclude the L^p estimate for $\nabla\rho$. For the inequality on ρ_t, let us observe that, thanks to inequalities (3.43) and (3.45), the transport equation implies that,

$$\|\rho_t(t)\|_{L^p} \leqslant \|u(t)\|_{L^\infty}\|\nabla\rho(t)\|_{L^p} \lesssim \|\nabla\rho_0\|_{L^p} E_0^{\frac14} E_2^{\frac14}\langle t\rangle_T^{-(\frac32)},$$

which is exactly the required inequality. In order to prove inequality (3.47), let us differentiate twice the transport equation which gives

$$\partial_t\partial_j\partial_k\rho + u\cdot\nabla\partial_j\partial_k\rho = -\partial_k u\cdot\nabla\partial_j\rho - \partial_j u\cdot\nabla\partial_k\rho - \partial_j\partial_k u\cdot\nabla\rho. \tag{3.50}$$

Let us observe that

$$\begin{aligned}\|\partial_k u(t)\cdot\nabla\partial_j\rho(t)\|_{L^2} &\leqslant \|\nabla u(t)\|_{L^\infty}\|\nabla^2\rho(t)\|_{L^2} \quad \text{and}\\ \|\partial_j\partial_k u(t)\cdot\nabla\rho(t)\|_{L^2} &\leqslant \|\nabla^2 u(t)\|_{L^2}\|\nabla\rho(t)\|_{L^\infty},\end{aligned}$$

so that we obtain

$$\frac{d}{dt}\|\nabla^2\rho(t)\|_{L^2} \leqslant 2\|\nabla u(t)\|_{L^\infty}\|\nabla^2\rho(t)\|_{L^2} + \|\nabla^2 u(t)\|_{L^2}\|\nabla\rho(t)\|_{L^\infty}. \tag{3.51}$$

Gronwall lemma along with Proposition 3.2 gives inequality (3.47). Finally it follows from inequality (3.49) that

$$\|\nabla\rho_t(t)\|_{L^2} \leqslant \|u(t)\|_{L^\infty}\|\nabla^2\rho(t)\|_{L^2} + \|\nabla u(t)\|_{L^2}\|\nabla\rho(t)\|_{L^\infty}.$$

Then inequality (3.48) follows from inequalities (3.43), (3.47) and (3.18). □

Let us remark that before inequality (3.45), we never use any regularity property for the density ρ. From now on, we shall do it in order to estimate the third derivatives of the velocity field.

Proposition 3.3. *For any T greater than or equal to $T_3(\rho_0, u_0) \overset{\text{def}}{=} \max\{T_2, E_2/E_3\}$, we have under the assumptions of Theorem 3.1,*

$$\|\nabla u_t(t)\|_{L^2}^2 + \int_t^\infty (\|u_{tt}(t')\|_{L^2}^2 + \|\nabla^2 u_t(t')\|_{L^2}^2 + \|\nabla\partial_t\Pi(t')\|_{L^2}^2)\,dt' \lesssim E_3\langle t\rangle_T^{-5}, \tag{3.52}$$

and for any non-negative t_0 and any $t \geqslant t_0$,

$$\int_{t_0}^t \langle t'\rangle_T^{5_-}(\|u_{tt}(t')\|_{L^2}^2 + \|\nabla^2 u_t(t')\|_{L^2}^2 + \|\nabla\partial_t\Pi(t')\|_{L^2}^2)\,dt' \lesssim (1+T)(1+\|\nabla\rho_0\|_{L^\infty}^2)E_3. \tag{3.53}$$

Here and in all that follows, $\mathfrak{a}_-$ denotes any number strictly less than $\mathfrak{a}$.

Proof. Relation (3.34) applies with $v = u$ and $f \equiv 0$ claims exactly that u_t is a solution of (LINS2D) with the external force

$$\widetilde{f} \overset{\text{def}}{=} -\rho_t u_t - \rho_t u\cdot\nabla u - \rho u_t\cdot\nabla u.$$

In order to apply Lemma 3.1, we have to estimate $\|\widetilde{f}(t)\|_{L^2}$. Hölder inequality and interpolation inequality allow to write

$$\begin{aligned}\|\widetilde{f}\|_{L^2}^2 &\leqslant 2\|\rho_t\|_{L^\infty}^2\|u_t\|_{L^2}^2 + 2\|\rho_t\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + 4\|u_t\|_{L^4}^2\|\nabla u\|_{L^4}^2\\ &\lesssim \|\rho_t\|_{L^\infty}^2\|u_t\|_{L^2}^2 + \|\rho_t\|_{L^\infty}^2\|u\|_{L^\infty}^2\|\nabla u\|_{L^2}^2 + \|\nabla u_t\|_{L^2}\|u_t\|_{L^2}\|\nabla^2 u\|_{L^2}\|\nabla u\|_{L^2}.\end{aligned}$$

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Using Corollary 3.2 and Proposition 3.2, we get

$$\|\widetilde{f}(t)\|_{L^2}^2 \lesssim \|\nabla\rho_0\|_{L^\infty}^2 E_2^{\frac{1}{2}}\langle t\rangle_T^{-3}\|u_t(t)\|_{L^2}^2 + \|\nabla\rho_0\|_{L^\infty}^2 E_2\langle t\rangle_T^{-6}\|\nabla u(t)\|_{L^2}^2 + E_2\langle t\rangle_T^{-4}\|\nabla u_t(t)\|_{L^2}\|\nabla u(t)\|_{L^2}. \quad (3.54)$$

Cauchy–Schwarz inequality gives

$$\begin{aligned}\int_{t_0}^t \|\widetilde{f}(t')\|_{L^2}^2\,dt' \lesssim{}& \|\nabla\rho_0\|_{L^\infty}^2 E_2^{\frac{1}{2}}\langle t_0\rangle_T^{-3}\int_{t_0}^t \|u_t(t')\|_{L^2}^2\,dt'\\ &+ \|\nabla\rho_0\|_{L^\infty}^2 E_2\langle t_0\rangle_T^{-6}\int_{t_0}^t \|\nabla u(t')\|_{L^2}^2\,dt'\\ &+ E_2\langle t_0\rangle_T^{-4}\left(\int_{t_0}^t \|\nabla u_t(t')\|_{L^2}^2\,dt'\right)^{\frac{1}{2}}\left(\int_{t_0}^t \|\nabla u(t')\|_{L^2}^2\,dt'\right)^{\frac{1}{2}}.\end{aligned}$$

Applying (3.12) and Corollary 3.1 leads to

$$\int_{t_0}^t \|\widetilde{f}(t')\|_{L^2}^2\,dt' \lesssim (\|\nabla\rho_0\|_{L^\infty}^2 E_2^{\frac{1}{2}}(E_1+E_2^{\frac{1}{2}}) + E_2^{\frac{3}{2}})\langle t_0\rangle_T^{-6} \lesssim E_3\langle t_0\rangle_T^{-6}. \quad (3.55)$$

Applying again (3.12) gives

$$\begin{aligned}&\|\nabla u_t(t)\|_{L^2}^2 + \int_{t_0}^\infty (\|u_{tt}(t')\|_{L^2}^2 + \|\nabla^2 u_t(t')\|_{L^2}^2 + \|\nabla\partial_t\Pi(t')\|_{L^2}^2)\,dt'\\ &\lesssim \|\nabla u_t(t_0)\|_{L^2}^2 + \int_{t_0}^\infty \|\widetilde{f}(t')\|_{L^2}^2\,dt'. \quad (3.56)\end{aligned}$$

In particular, (3.55) and (3.56) ensure that

$$\|\nabla u_t(t)\|_{L^2}^2 \lesssim E_3. \quad (3.57)$$

While it follows from (3.14) and (3.55) that

$$\begin{aligned}t\|\nabla u_t(t)\|_{L^2}^2 &\lesssim \|u_t(t/2)\|_{L^2}^2 + t\int_{t/2}^\infty \|\widetilde{f}(t')\|_{L^2}^2\,dt'\\ &\lesssim E_2\langle t\rangle_T^{-4} + E_3 t\langle t\rangle_T^{-6},\end{aligned}$$

which together with (3.57) implies that if T greater than or equal to E_2/E_3

$$\|\nabla u_t(t)\|_{L^2}^2 \lesssim \max(E_3, E_2/T)\langle t\rangle_T^{-5} + E_3\langle t\rangle_T^{-6} \lesssim E_3\langle t\rangle_T^{-5}.$$

Resuming the above estimate and (3.55) into (3.56) gives

$$\int_t^\infty (\|u_{tt}(t')\|_{L^2}^2 + \|\nabla^2 u_t(t')\|_{L^2}^2 + \|\nabla\partial_t\Pi(t')\|_{L^2}^2)\,dt' \lesssim E_3\langle t\rangle_T^{-5}.$$

This proves inequality (3.52). (3.52) together with (3.54) implies that

$$\|\widetilde{f}(t)\|_{L^2}^2 \lesssim (1+\|\nabla\rho_0\|_{L^\infty}^2)E_3\langle t\rangle_T^{-7}.$$

In order to prove (3.53), we apply (3.13) for any s less than 5 and v equal to u_t to get

$$\langle t\rangle_T^{5-}\|\nabla u_t(t)\|_{L^2}^2+\int_{t_0}^t\langle t'\rangle_T^{5-}(\|u_{tt}(t')\|_{L^2}^2+\|\nabla^2 u_t(t')\|_{L^2}^2+\|\nabla\partial_t\Pi(t')\|_{L^2}^2)\,dt'$$
$$\lesssim\langle t_0\rangle_T^{5-}\|\nabla u_t(t_0)\|_{L^2}^2+\int_{t_0}^t\langle t'\rangle_T^{5-}\|\widetilde{f}(t')\|_{L^2}^2\,dt'+\int_{t_0}^t\langle t'\rangle_T^{4-}\|\nabla u_t(t')\|_{L^2}^2\frac{dt'}{T},$$

which together with the fact that

$$\int_{t_0}^t\langle t'\rangle^{5-}\|\widetilde{f}(t')\|_{L^2}^2\,dt'\lesssim(1+\|\nabla\rho_0\|_{L^\infty}^2)E_3T\quad\text{and}$$
$$\int_{t_0}^t\langle t'\rangle_T^{4-}\|\nabla u_t(t')\|_{L^2}^2\frac{dt'}{T}\lesssim E_3\int_{t_0}^t\langle t'\rangle_T^{-1+}\frac{dt'}{T}\lesssim E_3$$

leads to estimate (3.53). This completes the proof of the proposition. □

Now let us translate the control of $\|\nabla u_t(t)\|_{L^2}$ in term of control of $\|\nabla^3 u(t)\|_{L^2}$.

Proposition 3.4. *Under the hypothesis of Theorem 3.1, for any $T\geqslant T_3(\rho_0,u_0)$, we have*

$$\|\nabla^3 u(t)\|_{L^2}^2+\|\nabla^2\Pi(t)\|_{L^2}^2\lesssim(1+\|\nabla\rho_0\|_{L^2})E_3\langle t\rangle_T^{-5}\log^2\langle t\rangle_T\quad\textit{and}\tag{3.58}$$
$$\|\nabla u(t)\|_{L^\infty}\lesssim(1+\|\nabla\rho_0\|_{L^2})^{\frac14}E_1^{\frac14}E_3^{\frac14}\langle t\rangle_T^{-2}\log^{\frac12}\langle t\rangle_T.\tag{3.59}$$

Proof. By differentiation of the momentum equation of (INS2D) with respect to the space variables, we get, by using Leibniz formula, that

$$\Delta\partial_j u-\partial_j\nabla\Pi=\rho\,\partial_j u_t+\partial_j\rho\,u_t+\partial_j\rho\,u\cdot\nabla u+\rho\,\partial_j u\cdot\nabla u+\rho u\cdot\nabla\partial_j u.$$

Applying Lemma 3.2 with $v=\partial_j u$ and $f=\partial_j\rho\,u_t+\partial_j\rho\,u\cdot\nabla u+\rho\,\partial_j u\cdot\nabla u$ gives

$$\|\Delta\partial_j u\|_{L^2}+\|\partial_j\nabla\Pi\|_{L^2}\leqslant C(\|\nabla u_t\|_{L^2}+\|f\|_{L^2}+\|u\|_{L^4}^2\|\nabla^2 u\|_{L^2}).$$

In view of Propositions 3.1 and 3.3, we infer that

$$\|\Delta\partial_j u(t)\|_{L^2}+\|\partial_j\nabla\Pi(t)\|_{L^2}\lesssim E_3^{\frac12}\langle t\rangle_T^{-(\frac52)}+E_1^{\frac12}E_2^{\frac12}\langle t\rangle_T^{-(\frac92)}+\|f(t)\|_{L^2}$$
$$\lesssim E_3^{\frac12}\langle t\rangle_T^{-(\frac52)}+\|f(t)\|_{L^2}.\tag{3.60}$$

Let us estimate $\|f(t)\|_{L^2}$. We write that

$$\|\partial_j\rho\,u\cdot\nabla u\|_{L^2}\leqslant\|\nabla\rho\|_{L^\infty}\|u\|_{L^\infty}\|\nabla u\|_{L^2}\quad\text{and}$$
$$\|\rho\,\partial_j u\cdot\nabla u\|_{L^2}\leqslant\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2}.$$

Applying Propositions 3.1 and 3.3, we infer that

$$\|\partial_j\rho\,u\cdot\nabla u\|_{L^2}+\|\rho\,\partial_j u\cdot\nabla u\|_{L^2}\lesssim(\|\nabla\rho_0\|_{L^\infty}E_1^{\frac12}E_2^{\frac14}+E_1^{\frac12}E_2^{\frac12})\langle t\rangle_T^{-3}$$
$$\lesssim E_3^{\frac12}\langle t\rangle_T^{-3}.\tag{3.61}$$

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The linear term $\partial_j \rho\, u_t$ is more delicate to estimate. Let us write that, using Hölder inequality and interpolation inequality, for any ϵ in $]0,1[$,

$$\begin{aligned}\|\partial_j \rho u_t\|_{L^2} &\leqslant \|\nabla\rho\|_{L^{\frac{2}{1-\epsilon}}}\|u_t\|_{L^{\frac{2}{\epsilon}}}\\ &\leqslant \frac{C}{\epsilon}\|\nabla\rho\|_{L^{\frac{2}{1-\epsilon}}}\|u_t\|_{L^2}^{\epsilon}\|\nabla u_t\|_{L^2}^{1-\epsilon}.\end{aligned}$$

Using Propositions 3.2 and 3.3, and estimates (3.45), we get that, for any ϵ in $]0,1[$,

$$\|\partial_j \rho\, u_t\|_{L^2} \leqslant (\|\nabla\rho_0\|_{L^\infty}^2 E_2)^{\frac{\epsilon}{2}}(\|\nabla\rho_0\|_{L^2}^2 E_3)^{\frac{1-\epsilon}{2}}\langle t\rangle_T^{-\frac{5}{2}} \times \frac{1}{\epsilon}\langle t\rangle_T^{\frac{\epsilon}{2}}.$$

By convexity inequality we get that for any ϵ in $]0,1[$,

$$(\|\nabla\rho_0\|_{L^\infty}^2 E_2)^{\frac{\epsilon}{2}}(\|\nabla\rho_0\|_{L^2}^2 E_3)^{\frac{1-\epsilon}{2}} \leqslant \|\nabla\rho_0\|_{L^\infty} E_2^{\frac{1}{2}} + \|\nabla\rho_0\|_{L^2} E_3^{\frac{1}{2}} \leqslant (1+\|\nabla\rho_0\|_{L^2})E_3^{\frac{1}{2}}.$$

Thus, for any ϵ in $]0,1[$,

$$\|\partial_j \rho u_t\|_{L^2} \lesssim (1+\|\nabla\rho_0\|_{L^2})E_3^{\frac{1}{2}}\langle t\rangle_T^{-\frac{5}{2}} \times \frac{1}{\epsilon}\langle t\rangle_T^{\frac{\epsilon}{2}}. \tag{3.62}$$

Choosing ϵ equal to $\log^{-1}\langle t\rangle_T$ in (3.62), and then substituting the resulting inequality and inequality (3.61) into (3.60) leads to (3.58).

Finally (3.59) follows from interpolation inequality (3.38), and (3.18), (3.58). This finishes the proof of Proposition 3.4. □

By summarizing Propositions 3.1, 3.2, 3.3 and 3.4, we conclude the proof of Theorem 3.1.

3.4. Decay of solutions to (2.2)

Applying Theorem 3.1 to the System (2.2) leads to the following theorem:

Theorem 3.2. *Let $(\rho^{\mathrm{h}}, v^{\mathrm{h}}, \nabla_{\mathrm{h}}\Pi^{\mathrm{h}})$ be the smooth solution of (2.2). Then under the assumptions of Theorem 1.2, we have*

$$\begin{aligned}&\langle t\rangle\|v^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \langle t\rangle^{\frac{3}{2}}(\|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \|v^{\mathrm{h}}(t,\cdot,z)\|_{L^\infty_{\mathrm{h}}})\\ &\quad + \langle t\rangle^2(\|v^{\mathrm{h}}_t(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \|\nabla^2_{\mathrm{h}} v^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \|\nabla_{\mathrm{h}}\Pi^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}})\\ &\quad + \langle t\rangle^{\frac{5}{2}}\log^{-1}\langle t\rangle(\|\nabla^3_{\mathrm{h}} v^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \|\nabla^2_{\mathrm{h}}\Pi^{\mathrm{h}}(t,\cdot,z)\|_{L^2_{\mathrm{h}}})\\ &\quad + \langle t\rangle^{\frac{5}{2}}\|\nabla_{\mathrm{h}} v^{\mathrm{h}}_t(t,\cdot,z)\|_{L^2_{\mathrm{h}}} + \langle t\rangle^2\log^{-\frac{1}{2}}\langle t\rangle\|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t,\cdot,z)\|_{L^\infty_{\mathrm{h}}} \leqslant \mathcal{C}_0 h(z),\end{aligned} \tag{3.63}$$

and

$$\int_{t_0}^t \langle t'\rangle_T^{5-}(\|\partial_t^2 v^{\mathrm{h}}(t')\|^2_{L^2_{\mathrm{h}}} + \|\nabla^2_{\mathrm{h}} v^{\mathrm{h}}_t(t')\|^2_{L^2_{\mathrm{h}}} + \|\nabla_{\mathrm{h}}\partial_t\Pi^{\mathrm{h}}(t')\|^2_{L^2_{\mathrm{h}}})\,dt' \lesssim \mathcal{C}_0 h^2(z). \tag{3.64}$$

We also have

$$\|\nabla^4_{\mathrm{h}} v^{\mathrm{h}}(\cdot,\cdot,z)\|_{L^1_t(L^2_{\mathrm{h}})} \leqslant \mathcal{C}_0 h(z) \quad \textit{and} \tag{3.65}$$

$$\|\varrho^h(t,\cdot,z)\|_{H^4_{\mathrm{h}}} + \langle t\rangle^{\frac{3}{2}}\|\rho_t(t,\cdot,z)\|_{H^3_{\mathrm{h}}} \leqslant \mathcal{C}_0\eta h(z). \tag{3.66}$$

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Here and in what follows, we always denote $\varrho^{\mathrm{h}} \overset{\mathrm{def}}{=} \rho^h - 1$ *and* $h(z)$ *to be a generic positive function which belongs to* $L^2_{\mathrm{v}} \cap L^\infty_{\mathrm{v}}$.

Proof. (3.63) and (3.64) follows directly from Theorem 3.1 and (3.53). In order to prove (3.65), we get, by applying (3.63) and [5, Theorem 3.14], that

$$\|\varrho(t)\|_{L^\infty_t(H^3_{\mathrm{h}})} \lesssim \|\varrho_0\|_{H^3_{\mathrm{h}}} \exp\left(C\int_0^t \|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t')\|_{H^2_{\mathrm{h}}}\,dt'\right) \leqslant \mathcal{C}_0\eta h(z). \tag{3.67}$$

Whereas we deduce from the momentum equation of (2.2) and the classical estimates on Stokes operator that

$$\|\nabla^4_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} + \|\nabla^3_{\mathrm{h}}\Pi^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} \lesssim \|\nabla^2_{\mathrm{h}}(\rho^{\mathrm{h}}\partial_t v^{\mathrm{h}})(t)\|_{L^2_{\mathrm{h}}} + \|\nabla^2_{\mathrm{h}}(\rho^{\mathrm{h}} v^{\mathrm{h}}\cdot\nabla_{\mathrm{h}} v^{\mathrm{h}})(t)\|_{L^2_{\mathrm{h}}},$$

which together with (3.63) and (3.67) ensures that

$$\|\nabla^4_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} + \|\nabla^3_{\mathrm{h}}\Pi^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} \leqslant \mathcal{C}_0(h(z)\langle t\rangle^{-\left(\frac{5}{2}\right)} + \|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}\partial_t v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} + \|\nabla^2_{\mathrm{h}}\partial_t v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}}). \tag{3.68}$$

Note that for any ϵ in $]0,1[$, we have

$$\begin{aligned}\|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}\partial_t v^{\mathrm{h}}\|_{L^2_{\mathrm{h}}} &\lesssim \|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}\|_{L_{\mathrm{h}}^{\frac{2}{1-\epsilon}}}\|\partial_t v^{\mathrm{h}}\|_{L_{\mathrm{h}}^{\frac{2}{\epsilon}}}\\ &\lesssim \frac{1}{\epsilon}\|\nabla^3_{\mathrm{h}}\rho^{\mathrm{h}}\|^{\epsilon}_{L^2_{\mathrm{h}}}\|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}\|^{1-\epsilon}_{L^2_{\mathrm{h}}}\|\partial_t v^{\mathrm{h}}\|^{\epsilon}_{L^2_{\mathrm{h}}}\|\nabla_{\mathrm{h}}\partial_t v^{\mathrm{h}}\|^{1-\epsilon}_{L^2_{\mathrm{h}}}.\end{aligned}$$

Applying (3.63), (3.67) and using a similar derivation of (3.62) gives

$$\|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}\partial_t v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} \leqslant \mathcal{C}_0 h(z)\langle t\rangle^{-\left(\frac{5}{2}\right)}\log\langle t\rangle.$$

Resuming the above estimate into (3.68) and using (3.64), we infer (3.65).

With (3.63) and (3.65), we deduce from [5, Theorem 3.14] that

$$\|\varrho^h\|_{L^\infty_t(H^4_{\mathrm{h}})} \leqslant \|\varrho_0\|_{H^4_{\mathrm{h}}} \exp\left(C\int_0^t \|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t')\|_{H^3_{\mathrm{h}}}\,dt'\right) \leqslant \mathcal{C}_0\eta h(z). \tag{3.69}$$

Then by taking one more horizontal derivative to (3.50) and using (3.63) and (3.69), we get

$$\begin{aligned}\|\nabla^3_{\mathrm{h}}\partial_t\rho^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} \leqslant{}& \|v^{\mathrm{h}}(t)\|_{L^\infty_{\mathrm{h}}}\|\nabla^4_{\mathrm{h}}\rho^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}} + 3\|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{L^\infty_{\mathrm{h}}}\|\nabla^3_{\mathrm{h}}\rho^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}}\\ &+ 3\|\nabla^2_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}}\|\nabla^2_{\mathrm{h}}\rho^{\mathrm{h}}(t)\|_{L^\infty_{\mathrm{h}}} + \|\nabla^3_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{L^2_{\mathrm{h}}}\|\nabla_{\mathrm{h}}\rho^{\mathrm{h}}(t)\|_{L^\infty_{\mathrm{h}}}\\ \leqslant{}& \mathcal{C}_0\eta h(z)\langle t\rangle^{-\left(\frac{3}{2}\right)}.\end{aligned}$$

This together with (3.46) and (3.69) ensures (3.66). This finishes the proof of Theorem 3.2. □

4. Estimates of v^{h} in terms of anisotropic Besov norms

In general, we have the following theorem concerning the decay estimates of solutions to (2.2), namely, $(\partial_z^\ell v^{\mathrm{h}}, \partial_z^\ell \Pi^{\mathrm{h}})$, for $\ell = 1, 2$, share the same decay properties as $(v^{\mathrm{h}}, \Pi^{\mathrm{h}})$ itself.

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Theorem 4.1. *Under the assumptions of Theorem 1.2, System (2.2) has a unique global solution* $(\rho^{\rm h}, v^{\rm h}, \nabla_{\rm h}\Pi^{\rm h})$ *so that there hold*

$$\begin{aligned}\sum_{\ell=0}^{2}\Big(&\langle t\rangle\|\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}+\langle t\rangle^{\frac32}(\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}+\|\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^\infty_{\rm h}})\\&+\langle t\rangle^2(\|\partial_z^\ell v^{\rm h}_t(t,\cdot,z)\|_{L^2_{\rm h}}+\|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}+\|\nabla_{\rm h}\partial_z^\ell\Pi^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}})\\&+\langle t\rangle^{\frac52}\log^{-1}\langle t\rangle(\|\nabla_{\rm h}^3\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}}+\|\nabla_{\rm h}^2\partial_z^\ell\Pi^{\rm h}(t,\cdot,z)\|_{L^2_{\rm h}})\\&+\langle t\rangle^{\frac52}\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}_t(t,\cdot,z)\|_{L^2_{\rm h}}+\langle t\rangle^2\log^{-\frac12}\langle t\rangle\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t,\cdot,z)\|_{L^\infty_{\rm h}}\Big)\leqslant\mathcal{C}_0h(z),\end{aligned}\tag{4.1}$$

and

$$\sum_{\ell=0}^{2}\int_0^t\langle t'\rangle^{5-}(\|\partial_t^2\partial_z^\ell v^{\rm h}(t')\|^2_{L^2_{\rm h}}+\|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}_t(t')\|^2_{L^2_{\rm h}}+\|\nabla_{\rm h}\partial_z^\ell\partial_t\Pi^{\rm h}(t')\|^2_{L^2_{\rm h}})\,dt'\leqslant\mathcal{C}_0h^2(z).\tag{4.2}$$

We also have

$$\begin{aligned}&\|\varrho^{\rm h}(t,\cdot,z)\|_{H^4_{\rm h}}+\|\rho^{\rm h}_z(t,\cdot,z)\|_{H^3_{\rm h}}+\|\rho^{\rm h}_{zz}(t,\cdot,z)\|_{H^2_{\rm h}}\\&\quad+\langle t\rangle^{\frac32}(\|\rho^{\rm h}_t(t,\cdot,z)\|_{H^3_{\rm h}}+\|\partial_z\rho^{\rm h}_t(t,\cdot,z)\|_{H^2_{\rm h}}+\|\partial_z^2\rho^{\rm h}_t(t,\cdot,z)\|_{H^1_{\rm h}})\leqslant\mathcal{C}_0\eta h(z).\end{aligned}\tag{4.3}$$

Let us remark that: except cumbersome calculations, the proof of the above theorem for the case: $\ell=1,2$, follows exactly the same line as that of Theorem 3.1. For a clear presentation, we choose to skip the details here. Instead we just outline the proof for the case when $\ell=1$ (one may check [14, §§ 4 and 5] for details).

In view of (2.2), the quantity $(\rho^{\rm h}_z, v^{\rm h}_z, \Pi^{\rm h}_z)\stackrel{\rm def}{=}(\partial_z\rho^{\rm h}, \partial_z v^{\rm h}, \partial_z\Pi^{\rm h})$ satisfies the system in $\mathbb{R}^+\times\mathbb{R}^2$

$$(\text{D1INS2D})\quad\begin{cases}\partial_t\rho^{\rm h}_z+v^{\rm h}\cdot\nabla_{\rm h}\rho^{\rm h}_z=-v^{\rm h}_z\cdot\nabla_{\rm h}\rho^{\rm h},\\ \rho^{\rm h}\partial_t v^{\rm h}_z+\rho^{\rm h}v^{\rm h}\cdot\nabla_{\rm h}v^{\rm h}_z-\Delta v^{\rm h}_z+\nabla_{\rm h}\Pi^{\rm h}_z=f_1+L(t)v^{\rm h}_z,\\ \operatorname{div}_{\rm h}v^{\rm h}_z=0,\\ (\rho_z, v^{\rm h}_z)|_{t=0}=(\eta\partial_z\varsigma_0, \partial_z v^{\rm h}_0).\end{cases}\quad\text{with}$$

$$f_1=-\rho^{\rm h}_z v^{\rm h}_t-\rho^{\rm h}_z v^{\rm h}\cdot\nabla_{\rm h}v^{\rm h}\quad\text{and}\quad L(t)w\stackrel{\rm def}{=}-\rho^{\rm h}w\cdot\nabla_{\rm h}v^{\rm h}.$$

The external force f_1 contains term with $\rho^{\rm h}_z$. We want of course global estimate. But the control of L^p norm of $\rho^{\rm h}_z$ demands the control of $v^{\rm h}_z$ in $L^1(\mathbb{R}^+;L^\infty)$ which will be proved at the end. Thus we argue with a continuation argument. More precisely, we shall first prove the decay estimates for $v^{\rm h}_z$ that are valid for t less than $T^\star_1$ defined by

$$T^\star_1\stackrel{\rm def}{=}\sup\{t\,/\,\|\rho^{\rm h}_z\|_{L^\infty([0,t];L^2_{\rm h}\cap L^\infty_{\rm h})}\leqslant 1\}.\tag{4.4}$$

The first step of the study is the proof that $\partial_z v^{\rm h}$ has the same decay property as $v^{\rm h}$ for the $L^2_{\rm h}$ norm, namely,

$$\|v^{\rm h}_z(t)\|^2_{L^2_{\rm h}}\leqslant\mathcal{C}_0h^2(z)\langle t\rangle^{-2}\quad\text{for }t\leqslant T^\star_1.\tag{4.5}$$

As in the proof of Proposition 3.1, the main idea to prove (4.5) is still based on Wiegner's approach in [33]. We first notice that

$$\|L(t)\|_{\mathcal{L}(L^2_{\rm h})} \leqslant 2\|\nabla_{\rm h} v^{\rm h}(t)\|_{L^\infty}. \tag{4.6}$$

Thus thanks to (3.63), $\|L(t)\|_{\mathcal{L}(L^2_{\rm h})}$ is integrable on $\mathbb{R}^+$. According to the remark at the beginning of § 3.1, we introduce

$$\widetilde{v}^{\rm h}_z(t) \stackrel{\rm def}{=} v^{\rm h}_z(t) \exp\left(-\int_0^t \|L(t')\|_{L^\infty}\,dt'\right).$$

Then the energy estimate (3.7) applied to the System (D1INS2D) gives

$$\frac{1}{2}\frac{d}{dt}\|\sqrt{\rho}^{\rm h}\widetilde{v}^{\rm h}_z(t)\|^2_{L^2_{\rm h}} + \|\nabla_{\rm h}\widetilde{v}^{\rm h}_z\|^2_{L^2_{\rm h}} \leqslant (f_1|\widetilde{v}^{\rm h}_z)_{L^2_{\rm h}}.$$

Then the key point of the proof will be again the estimate of $\|\widetilde{v}^{\rm h}_{z,\flat}(t)\|_{L^2_{\rm h}}$, which needs the decay estimate (3.63) and the differentiated Identity of (3.22) with respect to the parameter z.

With (4.5), we apply energy estimates of § 3.1 to get the decay of higher order derivatives of $v^{\rm h}_z$, which implies that $T_1^\star$ given by relation (4.4) equals to ∞, and there hold (4.1) and (4.2) for $\ell = 1$. Then as in the proof of (3.66), we can use [5, Theorem 3.14] to prove

$$\|\rho^{\rm h}_z(t,\cdot,z)\|_{H^3_{\rm h}} + \langle t\rangle^{\frac{3}{2}}\|\partial_t\rho^{\rm h}_z(t,\cdot,z)\|_{H^2_{\rm h}} \leqslant \mathcal{C}_0\eta h(z). \tag{4.7}$$

This leads to Theorem 4.1 for $\ell = 1$.

Now we shall transform the above decay estimates of $v^{\rm h}$ to the L^1 or L^2 in time estimate of the Besov norms to $v^{\rm h}$. For the convenience of the readers, we recall the following anisotropic Bernstein type lemma from [11, 26]:

Lemma 4.1. *Let $\mathcal{B}_{\rm h}$ (respectively $\mathcal{B}_{\rm v}$) a ball of $\mathbb{R}^2_{\rm h}$ (respectively $\mathbb{R}_{\rm v}$), and $\mathcal{C}_{\rm h}$ (respectively $\mathcal{C}_{\rm v}$) a ring of $\mathbb{R}^2_{\rm h}$ (respectively $\mathbb{R}_{\rm v}$); let $1 \leqslant p_2 \leqslant p_1 \leqslant \infty$ and $1 \leqslant q_2 \leqslant q_1 \leqslant \infty$. Then there holds:*

If the support of $\widehat{a}$ is included in $2^k\mathcal{B}_{\rm h}$, then

$$\|\partial^\alpha_{\rm h} a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})} \lesssim 2^{k\left(|\alpha|+2\left(\frac{1}{p_2}-\frac{1}{p_1}\right)\right)}\|a\|_{L^{p_2}_{\rm h}(L^{q_1}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{B}_{\rm v}$, then

$$\|\partial^\beta_z a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})} \lesssim 2^{\ell\left(\beta+(\frac{1}{q_2}-\frac{1}{q_1})\right)}\|a\|_{L^{p_1}_{\rm h}(L^{q_2}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^k\mathcal{C}_{\rm h}$, then

$$\|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})} \lesssim 2^{-kN}\sup_{|\alpha|=N}\|\partial^\alpha_{\rm h} a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}.$$

If the support of $\widehat{a}$ is included in $2^\ell\mathcal{C}_{\rm v}$, then

$$\|a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})} \lesssim 2^{-\ell N}\|\partial^N_z a\|_{L^{p_1}_{\rm h}(L^{q_1}_{\rm v})}.$$

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In view of Definition 2.1, as a corollary of Lemma 4.1, we have the following inequality, if $1 \leqslant p_2 \leqslant p_1$,

$$\|a\|_{\mathcal{B}_{p_1}^{s_1-2\left(\frac{1}{p_2}-\frac{1}{p_1}\right), s_2-\left(\frac{1}{p_2}-\frac{1}{p_1}\right)}} \lesssim \|a\|_{\mathcal{B}_{p_2}^{s_1,s_2}}. \tag{4.8}$$

To consider the product of a distribution in the isotropic Sobolev space with a distribution in the anisotropic Besov space, we need the following interpolation inequalities:

Lemma 4.2. *We have the following interpolation inequality for the L^∞ norm.*

$$\|f\|_{L^\infty} \lesssim \|f\|^{\frac{2}{3}}_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}} \|\nabla_{\mathrm{h}} f\|^{\frac{1}{3}}_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}. \tag{4.9}$$

Moreover, let $1 < q \leqslant p \leqslant \infty$, $-2/q + 2/p < s_1 < 1 - (2/q - 2/p)$ and s_2 in $]0,1[$, one has

$$\begin{aligned} \|f\|_{\mathcal{B}_p^{s_1,s_2}} \leqslant\ & C_{p,q} \|f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}^{\left(1-s_1-2\left(\frac{1}{q}-\frac{1}{p}\right)\right)(1-s_2)} \|\partial_z f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}^{\left(1-s_1-2\left(\frac{1}{q}-\frac{1}{p}\right)\right)s_2} \\ & \times \|\nabla_{\mathrm{h}} f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}^{\left(s_1+2\left(\frac{1}{q}-\frac{1}{p}\right)\right)(1-s_2)} \|\nabla_{\mathrm{h}}\partial_z f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}^{\left(s_1+2\left(\frac{1}{q}-\frac{1}{p}\right)\right)s_2}. \end{aligned} \tag{4.10}$$

Proof. In order to prove the first inequality, let us write according to Lemma 4.1 that

$$\begin{aligned} \|f\|_{L^\infty} &\leqslant \sum_{(k,\ell)\in\mathbb{Z}^2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^\infty} \\ &\lesssim 2^{\frac{K}{2}} \sum_{\substack{k<K\\ \ell\in\mathbb{Z}}} 2^{k\left(-\frac{1}{2}+\frac{2}{p}\right)} 2^{\frac{\ell}{p}} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^p} + 2^{-K} \sum_{\substack{k\geqslant K\\ \ell\in\mathbb{Z}}} 2^{k\frac{2}{p}} 2^{\frac{\ell}{p}} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} \nabla_{\mathrm{h}} f\|_{L^p} \\ &\lesssim 2^{\frac{K}{2}} \|f\|_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}} + 2^{-K} \|\nabla_{\mathrm{h}} f\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}. \end{aligned}$$

The appropriate choice of K ensures (4.9). Let us prove the second one. According to Definition 2.1, we have

$$\|f\|_{\mathcal{B}_p^{s_1,s_2}} = \sum_{k,\ell\in\mathbb{Z}^2} 2^{ks_1} 2^{\ell s_2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^p}.$$

For any integers K, L_1, which will be chosen late, we get, by applying Lemma 4.1, that

$$\begin{aligned} \sum_{k\leqslant K, \ell\leqslant L_1} 2^{ks_1} 2^{\ell s_2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^p} &\lesssim \sum_{k\leqslant K, \ell\leqslant L_1} 2^{k\left(s_1+\frac{2}{q}-\frac{2}{p}\right)} 2^{\ell s_2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})} \\ &\lesssim 2^{K\left(s_1+\frac{2}{q}-\frac{2}{p}\right)} 2^{L_1 s_2} \|f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}, \end{aligned}$$

by using the fact that s_1 is greater than $-2/q + 2/p$ and s_2 is positive.

Similarly since s_1 is greater than $-2/q + 2/p$ and s_2 is less than 1, one has

$$\begin{aligned} \sum_{k\leqslant K, \ell> L_1} 2^{ks_1} 2^{\ell s_2} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L^p} &\lesssim \sum_{k\leqslant K, \ell> L_1} 2^{k\left(s_1+\frac{2}{q}-\frac{2}{p}\right)} 2^{-\ell(1-s_2)} \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} \partial_z f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})} \\ &\lesssim 2^{K\left(s_1+\frac{2}{q}-\frac{2}{p}\right)} 2^{-L_1(1-s_2)} \|\partial_z f\|_{L^p_{\mathrm{v}}(L^q_{\mathrm{h}})}. \end{aligned}$$

Along the same line, since s_1 is less than $1-(2/q-2/p)$ and s_2 is in $]0,1[$, for some integer L_2 to be chosen hereafter, we write

$$\begin{aligned}\sum_{k>K,\ell\leqslant L_2} 2^{ks_1}2^{\ell s_2}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p} &\lesssim \sum_{k>K,\ell\leqslant L_2} 2^{-k\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}2^{\ell s_2}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)}\\ &\lesssim 2^{-K\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}2^{L_2 s_2}\|\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)},\end{aligned}$$

and

$$\begin{aligned}\sum_{k>K,\ell> L_2} 2^{ks_1}2^{\ell s_2}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v} f\|_{L^p} &\lesssim \sum_{k>K,\ell> L_2} 2^{-k\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}2^{-\ell(1-s_2)}\|\Delta_k^{\rm h}\Delta_\ell^{\rm v}\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}\\ &\lesssim 2^{-K\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}2^{-L_2(1-s_2)}\|\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}.\end{aligned}$$

As a consequence, we obtain

$$\begin{aligned}\|f\|_{\mathcal{B}_p^{s_1,s_2}} \lesssim\ & 2^{K\left(s_1+\frac{2}{q}-\frac{2}{p}\right)}2^{L_1 s_2}(\|f\|_{L_{\rm v}^p(L_{\rm h}^q)}+2^{-L_1}\|\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)})\\ &+2^{-K\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}2^{L_2 s_2}(\|\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)}+2^{-L_2}\|\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}).\end{aligned}$$

Taking L_1, L_2 in the above inequality so that

$$2^{L_1}\sim\frac{\|\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}}{\|f\|_{L_{\rm v}^p(L_{\rm h}^q)}}\quad\text{and}\quad 2^{L_2}\sim\frac{\|\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}}{\|\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)}},$$

we get

$$\begin{aligned}\|f\|_{\mathcal{B}_p^{s_1,s_2}} \lesssim\ & 2^{K\left(s_1+\frac{2}{q}-\frac{2}{p}\right)}\|f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{1-s_2}\|\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{s_2}\\ &+2^{-K\left(1-s_1-\frac{2}{q}+\frac{2}{p}\right)}\|\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{1-s_2}\|\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{s_2}.\end{aligned}$$

Taking K in the above inequality so that

$$2^K\sim\frac{\|\nabla_{\rm h} f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{1-s_2}\|\nabla_{\rm h}\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{s_2}}{\|f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{1-s_2}\|\partial_z f\|_{L_{\rm v}^p(L_{\rm h}^q)}^{s_2}}$$

gives rise to (4.10). This finishes the proof of Lemma 4.2. □

Lemma 4.3. *Let p be in $]2,\infty[$, s_1 and s_2 in $]0,1[$ and s' in $]2/p-1,2/p[$. Let $(\rho^{\rm h}, v^{\rm h}, \nabla_{\rm h}\Pi^{\rm h})$ be the global unique solution of (2.2). Then under the assumptions of Theorem 1.2, we have*

$$\|\varrho^{\rm h}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{s_1,s_2}\cap\mathcal{B}_2^{2+s_1,s_2})}+\|\rho_z^{\rm h}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{s_1,s_2}\cap\mathcal{B}_2^{1+s_1,s_2})}\leqslant\mathcal{C}_0\eta, \tag{4.11}$$

$$\|\rho_t^{\rm h}(t)\|_{\mathcal{B}_2^{s_1,s_2}\cap\mathcal{B}_2^{1+s_1,s_2}}+\|\partial_z\rho_t^{\rm h}(t)\|_{\mathcal{B}_2^{s_1,s_2}}\leqslant\mathcal{C}_0\eta\langle t\rangle^{-(\frac{3}{2})}, \tag{4.12}$$

$$\|v_t^{\rm h}(t)\|_{\mathcal{B}_2^{s_1,s_2}}+\|\partial_z v_t^{\rm h}(t)\|_{\mathcal{B}_2^{s_1,s_2}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(2+\frac{s_1}{2}\right)}, \tag{4.13}$$

$$\|v^{\rm h}(t)\|_{\mathcal{B}_p^{s_1,s_2}}+\|\partial_z v^{\rm h}(t)\|_{\mathcal{B}_p^{s_1,s_2}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{3}{2}+\frac{s_1}{2}-\frac{1}{p}\right)}, \tag{4.14}$$

$$\|v^{\rm h}(t)\|_{\mathcal{B}_p^{1+s_1,s_2}}+\|\partial_z v^{\rm h}(t)\|_{\mathcal{B}_p^{1+s_1,s_2}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(2+\frac{s_1}{2}-\frac{1}{p}\right)}\log^{\left(1-\frac{2}{p}\right)s_1}\langle t\rangle \quad \textit{and} \tag{4.15}$$

$$\sum_{\ell=0}^{1}(\|\partial_z^\ell v^{\rm h}(t)\|_{\mathcal{B}_p^{2+s',s_2}}+\|\nabla_{\rm h}\partial_z^\ell\Pi^{\rm h}(t)\|_{\mathcal{B}_p^{s',s_2}})\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{5}{2}+\frac{s'}{2}-\frac{1}{p}\right)}\log^{\left(1+s'-\frac{2}{p}\right)}\langle t\rangle. \tag{4.16}$$

Proof. The inequalities (4.11)–(4.13) follow directly from Lemma 4.2 and from inequalities (4.3) and (4.1). Whereas note that in two space dimension, there holds

$$\forall p\in\,]2,\infty[,\quad \|f\|_{L^p_{\rm h}}\lesssim \|f\|_{L^2_{\rm h}}^{\frac{2}{p}}\|\nabla_{\rm h}f\|_{L^2_{\rm h}}^{1-\frac{2}{p}}. \tag{4.17}$$

Then in view of (4.1), we infer for ℓ in $\{0,1,2\}$,

$$\begin{aligned}
\|\partial_z^\ell v^{\rm h}(t)\|_{L^p}&\lesssim \|\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\frac{2}{p}}\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{1-\frac{2}{p}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{3}{2}-\frac{1}{p}\right)},\\
\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t)\|_{L^p}&\lesssim \|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\frac{2}{p}}\|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{1-\frac{2}{p}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(2-\frac{1}{p}\right)}\quad\text{and}\\
\|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t)\|_{L^p}&\lesssim \|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\frac{2}{p}}\|\nabla_{\rm h}^3\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{1-\frac{2}{p}}\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{5}{2}-\frac{1}{p}\right)}\log^{1-\frac{2}{p}}\langle t\rangle.
\end{aligned}$$

Hence, by virtue of Lemma 4.2, we infer for ℓ in $\{0,1\}$ and $s_1,s_2\in\,]0,1[$,

$$\begin{aligned}
\|\partial_z^\ell v^{\rm h}(t)\|_{\mathcal{B}_p^{s_1,s_2}}&\leqslant \|\partial_z^\ell v^{\rm h}(t)\|_{L^p}^{(1-s_1)(1-s_2)}\|\partial_z^{\ell+1}v^{\rm h}(t)\|_{L^p}^{(1-s_1)s_2}\\
&\qquad\times\|\nabla_{\rm h}\partial_z^\ell v^{\rm h}(t)\|_{L^p}^{s_1(1-s_2)}\|\nabla_{\rm h}\partial_z^{\ell+1}v^{\rm h}(t)\|_{L^p}^{s_1s_2}\\
&\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{3}{2}-\frac{1}{p}\right)(1-s_1)}\times\langle t\rangle^{-\left(2-\frac{1}{p}\right)s_1}.
\end{aligned}$$

This proves (4.14). A similar argument yields (4.15).

Since s' is in $]2/p-1,2/p[$, by applying Lemma 4.2 and (4.1), we get for ℓ in $\{0,1\}$,

$$\begin{aligned}
\|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t)\|_{\mathcal{B}_p^{s',s_2}}&\leqslant \|\nabla_{\rm h}^2\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\left(\frac{2}{p}-s'\right)(1-s_2)}\|\nabla_{\rm h}^2\partial_z^{\ell+1}v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\left(\frac{2}{p}-s'\right)s_2}\\
&\qquad\times\|\nabla_{\rm h}^3\partial_z^\ell v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\left(1+s'-\frac{2}{p}\right)(1-s_2)}\|\nabla_{\rm h}^3\partial_z^{\ell+1}v^{\rm h}(t)\|_{L^p_{\rm v}(L^2_{\rm h})}^{\left(1+s'-\frac{2}{p}\right)s_2}\\
&\leqslant \mathcal{C}_0\langle t\rangle^{-\left(\frac{5}{2}+\frac{1}{2}\left(s'-\frac{2}{p}\right)\right)}\log^{\left(1+s'-\frac{2}{p}\right)}\langle t\rangle.
\end{aligned}$$

The same estimate holds for $\nabla_{\rm h}\partial_z^\ell\Pi^{\rm h}$. This leads to (4.16), and the proof of the lemma is complete. □

Remark 4.1. It is easy to observe that $a^{\rm h}$ satisfies the same estimate as (4.11), that is

$$\|a^{\rm h}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{s_1,s_2}\cap\mathcal{B}_2^{2+s_1,s_2})}+\|\partial_z a^{\rm h}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{s_1,s_2}\cap\mathcal{B}_2^{1+s_1,s_2})}\leqslant \mathcal{C}_0\eta, \tag{4.18}$$

for any $s_1,s_2\in\,]0,1[$.

Let us now turn to the proof of Proposition 2.1.

Proof of Proposition 2.1. It follows from Lemma 4.3 and interpolation inequality in Besov spaces that for ℓ in $\{0, 1\}$

$$\begin{aligned}
&\|\partial_z v^{\mathrm{h}}(t)\|_{\mathcal{B}_2^{\frac{3}{4},\frac{3}{4}}} \leqslant \mathcal{C}_0\langle t\rangle^{-(\frac{11}{8})},\\
&\|\partial_z^{\ell} v^{\mathrm{h}}(t)\|_{\mathcal{B}_2^{1,\frac{1}{2}}} \lesssim \|\partial_z^{\ell} v^{\mathrm{h}}(t)\|^{\frac{1}{2}}_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}} \|\nabla_{\mathrm{h}}\partial_z^{\ell} v^{\mathrm{h}}(t)\|^{\frac{1}{2}}_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}} \leqslant \mathcal{C}_0\langle t\rangle^{-(\frac{3}{2})} \quad \text{and}\\
&\|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t)\|_{\mathcal{B}_2^{1,\frac{1}{2}}} \lesssim \|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t)\|^{\frac{1}{2}}_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}} \|\nabla_{\mathrm{h}}^2 v^{\mathrm{h}}(t)\|^{\frac{1}{2}}_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}} \leqslant \mathcal{C}_0\langle t\rangle^{-2}\log^{\frac{1}{4}}\langle t\rangle.
\end{aligned} \tag{4.19}$$

This implies

$$\begin{aligned}
&\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} + \|\partial_z v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} + \|\partial_z v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}\\
&\quad + \|\partial_z v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{3}{4},\frac{3}{4}})} + \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} \leqslant \mathcal{C}_0.
\end{aligned} \tag{4.20}$$

It remains to handle $\|\partial_z \Pi^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}$. Indeed it is easy to observe from (2.1) that

$$\Delta_{\mathrm{h}}\Pi^{\mathrm{h}} = -\mathrm{div}_{\mathrm{h}}(a^{\mathrm{h}}(\nabla_{\mathrm{h}}\Pi^{\mathrm{h}} - \Delta_{\mathrm{h}} v^{\mathrm{h}})) - \mathrm{div}_{\mathrm{h}}\,\mathrm{div}_{\mathrm{h}}(v^{\mathrm{h}}\otimes v^{\mathrm{h}}), \tag{4.21}$$

from which, we deduce from the law of product (2.7) that

$$\begin{aligned}
\|\Pi^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} &\leqslant C(\|v^{\mathrm{h}}\otimes v^{\mathrm{h}}\|_{L^1\mathbb{R}^+;(\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}\\
&\quad + \|a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}(\|\Pi^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} + \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})})).
\end{aligned}$$

Whereas it follows from (4.18) that

$$\|a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} + \|\partial_z a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} \leqslant \mathcal{C}_0\eta. \tag{4.22}$$

When we take η so small that $C\mathcal{C}_0\eta \leqslant \frac{1}{2}$, (4.22) implies

$$\begin{aligned}
\|\Pi^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} &\leqslant C\Bigg(\|a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}\|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}\\
&\quad + \|v^{\mathrm{h}}\|_{L^\infty_t(\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}\|v^{\mathrm{h}}\|_{L^1(\mathcal{B}_2^{1,\frac{1}{2}})}\Bigg) \leqslant \mathcal{C}_0.
\end{aligned} \tag{4.23}$$

Similar to the proof of (4.23), we also have

$$\begin{aligned}
\|\Pi^{\mathrm{h}}_z\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} &\leqslant C\Bigg(\|\partial_z a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}(\|\Pi^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} + \|\nabla_{\mathrm{h}} v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})})\\
&\quad + \|a^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}\|\nabla_{\mathrm{h}}\partial_z v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} + \|\partial_z v^{\mathrm{h}}\|_{L^\infty_t(\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}\|v^{\mathrm{h}}\|_{L^1(\mathcal{B}_2^{1,\frac{1}{2}})}\Bigg).
\end{aligned}$$

Hence by virtue of Lemma 4.3, (4.22) and (4.23), we infer

$$\|\Pi^{\mathrm{h}}_z\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} \leqslant \mathcal{C}_0.$$

Together with (4.20), we complete the proof of Proposition 2.1. □

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5. The equation on w_ε and estimates of some error terms

The purpose of this section is to study the equation that determines the correction term in $u_{\varepsilon,\text{app}}$. Let us recall it.

$$\begin{cases} \partial_t w_\varepsilon^{\text{h}} - \Delta_\varepsilon w_\varepsilon^{\text{h}} = -\nabla_{\text{h}} \Pi_\varepsilon^1 \\ \partial_t w_\varepsilon^3 - \Delta_\varepsilon w_\varepsilon^3 = -\varepsilon^2 \partial_z \Pi_\varepsilon^1 + \partial_z \Pi_L^{\text{h}} \quad \text{with } \Pi_L^{\text{h}} \stackrel{\text{def}}{=} -\Delta_{\text{h}}^{-1} \operatorname{div}_{\text{h}} \partial_t (\varrho^{\text{h}} v^{\text{h}}). \\ \operatorname{div} w_\varepsilon = 0 \quad \text{and} \quad w_{\varepsilon|t=0} = 0. \end{cases} \tag{5.1}$$

We have the following proposition.

Proposition 5.1. *Let $(w_\varepsilon, \Pi_\varepsilon^1)$ be the unique solution of the above system, then we have*

$$\begin{aligned} &\|(\varepsilon w_\varepsilon^{\text{h}}, w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})} + \|\nabla_\varepsilon(\varepsilon w_\varepsilon^{\text{h}}, w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})} + \|\nabla_\varepsilon(\varepsilon w_\varepsilon^{\text{h}}, w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})} \\ &\quad \leqslant C\|\varrho_0 v_0^{\text{h}}\|_{\mathcal{B}_2^{-1,\frac32}} + C\|\varrho^{\text{h}} v^{\text{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac32}) \cap L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac32})}, \end{aligned}$$

and

$$\varepsilon\|(\varepsilon w_\varepsilon^{\text{h}}, w_\varepsilon^3)\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})} \leqslant C\|\varrho_0 v_0^{\text{h}}\|_{\mathcal{B}_2^{0,\frac12}} + C\|\varrho^{\text{h}} v^{\text{h}}\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}.$$

Moreover, for any positive α less than 1, we have

$$\|\Delta_\varepsilon(\varepsilon w_\varepsilon^{\text{h}}, w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}^{\alpha,\frac12})} + \|\varepsilon\nabla_\varepsilon \Pi_\varepsilon^1\|_{L^1(\mathbb{R}^+;\mathcal{B}^{\alpha,\frac12})} \leqslant C_\alpha \|\partial_t(\varrho^{\text{h}} v^{\text{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac32})}.$$

Proof. Let us first compute Π_ε^1. Applying the divergence operator to the System (5.1) gives

$$-\Delta_\varepsilon \Pi_\varepsilon^1 + \partial_z^2 \Pi_L^{\text{h}} = 0. \tag{5.2}$$

This together with (5.1) gives

$$\begin{aligned} w_\varepsilon^{\text{h}} &= \int_0^t e^{(t-t')\Delta_\varepsilon} \nabla_{\text{h}} \Delta_\varepsilon^{-1} \partial_z^2 \Delta_{\text{h}}^{-1} \operatorname{div}_{\text{h}} \partial_t (\varrho^{\text{h}} v^{\text{h}})(t')\, dt' \quad \text{and} \\ w_\varepsilon^3 &= -\int_0^t e^{(t-t')\Delta_\varepsilon} \partial_z \Delta_\varepsilon^{-1} \operatorname{div}_{\text{h}} \partial_t (\varrho^{\text{h}} v^{\text{h}})(t')\, dt'. \end{aligned} \tag{5.3}$$

By integration by parts, we get

$$\begin{aligned} w_\varepsilon^{\text{h}} &= \nabla_{\text{h}} \Delta_\varepsilon^{-1} \partial_z^2 \Delta_{\text{h}}^{-1} \operatorname{div}_{\text{h}} (\varrho^{\text{h}} v^{\text{h}})(t) - e^{t\Delta_\varepsilon} \nabla_{\text{h}} \Delta_\varepsilon^{-1} \partial_z^2 \Delta_{\text{h}}^{-1} \operatorname{div}_{\text{h}} (\varrho_0 v_0^{\text{h}}) \\ &\quad + \int_0^t e^{(t-t')\Delta_\varepsilon} \nabla_{\text{h}} \partial_z^2 \Delta_{\text{h}}^{-1} \operatorname{div}_{\text{h}} (\varrho^{\text{h}} v^{\text{h}})(t')\, dt' \quad \text{and} \\ w_\varepsilon^3 &= -\partial_z \Delta_\varepsilon^{-1} \operatorname{div}_{\text{h}} (\varrho^{\text{h}} v^{\text{h}})(t) + e^{t\Delta_\varepsilon} \partial_z \Delta_\varepsilon^{-1} \operatorname{div}_{\text{h}} (\varrho_0 v_0^{\text{h}}) \\ &\quad - \int_0^t e^{(t-t')\Delta_\varepsilon} \partial_z \operatorname{div}_{\text{h}} (\varrho^{\text{h}} v^{\text{h}})(t')\, dt'. \end{aligned} \tag{5.4}$$

Written in term of Fourier transform with the notation $\xi = (\xi_h, \zeta)$ and using the fact that $\mathrm{div}_h v^h = 0$,

$$|(\varepsilon \widehat{w}_\varepsilon^h(t,\xi), \widehat{w}_\varepsilon^3(t,\xi))| \leqslant |\xi_h|^{-1}|\mathcal{F}(\partial_z(\varrho^h v^h))(t,\xi)| + e^{-t(|\xi_h^2+\varepsilon^2\zeta^2)}|\xi_h|^{-1}|\mathcal{F}(\partial_z(\varrho_0 v_0^h))(\xi)| + \int_0^t e^{-(t-t')(|\xi_h^2+\varepsilon^2\zeta^2)}(\varepsilon|\zeta|+|\xi_h|)|\zeta|\,|\mathcal{F}(\varrho^h v^h)(t',\xi)|\,dt'. \quad (5.5)$$

Applying the cutoff operator in the frequency space in the horizontal and the vertical directions gives, for any r in $[1,2]$,

$$2^{k\frac{2}{r}+\frac{\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varepsilon w_\varepsilon^h, w_\varepsilon^3)(t)\|_{L^2} \leqslant C2^{k\left(\frac{2}{r}-1\right)+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho^h v^h)(t)\|_{L^2} + \mathcal{W}_{k,\ell}(t)$$

with

$$\mathcal{W}_{k,\ell}(t) \stackrel{\mathrm{def}}{=} e^{-ct(2^{2k}+\varepsilon^2 2^{2\ell})}2^{k\left(\frac{2}{r}-1\right)+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho_0 v_0^h)\|_{L^2} + \int_0^t e^{-c(t-t')(2^{2k}+\varepsilon^2 2^{2\ell})}(2^k+\varepsilon 2^\ell)2^{k\frac{2}{r}+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho^h v^h)(t')\|_{L^2}\,dt'.$$

As r is in $[1,2]$, we get by convolution inequality,

$$\begin{aligned}\|\mathcal{W}_{k,\ell}\|_{L^r(\mathbb{R}^+;L^2)} &\leqslant C2^{-k+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho_0 v_0^h)\|_{L^2} + (2^k+\varepsilon 2^\ell)^{1-\frac{2}{r}}2^{k\frac{2}{r}+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho^h v^h)\|_{L^1(\mathbb{R}^+;L^2)}\\ &\leqslant C2^{-k+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho_0 v_0^h)\|_{L^2} + 2^{k+\frac{3\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varrho^h v^h)\|_{L^1(\mathbb{R}^+;L^2)}.\end{aligned}$$

By summation with respect to the indices k and ℓ, we get thanks to the Minkowski inequality,

$$\sum_{k,\ell}\|\mathcal{W}_{k,\ell}\|_{L^r(\mathbb{R}^+;L^2)} \leqslant C\|\varrho_0 v_0^h\|_{\mathcal{B}_2^{-1,\frac{3}{2}}} + C\|\varrho^h v^h\|_{L^1(\mathbb{R}^+;\mathcal{B}^{1,\frac{3}{2}})}.$$

By definition of the $\mathcal{B}_p^{s,s'}$ norms, we infer that, for r in $[1,2]$,

$$\begin{aligned}\|(\varepsilon w_\varepsilon^h, w_\varepsilon^3)\|_{L^r(\mathbb{R}^+;\mathcal{B}_2^{\frac{2}{r},\frac{1}{2}})} &\leqslant \sum_{(k,\ell)\in\mathbb{Z}^2}2^{k\frac{2}{r}+\frac{\ell}{2}}\|\Delta_k^h\Delta_\ell^v(\varepsilon w_\varepsilon^h, w_\varepsilon^3)\|_{L^r(\mathbb{R}^+;L^2)}\\ &\lesssim \|\varrho_0 v_0^h\|_{\mathcal{B}_2^{-1,\frac{3}{2}}} + \|\varrho^h v^h\|_{L^r(\mathbb{R}^+;\mathcal{B}_2^{\frac{2}{r}-1,\frac{3}{2}})\cap L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{3}{2}})}. \quad (5.6)\end{aligned}$$

Now let us estimate $\|\varepsilon\partial_z(\varepsilon w_\varepsilon^h, w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac{1}{2}})}$ and $\|\varepsilon\partial_z(\varepsilon w_\varepsilon^h, w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}$. As w is a divergence free vector field, we have

$$\begin{aligned}\varepsilon\|\partial_z w_\varepsilon^3\|_{L^2(\mathbb{R}^;\mathcal{B}_2^{0,\frac{1}{2}})} &\leqslant C\|\varepsilon\,\mathrm{div}_h w^h\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac{1}{2}})} \leqslant C\|\varepsilon w^h\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})},\\ \varepsilon\|\partial_z w_\varepsilon^3\|_{L^1(\mathbb{R}^;\mathcal{B}_2^{1,\frac{1}{2}})} &\leqslant C\|\varepsilon\,\mathrm{div}_h w^h\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})} \leqslant C\|\varepsilon w_\varepsilon^h\|_{L^1(\mathbb{R}^;\mathcal{B}_2^{2,\frac{1}{2}})}.\end{aligned}$$

Then inequality (5.6) applied with r equal to 2 and r equal to 1 gives

$$\varepsilon\|\partial_z w_\varepsilon^3\|_{L^2(\mathbb{R}^;\mathcal{B}_2^{0,\frac{1}{2}})} + \|\varepsilon\partial_z w_\varepsilon^3\|_{L^1(\mathbb{R}^;\mathcal{B}_2^{1,\frac{1}{2}})} \leqslant C\|\varrho_0 v_0^h\|_{\mathcal{B}_2^{-1,\frac{3}{2}}} + C\|\varrho^h v^h\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac{3}{2}})\cap L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{3}{2}})}. \quad (5.7)$$

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Now let us estimate $\|\varepsilon^2\partial_z w^{\mathrm{h}}_\varepsilon\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{1}{2}})}$. From equality (5.4), we infer that

$$\begin{aligned}\varepsilon\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)(t)\|_{L^2} \lesssim \|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho^{\mathrm{h}}v^{\mathrm{h}})(t)\|_{L^2} + e^{-ct(2^{2k}+\varepsilon^2 2^{2\ell})}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho_0 v_0^{\mathrm{h}})\|_{L^2}\\ + \int_0^t e^{-c(t-t')(2^{2k}+\varepsilon^2 2^{2\ell})}(\varepsilon 2^\ell + 2^k)\varepsilon 2^\ell\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho^{\mathrm{h}}v^{\mathrm{h}})(t')\|_{L^2}\,dt'.\end{aligned} \tag{5.8}$$

Taking the L^1 norm in time gives

$$\begin{aligned}2^{k+\frac{3\ell}{2}}\varepsilon\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^1(\mathbb{R}^+;L^2)} \lesssim 2^{-k+\frac{3\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho_0 v_0^{\mathrm{h}})\|_{L^2}\\ + 2^{k+\frac{3\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;L^2)}.\end{aligned}$$

By summation with respect to the indices k and ℓ, we infer that

$$\varepsilon\|(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{3}{2}})} \lesssim \|\varrho_0 v_0^{\mathrm{h}}\|_{\mathcal{B}_2^{-1,\frac{3}{2}}} + \|\varrho^{\mathrm{h}}v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac{3}{2}})}.$$

Together with inequalities (5.6) and (5.7), this gives the first inequality of the proposition.

To prove the second inequality, we get, by taking the L^4 norm in time of (5.8), that

$$2^{\frac{k}{2}}2^{\frac{\ell}{2}}\varepsilon\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^4(\mathbb{R}^+;L^2)} \lesssim 2^{\frac{\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho_0 v_0^{\mathrm{h}})\|_{L^2} + 2^{\frac{k}{2}}2^{\frac{\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^4(\mathbb{R}^+;L^2)}.$$

Summing up the above inequality with respect to the indices k and ℓ yields

$$\varepsilon\|(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} \leqslant C\|\varrho_0 v_0^{\mathrm{h}}\|_{\mathcal{B}_2^{0,\frac{1}{2}}} + C\|\varrho^{\mathrm{h}}v^{\mathrm{h}}\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})}.$$

This proves the second inequality of the proposition.

Let us prove the third inequality. Using (5.3), we can write that

$$|\mathcal{F}\Delta_\varepsilon(\varepsilon\widehat{w}^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)(t,\xi)| \leqslant C\int_0^t e^{-(t-t')(|\xi_{\mathrm{h}}|^2+\varepsilon^2\zeta^2)}(\varepsilon|\zeta| + |\xi_{\mathrm{h}}|)|\zeta||\mathcal{F}(\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}}))(t',\xi)|\,dt'.$$

Then applying the cutoff operators in both horizontal and vertical frequencies gives

$$\begin{aligned}&2^{k\alpha+\frac{\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\Delta_\varepsilon(\varepsilon\widehat{w}^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)(t)\|_{L^2}\\ &\quad\leqslant \int_0^t e^{-c(t-t')(2^{2k}+\varepsilon^2 2^{2\ell})}(\varepsilon 2^\ell + 2^k)2^{k\alpha+\frac{3\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})(t')\|_{L^2}\,dt'.\end{aligned}$$

Using Young's inequality, we get

$$2^{k\alpha+\frac{\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\Delta_\varepsilon(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^1(\mathbb{R}^+;L^2)} \leqslant C2^{k(-1+\alpha)+\frac{3\ell}{2}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;L^2)}.$$

By summation the above inequality with respect to the indices k and ℓ, we get

$$\|\Delta_\varepsilon(\varepsilon w^{\mathrm{h}}_\varepsilon, w^3_\varepsilon)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\alpha,\frac{1}{2}})} \leqslant C_\alpha\|\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac{3}{2}})}. \tag{5.9}$$

In order to get the estimates on the pressure term Π^1_ε, let us observe that Relation (5.2) implies that

$$\varepsilon\nabla_\varepsilon\Pi^1_\varepsilon = \begin{pmatrix}-\varepsilon\nabla_{\mathrm{h}}\Delta_\varepsilon^{-1}\partial_z\,\mathrm{div}_h\Delta_{\mathrm{h}}^{-1}\partial_z\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\\ \varepsilon^2\Delta_\varepsilon^{-1}\partial_z^2\,\mathrm{div}_{\mathrm{h}}\Delta_{\mathrm{h}}^{-1}\partial_z\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\end{pmatrix}.$$

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As we have

$$\frac{\varepsilon|\zeta|}{|\xi_{\mathrm{h}}|^2+\varepsilon^2\zeta^2}\leqslant\frac{1}{|\xi_{\mathrm{h}}|}\quad\text{and}\quad\frac{\varepsilon^2|\zeta|^2}{|\xi_{\mathrm{h}}|^2+\varepsilon^2\zeta^2}\leqslant 1$$

we infer that, for any α in $]0,1[$,

$$\varepsilon\|\nabla_\varepsilon\Pi^1_\varepsilon\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\alpha,\frac12})}\leqslant\|\partial_z\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac12})}.$$

Then the proposition is proved. □

Let us now turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. It follows from the law of product (2.7) and Lemma 4.3 that

$$\|\varrho_0v_0^{\mathrm{h}}\|_{\mathcal{B}_2^{0,\frac12}}\lesssim\|\varrho_0\|_{\mathcal{B}_2^{\frac12,\frac12}}\|v_0^{\mathrm{h}}\|_{\mathcal{B}_2^{\frac12,\frac12}}\leqslant\mathcal{C}_0\eta,$$

$$\|\varrho^{\mathrm{h}}v^{\mathrm{h}}\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\lesssim\|\varrho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac34,\frac12})}\|v^{\mathrm{h}}\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac34,\frac12})}\leqslant\mathcal{C}_0\eta,$$

and

$$\begin{aligned}\|\varrho^{\mathrm{h}}v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac32})}\leqslant{}&\|\varrho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|\partial_zv^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\\&+\|\partial_z\rho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\leqslant\mathcal{C}_0\eta.\end{aligned}$$

Similarly, we deduce from law of product (2.7) and (4.19) that

$$\begin{aligned}\|\varrho^{\mathrm{h}}v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac32})}\leqslant{}&\|\varrho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\|\partial_zv^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\\&+\|\partial_z\rho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\|v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\leqslant\mathcal{C}_0\eta.\end{aligned}$$

Hence by virtue of the first two inequalities of Proposition 5.1 and the remark following (1.7), we conclude the first inequality of Proposition 2.2.

On the other hand, for any α in $]0,1[$, we get, by applying the law of product (2.7), that

$$\begin{aligned}\|\partial_z(\partial_t\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac12})}\lesssim{}&\|\partial_t\partial_z\rho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\|v^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\\&+\|\partial_t\rho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\|\partial_zv^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})},\end{aligned}$$

and

$$\begin{aligned}\|\partial_z(\varrho^{\mathrm{h}}\partial_tv^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac12})}\lesssim{}&\|\partial_z\rho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\|\partial_tv^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\\&+\|\varrho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})}\|\partial_z\partial_tv^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{\frac{\alpha}{2},\frac12})},\end{aligned}$$

so that by applying Lemma 4.3, we obtain

$$\|\partial_t(\varrho^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac32})}\leqslant\|\partial_z(\rho_t^{\mathrm{h}}v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac12})}+\|\partial_z(\varrho^{\mathrm{h}}v_t^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{-1+\alpha,\frac12})}\leqslant\mathcal{C}_0.$$

This together with the third inequality of Proposition 5.1 leads to the second inequality of Proposition 2.2. □

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Corollary 5.1. *Under the assumptions of Theorem 1.2, there holds (2.13).*

Proof. We first deduce from the law of product (2.7) that

$$\|v^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}(\varepsilon w_\varepsilon^{\mathrm{h}},w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})}\lesssim\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\|(\varepsilon w_\varepsilon^{\mathrm{h}},w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})},$$
$$\|\varepsilon^2 w_\varepsilon\cdot\nabla(\varepsilon w_\varepsilon^{\mathrm{h}},w_\varepsilon^3)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})}\lesssim\|\varepsilon w_\varepsilon\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\|\nabla_\varepsilon(\varepsilon w_\varepsilon^{\mathrm{h}},w_\varepsilon^3)\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})},$$

and

$$\begin{aligned}\|\varepsilon w_\varepsilon\cdot\nabla(v^{\mathrm{h}},0)\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})}&\lesssim\|\varepsilon w_\varepsilon^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\\&\quad+\varepsilon\|w_\varepsilon^3\|_{L^4(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|\partial_z v^{\mathrm{h}}\|_{L^{\frac43}(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}.\end{aligned}$$

Moreover, it follows from (2.8) and (2.7) that

$$\begin{aligned}\|\partial_z\Pi_Q^{\mathrm{h}}\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})}&\lesssim\|\partial_z(\rho^{\mathrm{h}}v^{\mathrm{h}}\otimes v^{\mathrm{h}})\|_{L^1(\mathbb{R}^+;\mathcal{B}_2^{0,\frac12})}\\&\lesssim\|\rho_z^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}\\&\quad+(1+\|\varrho^{\mathrm{h}}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})})\|v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{\frac12,\frac12})}\|\partial_z v^{\mathrm{h}}\|_{L^2(\mathbb{R}^+;\mathcal{B}_2^{1,\frac12})}.\end{aligned}$$

Hence by virtue of (2.12), Proposition 2.2, and Lemma 4.3, we conclude the proof of (2.13). □

6. The control of the term b_ε

The purpose of this section is the control of the term b_ε which satisfies equation (2.18) as described by Proposition 2.4. Namely we want to decompose the solution b_ε of equation (2.18) which is

$$\partial_t b_\varepsilon+u_\varepsilon\cdot\nabla b_\varepsilon=-R_\varepsilon\cdot\nabla[a^{\mathrm{h}}]_\varepsilon-\varepsilon[w_\varepsilon\cdot\nabla a^{\mathrm{h}}]_\varepsilon\quad\text{with } b_{\varepsilon|t=0}=0\tag{6.1}$$

as $b_\varepsilon=\overline{b}_\varepsilon+\widetilde{b}_\varepsilon$ such that

$$\|\overline{b}_\varepsilon(t)\|_{\mathcal{B}_p^{\frac2p,\frac1p}}\leqslant\mathcal{C}_0\eta(1+\mathcal{R}_0)\langle t\rangle^{\frac12}\quad\text{and}\quad\|\widetilde{b}_\varepsilon(t)\|_{L^p}\leqslant\mathcal{C}_0\varepsilon^{1-\frac1p}(1+\mathcal{R}_0)^2\langle t\rangle.$$

In order to do it, let us introduce the following decomposition of $b_\varepsilon=b_{1,\varepsilon}+b_{2,\varepsilon}+b_{3,\varepsilon}$ with

$$\begin{aligned}\partial_t b_{1,\varepsilon}+u_\varepsilon\cdot\nabla b_{1,\varepsilon}&=-\varepsilon R_\varepsilon^3[\partial_z a^{\mathrm{h}}]_\varepsilon,\\\partial_t b_{2,\varepsilon}+[v^{\mathrm{h}}]_\varepsilon\cdot\nabla_{\mathrm{h}}b_{2,\varepsilon}&=-R_\varepsilon^{\mathrm{h}}\cdot\nabla_{\mathrm{h}}[a^{\mathrm{h}}]_\varepsilon-\varepsilon[w_\varepsilon\cdot\nabla a^{\mathrm{h}}]_\varepsilon\quad\text{and}\\\partial_t b_{3,\varepsilon}+u_\varepsilon\cdot\nabla b_{3,\varepsilon}&=-\varepsilon((\varepsilon w_\varepsilon^{\mathrm{h}},w_\varepsilon^3)+R_\varepsilon)\cdot\nabla b_{2,\varepsilon}.\end{aligned}\tag{6.2}$$

Let us first estimate $\|b_{1,\varepsilon}(t)\|_{L^p}$. As the vector field u_ε is divergence free, we have

$$\|b_{1,\varepsilon}(t)\|_{L^p}\leqslant\varepsilon^{1-\frac1p}\int_0^t\|R_\varepsilon^3(t')\|_{L^\infty}\|\partial_z a^{\mathrm{h}}(t')\|_{L^p}\,dt'.$$

Interpolation inequality (4.9) and the induction hypothesis (2.14) implies that, for any t less than $\overline{T}_\varepsilon$, we have

$$\begin{aligned}\int_0^t \|R(t')\|_{L^\infty}\,dt' &\leqslant \int_0^t \|R(t')\|^{\frac{2}{3}}_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}} \|\nabla_{\rm h} R(t')\|^{\frac{1}{3}}_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\,dt' \\ &\leqslant t^{\frac{1}{2}}\left(\int_0^t \|R(t')\|^4_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}}\,dt'\right)^{\frac{1}{4}\times\frac{2}{3}} \left(\int_0^t \|\nabla_{\rm h} R(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\,dt'\right)^{\frac{1}{3}} \\ &\leqslant \mathcal{R}_0 t^{\frac{1}{2}}. \end{aligned}\tag{6.3}$$

Then using the estimate (4.3), we infer that, for any t less than $\overline{T}_\varepsilon$,

$$\|b_{1,\varepsilon}(t)\|_{L^p} \leqslant \mathcal{C}_0\eta\varepsilon^{1-\frac{1}{p}} t^{\frac{1}{2}}\mathcal{R}_0. \tag{6.4}$$

In order to estimate $\|b_{3,\varepsilon}(t)\|_{L^p}$, we need to estimate $\|\nabla b_{2,\varepsilon}(t)\|_{L^p}$. Let us observe that

$$\begin{aligned}\partial_t\nabla b_{2,\varepsilon} + [v^{\rm h}]_\varepsilon\cdot\nabla_{\rm h}\nabla b_{2,\varepsilon} &= -\nabla R^{\rm h}_\varepsilon\cdot\nabla_{\rm h}[a^{\rm h}]_\varepsilon - R^{\rm h}_\varepsilon\cdot\nabla_{\rm h}\nabla[a^{\rm h}]_\varepsilon \\ &\quad - \nabla[v^{\rm h}]_\varepsilon\cdot\nabla_{\rm h} b_{2,\varepsilon} - \varepsilon[\nabla_\varepsilon w_\varepsilon\cdot\nabla a^{\rm h}]_\varepsilon - \varepsilon[w_\varepsilon\cdot\nabla\nabla_\varepsilon a^{\rm h}]_\varepsilon.\end{aligned}$$

Using the fact that $v^{\rm h}$ is divergence free, we get,

$$\begin{aligned}\frac{d}{dt}\|\nabla b_{2,\varepsilon}(t)\|_{L^p} &\leqslant \varepsilon^{-\frac{1}{p}}\Big((\|\nabla R^{\rm h}_\varepsilon\|_{L^\infty} + \|\varepsilon\nabla_\varepsilon w_\varepsilon\|_{L^\infty})\|\nabla a^{\rm h}\|_{L^p} \\ &\quad + (\|R^{\rm h}_\varepsilon\|_{L^\infty} + \|\varepsilon w_\varepsilon\|_{L^\infty})\|\nabla^2 a^{\rm h}\|_{L^p}\Big) + \|\nabla v^{\rm h}\|_{L^\infty}\|\nabla_{\rm h} b_{2,\varepsilon}\|_{L^p}.\end{aligned}\tag{6.5}$$

Estimate (4.3) together with Sobolev embedding implies that

$$\forall t < \overline{T}_\varepsilon,\quad \|\nabla a^{\rm h}(t)\|_{L^p} \leqslant \eta\,\mathcal{C}_0 \quad\text{and}\quad \|\nabla^2 a^{\rm h}(t)\|_{L^p} \leqslant \eta\,\mathcal{C}_0. \tag{6.6}$$

Induction hypothesis (2.14) and Proposition 2.2 implies that

$$\int_0^{\overline{T}_\varepsilon} (\|\nabla R^{\rm h}_\varepsilon(t)\|_{L^\infty} + \|\varepsilon\nabla_\varepsilon w_\varepsilon(t)\|_{L^\infty})\,dt \lesssim \mathcal{R}_0 + \mathcal{C}_0.$$

Together with (6.6) this implies that

$$\int_0^{\overline{T}_\varepsilon} (\|\nabla R^{\rm h}_\varepsilon(t)\|_{L^\infty} + \|\varepsilon\nabla_\varepsilon w_\varepsilon(t)\|_{L^\infty})\|\nabla a^{\rm h}(t)\|_{L^p}\,dt \lesssim \eta\,\mathcal{C}_0(1+\mathcal{R}_0). \tag{6.7}$$

Proposition 2.2 yields that

$$\int_0^t \|\varepsilon w_\varepsilon(t')\|_{L^\infty}\,dt' \lesssim t^{\frac{1}{2}}\|(\varepsilon w^{\rm h}_\varepsilon, w^3_\varepsilon)\|_{L^2(\mathbb{R}^+;\mathcal{B}^{1,\frac{1}{2}})} \leqslant \mathcal{C}_0 t^{\frac{1}{2}}. \tag{6.8}$$

Proposition 2.1 claims in particular that $\|\nabla v^{\rm h}(t)\|_{L^\infty}$ is an integrable function on R^+ the integral of which is less than some $\mathcal{C}_0$. Applying Gronwall's Lemma to (6.5) and using the estimates (6.3), (6.7) and (6.8), we get for any t less than $\overline{T}_\varepsilon$,

$$\|\nabla b_{2,\varepsilon}(t)\|_{L^p} \leqslant \mathcal{C}_0\varepsilon^{-\frac{1}{p}}(1+\mathcal{R}_0)\langle t\rangle^{\frac{1}{2}}.$$

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Now let us consider the equation on $b_{3,\varepsilon}$ in (6.2). As u_ε is divergence free, we get, by applying again the estimates (6.3) and (6.8), that

$$\begin{aligned}\|b_{3,\varepsilon}(t)\|_{L^p} &\leqslant \varepsilon \int_0^t (\|R_\varepsilon(t')\|_{L^\infty} + \|\varepsilon w_\varepsilon^{\mathrm{h}}(t')\|_{L^\infty} + \|w_\varepsilon^3(t')\|_{L^\infty})\|\nabla b_{2,\varepsilon}(t')\|_{L^p}\, dt' \\ &\leqslant \mathcal{C}_0 \varepsilon^{1-\frac{1}{p}}(1+\mathcal{R}_0)^2\langle t\rangle. \end{aligned} \tag{6.9}$$

Defining $\widetilde{b}_\varepsilon = b_{1,\varepsilon} + b_{3,\varepsilon}$ ensures the inequality (2.20) of Proposition 2.4. As there is no power of ε in the right-hand side of the equation on $b_{2,\varepsilon}$ we must use another norm to measure the size of $b_{2,\varepsilon}$. The fact that the convection vector field involved in the equation of $b_{2,\varepsilon}$ has no vertical component will allow us to propagate the anisotropic regularity thanks to the following lemma.

Proposition 6.1. *Given a smooth vector field v^{h} with $\operatorname{div}_{\mathrm{h}} v^{\mathrm{h}} = 0$, we consider the following transport equation with a parameter z*

$$\begin{cases} \partial_t b(t, x_{\mathrm{h}}, z) + v^{\mathrm{h}}(t, x_{\mathrm{h}}, z) \cdot \nabla_{\mathrm{h}} b(t, x_{\mathrm{h}}, z) = g(t, x_{\mathrm{h}}, z), \\ b(0, x_{\mathrm{h}}, z) = b_0(x_{\mathrm{h}}, z). \end{cases} \tag{6.10}$$

Let p be in $]2, 4[$. Let us define

$$\mathcal{V}_p(t) \overset{\text{def}}{=} \sup_{z'\in\mathbb{R}} \int_0^t \|\nabla_{\mathrm{h}} v^{\mathrm{h}}(t', \cdot, z')\|_{(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}}}\, dt'.$$

Then for s in $]0, 2/p]$, we have

$$\begin{aligned}\exp(-C\mathcal{V}_p(t))\|b\|_{L_t^\infty(\mathcal{B}_p^{s,\frac{1}{p}})} &\leqslant C\|b_0\|_{\mathcal{B}_p^{s,\frac{1}{p}}} + C\|g\|_{L_t^1(\mathcal{B}_p^{s,\frac{1}{p}})} \\ &\quad + C\|v^{\mathrm{h}}\|_{L_t^1(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}(\|\nabla_{\mathrm{h}} b_0\|_{L_{\mathrm{v}}^\infty(\mathcal{B}_p^s)_{\mathrm{h}}} + \|\nabla_{\mathrm{h}} g\|_{L_{\mathrm{v}}^\infty(L_t^1(\mathcal{B}_p^s)_{\mathrm{h}})}). \end{aligned} \tag{6.11}$$

Proof. Let us first observe that [5, Theorem 3.14] implies that for any σ in $[0, 1+2/p]$, we have, for any z in $\mathbb{R}$

$$\|b(t, \cdot, z)\|_{(\mathcal{B}_p^\sigma)_{\mathrm{h}}} \leqslant \left(\|b_0(\cdot, z)\|_{(\mathcal{B}_p^\sigma)_{\mathrm{h}}} + \int_0^t \|g(t', \cdot, z)\|_{(\mathcal{B}_p^\sigma)_{\mathrm{h}}}\, dt'\right)\exp(C\mathcal{V}_p(t)). \tag{6.12}$$

Let us define $(\tau_{-z} b)(x_{\mathrm{h}}, z') \overset{\text{def}}{=} b(x_{\mathrm{h}}, z'+z)$ and $\delta_{-z} a \overset{\text{def}}{=} \tau_{-z} a - a$. Then in view of (6.10), one has

$$\partial_t \tau_{-z} b + \tau_{-z} v^{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \tau_{-z} b = \tau_{-z} g.$$

Subtracting (6.10) from the above equation, we get

$$\partial_t \delta_{-z} b + v^{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \delta_{-z} b + \delta_{-z} v^{\mathrm{h}} \cdot \nabla_{\mathrm{h}} \tau_{-z} b = \delta_{-z} g. \tag{6.13}$$

Applying again [5, Theorem 3.14], we infer that

$$\begin{aligned}\exp(-C\mathcal{V}_p(t))\|\delta_{-z} b(t, \cdot, z')\|_{(\mathcal{B}_p^s)_{\mathrm{h}}} &\leqslant \|\delta_{-z} b_0(\cdot, z')\|_{(\mathcal{B}_p^s)_{\mathrm{h}}} \\ + C\int_0^t \|\delta_{-z} g(t', \cdot, z')\|_{(\mathcal{B}_p^s)_{\mathrm{h}}}\, dt' &+ \int_0^t \|\delta_{-z} v^{\mathrm{h}}(t', \cdot, z') \cdot \nabla_{\mathrm{h}} \tau_{-z} b(t')\|_{(\mathcal{B}_p^s)_{\mathrm{h}}}\, dt'. \end{aligned} \tag{6.14}$$

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The law of product in $\mathbb{R}^2$, which claims that for s in $[0,2/p]$, $\|ab\|_{\mathcal{B}^s_p} \leqslant C\|a\|_{\mathcal{B}_p^{\frac{2}{p}}}\|b\|_{\mathcal{B}^s_p}$, together with inequality (6.12) implies that

$$
\begin{aligned}
I(t,z,z') &\stackrel{\text{def}}{=} \int_0^t \|\delta_{-z}v^{\mathrm{h}}(t',\cdot,z')\cdot\nabla_{\mathrm{h}}\tau_{-z}b(t',\cdot,z')\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}\,dt'\\
&\leqslant \sup_{\substack{t'\in[0,t]\\ z'\in\mathbb{R}}}\|\nabla_{\mathrm{h}}b(t',\cdot,z')\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}\int_0^t\|\delta_{-z}v^{\mathrm{h}}(t',\cdot,z')\|_{(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}}}\,dt'\\
&\leqslant \biggl(\|\nabla_{\mathrm{h}}b_0\|_{L^\infty_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}}+\sup_{z\in\mathbb{R}}\int_0^t\|\nabla_{\mathrm{h}}g(t',\cdot,z)\|_{(\mathcal{B}^\sigma_p)_{\mathrm{h}}}\,dt'\biggr)\exp(C\mathcal{V}_p(t))\\
&\qquad\times\int_0^t\|\delta_{-z}v^{\mathrm{h}}(t',\cdot,z')\|_{(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}}}\,dt'.
\end{aligned}
$$

Plugging this into inequality (6.14) gives

$$
\begin{aligned}
\exp(-C\mathcal{V}_p(t))\|\delta_{-z}b(t,\cdot,z')\|_{(\mathcal{B}^s_p)_{\mathrm{h}}} &\leqslant \|\delta_{-z}b_0(\cdot,z')\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}+C\int_0^t\|\delta_{-z}g(t',\cdot,z')\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}\,dt'\\
+\biggl(\|\nabla_{\mathrm{h}}b_0\|_{L^\infty_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}}+\sup_{z\in\mathbb{R}}\int_0^t\|\nabla_{\mathrm{h}}g(t',\cdot,z)\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}\,dt'\biggr)&\int_0^t\|\delta_{-z}v^{\mathrm{h}}(t',\cdot,z')\|_{(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}}}\,dt'.
\end{aligned}
$$

Taking L^p norm of the above inequality with respect to the vertical variable z' yields

$$
\begin{aligned}
\exp(-C\mathcal{V}_p(t))\|\delta_{-z}b(t)\|_{L^p_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}} &\leqslant \|\delta_{-z}b_0\|_{L^p_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}}+C\int_0^t\|\delta_{-z}g(t')\|_{L^p_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}}\,dt'\\
+C\biggl(\|\nabla_{\mathrm{h}}b_0\|_{L^\infty_{\mathrm{v}}(\mathcal{B}^s_p)_{\mathrm{h}}}+\sup_{z\in\mathbb{R}}\int_0^t\|\nabla_{\mathrm{h}}g(t',\cdot,z)\|_{(\mathcal{B}^s_p)_{\mathrm{h}}}\,dt'\biggr)&\int_0^t\|\delta_{-z}v^{\mathrm{h}}(t')\|_{L^p_{\mathrm{v}}(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}}}\,dt'.
\end{aligned}
$$

Dividing the above inequality by $|z|^{\frac{1}{p}}$ and taking the L^p norm of the resulting inequality with the measure $\frac{dz}{|z|}$ over $\mathbb{R}$, we obtain inequality (6.11) and thus the proposition. □

Continuation of the proof to Proposition 2.4. Let us observe that Proposition 6.1 implies that for t less than $\overline{T}_\varepsilon$,

$$
\begin{aligned}
\|b_{2,\varepsilon}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{C}_0\biggl(&\|R^{\mathrm{h}}_\varepsilon\cdot\nabla_{\mathrm{h}}[a^{\mathrm{h}}]_\varepsilon\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}+\varepsilon\|w_\varepsilon\cdot\nabla a^{\mathrm{h}}\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\\
&+\|v^{\mathrm{h}}\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}(\|\nabla_{\mathrm{h}}(R^{\mathrm{h}}_\varepsilon\cdot\nabla_{\mathrm{h}}[a^{\mathrm{h}}]_\varepsilon)\|_{L^\infty_{\mathrm{v}}(L^1_t(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}})}\\
&+\varepsilon\|\nabla_{\mathrm{h}}(w_\varepsilon\cdot\nabla a^{\mathrm{h}})\|_{L^\infty_{\mathrm{v}}(L^1_t(\mathcal{B}_p^{\frac{2}{p}})_{\mathrm{h}})})\biggr).
\end{aligned}
$$

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Hence it follows from the law of product (2.7), Proposition 2.2 and the induction hypothesis (2.14) that

$$\begin{aligned}\|b_{2,\varepsilon}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \leqslant \Big(&\|R^{\rm h}_\varepsilon\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} + \|\varepsilon w_\varepsilon\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} + \|\nabla_{\rm h} R^{\rm h}_\varepsilon\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\\ &+ \|\varepsilon\nabla_{\rm h} w_\varepsilon\|_{L^1_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\Big)\Big(\|\nabla a^{\rm h}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} + \|\nabla a^{\rm h}\|_{L^\infty_{\rm v}(L^\infty_t(\mathcal{B}_p^{1+\frac{2}{p}})_{\rm h})}\Big)\\ \leqslant{}& \mathcal{C}_0(1+\mathcal{R}_0)\langle t\rangle^{\frac{1}{2}}\Big(\|\nabla a^{\rm h}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} + \|\nabla a^{\rm h}\|_{L^\infty_{\rm v}(L^\infty_t(\mathcal{B}_p^{1+\frac{2}{p}})_{\rm h})}\Big). \qquad (6.15)\end{aligned}$$

Yet it follows from Lemma 4.1, (4.18) and (4.3) that

$$\|\nabla a^{\rm h}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \lesssim \|\nabla a^{\rm h}\|_{L^\infty_t(\mathcal{B}_2^{1,\frac{1}{2}})} \lesssim \|\nabla a^{\rm h}\|^{\frac{1}{2}}_{L^\infty_t(\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} \|\nabla_{\rm h}\nabla a^{\rm h}\|^{\frac{1}{2}}_{L^\infty_t(\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}})} \leqslant \mathcal{C}_0\eta,$$

$$\|\nabla a^{\rm h}\|_{L^\infty_{\rm v}(L^\infty_t(\mathcal{B}_p^{1+\frac{2}{p}})_{\rm h})} \lesssim \|\nabla a^{\rm h}\|_{L^\infty_{\rm v}(L^\infty_t(\mathcal{B}^2_2)_{\rm h})} \lesssim \|\nabla a^{\rm h}\|^{\frac{1}{2}}_{L^\infty_{\rm v}(L^\infty_t(\dot H^1_{\rm h}))} \|\nabla a^{\rm h}\|^{\frac{1}{2}}_{L^\infty_{\rm v}(L^\infty_t(\dot H^3_{\rm h}))} \leqslant \mathcal{C}_0\eta.$$

This implies that for all t less than $\overline{T}_\varepsilon$

$$\|b_{2,\varepsilon}\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{C}_0\eta(1+\mathcal{R}_0)\langle t\rangle^{\frac{1}{2}}.$$

Taking $\overline{b}_\varepsilon = b_{2,\varepsilon}$ leads to (2.19). We thus complete the proof of Proposition 2.4. □

Corollary 6.1. *Under the assumptions of Theorem 1.2, the inequalities of Assertion (2.21) holds namely*

$$\|E^{4,1}_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})} \leqslant \mathcal{C}_0\eta(1+\mathcal{R}_0) \quad \textit{and} \quad \|E^{4,2}_\varepsilon\|_{L^1_T(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})} \leqslant \mathcal{C}_0\varepsilon^{1-\frac{1}{p}}(1+\mathcal{R}_0)^2,$$

for $p\in\,]3,4[$ *and* $\delta\in\,]0,1-3/p[$.

Proof. We deduce from a similar derivation of (4.23) that

$$\|\Pi^{\rm h}(t)\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}} \leqslant C\Big(\|a^{\rm h}\|_{L^\infty(\mathbb{R}^+;\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\|\nabla_{\rm h} v^{\rm h}(t)\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}} + \|v^{\rm h}(t)\|^2_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\Big),$$

which together with (4.15) and (4.18) ensures that

$$\|\Delta_{\rm h} v^{\rm h}(t)\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}} + \|\nabla_{\rm h}\Pi^{\rm h}(t)\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}} \leqslant \mathcal{C}_0\langle t\rangle^{-2}\log^{\left(1-\frac{2}{p}\right)\frac{2}{p}}\langle t\rangle. \qquad (6.16)$$

Moreover, as p belongs to $]3,4[$ and δ to $]0,1-3/p[$, we have $-1+\delta+5/p\in\,]0,2/p[$ and $-\delta+1/p\in\,]0,1[$. Then it follows from inequality (4.16) that

$$\begin{aligned}\int_{\mathbb{R}^+}\langle t\rangle\Big(&\|\Delta_{\rm h} v^{\rm h}(t)\|_{\mathcal{B}_p^{-1+\delta+\frac{5}{p},-\delta+\frac{1}{p}}} + \|\nabla_{\rm h}\Pi^{\rm h}(t)\|_{\mathcal{B}_p^{-1+\delta+\frac{5}{p},-\delta+\frac{1}{p}}}\Big)\,dt\\ &\leqslant \int_{\mathbb{R}^+}\langle t\rangle^{-\left(1+\frac{\delta}{2}+\frac{3}{2p}\right)}\log^{\delta+\frac{3}{p}}\,dt \leqslant \mathcal{C}_0. \qquad (6.17)\end{aligned}$$

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On the other hand, it follows from the law of product (2.7) that

$$\|\bar{b}([\Delta_{\rm h}v^{\rm h}]_\varepsilon+[\nabla_{\rm h}\Pi^{\rm h}]_\varepsilon)\|_{L^1_t(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})}$$
$$\lesssim \int_0^t \|\bar{b}(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}(\|\Delta_{\rm h}v^{\rm h}(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}+\|\nabla_{\rm h}\Pi^{\rm h}(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}})\,dt',$$

from which, (2.19) and (6.16), we infer the first inequality of the corollary. Similarly, again p belongs to $]3,4[$ and δ to $]0,1-3/p[$, the law of product (2.7) ensures that

$$\|\tilde{b}([\Delta_{\rm h}v^{\rm h}]_\varepsilon+[\nabla_{\rm h}\Pi^{\rm h}]_\varepsilon)\|_{L^1_t(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})}$$
$$\lesssim \int_0^t \|\tilde{b}(t')\|_{L^p}(\|\Delta_{\rm h}v^{\rm h}(t')\|_{\mathcal{B}_p^{-1+\delta+\frac{5}{p},-\delta+\frac{1}{p}}}+\|\nabla_{\rm h}\Pi^{\rm h}(t')\|_{\mathcal{B}_p^{-1+\delta+\frac{5}{p},-\delta+\frac{1}{p}}})\,dt',$$

which together with (2.20) and (6.17) gives rise to the second inequality of the corollary. □

7. Conclusion of the proof of the main theorem

Proof of Theorem 1.2. Let us first observe that law of products implies that, if $p\in\,]3,4[$ and $\|a\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}$ is less than c_p, which is the case for $\|a_\varepsilon(t)\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}$ for $t\leqslant \overline{T}_\varepsilon$ thanks to Corollary 2.1 provided that η is sufficiently small, $\mathbb{P}_a$ given by Definition 2.2 maps continuously from $\mathcal{B}_p^{s_1,s_2}$ into itself for any s_1 in $]-2/p,2/p]$ and s_2 in $]-1/p,1/p]$, which reads

$$\|\mathbb{P}_a g\|_{\mathcal{B}_p^{s_1,s_2}}\lesssim \|g\|_{\mathcal{B}_p^{s_1,s_2}}. \tag{7.1}$$

Let us now fix p in $]3,4[$ and δ in $]0,1-3/p[$, which is determined by (2.16). For t less than $\overline{T}_\varepsilon$ defined by relation (2.14), we denote

$$g_\lambda(t)\stackrel{\text{def}}{=}g(t)\exp\Bigl(-\lambda\int_0^t U_{\varepsilon,\text{app}}(t')\,dt'\Bigr)\quad\text{with } U_{\varepsilon,\text{app}}(t)\stackrel{\text{def}}{=}\|u_{\varepsilon,\text{app}}(t)\|^4_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}}$$
$$+\|u_{\varepsilon,\text{app}}(t)\|^2_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}+\|\partial_z v^{\rm h}(t)\|^{\frac{4}{3}}_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}}+\varepsilon\|(\varepsilon\partial_z w^{\rm h},\partial_z w^3)(t)\|^2_{\mathcal{B}_2^{0,\frac{1}{2}}}. \tag{7.2}$$

Then we deduce from equality (2.24) that

$$R_{\varepsilon,\lambda}(t)=\int_0^t \exp\Bigl(-\lambda\int_{t'}^t U_{\varepsilon,\text{app}}(t'')\,dt''\Bigr)e^{(t-t')\Delta}\mathbb{P}_{a_\varepsilon}(a_\varepsilon\Delta R_{\varepsilon,\lambda}$$
$$-\operatorname{div}(u_{\varepsilon,\text{app}}\otimes R_{\varepsilon,\lambda}+R_{\varepsilon,\lambda}\otimes u_{\varepsilon,\text{app}}+\varepsilon R_\varepsilon\otimes R_{\varepsilon,\lambda})-E_{\varepsilon,\lambda})(t')\,dt'.$$

So that for the norm $\|\cdot\|_{X(t)}$ given by Definition 2.1, we have

$$\|R_{\varepsilon,\lambda}\|_{X(t)}\leqslant\Bigl\|\exp\Bigl(-\lambda\int_{t'}^t U_{\varepsilon,\text{app}}(t'')\,dt''\Bigr)\mathbb{P}_{a_\varepsilon}\bigl(a_\varepsilon\Delta R_{\varepsilon,\lambda}$$
$$-\operatorname{div}(u_{\varepsilon,\text{app}}\otimes R_{\varepsilon,\lambda}+R_{\varepsilon,\lambda}\otimes u_{\varepsilon,\text{app}}+\varepsilon R_\varepsilon\otimes R_{\varepsilon,\lambda})-E_{\varepsilon,\lambda})\Bigr\|_{\mathcal{F}_p(t)}. \tag{7.3}$$

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It is easy to observe from inequality (2.6), the law of product (2.7) and Corollary 2.1 that

$$\begin{aligned}\|\mathbb{P}_{a_\varepsilon}(a_\varepsilon\Delta R_{\varepsilon,\lambda})\|_{\mathcal{F}_p(t)} &\leqslant C\Big(\|a_\varepsilon\Delta_{\rm h}R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})}+\|a_\varepsilon\partial_3^2R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})}\Big)\\ &\leqslant C\|a_\varepsilon\|_{L^\infty_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\Big(\|R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{1+\frac{2}{p},\frac{1}{p}})}+\|\partial_3^2R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})}\Big)\\ &\leqslant \mathcal{C}_0\eta\exp(C\mathcal{R}_0)\Big(\|R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{1+\frac{2}{p},\frac{1}{p}})}+\|\partial_3^2R_{\varepsilon,\lambda}\|_{L^1_t(\mathcal{B}_p^{-1+\delta+\frac{3}{p},-\delta})}\Big).\end{aligned}$$

Along the same lines, we get

$$\begin{aligned}&\|\mathbb{P}_{a_\varepsilon}\operatorname{div}(\varepsilon R_\varepsilon\otimes R_{\varepsilon,\lambda}))\|_{\mathcal{F}_p(t)}\\ &\quad\lesssim\|\mathbb{P}_{a_\varepsilon}(\varepsilon R_\varepsilon^{\rm h}\cdot\nabla_{\rm h}R_{\varepsilon,\lambda}+\varepsilon R_\varepsilon^3\partial_3R_{\varepsilon,\lambda})\|_{L^1_t(\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}})}\\ &\quad\lesssim\varepsilon\Big(\|R_\varepsilon\|_{L^2_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}\|R_{\varepsilon,\lambda}\|_{L^2_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}+\|R_\varepsilon\|_{L^4_t(\mathcal{B}_p^{-\frac12+\frac{2}{p},\frac{1}{p}})}\|\partial_3R_{\varepsilon,\lambda}\|_{L^{\frac43}_t(\mathcal{B}_p^{-\frac12+\frac{2}{p},\frac{1}{p}})}\Big).\end{aligned}$$

Using inequality (2.6), we deduce from inequalities (2.13), (2.16), (2.17) and (2.21) that

$$\|\mathbb{P}_{a_\varepsilon}E_\varepsilon\|_{\mathcal{F}_p(t)}\leqslant\mathcal{C}_0(1+(\eta+\varepsilon^{1-\delta-\frac{1}{p}})\exp(C\mathcal{R}_0)).$$

Now let us turn to the estimates of the last two terms in the right-hand side of inequality (7.3). We deduce again from inequality (2.6) that

$$\begin{aligned}&\Big\|\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\operatorname{div}_{\rm h}(u_{\varepsilon,\rm app}\otimes R^{\rm h}_{\varepsilon,\lambda}+R_{\varepsilon,\lambda}\otimes u^{\rm h}_{\varepsilon,\rm app})\Big\|_{\mathcal{F}_p(t)}\\ &\quad\lesssim\int_0^t\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\|R_{\varepsilon,\lambda}\otimes u_{\varepsilon,\rm app}(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\,dt'\\ &\quad\lesssim\int_0^t\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\|u_{\varepsilon,\rm app}(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\|R_{\varepsilon,\lambda}(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\,dt'\\ &\quad\lesssim\Big(\int_0^t\exp\Big(-2\lambda\int_{t'}^tU_{\varepsilon,\rm app}(\tau)\,dt''\Big)\|u_{\varepsilon,\rm app}(t')\|^2_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\,dt'\Big)^{\frac12}\|R_{\varepsilon,\lambda}\|_{L^2_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})},\end{aligned}$$

which together with (7.2) ensures that

$$\Big\|\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\operatorname{div}_{\rm h}(u_{\varepsilon,\rm app}\otimes R^{\rm h}_{\varepsilon,\lambda}+R_{\varepsilon,\lambda}\otimes u^{\rm h}_{\varepsilon,\rm app})\Big\|_{\mathcal{F}_p(t)}\lesssim\frac{1}{\lambda^{\frac12}}\|R_{\varepsilon,\lambda}\|_{L^2_t(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}.$$

Along the same lines, we have

$$\begin{aligned}&\Big\|\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\partial_3(u_{\varepsilon,\rm app}\otimes R^3_{\varepsilon,\lambda}+R_{\varepsilon,\lambda}\otimes u^3_{\varepsilon,\rm app})\Big\|_{\mathcal{F}_p(t)}\\ &\lesssim\int_0^t\exp\Big(-\lambda\int_{t'}^tU_{\varepsilon,\rm app}(t'')\,dt''\Big)\|(\partial_3R_{\varepsilon,\lambda}\otimes u_{\varepsilon,\rm app}(t')+R_{\varepsilon,\lambda}\otimes\partial_3u_{\varepsilon,\rm app})(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\,dt',\end{aligned}$$

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By the definition of $u_{\varepsilon,\mathrm{app}}$ given by (2.10), we infer

$$\begin{aligned}&\int_0^t \exp\left(-\lambda\int_{t'}^t U_{\varepsilon,\mathrm{app}}(t'')\,dt''\right)\|R_{\varepsilon,\lambda}\otimes\partial_3 u_{\varepsilon,\mathrm{app}}(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\,dt'\\&\lesssim\int_0^t \exp\left(-\lambda\int_{t'}^t U_{\varepsilon,\mathrm{app}}(t'')\,dt''\right)\Big(\varepsilon\|\partial_z v^{\mathrm{h}}(t')\|_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}}\|R_{\varepsilon,\lambda}(t')\|_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}}\\&\quad+\varepsilon^2\|(\varepsilon\partial_z w^{\mathrm{h}},\partial_z w^3)\|_{\mathcal{B}_2^{0,\frac{1}{2}}}\|R_{\varepsilon,\lambda}(t')\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}}\Big)\,dt'.\end{aligned}$$

Then due to Definition (7.2) of $U_{\varepsilon,\mathrm{app}}$, Hölder inequality implies that

$$\begin{aligned}&\int_0^t \exp\left(-\lambda\int_{t'}^t U_{\varepsilon,\mathrm{app}}(t'')\,dt''\right)\|R_{\varepsilon,\lambda}\otimes\partial_3 u_{\varepsilon,\mathrm{app}}(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\,dt'\\&\lesssim\frac{1}{\lambda^{\frac{3}{4}}}\|R_{\varepsilon,\lambda}\|_{L_t^4(\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}})}+\frac{1}{\lambda^{\frac{1}{2}}}\|R_{\varepsilon,\lambda}\|_{L_t^2(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})}.\end{aligned}$$

Following the same lines we get

$$\int_0^t \exp\left(-\lambda\int_{t'}^t U_{\varepsilon,\mathrm{app}}(t'')\,dt''\right)\|\partial_3 R_{\varepsilon,\lambda}\otimes u_{\varepsilon,\mathrm{app}}(t')\|_{\mathcal{B}_p^{-1+\frac{2}{p},\frac{1}{p}}}\,dt'\lesssim\frac{1}{\lambda^{\frac{1}{4}}}\|\partial_3 R_{\varepsilon,\lambda}\|_{L_t^{\frac{4}{3}}(\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}})}.$$

Substituting the above estimates into inequality (7.3) and using the definition of the norm $\|\cdot\|_{X(t)}$ given by Definition 2.1, we infer that, for any λ greater than 1,

$$\begin{aligned}\|R_{\varepsilon,\lambda}\|_{X(t)}&\leqslant\mathcal{C}_0(1+(\eta+\varepsilon^{1-\delta-\frac{1}{p}})\exp(C\mathcal{R}_0))+\left(\mathcal{C}_0\eta\exp(C\mathcal{R}_0)+\frac{C}{\lambda^{\frac{1}{4}}}\right)\|R_{\varepsilon,\lambda}\|_{X(t)}\\&\quad+C\varepsilon(\|R_\varepsilon\|_{L_t^4(\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}})}+\|R_\varepsilon\|_{L_t^2(\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}})})\|R_{\varepsilon,\lambda}\|_{X(t)},\end{aligned}$$

which together with the induction assumption (2.14) ensures that for t less than $\overline{T}_\varepsilon$ and for any λ greater than 1,

$$\left(1-C\left(\mathcal{C}_0\eta\exp(C\mathcal{R}_0)+\frac{1}{\lambda^{\frac{1}{4}}}+\varepsilon\mathcal{R}_0\right)\right)\|R_{\varepsilon,\lambda}\|_{X(t)}\leqslant\mathcal{C}_0(1+(\eta+\varepsilon^{1-\delta-\frac{1}{p}})\exp(C\mathcal{R}_0)).\tag{7.4}$$

Let us take ε,η sufficiently small and λ sufficiently large so that

$$\eta<\frac{1}{6C\mathcal{C}_0},\quad\frac{C}{\lambda^{\frac{1}{4}}}\leqslant\frac{1}{6}\quad\text{and}\quad\mathcal{R}_0\leqslant\frac{1}{C}\min\{-\ln(2\varepsilon^{1-\delta-1/p}),-\ln(2\eta),-\ln(6C\mathcal{C}_0\eta),1/6\varepsilon\}.$$

Then we deduce from inequality (7.4) that $\|R_{\varepsilon,\lambda}\|_{X(t)}\leqslant 4\mathcal{C}_0$, from which and from inequality (7.2), we infer

$$\|R_\varepsilon\|_{X(t)}\leqslant\|R_{\varepsilon,\lambda}\|_{X(t)}\exp\left(\lambda\int_0^t U_{\varepsilon,\mathrm{app}}(t')\,dt'\right)\leqslant 4\mathcal{C}_0\exp\left(\lambda\int_0^t U_{\varepsilon,\mathrm{app}}(t')\,dt'\right).$$

However, let us notice from equality (2.10) and Lemma 4.1 that

$$\|u_{\varepsilon,\mathrm{app}}(t)\|_{\mathcal{B}_p^{-\frac{1}{2}+\frac{2}{p},\frac{1}{p}}} \lesssim \|v^{\mathrm{h}}(t)\|_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}} + \varepsilon\|(\varepsilon w^{\mathrm{h}}, w^3)(t)\|_{\mathcal{B}_2^{\frac{1}{2},\frac{1}{2}}},$$

$$\|u_{\varepsilon,\mathrm{app}}(t)\|_{\mathcal{B}_p^{\frac{2}{p},\frac{1}{p}}} \lesssim \|v^{\mathrm{h}}(t)\|_{\mathcal{B}_2^{1,\frac{1}{2}}} + \varepsilon\|(\varepsilon w^{\mathrm{h}}, w^3)(t)\|_{\mathcal{B}_2^{1,\frac{1}{2}}},$$

which together with Proposition 2.2, (4.14) and (4.19) ensures that

$$\int_{\mathbb{R}^+} U_{\varepsilon,\mathrm{app}}(t')\,dt' \leqslant \mathcal{C}_0 \quad \text{and} \quad \|R_\varepsilon\|_{X(t)} \leqslant 4\mathcal{C}_0\exp(\mathcal{C}_0\lambda) \stackrel{\mathrm{def}}{=} \mathcal{C}_0'. \tag{7.5}$$

We take $\mathcal{R}_0 = 2\mathcal{C}_0'$ and take ε, η so small that

$$\eta < \frac{1}{6C\mathcal{C}_0} \quad \text{and} \quad 2\mathcal{C}_0' \leqslant \frac{1}{C}\min\{-\ln(2\varepsilon^{1-\delta-1/p}), -\ln(2\eta), -\ln(6C\mathcal{C}_0\eta), 1/6\varepsilon\}. \tag{7.6}$$

Then we deduce from inequality (7.5) that

$$\forall t \leqslant \overline{T}_\varepsilon, \quad \|R_\varepsilon\|_{X(t)} \leqslant \frac{\mathcal{R}_0}{2}.$$

The necessary condition for blowup implies that $\overline{T}_\varepsilon$ equals to infinity. This completes the proof of Theorem 1.2. □

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Appendix A. The proof of (2.6)

Proof of (2.6). For $j = 0, 1, 2$, we get, by applying [5, Lemma 2.4], that

$$\begin{aligned}\Big\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}}\int_0^t e^{(t-t')\Delta}\partial_3^j f(t')\,dt'\Big\|_{L_T^q(L^p)} &\lesssim 2^{j\ell}\Big\|\int_0^t e^{-c(t-t')(2^{2k}+2^{2\ell})}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f(t')\|_{L^p}\,dt'\Big\|_{L_T^q}\\ &\lesssim \frac{2^{j\ell}}{(2^{2k}+2^{2\ell})^{\frac{1}{q}}}\|\Delta_k^{\mathrm{h}}\Delta_\ell^{\mathrm{v}} f\|_{L_T^1(L^p)}\\ &\lesssim d_{k,\ell}\frac{2^{-k\alpha}2^{\ell(j-\beta)}}{(2^{2k}+2^{2\ell})^{\frac{1}{q}}}\|f\|_{L_T^1(\mathcal{B}_p^{\alpha,\beta})},\end{aligned}$$

where $(d_{k,\ell})_{k,\ell\in\mathbb{Z}^2}$ denotes a generic element of $\ell^1(\mathbb{Z}^2)$ so that $\sum_{k,\ell\in\mathbb{Z}^2} d_{k,\ell} = 1$. This together with Definition 2.1 ensures that

$$\Big\|\int_0^t e^{(t-t')\Delta}\partial_3^j f(t')\,dt'\Big\|_{L_T^q(\mathcal{B}_p^{s,s'})} \lesssim \sum_{k,\ell\in\mathbb{Z}^2} d_{k,\ell}\frac{2^{k(s-\alpha)}2^{\ell(s'+j-\beta)}}{(2^{2k}+2^{2\ell})^{\frac{1}{q}}}\|f\|_{L_T^1(\mathcal{B}_p^{\alpha,\beta})}.$$

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In the particular case when

$$\alpha \leqslant s, \quad \beta \leqslant s' + j \quad \text{and} \quad \alpha + \beta = s + s' + j - \frac{2}{q},$$

we have

$$\left\| \int_0^t e^{(t-t')\Delta} \partial_3^j f(t')\, dt' \right\|_{L_T^q(\mathcal{B}_p^{s,s'})} \lesssim \|f\|_{L_T^1(\mathcal{B}_p^{\alpha,\beta})}. \tag{A.1}$$

This together with the definition of the norm $\|\cdot\|_{\mathcal{F}_p(T)}$ given by Definition 2.1 leads to (2.6). □

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