THE j-DIFFERENTIAL AND ITS INTEGRAL

by W. H. INGRAM
(Received 10th December, 1961)

The reciprocity, $\boldsymbol{dy} = f(x)\boldsymbol{dg}(x) \leftrightarrow y(x) - y(a) = \int_a^x f\boldsymbol{dg}$, has been shown to hold when $g(x)$ is in B^* [*Proc. Edin. Math. Soc.*, **12** (1960), 85]. The purpose of the present note is to show that it holds more generally and in particular when $g(x)$ is in a subclass B'', of functions of bounded variation B, such that $B^* \subset B'' \subset B' \subset B$; it is assumed that $f(x)$ is any function in D_1 and that $f(x)$ and $g(x)$ are both continuous on the left and possibly with simultaneous points of discontinuity.

Definition. By B'' is to be understood that subclass of B' such that if $g(x)$ is in B'' and $v_{jg}(x) = \lim_\sigma \Sigma_{\sigma/[ax)} | jg |$ is the total variation of $jg(x)$ over $[ax]$, then $\lim_{\delta \to +0}(v_{jg}(x+\delta) - v_{jg}(x+))/\delta = 0$ at all points x, $a \leqq x < b$.

The theorems of the original paper remain true when B'' is substituted for B^*, and in fact, if $g(x)$ has only n points of discontinuity in (ab), when B' is substituted for B^*. The demonstration is simplified when the integral

$$\lim_\sigma \Sigma_{\sigma/[ab)} f\boldsymbol{dg} = (RJDS\sigma)\int_a^b f\boldsymbol{dg},$$

to be called the *right jump-differential Stieltjes σ-integral*, is given autonomous status.

Theorem I′. *When $f(x)$ is in D_1 and $g(x)$ is in B', the $RJDS\sigma$-integral exists.*

Proof. $\Sigma_{\sigma_i} | jg |$, $i = 1, 2, \ldots$, is an increasing sequence, since the σ's proceed by inclusion, and is bounded on $[ab]$ since $g(x)$ is in B' and therefore in B, and $f(x+)g'^{+}(x+)$ is in D_1; hence

$$\lim_\sigma \Sigma_{\sigma/[ab]}\{\bar{f}jg + f(x+)g'^{+}(x+)dx\} = \int_a^b f\boldsymbol{dg}$$

exists as the sum of the two limits.

The elementary integral properties of paragraph 3 of the original paper hold for this integral if by $\int_a^x \boldsymbol{dg}$ we understand the special case of $\int_a^x f\boldsymbol{dg}$ with $f(x) = 1$, viz., $\int_a^x \boldsymbol{dg} = g(x) - g(a)$. The mean-value lemma does not hold for the $RJDS\sigma$-integral, but the *supplementary relations*

$$(1)\ \int_x^{x+} f\boldsymbol{dg} = \bar{f}jg, \quad (2)\ \lim_{\delta\to+0}\frac{1}{\delta}\int_{x+}^{x+\delta} f\boldsymbol{dg} = f(x+)g'^{+}(x+),$$

https://doi.org/10.1017/S0013091500014528 Published online by Cambridge University Press

hold when $g(x)$ is in B''. On the one hand, we have the equation

$$\int_{x_0}^{x_0+\delta} \boldsymbol{fdg} = \lim_\sigma \Sigma_{\sigma/[x_0,\, x_0+\delta)\smile} jg + (LM)\int_{x_0}^{x_0+\delta} f(x+)g'^{+}(x+)dx$$

of which the second term on the right vanishes with δ, by the mean-value lemma, and of which the first term is equal to

$$\bar{f}(x_0)jg(x_0) + \lim_\sigma \Sigma_{\sigma/(x_0,\, x_0+\delta)}\bar{f}jg.$$

The second term here vanishes with δ because $\Sigma_{\sigma/(x_0,\, x_0+\delta)} \mid jg \mid$ is a decreasing function of δ with limit zero because, with limit $k>0$, $\mid jg \mid$ would be greater than $k/2$ at an infinity of points which is impossible for a function in B; thus " 1 " holds for any function $g(x)$ in B'. On the other hand, there is the equation

$$\lim_{\delta\to+0}\frac{1}{\delta}\int_{x_0+}^{x_0+\delta} \boldsymbol{fdg} = \lim_\delta \frac{1}{\delta}\lim_\sigma \Sigma_{\sigma/(x_0,\, x_0+\delta)}\bar{f}jg$$
$$+\lim_\delta \frac{1}{\delta}\int_{x_0}^{x_0+\delta} f(x+)g'^{+}(x+)dx$$

where the second term on the right, by the mean-value lemma, is $f(x_0+)g'^{+}(x_0+)$ and where the first term is zero either because $g(x)$ in B' has only n points of discontinuity or, otherwise, because $g(x)$ is in B''.

Theorem II′. *Theorem II remains true when $g(x)$ is in B''.*

Since the $RJDS\sigma$-integral has the same j-differential properties as has the LM-integral at each point x, $a \leqq x < b$, the two integrals are equivalent.

Integration-by-parts. Because of Theorem II′, if $f(x)$ and $g(x)$ are in B'', then

$$\int_a^x \boldsymbol{fdg} + \int_a^x \boldsymbol{gdf} = [fg]_a^x.$$

This follows also from the equation $\boldsymbol{fdg} + \boldsymbol{gdf} = \boldsymbol{d}(\boldsymbol{fg})$. Moreover, from the definitions in paragraph 1 of the original paper,

$$\int_a^x \boldsymbol{fdg} + \int_a^x \boldsymbol{gdf} = \lim_\sigma \Sigma_{\sigma/[ax)} jp + \int_a^x p'^{+}(x+)dx = p(x)\,\Big|_a^x,$$

where $p(x) = f(x)g(x)$.

Theorems III and IV remain valid when $g(x)$ is in B''.

150 Claremont Avenue
New York 27