*Ergod. Th. & Dynam. Sys.*, (2021), **41**, 3704–3721 © The Author(s), 2020. Published by Cambridge University Press.
doi:10.1017/etds.2020.108 

# Mean equicontinuity and mean sensitivity on cellular automata

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(*Received* 12 *April* 2020 *and accepted in revised form* 19 *August* 2020)

*Abstract.* We show that a cellular automaton (or shift-endomorphism) on a transitive subshift is either almost equicontinuous or sensitive. On the other hand, we construct a cellular automaton on a full shift (hence a transitive subshift) that is neither almost mean equicontinuous nor mean sensitive.

Key words: cellular automata, mean equicontinuity, mean sensitivity, almost equicontinuity, almost mean equicontinuity
2020 Mathematics Subject Classification: 37B05, 37B10, 37B15 (Primary)

## 1. *Introduction*

Sensitivity to initial conditions (or simply sensitivity) is one of the classical notions of chaos in dynamical systems. It was introduced for topological dynamical systems by Guckenheimer [13]. By a *topological dynamical system (TDS)* we mean a pair $(X, T)$ such that $X$ is a compact metric space (with metric $d$) and $T : X \to X$ is continuous. A TDS is *sensitive* if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ and $n > 0$ such that $d(T^n x, T^n y) > \varepsilon$. In contrast to sensitivity, there is the notion of equicontinuity (or Lyapunov stability); a TDS is equicontinuous if $\{T^n\}_{n\in\mathbb{N}}$ is an equicontinuous family. Using sensitivity and equicontinuity, one can classify transitive topological dynamical systems (see Definition 2.2). Akin, Auslander and Berg proved that any transitive TDS is either sensitive or almost equicontinuous [1] (a generalization of the Auslander–Yorke dichotomy [2]). Nonetheless, this classification has some limitations, because sensitivity is not a very strong form of chaos (for example: every non-finite subshift is sensitive; for cellular automata (CA), equicontinuity is strongly connected to local periodicity [8]). Inspired by the notion of mean equicontinuity (or mean Lyapunov stability) first studied by Fomin [6] and Oxtoby [20], the notion of mean sensitivity was introduced [**9, 17**]. A TDS is *mean sensitive* if there exists $\varepsilon > 0$ such that for every non-empty open set $U$ there exist $x, y \in U$ such that $d(T^n x, T^n y) > \varepsilon$ for any $n$ in a set with density greater than $\varepsilon$. The key difference is 'how many' $n$ satisfy the condition. For example, the Sturmian subshift is sensitive but not mean sensitive [**9**]. Similar to the classic case, one can classify transitive TDSs using the mean notions, that is, a transitive TDS is either mean sensitive or almost mean equicontinuous [**9, 17**]. Mean equicontinuity and sensitivity have been studied in other recent papers (for instance, [**5, 7, 10, 11, 14**] or the survey [**18**]) and are strongly related to (measurable) discrete spectra, properties of the maximal equicontinuous factor and quasicrystals.

CA are dynamical systems defined on full shifts $A^{\mathbb{Z}}$ (or more generally on subshifts). They have been used to model phenomena that are based on local rules in physics, biology and computer science. The notion of sensitivity in CA has been studied in many papers (for example, [**4, 8, 12, 15, 19, 21**]). In particular, Kůrka proved that any CA (not necessarily transitive) is either sensitive or almost equicontinuous [**16**]. One of the main ingredients of this proof is that the full shift is transitive (with respect to the shift map). Hence, this statement can be generalized to any shift endomorphism on a transitive subshift (see Proposition 2.8). Thus, it is natural to ask whether, as in the transitive topological dynamics case, a similar dichotomy to Kůrka's holds for the mean versions on CA (on transitive subshifts).

In this paper, we provide the first examples of the study of mean equicontinuity/sensitivity on CA. First, we construct an almost mean equicontinuous CA that is not almost equicontinuous (Theorem 3.14). Second, we construct a CA that is neither mean sensitive nor almost mean equicontinuous (Theorem 3.14). Thus, Kůrka's dichotomy does not hold for the mean notions on CA. In conclusion, CA can be divided in the following four disjoint non-empty classes (Theorem 3.14 and Theorem 4.4): almost equicontinuous, almost mean equicontinuous but not almost equicontinuous, neither almost mean equicontinuous nor mean sensitive, and mean sensitive.

## 2. *Definitions and preliminaries*

*Definition 2.1.* Let $S \subseteq \mathbb{Z}_{\geq 0}$. We define the *upper density* of $S$ by

$$\overline{D}(S) = \limsup_{n\to\infty} \frac{\sharp(S \cap \{0, \ldots, n-1\})}{n}.$$

A *TDS* is a pair $(X, T)$ where $X$ is a compact metric space (with metric $d$) and $T : X \to X$ is continuous.

Transitivity is a topological form of ergodicity.

*Definition 2.2.* Let $(X, T)$ be a TDS. We say that $(X, T)$ is *transitive* if for every pair of non-empty open sets $U$ and $V$ there exists $n > 0$ such that $T^{-n}U \cap V \neq \emptyset$.

*Definition 2.3.*

(1) Given a finite non-singular set $A$ (called an alphabet), we define the *A-full shift* as $A^{\mathbb{Z}}$. If $X$ is the $A$-full shift for some finite $A$, we say that $X$ is a *full shift*.

(2) Given $x \in A^{\mathbb{Z}}$, we represent the $i$th coordinate of $x$ as $x_i$. Also, given $i, j \in \mathbb{Z}$ with $i < j$, we define the finite word $x_{[i,j]} = x_i \ldots x_j$.

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(3) We endow any full shift with the metric

$$d(x, y) = \begin{cases} 2^{-i} & \text{if } x \neq y \text{ where } i = \min\{|j| : x_j \neq y_j\}; \\ 0 & \text{otherwise.} \end{cases}$$

This metric generates the same topology as the product topology.

(4) For any full shift $A^{\mathbb{Z}}$, we define the shift map $\sigma : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ by $\sigma(x)_i = x_{i+1}$. The shift map is continuous (with respect to the previously defined metric).

(5) We say $X$ is a *subshift* (or shift space) if $X \subseteq A^{\mathbb{Z}}$ is closed and $\sigma$-invariant.

Typically, CA are defined on a full shift. We give a more general definition. These systems are also known as *shift-endomorphisms* or *sliding block-codes*.

*Definition 2.4.* We say that $(X, T)$ is a *cellular automaton (CA)* if $X$ is a subshift and $T : X \to X$ is continuous and commutes with $\sigma$, i.e., $\sigma \circ T = T \circ \sigma$.

As mentioned in the Introduction, CA can be described using local rules. Note that $Tx_i$ represents the $i$th coordinate of the point $Tx$.

THEOREM 2.5. (Curtis–Hedlund–Lyndon) *Let $X$ be a subshift and $T : X \to X$ a function. Then, $(X, T)$ is a CA if and only if there exist integers $m \leq a$ and a (local) function $f : A^{a-m+1} \to A$ such that for any $x \in X$ and any $i \in \mathbb{Z}$,*

$$Tx_i = f(x_{[i+m,i+a]}).$$

2.1. *Sensitivity, equicontinuity and dichotomies.* A subset of a topological space is *residual* (or comeagre) if it is the intersection of a countable number of dense open sets.

*Definition 2.6.* Let $(X, T)$ be a TDS and $x \in X$.

(1) The point $x$ is an *equicontinuity point* if

$$\text{for all } \varepsilon > 0, \text{ there exists } \delta > 0 \text{ such that for all } y \in B_\delta(x)$$
$$\text{and all } n \geq 0,\ d(T^n x, T^n y) < \varepsilon.$$

The set of equicontinuity points of $(X, T)$ is denoted by $EQ$.

(2) $(X, T)$ is *equicontinuous* if $EQ = X$.

(3) $(X, T)$ is *almost equicontinuous* if $EQ$ is a residual set.

(4) $(X, T)$ is *sensitive* if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ and $n \neq 0$ such that

$$d(T^n x, T^n y) > \varepsilon.$$

Sensitivity and almost equicontinuity can be used to classify transitive topological dynamical systems.

THEOREM 2.7. (Akin–Auslander–Berg dichotomy [1]) *Transitive topological dynamical systems are sensitive if and only if they are not almost equicontinuous.*

A CA satisfies the same dichotomy without assuming transitivity. This result is proved in [16] for CA on a full shift. Using the same technique, we prove the result for CA on transitive subshifts.

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PROPOSITION 2.8. *Let $(X, \sigma)$ be a transitive subshift and $(X, T)$ a CA. Then, $(X, T)$ is almost equicontinuous if and only if is not sensitive.*

*Proof.* $\Rightarrow$: Assume that $(X, T)$ is almost equicontinuous. This means that for every open subset $U \subseteq X$, there exists $x \in U$ such that for all $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$, then $d(T^n x, T^n y) < \varepsilon$ for all $n \geq 0$.

Let $\varepsilon > 0$. Observe that (using $\varepsilon/2$) there exists $\delta > 0$ such that for all $y, z \in B_\delta(x)$ and all $n \geq 0$,

$$\begin{aligned} d(T^n y, T^n z) &\leq d(T^n y, T^n x) + d(T^n x, T^n z) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, $(X, T)$ is not sensitive.

$\Leftarrow$: Assume that $(X, T)$ is not sensitive, that is, for all $\varepsilon > 0$ there exists an open set $U \subseteq X$ such that for all $x, y \in U$ and for all $n \geq 0$, $d(T^n x, T^n y) < \varepsilon$. Now, since $T$ is uniformly continuous, for $\varepsilon = 1$, there exists $r \geq 0$ such that if $d(x, y) = 2^{-r}$, then $d(Tx, Ty) < 1$. This implies that for all $x, y \in X$ such that $x_{[-r,r]} = y_{[-r,r]}$, $Tx_0 = Ty_0$. Hence, for all $m \geq 0$, there exist $d \geq r$ and $w \in A^{2d+1}$ (given by $U$) such that for all $x, y \in X$ with $x_{[-d,d]} = w = y_{[-d,d]}$ and all $n \geq 0$,

$$T^n x_{[-m,m]} = T^n y_{[-m,m]}.$$

Then, there is $p \in \{0, \ldots, |w| - r\}$ such that for all $x, y \in X$ satisfying $x_{[0,|w|-1]} = w = y_{[0,|w|-1]}$,

$$T^n x_{[p,p+r-1]} = T^n y_{[p,p+r-1]}$$

for all $n \in \mathbb{N}$.

For every $k \geq 0$ we define the set

$$\Omega_k = \{x \in X : \exists i \leq -k, x_{[i,i+|w|-1]} = w \wedge \exists j \geq k, x_{[j,j+|w|-1]} = w\}.$$

The sets $\Omega_k$ are clearly open. Furthermore, the transitivity of $(X, T)$ implies that $\Omega_K$ are non-empty and dense, for every $k \geq 0$. Therefore, $\bigcap_{k \geq 0} \Omega_k$ is a residual set. We now show that for every $m \geq 0$, there exists $k_m \geq 0$ such that

$$\Omega_{k_m} \subseteq EQ_{2^{-m}} := \{x \in X : \exists \delta,\ \forall y, z \in B_\delta(x),\ \forall n \geq 0,\ d(T^n y, T^n z) < 2^{-m}\}.$$

Observe that for all $x, y \in \Omega_k$,

$$T^n x_{[i+p,i+p+r-1]} = T^n x_{[j+p,j+p+r-1]} \quad \text{and} \quad T^n y_{[i+p,i+p+r-1]} = T^n y_{[j+p,j+p+r-1]}.$$

If $x_{[i,j+|w|]} = y_{[i,j+|w|]}$, then for all $n \geq 0$ we obtain

$$T^n x_{[i+p,j+p+r-1]} = T^n y_{[i+p,j+p+r-1]}.$$

Therefore, for every $m \geq 0$, there exists a $k_m \geq 0$ sufficiently large that $\Omega_{k_m} \subseteq EQ_{2^{-m}}$. Hence, $\bigcap_{k_m \geq 0} \Omega_{k_m} \subseteq \bigcap_{m \geq 0} EQ_{2^{-m}}$. This makes $\bigcap_{m \geq 0} EQ_{2^{-m}}$ a residual set. Since $EQ = \bigcap_{m \geq 0} EQ_{2^{-m}}$, we conclude that $(X, T)$ is almost equicontinuous. $\square$

A TDS is *minimal* if every orbit is dense. The Auslander–Yorke dichotomy states that a minimal TDS is either equicontinuous or sensitive [2]. Now, consider the proof of

Proposition 2.8. Note that if $(X, \sigma)$ is minimal then $\Omega_k = X$. With this observation we obtain the following result.

PROPOSITION 2.9. *Let $(X, \sigma)$ be a minimal subshift and $(X, T)$ a CA. Then, $(X, T)$ is equicontinuous if and only if is not sensitive.*

We now study the mean versions of equicontinuity and sensitivity (mean equicontinuity is weaker than equicontinuity, and sensitivity is weaker than mean sensitivity).

*Definition 2.10.* Let $(X, T)$ be a TDS and $x \in X$.

(1) The point $x$ is a *mean equicontinuity point* if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $d(x, y) < \delta$, then

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} \leq \varepsilon.$$

We denote the set of mean equicontinuity points by $EQ^M$.

(2) $(X, T)$ is *mean equicontinuous* if $X = EQ^M$.

(3) $(X, T)$ is *almost mean equicontinuous* if $EQ^M$ is residual.

(4) $(X, T)$ is *mean sensitive* if there exists $\varepsilon > 0$ such that for every non-empty open set $U \subseteq X$ there exist $x, y \in U$ such that

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} > \varepsilon.$$

Clearly, every almost equicontinuous TDS is almost mean equicontinuous. There exist many almost mean equicontinuous TDSs that are not almost equicontinuous [11, 17]; none of these examples is a CA. We will later construct an almost mean equicontinuous CA that is not almost equicontinuous.

PROPOSITION 2.11. [9, Lemma 5] *Let $(X, T)$ be a TDS and $\varepsilon > 0$. Define*

$$EQ^M_\varepsilon = \left\{ x \in X : \exists \delta > 0,\ \forall y, z \in B_\delta(x),\ \limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i y, T^i z)}{n+1} < \varepsilon \right\}.$$

*Then, $EQ^M_\varepsilon$ is open and $EQ^M = \bigcap_{m>0} EQ^M_{1/m}$. Furthermore, $EQ^M$ is dense if and only if it is a residual set.*

The Akin–Auslander–Berg dichotomy can also be stated for the mean versions of equicontinuity and sensitivity.

THEOREM 2.12. (Mean Akin–Auslander–Berg dichotomy [9, 17]) *Transitive topological dynamical systems are mean sensitive if and only if they are not almost mean equicontinuous.*

In view of the previous results in this section, it is natural to ask whether there is a mean version of Theorem 2.8. Later, we show that this question has a negative answer. First, we give a more concrete characterization of mean equicontinuity on CA. The following proposition uses standard tools to connect density and averages.

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PROPOSITION 2.13. *Let $(X, T)$ be a CA and $x \in X$. Then $x$ is a mean equicontinuity point if and only if for every $m \geq 0$ there exists $m' \geq 0$ such that for every $y \in B_{2^{-m'}}(x)$, the set*

$$S_j := \{i \in \mathbb{Z}_{\geq 0} : T^i x_j \neq T^i y_j \vee T^i x_{-j} \neq T^i y_{-j}\}$$

*satisfies*

$$\overline{D}(S_j) \leq \frac{1}{2^{m+2}},$$

*for all* $0 \leq j \leq m+1$.

*Proof.* $\Rightarrow$: Assume on the contrary that there exists $m \geq 0$ such that for all $m' \geq 0$ there exists $y \in B_{2^{-m'}}(x)$ such that $\overline{D}(S_l) > 1/2^m$ for some $0 \leq l \leq m+1$. Therefore, there exists $(n_j)_{j\geq 0} \subseteq \mathbb{Z}_{\geq 0}$ such that

$$\lim_{n_j\to\infty} \frac{\sharp(S_l \cap \{0, \ldots, n_j\})}{n_j + 1} > 2^{-m}.$$

Observe that

$$\lim_{n_j\to\infty} \frac{\sharp(S_l \cap \{0, \ldots, n_j\})}{n_j + 1} = \lim_{n_j\to\infty} \frac{\sum_{i\in S_l\cap\{0,\ldots,n_j\}} d(T^i x, T^i y)}{n_j + 1}.$$

Then, we obtain

$$\lim_{n_j\to\infty} \frac{\sum_{i=0}^{n_j} d(T^i x, T^i y)}{n_j + 1} > 2^{-m}.$$

Hence,

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n + 1} > 2^{-m}.$$

Therefore, $x$ is not a mean equicontinuity point.

$\Leftarrow$: For every $x, y \in A^{\mathbb{Z}}$ and every pair of integers $n, k \geq 0$, define the set

$$C_{n,k} := \{0 \leq i \leq n : T^i x_{[-k,k]} \neq T^i y_{[-k,k]}\}.$$

Observe that:

(1) for every $k \geq 0$, $C_{n,k} \subseteq C_{n,k+1}$;

(2) for every $k \geq 0$,

$$\begin{aligned} C_{n,k+1} \backslash C_{n,k} = \{i \in [0, n] : T^i x_{[-k,k]} \\ = T^i y_{[-k,k]} \wedge T^i x_{[-(k+1),k+1]} \neq T^i y_{[-(k+1),k+1]}\}. \end{aligned}$$

Now, assume that for every $m \geq 0$ there exists $m' \geq 0$ such that for every $y \in B_{1/2^{m'}}(x)$, the set

$$S_j = \{i \geq 0 : T^i x_j \neq T^i y_j \vee T^i x_{-j} \neq T^i y_{-j}\}$$

satisfies $\overline{D}(S_j) \leq 1/2^{m+2}$, for every $0 \leq j \leq m+1$. Then,

$$\begin{aligned} &\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T^i x, T^i y)}{n+1} \\ &= \limsup_{n\to\infty} \frac{\sharp(C_{n,0}) + \sum_{i=1}^{\infty} (1/2^i)\sharp(C_{n,i} \backslash C_{n,i-1})}{n+1} \end{aligned}$$

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$$
\begin{aligned}
&= \sum_{n\to\infty} \frac{\sum_{i=0}^{m+1}(1/2^i)\sharp(S_i \cap [0,n]) + \sum_{i=m+2}^{\infty}(1/2^i)\sharp(C_{n,i}\backslash C_{n,i-1})}{n+1} \\
&\leq \sum_{i=0}^{m+1} \frac{1}{2^i}\frac{1}{2^{m+2}} + \limsup_{n\to\infty} \frac{\sum_{i=m+2}^{\infty}(1/2^i)\sharp(C_{n,i}\backslash C_{n,i-1})}{n+1} \\
&\leq \sum_{i=0}^{m+1} \frac{1}{2^i}\frac{1}{2^{m+2}} + \sum_{i=m+2}^{\infty}\frac{1}{2^i} \\
&\leq \frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} \\
&= \frac{1}{2^m}.
\end{aligned}
$$

This implies that $x$ is a mean equicontinuity point. □

Mean equicontinuity of CA should not be confused with equicontinuity with respect to the Besicovitch pseudometric studied in [3].

### 3. *Example 1: Pacman CA*

In this section, we construct a CA that is almost mean equicontinuous but not almost equicontinuous. First, we give the formal definition of the CA, then we give the heuristics of the map so that the reader gains intuition and, finally, we approach the result using a series of technical lemmas. We remind the reader that $Tx_i$ represents the $i$th coordinate of the point $Tx$.

Let $A = \{\square, \mathrm{W}, \mathrm{G}, \mathrm{G}^{*}, \mathrm{P}, \mathrm{W}_{\mathrm{G}}\}$. We define the function $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ locally as follows:

$$
Tx_i = \begin{cases}
\square & \text{if } (x_{i-1} \in \{\square, \mathrm{G}, \mathrm{G}^{*}\} \wedge [(x_i \in \{\square, \mathrm{G}, \mathrm{G}^{*}\} \wedge x_{i+1} \in \{\square, \mathrm{W}, \mathrm{P}\}) \\
& \quad \vee (x_i = \mathrm{P} \wedge x_{i+1} \notin \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\})]) \\
& \quad \vee (x_{i-1} \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge [(x_i \in \{\square, \mathrm{G}^{*}\} \wedge x_{i+1} \in \{\square, \mathrm{W}, \mathrm{P}\}) \\
& \quad \vee (x_i = \mathrm{P} \wedge x_{i+1} \notin \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\})]), \\
\mathrm{W} & \text{if } x_i \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge x_{i+1} \notin \{\mathrm{G}^{*}, \mathrm{W}_{\mathrm{G}}\}, \\
\mathrm{G} & \text{if } (x_{i-1} \in \{\square, \mathrm{G}, \mathrm{G}^{*}\} \wedge x_i \in \{\square, \mathrm{G}, \mathrm{G}^{*}\} \wedge x_{i+1} \in \{\mathrm{G}, \mathrm{W}_{\mathrm{G}}\}) \\
& \quad \vee (x_{i-1} \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge x_i \in \{\square, \mathrm{G}^{*}\} \wedge x_{i+1} = \mathrm{W}_{\mathrm{G}}), \\
\mathrm{G}^{*} & \text{if } (x_{i+1} = \mathrm{G}^{*} \wedge [(x_{i-1} \in \{\square, \mathrm{G}, \mathrm{G}^{*}\} \wedge x_i \in \{\square, \mathrm{G}, \mathrm{G}^{*}\}) \\
& \quad \vee (x_{i-1} \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge x_i \in \{\square, \mathrm{G}^{*}\})]) \\
& \quad \vee (x_{i-1} = \mathrm{P} \wedge x_i \notin \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge x_{i+1} \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\}) \\
& \quad \vee (x_i = \mathrm{P} \wedge x_{i+1} \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\}), \\
\mathrm{P} & \text{if } (x_{i-1} = \mathrm{W} \wedge [(x_i \in \{\square, \mathrm{G}^{*}\} \wedge x_{i+1} = \mathrm{G}) \vee x_i = \mathrm{G}]) \\
& \quad \vee (x_{i-1} = \mathrm{P} \wedge x_i, x_{i+1} \notin \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\}), \text{ and} \\
\mathrm{W}_{\mathrm{G}} & \text{if } x_i \in \{\mathrm{W}, \mathrm{W}_{\mathrm{G}}\} \wedge x_{i+1} \in \{\mathrm{G}^{*}, \mathrm{W}_{\mathrm{G}}\}.
\end{cases}
$$

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This CA has memory and anticipation 1. We define the members of the alphabet as follows:

- [space] empty space;
- [door] empty door;
- [pacman] pacman;
- [ghost] ghost;
- [keymaster] keymaster ghost;
- [door-ghost] door with ghost.

We now explain the heuristics of this map so the reader gains intuition regarding the dynamics. The reader does not need to know all the rules of the game *Pacman*, only that pacmans eat blue ghosts.

- A door always stays fixed in the same place (a ghost might cross it); that is, $x_i \in$ {[door], [door-ghost]} if and only if $Tx_i \in$ {[door], [door-ghost]}.
- Pacmans [pacman] move to the right (one position per unit of time) if there is no door; that is, if $x_i =$ [pacman] and $x_{i+1}, x_{i+2} \notin$ {[door], [door-ghost]}, then $Tx_{i+1} =$ [pacman].
- If a pacman encounters a door (on the right), it is transformed into a keymaster ghost [keymaster]; that is, if $x_i =$ [pacman] and $x_j \in$ {[door], [door-ghost]} with $j \in \{i+1, i+2\}$, then $Tx_{j-1} =$ [keymaster].
- Ghosts ([ghost], [keymaster]) always move to the left (one position per unit of time) if there is no pacman or door on the left; that is, if $x_i =$ [ghost]($x_i =$ [keymaster]), $x_{i-1} \in$ {[space], [ghost], [keymaster]} and $x_{i-2} \in$ {[space], [ghost], [keymaster]}($x_{i-2} \neq$ [pacman]), then $Tx_{i-1} =$ [ghost]($Tx_{i-1} =$ [keymaster]).
- If a ghost or keymaster ghost encounters a pacman (on the left) it will disappear (get eaten); that is,
  - if $x_i \in$ {[ghost], [keymaster]} and $x_{i-1} =$ [pacman], then $Tx_{i-1} \notin$ {[ghost], [keymaster]};
  - if $x_i \in$ {[ghost], [keymaster]}, $x_{i-2} =$ [pacman] and $x_{i-1} \notin$ {[door], [door-ghost]}, then $Tx_{i-1} =$ [pacman].
- If a ghost [ghost] encounters a door, it transforms into a pacman; that is,
  - if $x_i =$ [ghost] and $x_{i-1} \in$ {[door], [door-ghost]}, then $Tx_i =$ [pacman];
  - if $x_i =$ [ghost], $x_{i-2} =$ {[door], [door-ghost]} and $x_{i-1} \notin$ {[door], [door-ghost], [pacman]}, then $Tx_{i-1} =$ [pacman].
- If a keymaster ghost encounters a door, it will enter the door, lose its key and (in the following step) proceed to the left; that is, if $x_i =$ [keymaster] and $x_{i-1} \in$ {[door], [door-ghost]}, then $Tx_{i-1} =$ [door-ghost], and
  - if $x_{i-3} =$ [space], then $T^2x_{i-2} =$ [ghost];
  - if $x_{i-3} =$ [pacman], then $T^2x_{i-2} =$ [keymaster].

When describing a point in $A^{\mathbb{Z}}$, we use a point (.) to indicate the 0th coordinate; for example, if $x =$ $^{\infty}$[space].[pacman][space]$^{\infty}$ then $x_0 =$ [pacman] and $x_i =$ [space] for every $i \neq 0$. We now provide some examples of how the Pacman CA works. Note that time flows downward in the diagrams.

*Example 3.1.* Let $m \geq 2$, $w =$ [door] [space]$^m$ [door]. We show a section of the orbit of $x :=$ $^{\infty}$[space].$w$[keymaster]$^{\infty}$. In this example, the space between two doors acts as a ‘filter’, because many ghosts disappear.

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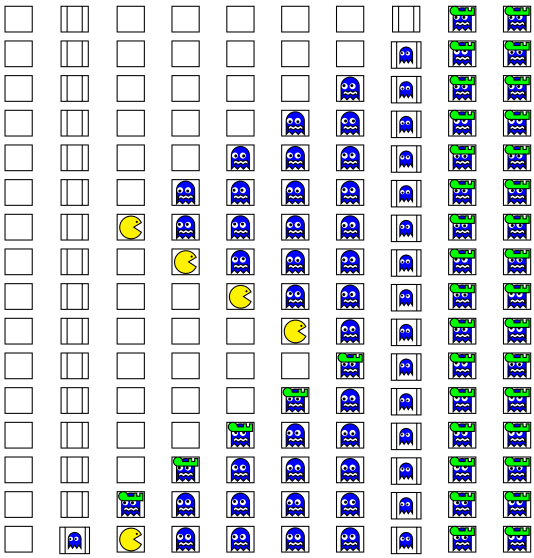
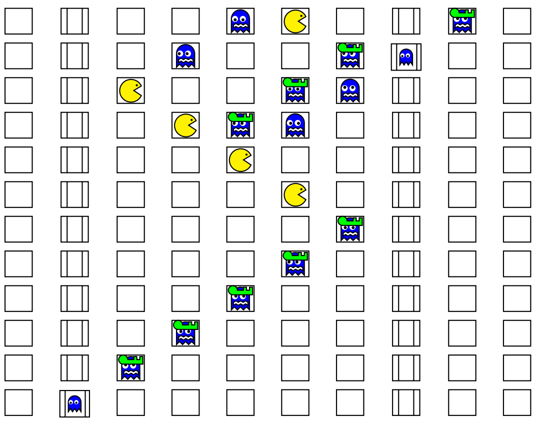


*Example 3.2.* Let $w =$  $\square\ \square$   $\square$  . We show a section of the orbit of $x = {}^{\infty}\square.w\square^{\infty}$.



We now prove a series of technical lemmas. If there is one  to the right of a pattern with empty spaces and empty doors, then  will 'cross' all the doors eventually. We state this fact formally in the next lemma.

LEMMA 3.3. *Let $m \geq 1$, and $w \in \{\square,$ $\}^m$ such that $w_0 =$  $= w_{m-1}$ and $w_i = \square$ for all $0 < i < m-1$. Set $x = {}^{\infty}\square.w$ $\square^{\infty}$. Then, there exists $N > 0$ such that $T^N x_{[0,m-1]} = w$.*

*Proof.* Assume that $m \geq 4$. From the definition of $T$, we have the following implications:

- $Tx_{m-1} =$  $\wedge\ Tx_i \in \{\square,$ $\}$ for all $i \neq m-1$;
- $T^{m-j}x_j =$  for $1 < j < m-1 \wedge T^{m-j}x_i \in \{\square,$ $\}$ for all $i \neq j$;

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- $T^{m-2+j}x_j = \text{[pacman]}$ for $1 \leq j < m-2 \wedge T^{m-2+j}x_i \in \{\text{[empty]}, \text{[door]}\}$ for all $i \neq j$;
- $T^{3m-6-j}x_j = \text{[green ghost]}$ for $1 \leq j \leq m-2 \wedge T^{3m-6-j}x_i \in \{\text{[empty]}, \text{[door]}\}$ for all $i \neq j$;
- $T^{3m-6}x_0 = \text{[ghost in door]} \wedge T^{3m-6}x_i \in \{\text{[empty]}, \text{[door]}\}$ for all $i \neq 0$.

Then, for $N = 3m-5$, we have that $T^N x_{[0,m-1]} = w$. The case when $1 \leq m \leq 3$ is easy to check. □

*Remark 3.4.* Note that if $x_i = \text{[door]}$ and $x_i + 1 = \text{[empty]}$ then $Tx_i = \text{[door]}$.

Using Remark 3.4 and Lemma 3.3 we obtain the following result.

LEMMA 3.5. *Let $m > 0$, $w \in A^m$ and $x = {}^{\infty}\text{[empty]}.w\text{[door]}\,\text{[empty]}^{\infty}$. There exists $N > 0$ such that for all $n \geq N$,*

$$T^n x_i \in \{\text{[empty]}, \text{[door]}\} \quad \textit{for all } i \geq 0.$$

In the proof of Lemma 3.3 we describe the 'trajectory' of [green ghost] from the start until it crosses the doors. In the following lemma we describe a similar trajectory, but this time we do it backwards in time.

LEMMA 3.6. *Let $m \geq 2$, $v \in A^{\mathbb{N}}$ and $x := {}^{\infty}\text{[empty]}.\text{[door]}\,\text{[empty]}^m\,\text{[door]}v$. If $N \geq 3m$ and $T^N x_0 = \text{[ghost in door]}$, then:*

- $T^{N-3m+1}x_{m+1} = \text{[ghost in door]}$;
- $T^{N-2(m-1)-j}x_j = \text{[ghost in door]}$ *for* $2 \leq j \leq m$;
- $T^{N-m-j}x_{m-j} = \text{[pacman]}$ *for* $1 \leq j \leq m-1$; *and*
- $T^{N-j}x_j = \text{[green ghost]}$ *for* $1 \leq j \leq m$.

*Proof.* Assume the hypothesis of the lemma. By checking the rules of $T$ one can see that if $T^N x_0 = \text{[ghost in door]}$ and $x_1 \neq \text{[door]}$ then necessarily $T^{N-1}x_1 = \text{[green ghost]}$. We can go back step by step to obtain the result. □

Using Lemma 3.6 we can see that if $T^N x_0 = \text{[ghost in door]} = T^{N'}x_0$, then $N$ and $N'$ cannot be near.

LEMMA 3.7. *Let $m \geq 0$, $v \in A^{\mathbb{N}}$ and $x := {}^{\infty}\text{[empty]}.\text{[door]}\,\text{[empty]}^m\text{[door]}v$. If $N' > N \geq 3m$ are such that $T^N x_0 = \text{[ghost in door]} = T^{N'}x_0$, then:*

- *if $0 \leq m \leq 1$, then $N' - N > 2m$; and*
- *if $m \geq 2$, then $N' - N \geq 2m - 1$.*

*Proof.* The case $0 \leq m \leq 1$ is trivial.

Let $m \geq 2$, and $N' > N \geq 3m$ such that $T^N x_0 = \text{[ghost in door]} = T^{N'}x_0$. Assume that $N' - N < 2m - 1$. From Lemma 3.6 we have that:

- $T^{N-j}x_j = \text{[green ghost]}$ for $1 \leq j \leq m$; and
- $T^{N'-m-j'}x_{m-j'} = \text{[pacman]}$ for $1 \leq j' \leq m-1$.

First, suppose that $N' - N$ is even. Let $j = m - (N' - N)/2$ and $j' = (N' - N)/2$. By the assumption on $N$, $N'$ and $m$ it follows that $1 \leq j \leq m$, $1 \leq j' \leq m-1$ and

$$T^{N-j}x_j = T^{N'-m-j'}x_{m-j},$$

a clear contradiction.

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Now suppose $N' - N$ is odd. Let $j = m - \lceil (N' - N)/2 \rceil$ and $j' = \lceil (N' - N)/2 \rceil$. By the assumption on $N$, $N'$ and $m$ it follows that $1 \leq j \leq m$, $1 \leq j' \leq m - 1$,

$$T^{N-j} x_j = \quad \text{and} \quad T^{N'-m-j'} x_{m-j'} = .$$

Therefore,

$$T^{N-j} x_{[j,j+1]} = .$$

This is also a contradiction because is not on the image of $T$. $\square$

In Example 3.1, we see that in an infinite right-tail of keymaster ghosts, some get eaten and some cross the doors. It is natural to ask the frequency of that cross doors. The next lemma answers this question.

LEMMA 3.8. *Let $w = \square\square^{m}\square$, with $m \geq 0$ and $x = {}^{\infty}\square.w{}^{\infty}$.*

*If $0 \leq m \leq 1$, then:*

- $T^{3m-2} x_0 = $;
- $T^{3m-2+(2m+1)k} x_0 = $ *for $k \geq 0$; and*
- $T^i x_0 = \square$ *for all $3m - 2 + (2m+1)k < i < 3m - 2 + (2m+1)(k+1)$ and $k \geq 0$.*

*If $m \geq 2$, then:*

- $T^{3m} x_0 = $,
- $T^{3m+(2m-1)k} x_0 = $ *for $k \geq 0$; and*
- $T^i x_0 = \square$ *for all $3m + (2m-1)k < i < 3m + (2m-1)(k+1)$ and $k \geq 0$.*

*Proof.* The proof for $0 \leq m \leq 1$ is similar to the proof when $m \geq 2$. Thus, we only prove the result when $m \geq 2$. Using a similar argument to that used in the proof of Lemma 3.3, we obtain that $T^{3m} x_0 = $ and $T^i x_0 = \square$ for all $0 < i < 3m$. Also, we have that $T^{2m-1} x_{m+2} = $. Hence, $T^{5m-1} x_0 = $.

We proceed by induction on $k$. Assume that

$$T^{3m+(2m-1)l} x_0 = .$$

Next, let $k = l + 1$. By the induction hypothesis,

$$T^{2m-1+(2m-1)l} x_{(m+2)} = .$$

Hence, $T^{5m-1+(2m-1)l} x_0 = $. By simple calculations, we obtain

$$T^{3m+(2m-1)(l+1)} x_0 = .$$

The proof of $T^i x_0 = \square$, for all $3m + (2m-1)k < i < 3m + (2m-1)(k+1)$ and $k \geq 0$, follows immediately from Lemma 3.7. $\square$

We now prove that the set of equicontinuity points is empty.

PROPOSITION 3.9. *Let $m \geq 1$ and $w \in A^m$. Then there exist $x, y \in A^{\mathbb{Z}}$ such that*

$$x_{[0,m-1]} = w = y_{[0,m-1]}$$

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*and the set*

$$S = \{i \in \mathbb{Z}_{\geq 0} : T^i x_0 \neq T^i y_0\}$$

*is infinite.*

*Proof.* Let $m \geq 1$, $w \in A^m$ and $x = {}^{\infty}\square.w\square^{\infty}$. By Lemma 3.5, there exists $N > 0$ such that $T^n x_0 \in \{\square, \square\square\}$ for every $n \geq N$. Let $y = {}^{\infty}\square.w\square\square$ $^{\infty}$; then, by Lemma 3.8 the set $S$ is infinite. $\square$

Lemma 3.8 tells us the exact frequency of  crossing doors when we have a tail $^{\infty}$ to the right. If we do not have precise information on what is on the right, we may not have the exact frequency, as in Lemma 3.8. However, using Lemma 3.7, we can obtain an upper bound.

Now we explore a similar situation but with finitely many doors.

LEMMA 3.10. *Let $\{d_i\}_{i=0}^{n}$ be a finite set of non-negative integers, $v \in A^{\mathbb{N}}$,*

$$w = \square\square\square^{d_0}\square\square\square^{d_1} \cdots \square\square\square^{d_n}\square\square,$$

*$x = {}^{\infty}\square.wv$ and $0 \leq j < n + \sum_{i=0}^{n-1} d_i$. Assume that $T^N x_j =$*  *$= T^{N'} x_j$ for some $N, N' \geq 0$. We have that:*

- *if $0 \leq d_n \leq 1$, then $|N - N'| > 2d_n$; and*
- *if $d_n \geq 2$, then $|N - N'| > 2(d_n - 1)$.*

*Proof.* The case where $n = 0$ is a direct application of Lemma 3.7. We prove the other case by induction. Assume that for $n = p$ the result holds. Now, let $n = p + 1$ . By the induction hypothesis, if $x_j = \square\square$ for all $1 \leq j \leq p + 1 + \sum_{i=0}^{p} d_i$, and $T^N x_j =$  $= T^{N'} x_j$ for all $N, N' \geq 0$, then

$$\begin{array}{lll} \text{if } d_{p+1} \geq 2 & \text{then } |N - N'| \geq 2(d_{p+1} - 1) & \text{or} \\ \text{if } 0 \leq d_{p+1} \leq 1 & \text{then } |N - N'| \geq 2d_{p+1}. & \end{array}$$

Hence, all that is left to prove is that for $x_0 = \square\square$ and all $N, N' \geq 0$ such that $T^N x_0 =$  $= T^{N'} x_0$,

$$\begin{array}{lll} \text{if } d_{p+1} \geq 2 & \text{then } |N - N'| \geq 2(d_{p+1} - 1) & \text{or} \\ \text{if } 0 \leq d_{p+1} \leq 1 & \text{then } |N - N'| \geq 2d_{p+1}. & \end{array}$$

For $0 \leq d_{p+1} \leq 1$ the result is trivial. So, let us assume that $d_{p+1} \geq 2$. Also, let us assume that there exist $N, N' \geq 0$ such that $T^N x_0 =$  $= T^{N'} x_0$. This means that there exist $N_0, N_0' \geq 0$ such that $T^{N_0} x_{d_0+1} =$  $= T^{N_0'} x_{d_0+1}$ and $N_0' + r = N'$ and $N_0 + r = N$. Therefore,

$$2(d_{p+1} - 1) \leq |N' - N|. \qquad \square$$

For Lemma 3.11 it will be useful to consider the CA as a (vanishing) particle system, where ghosts and pacmans are particles.

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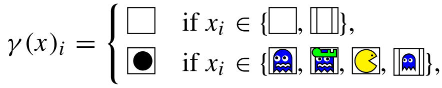
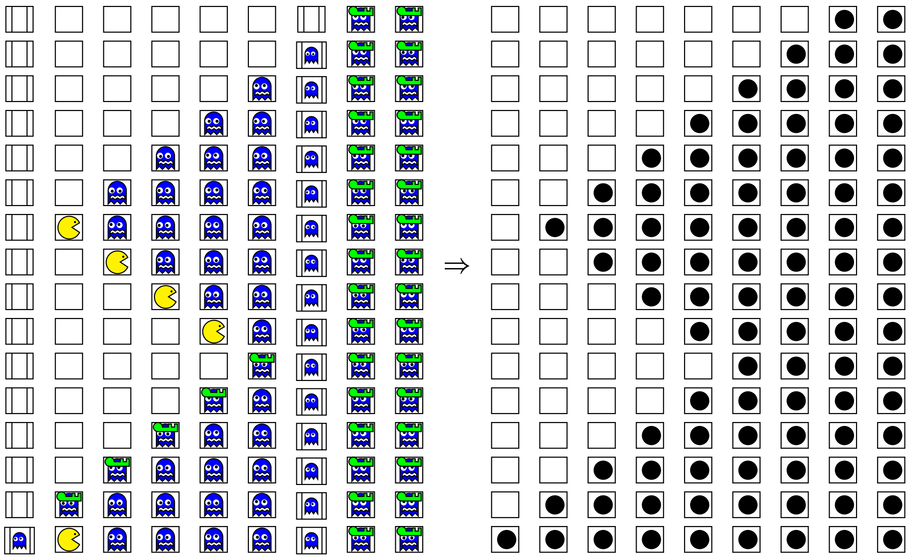
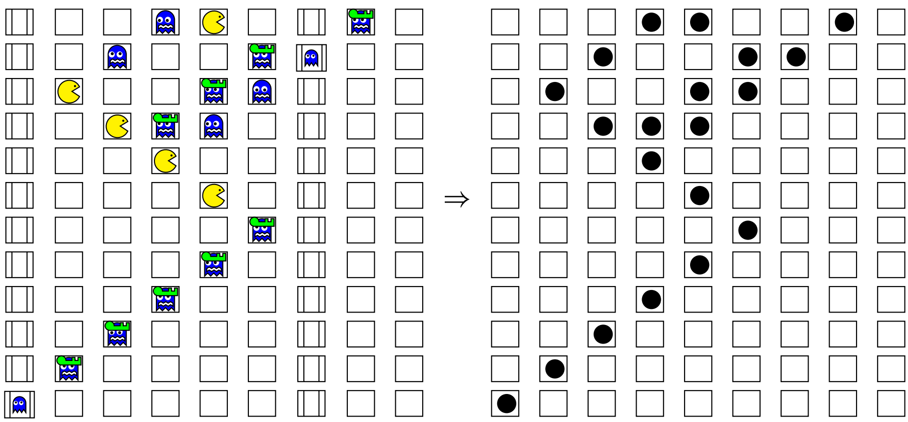
We define the *particle function* $\gamma : A^{\mathbb{Z}} \to \{□, ●\}^{\mathbb{Z}}$ as



where $x \in A^{\mathbb{Z}}$ and $i \in \mathbb{Z}$. Observe that with this function, Examples 3.1 and 3.2 turn out as follows.





Given $x, y \in X$, we define the sets

$$S_{+j} := \{i \geq 0 : T^i x_j \neq T^i y_j\}$$

and

$$S_{-j} := \{i \geq 0 : T^i x_{-j} \neq T^i y_{-j}\}.$$

Observe that $S_j = S_{+j} \cup S_{-j}$ (see Proposition 2.13 for the definition of $S_j$).

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LEMMA 3.11. *Let $d > 0$, $w = \boxminus\square^d\boxminus$, $x :=^\infty \square.w\square^\infty$, $v \in A^{\mathbb{N}}$ and $y =^\infty \square.wv$. If $1 \leq i \leq d$, then*

$$3\overline{D}(S_{+(d+1)}) \geq \overline{D}(S_{+i}).$$

*Proof.* Using the CA and the particle function we can define a *trajectory function* of an specific particle (a ghost/pacman) $p$. We will not construct this function explicitly, but we give its properties. Given a point $x \in X$ and a particle of that point, that is, $p \in \mathbb{Z}$ with $\gamma(x)_p = \boxdot$, we can define trajectory of that particle (all the way to infinity or until it disappears). This trajectory is a function $\tau_p : N \to \mathbb{Z}$ where $N \subset \mathbb{N}$ is the lifespan of the particle ($N = \mathbb{N}$ if it never disappears), $\tau_p(0) = p$ and $\tau_p(n)$ is the position at time $n$. We have that $|\tau_p(n) - \tau_p(n+1)| \leq 1$ for $n + 1 \in N$. Using the properties of $T$, it is not hard to see that for every $z \in \mathbb{Z}$ we have that $|\tau_p^{-1}(z)| \leq 3$, that is, a particle can be at a particular position at most three times. By Lemma 3.5, there exists $N > 0$ such that if for some $l > 0$, $T^{N+l}y_i \notin \{\square, \boxminus\}$, then there exists a unique $k \geq |w|$ such that $\gamma(y)_k = \boxdot$ and $\tau_k(N + l) = i$. Hence, for all $n \in S_{+i}$, there exist a unique $k_n \geq |w|$ and $m < n$ such that $\tau_{k_n}(n) = i$ and $\tau_{k_n}(m) = d + 1$. Since $y_{d+1} = \boxminus$, $|\tau_{k_n}^{-1}(d + 1)| = 1$. Define $P := \{z \in \mathbb{Z} : \gamma(y)_z = \boxdot\}$. Therefore,

$$\limsup_{n\to\infty} \frac{3\sharp[(\bigcup_{z\in P} \tau_z^{-1}(d+1)) \cap [0, n]]}{n+1} \geq \limsup_{n\to\infty} \frac{\sharp[(\bigcup_{z\in P} \tau_z^{-1}(i)) \cap [0, n]]}{n+1}. \tag{1}$$

Since $\gamma(x)_z = \square$ for all $z \in \mathbb{Z}$,

$$\bigcup_{z\in P} \tau_z^{-1}(i) = S_{+i} \quad \text{and}$$

$$\bigcup_{z\in P} \tau_z^{-1}(d+1) = S_{+(d+1)}.$$

Therefore, from (1), we conclude that

$$3\overline{D}(S_{+(d+1)}) \geq \overline{D}(S_{+i}).$$

□

PROPOSITION 3.12. *Let $x = \cdots \boxminus\square^{2^2}\boxminus\square^{2^1}\boxminus\square^{2^0}.\boxminus\square^{2^0}\boxminus\square^{2^1}\boxminus\square^{2^2}\cdots$. Then, $x$ is a mean equicontinuity point.*

*Proof.* We divide the proof in two parts.

Part 1: Let $m \geq 0$, $m' = m + 3 + \sum_{l=0}^{m+3} 2^l$ and $y \in X$, with

$$y_{[-k,k]} = \boxminus\square^{2^{m+3}}\cdots\boxminus\square^{2^2}\boxminus\square^{2^1}\boxminus\square^{2^0}.\boxminus\square^{2^0}\boxminus\square^{2^1}\boxminus\square^{2^2}\cdots\boxminus\square^{2^{m+3}}\boxminus,$$

for a certain $k$. By Lemma 3.10, if $x_j =$, where $0 \leq j \leq m + 3 + \sum_{l=0}^{m+2} 2^l$, then for all $N, N' \geq 0$ such that $T^N y_j = \boxed{☺} = T^{N'} y_j$, $|N - N'| \geq 2(2^{m+3} - 1)$. Now,

$$\limsup_{n\to\infty} \frac{\sharp(S_j \cap [0, n])}{n+1} \leq \limsup_{n\to n} \frac{\sharp(S_{+j} \cap [0, n])}{n+1} + \limsup_{n\to n} \frac{\sharp(S_{-j} \cap [0, n])}{n+1}.$$

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Define $N_0 = \min S_{+j}$. Observe that $\sharp(S_{+j} \cap [0, N_0])/(N_0 + 1) = 1/(N_0 + 1)$. Let $N_1 = \min S_{+j} \backslash \{N_0\}$. We have that $\sharp(S_{+j} \cap [0, N_1])/(N_1 + 1) = 2/(N_1 + 1) < 2/(N_0 + 2(2^{m+3} - 1) + 1)$. Following this construction, for every $r \geq 1$, we define $N_r = \min(S_{+j} \backslash \{N_l\}_{l=0}^{r-1})$. Observe that

$$\begin{aligned}\frac{\sharp(S_{+j} \cap [0, N_r])}{N_r + 1} &= \frac{r+1}{N_r + 1} \\ &< \frac{r+1}{r(2^{m+4} - 2 + 1/r)}.\end{aligned}$$

Since

$$\lim_{r\to\infty} \frac{r+1}{r} \frac{1}{2^{m+4} - 2 + 1/r} = \frac{1}{2^{m+4} - 2},$$

we have

$$\lim_{r\to\infty} \frac{\sharp(S_{+j} \cap [0, N_r])}{N_r + 1} \leq \frac{1}{2(2^{m+3} - 1)} < \frac{1}{2^{m+3}}.$$

Similarly, we obtain

$$\lim_{r\to\infty} \frac{\sharp(S_{-j} \cap [0, N_r])}{N_r + 1} < \frac{1}{2^{m+3}}.$$

Thus,

$$\lim_{r\to\infty} \frac{\sharp(S_{j} \cap [0, N_r])}{N_r + 1} \leq \frac{1}{2^{m+2}}.$$

Part 2: By Lemma 3.11 and Part 1, for all $j \in \mathbb{Z}$ with $x_j = \square$ and

$$-\left(m + 2 + \sum_{l=0}^{m+2} 2^l\right) \leq j \leq m + 2 + \sum_{l=0}^{m+2} 2^l,$$

we have

$$3\overline{D}(S_d) \geq \overline{D}(S_j),$$

where $d = m + 3 + \sum_{l=0}^{m+2} 2^l$. Since $\overline{D}(S_d) \leq \frac{1}{3}(1/2^{m+2})$, $\overline{D}(S_j) \leq 1/2^{m+2}$. Therefore, by Proposition 2.13, $x$ is a mean equicontinuity point. $\square$

The proof of Lemma 3.13 is very similar to that of Lemma 3.12.

LEMMA 3.13. *Let $m > 0$, $w \in A^m$ and*

$$x := \cdots \boxed{\|}\,\square^{2^2}\boxed{\|}\,\square^{2^1}\boxed{\|}\,\square^{2^0}.w\boxed{\|}\,\square^{2^0}\boxed{\|}\,\square^{2^1}\boxed{\|}\,\square^{2^2} \cdots .$$

*Then $x$ is a mean equicontinuity point.*

THEOREM 3.14. *$(A^{\mathbb{Z}}, T)$ has no equicontinuity points (hence is not almost equicontinuous). However, it is almost mean equicontinuous.*

*Proof.* The first statement follows immediately from Proposition 3.9.

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Now, let $x \in A^{\mathbb{Z}}$, $m \geq 0$ and $w = x_{[0,m]}$. We set

$$y := \cdots \text{[wall]}\,\text{[empty]}^{2^2}\,\text{[wall]}\,\text{[empty]}^{2^1}\,\text{[wall]}\,\text{[empty]}^{2^0}.w\,\text{[wall]}\,\text{[empty]}^{2^0}\,\text{[wall]}\,\text{[empty]}^{2^1}\,\text{[wall]}\,\text{[empty]}^{2^2} \cdots .$$

From Lemma 3.13, we conclude that $y$ is a mean equicontinuity point. Therefore, $(A^{\mathbb{Z}}, T)$ is almost mean equicontinuous. □

## 4. *Example 2: Pacman level 2 CA*

Let $A = \{\text{[empty]}, \text{[wall]}, \text{[ghost]}, \text{[monster]}, \text{[pacman]}, \text{[ghost-in-wall]}\}$, $A_2 = \{\text{[empty]}, \text{[cherry]}, \text{[banana]}\}$, and let $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the Pacman CA of §3. We define $T_2 : A_2^{\mathbb{Z}} \to A_2^{\mathbb{Z}}$ as

$$T_2 x_i = \begin{cases} \text{[empty]} & \text{if } x_i = \text{[empty]}; \\ \text{[cherry]} & \text{if } x_i = \text{[banana]}; \\ \text{[banana]} & \text{if } x_i = \text{[cherry]}. \end{cases}$$

Now, we define a skew product. We define $A_P := A \times A_2$ and the map $T_P : A_P^{\mathbb{Z}} \to A_P^{\mathbb{Z}}$ as

$$T_P x_i = \begin{cases} (T x_i, T_2 x_i) & \text{if } x_i \notin \big\{(\text{[monster]}, \text{[cherry]}), (\text{[ghost-in-wall]}, \text{[cherry]}), (\text{[ghost]}, \text{[cherry]})\big\}; \\ (T x_i, \text{[cherry]}) & \text{if } x_i \in \big\{(\text{[monster]}, \text{[cherry]}), (\text{[ghost-in-wall]}, \text{[cherry]}), (\text{[ghost]}, \text{[cherry]})\big\}. \end{cases}$$

LEMMA 4.1. *Let $m > 0$, $w \in A_P^m$ and*

$$x = {}^{\infty}(\text{[empty]}, \text{[empty]}).w(\text{[wall]}, \text{[empty]})(\text{[empty]}, \text{[empty]})^{\infty}.$$

*Then, there exists $N > 0$ such that for all $n \geq N$ and all $0 \leq i \leq |w|$,*

$$T_P^n x_i \in \big\{(p, q) : p \in \{\text{[empty]}, \text{[wall]}\} \wedge q \in A_2\big\}.$$

*Proof.* This proof follows immediately from Lemma 3.5. □

We want to show that $(A_P^{\mathbb{Z}}, T_P)$ is not almost mean equicontinuous. Using Proposition 2.11, we need to find a non-empty open set that does not contain any mean equicontinuity points.

LEMMA 4.2. *Let $m > 0$ and $w \in A_P^m$ such that $w_0 = (\text{[wall]}, \text{[cherry]})$. Then, there exist $x, y \in A_P^{\mathbb{Z}}$ such that*

$$x_{[0,|w|-1]} = y_{[0,|w|-1]} = w,$$

*and the set*

$$\mathbb{Z}_{n\geq 0} \backslash \{n \in \mathbb{Z}_{n\geq 0} : T_P^n x_0 \neq T_P^n y_0\}$$

*is finite.*

*Proof.* Let $w \in A_P^m$ as in the hypothesis of the lemma. Define

$$x := {}^{\infty}(\text{[empty]}, \text{[empty]}).w(\text{[wall]}, \text{[empty]})(\text{[monster]}, \text{[empty]})(\text{[empty]}, \text{[empty]})^{\infty}$$

and

$$y := {}^{\infty}(\text{[empty]}, \text{[empty]}).w(\text{[wall]}, \text{[empty]})(\text{[empty]}, \text{[empty]})^{\infty}.$$

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Using Lemma 4.1, we can assume, without loss of generality, that $w_i \in \{(p,q) : p \in$ {□, ◫} $\wedge\, q \in A_2\}$. Now, there exists $N > 0$ such that $T_P^N x_0 =$ (👻, $q$), where $q \in$ {🍒, 🍌}. Meanwhile, for all $i \geq 0$, we have that $T_P^i y_0 =$ (◫, $q$) with $q \in$ {🍒, 🍌}. There are two cases to prove.

Case 1: $T_P^N x_0 =$ (👻, 🍒). This implies that $T_P^{N+1} x_0 =$ (◫, 🍒). Meanwhile, $T_P^{N+1} y_0 =$ (◫, 🍌). Therefore, we can easily see that $T_P^{N+i} x_0 \neq T_P^{N+i} y_0$, for all $i > 0$.

Case 2: $T_P^N x =$ (👻, 🍌). Again we have that $T_P^{N+1} x_0 =$ (◫, 🍒). So, $T_P^{N+i} x_0 = T_P^{N+i} y_0$ for all $i \geq 0$. In this case, we redefine

$x := {}^{\infty}$(□, □)$.w$(◫, □)(□, □)(👾, □)(□, □)$^{\infty}$

and finish the proof with a similar argument as the one given in Case 1. □

LEMMA 4.3. *Let $x \in A_P^{\mathbb{Z}}$ such that $x_0 =$ (◫, 🍒). Then, $x$ is not a mean equicontinuity point.*

*Proof.* This lemma follows immediately from Lemma 4.2. □

Note that for all $\varepsilon > 0$, any $y \in B_\varepsilon(x)$, where $x_0 =$ (◫, 🍒), is not a mean equicontinuity point.

THEOREM 4.4. *$(A_P^{\mathbb{Z}}, T_P)$ is neither mean sensitive nor almost mean equicontinuous.*

*Proof.* Let us show that $(A_P^{\mathbb{Z}}, T_P)$ is not mean sensitive, that is, for every $\varepsilon > 0$ there exists a open set $U \subset A_P^{\mathbb{Z}}$ such that for every $x, y \in U$,

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T_P^i x, T_P^i y)}{n+1} < \varepsilon.$$

From Proposition 3.13, we have that the element

$x := \cdots$(◫, □)(□, □)$^{2^1}$(◫, □)(□, □)$^{2^0}.$(◫, □)(□, □)$^{2^0}$(◫, □)(□, □)$^{2^1}\cdots$

is a mean equicontinuity point. By Proposition 2.11, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in B_\delta(x)$,

$$\limsup_{n\to\infty} \frac{\sum_{i=0}^{n} d(T_P^i y, T_P^i z)}{n+1} < \varepsilon.$$

Therefore, $(A_P^{\mathbb{Z}}, T_P)$ is not mean sensitive.

The fact that $(A_P^{\mathbb{Z}}, T_P)$ is not almost mean equicontinuous follows immediately from Lemma 4.3. □

We finish the paper with a question. A minimal TDS is mean equicontinuous if and only if it is not mean sensitive [**9, 17**]. Considering Proposition 2.9, we ask the following.

*Question 4.5.* Does there exist a minimal subshift $(X, \sigma)$ and a CA $(X, T)$ that is neither mean equicontinuous nor mean sensitive?

*Acknowledgements.* The authors thank Rafael Alcaraz Barrera for valuable comments. The first author receives support from a CONACyT PhD fellowship, and the second author from the CONACyT Ciencia Básica project 287764.

## REFERENCES

**[1]** E. Akin, J. Auslander and K. Berg. When is a transitive map chaotic? *Convergence in Ergodic Theory and Probability*. Walter de Gruyter & Co., Berlin, 1996, pp. 25–40.
**[2]** J. Auslander and J. A. Yorke. Interval maps, factors of maps, and chaos. *Tohoku Math. J.* **32**(2) (1980), 177–188.
**[3]** F. Blanchard, E. Formenti and P. Kurka. Cellular automata in the Cantor, Besicovitch, and Weyl topological spaces. *Complex Syst.* **11**(2) (1997), 107–124.
**[4]** F. Blanchard and P. Tisseur. Some properties of cellular automata with equicontinuity points. *Ann. Inst. Henri Poincaré Probab. Stat.* **36**(5) (2000), 569–582.
**[5]** T. Downarowicz and E. Glasner. Isomorphic extensions and applications. *Topol. Methods Nonlinear Anal.* **48**(1) (2016), 321–338.
**[6]** S. Fomin. On dynamical systems with pure point spectrum. *Dokl. Akad. Nauk SSSR* **77**(4) (1951), 29–32 (in Russian).
**[7]** G. Fuhrmann, M. Gröger and D. Lenz. The structure of mean equicontinuous group actions. *Preprint*, 2018, arXiv:1812.10219.
**[8]** F. García-Ramos. Limit behaviour of $\mu$-equicontinuous cellular automata. *Theoret. Comput. Sci.* **623** (2016), 2–14.
**[9]** F. García-Ramos. Weak forms of topological and measure-theoretical equicontinuity: relationships with discrete spectrum and sequence entropy. *Ergod. Th. & Dynam. Sys.* **37**(4) (2017), 1211–1237.
**[10]** F. García-Ramos, T. Jäger and X. Ye. Mean equicontinuity, almost automorphy and regularity. *Israel J. Math.*, to appear.
**[11]** F. García-Ramos, J. Li and R. Zhang. When is a dynamical system mean sensitive? *Ergod. Th. & Dynam. Sys.* **39**(6) (2019), 1608–1636.
**[12]** R. H. Gilman. Classes of linear automata. *Ergod. Th. & Dynam. Sys.* **7**(1) (1987), 105–118.
**[13]** J. Guckenheimer. Sensitive dependence to initial conditions for one dimensional maps. *Comm. Math. Phys.* **70**(2) (1979), 133–160.
**[14]** W. Huang, J. Li, J.-P. Thouvenot, L. Xu and X. Ye. Bounded complexity, mean equicontinuity and discrete spectrum. *Ergod. Th. & Dynam. Sys.*, to appear.
**[15]** P. Kurka. Languages, equicontinuity and attractors in cellular automata. *Ergod. Th. & Dynam. Sys.*, **17**(2) (1997), 417–433.
**[16]** P. Kurka. *Topological and Symbolic Dynamics (Cours Specials, 11)*. Société Mathématique de France, Paris, 2003.
**[17]** J. Li, S. Tu and X. Ye. Mean equicontinuity and mean sensitivity. *Ergod. Th. & Dynam. Sys.* **35**(8) (2015), 2587–2612.
**[18]** J. Li, X. Ye and T. Yu. Mean equicontinuity, bounded complexity and applications. *Discrete Contin. Dyn. Syst. A*, **25**. , in press.
**[19]** B. Martin. Damage spreading and $\mu$-sensitivity on cellular automata. *Ergod. Th. & Dynam. Sys.* **27**(2) (2007), 545–565.
**[20]** J. C. Oxtoby. Ergodic sets. *Bull. Amer. Math. Soc.* **58**(2) (1952), 116–136.
**[21]** M. Sablik. Directional dynamics for cellular automata: a sensitivity to initial condition approach. *Theoret. Comput. Sci.* **400**(1–3) (2008), 1–18.

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