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PURE LOGIC OF ITERATED FULL GROUND

JON ERLING LITLAND
Department of Philosophy, University of Texas at Austin

Abstract. This article develops the Pure Logic of Iterated Full Ground (PLIFG), a logic of ground that can deal with claims of the form "ϕ grounds that (ψ grounds θ)"—what we call iterated grounding claims. The core idea is that some truths Γ ground a truth ϕ when there is an explanatory argument (of a certain sort) from premisses Γ to conclusion ϕ. By developing a deductive system that distinguishes between explanatory and nonexplanatory arguments we can give introduction rules for operators for factive and nonfactive full ground, as well as for a propositional "identity" connective. Elimination rules are then found by using a proof-theoretic inversion principle.

§1. Introduction. Fine (2012a,b), Correia (2010, 2014), Schnieder (2011), and Poggiolesi (2016, 2018) have developed logics of ground where ground is treated as a sentential operator (to be read: "BECAUSE"). All of these logics, however, have been logics of "simple" ground: they have nothing to say about claims of the form "ϕ grounds that (ψ grounds θ)"—what we may call *iterated grounding claims*. In the metaphysics literature, on the other hand, Bennett (2011), deRosset (2013a), and Dasgupta (2014b) have given accounts of iterated ground, but their accounts were not accompanied by logics of ground. This is perhaps not surprising—and not only because developing such a logic is a nontrivial matter. While one might accept that true grounding claims themselves need to be grounded, one might think that it is a substantive matter what the grounds of grounding claims are: if so, one has no right to expect a *logic* of iterated ground.

In previous work (Litland, 2017b) I argued that by linking ground to a type of "explanatory argument" one naturally arrives at a logic of iterated ground. The key upshot of the proposed logic was that true nonfactive grounding claims are *zero-grounded* in Fine's sense. Unfortunately, the resulting logic was not satisfactory by its own lights. The problem is that what the connection between ground and explanatory argument gives us is not just that true nonfactive grounding claims are zero-grounded, it gives us the stronger claim that the *only* ground they have is the empty ground. The system proposed in (Litland, 2017b) does not allow us to derive this stronger claim.

In this article I rectify these shortcomings by developing the Pure Logic of Iterated Full Ground, or PLIFG.[1] In PLIFG we can prove that the unique ground for true nonfactive grounding claims is the empty ground. Here is the plan. In §2 I describe the basic connection between ground and explanatory arguments. In §3 I lay down the structural constraints on explanatory arguments. In §4 I show how to give introduction rules for grounding operators. §5 is a philosophical interlude concerned with what it means to give a real

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[1] A *pure* logic of ground has no logical operators apart from the grounding operators; an *impure* logic of ground also deals with other operators (e.g., conjunction, disjunction, and negation).

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definition of the grounding operators. §6 uses an inversion principle to derive elimination rules for the grounding operators. In §7 I define PLIFG and show how to derive basic results about iterated ground. In §8 I compare PLIFG to Fine's Pure Logic of Ground. §9 concludes with some methodological remarks and open questions.

§2. Ground, explanation, and explanatory arguments.

2.1. Ground. The idea is that some truths Γ *ground* another truth ϕ[2] when it is *in virtue of* the truths Γ that it is the case that ϕ; or one might say that the truths Γ ground ϕ when the obtaining of the first truths *make it the case that* ϕ; or one might say that Γ grounds ϕ when its being the case that Γ *accounts for* its being the case that ϕ; or, finally, one might say that Γ grounds ϕ when it is *because* Γ that ϕ. These notions have been with philosophy since the beginning; what is distinctive of the recent literature on ground—following the seminal (Fine, 2001)—is that one has eschewed trying to give noncircular definitions of ground. One has accepted the notion of ground as a primitive, settling for elucidating it by presenting paradigm cases and laying down structural principles governing it. We will follow suit.

Several notions of ground have been distinguished in the literature; here our focus will be on strict, full, mediate ground—in both its factive and nonfactive varieties. These notions have the following features. First, they are *left-collective*: some propositions $\phi_0, \phi_1, \ldots$ may together ground ϕ, even though no ϕ_i by itself grounds ϕ.[3] Second, we are concerned with *full* ground: if Γ grounds ϕ then no proposition need be added to Γ in order to account for why ϕ is the case: its being the case that Γ fully accounts for its being the case that ϕ. Third, the notion of ground is that of *mediate* ground: we allow the propositions $\Gamma_0, \Gamma_1, \ldots, \Sigma$ to ground ϕ by way of Γ_0's grounding ϕ_0, Γ_1's grounding $\phi_1, \ldots$ and the propositions $\{\phi_0, \phi_1, \ldots\} \cup \Sigma$'s together grounding ϕ. Fourth, in the *factive* sense, if $\gamma_0, \gamma_1, \ldots$ ground ϕ, then each of $\gamma_0, \gamma_1, \ldots$ and ϕ is the case; in the *nonfactive* sense, we allow $\gamma_0, \gamma_1, \ldots$ to ground ϕ without any of $\gamma_0, \gamma_1, \ldots$ or ϕ being the case. Fifth, and finally, ground is *strict* in the following sense: if Γ grounds ϕ then it is impossible for each of the $\gamma \in \Gamma$ to be the case and for ϕ to contribute to explaining any $\gamma \in \Gamma$.[4] We will use $<$ as a sentential operator for factive ground and $\Rightarrow$ as a sentential operator for nonfactive ground. For now assume that they have the following grammar: they take any number of sentences on the left and a single sentence on the right.

2.2. Explanation and explanatory arguments. Most philosophers working on ground have accepted that there is an intimate connection between ground and explanation. Fine, e.g., holds that the grounds explain the grounded in the sense

> that there is no stricter or fuller account of that in virtue of which the explanandum holds. If there is a gap between the grounds and what is grounded, then it is not an explanatory gap. (Fine, 2012a, 39)

[2] In the interest of readability we will be abusing the use/mention distinction throughout. This should cause no confusion.

[3] Dasgupta (2014a) has suggested that the basic notion of ground is *bicollective*: some propositions $\gamma_0, \gamma_1, \ldots$ may ground some propositions $\delta_0, \delta_1, \ldots$ without any of the δ_i being grounded in any subcollection of $\gamma_0, \gamma_1, \ldots$. I believe it is possible to extend the present framework to deal with iterated bicollective ground, but this is not something I will pursue here. (For some attempts to model noniterated bicollective ground, see (Litland, 2016, 2018).)

[4] This formulation of strictness is more careful than usual; see 3.1.1. for the reasons why.

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Philosophers have, however, differed on what the intimate connection grounding and explanation is. Here we develop the consequences of assuming that the connection is the tightest possible: the claim that Γ grounds ϕ simply *is* the claim that Γ explains ϕ, in this special, "tightest" sense of explanation.[5] While this view hardly is uncontroversial[6] here we will simply assume this conception of ground and will not argue against other views on the relation between ground and explanation.[7]

What, then, is explanation? There is a long tradition connecting explanation in general with *explanatory arguments*. (This is the *deductive-nomological* model of explanation.) The idea in (Litland, 2017b) was to use this connection to throw light on ground. The ideas, in brief, were as follows:[8]

First, there is a connection between ground and explanatory arguments: Γ nonfactively grounds ϕ if there is a certain type of explanatory argument from premisses (exactly) Γ to conclusion ϕ. (Γ factively grounds ϕ if in addition Γ is the case.) Second, to determine the grounds of a grounding claim like $\Gamma \Rightarrow \phi$ or $\Gamma < \phi$ we have to determine what explanatory arguments end with those conclusions: the premisses of an explanatory argument ending in $\Gamma \Rightarrow \phi$ will be the grounds of $\Gamma \Rightarrow \phi$ (and similarly for $\Gamma < \phi$). Third, the basic notion of ground is nonfactive: the *immediate* grounds for a factive grounding claim $\Gamma < \phi$ are Γ and $\Gamma \Rightarrow \phi$ taken together. Fourth, given the connection between ground and explanatory arguments this means that there is an explanatory argument from premisses (all and only) Γ, $(\Gamma \Rightarrow \phi)$ to conclusion $\Gamma < \phi$. Fifth, we should understand the nonfactive grounding operator $\Rightarrow$ as follows: for $\Gamma \Rightarrow \phi$ to be the case simply is for there to be an explanatory argument from premisses (all and only) Γ to conclusion ϕ. Sixth, this suggests the following introduction rule for $\Rightarrow$:

$$\dfrac{\dfrac{\overline{\;\Gamma\;}^{\,1}}{\begin{array}{c}\mathcal{E}\\ \phi\end{array}}}{\Gamma \Rightarrow \phi}\;1, \Rightarrow\text{-I}$$

(Here $\mathcal{E}$ is an explanatory argument and Γ are all and only the premisses on which ϕ depends; in passing to the conclusion $\Gamma \Rightarrow \phi$ we discharge all the premisses.) Seventh, just like there are arguments from the empty set of premisses to some conclusion, there might be *explanatory* arguments from the empty set of premisses to some conclusion. Eight, if there is an explanatory argument to ϕ from the empty set of premisses this shows that ϕ is grounded: it is *zero-grounded* in the sense of (Fine, 2012a, 47–48). Ninth, and this is the crucial move, an argument ending with an application of $\Rightarrow$-introduction is *itself explanatory*.

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5 If $\Gamma < \phi$, then a full explanation of ϕ is simply Γ; Γ does not need to be supplemented with the fact that $\Gamma < \phi$ in order to fully explain ϕ. (There may, however, be some special cases where $\Gamma < \phi$ is a separate ground for ϕ—see (Litland, 2017a).)

6 Dasgupta (2014b) explicitly endorses it. Audi (2012, 687–688) and Schaffer (2012, 122) explicitly reject it.

7 For more discussion of the issues at stake here see (Bliss & Trogdon, 2016, §4).

8 Others have also noted the connection between ground and explanatory arguments. deRosset (2013a, 12–13) comes very close to the present ideas, but he focuses on factive ground and is not (there) trying to develop a logic of iterated ground. Poggiolesi (2018) proposes that we link ground to what she calls "complete and immediate formal explanations". For some further remarks on Poggiolesi's approach, see footnote 12. A deductive-nomological conception of ground has also been explored by Wilsch (2015a,b).

Suppose now that we have concluded $\Gamma \Rightarrow \phi$ by an application of $\Rightarrow$-introduction. Since an application of $\Rightarrow$-introduction discharges all the premisses on which the subconclusion ϕ depends this means that $\Gamma \Rightarrow \phi$ is the conclusion of an explanatory argument from the empty collection of premisses. In other words, $\Gamma \Rightarrow \phi$ is zero-grounded.

A nice feature of this framework is that the seemingly obscure idea of zero-grounding turns out to be unproblematic. To say that ϕ is zero-grounded is to say that there is an explanatory argument from no premisses to ϕ. The seemingly obscure distinction between being ungrounded and being zero-grounded—grounded, but in nothing at all!—is an instance of a familiar phenomenon: the distinction between being underivable and being derivable from the empty collection of premisses.

Another nice feature is that the framework subsumes the "superinternality" account proposed by deRosset (2013a) and Bennett (2011). Their proposal is that the grounds ground that the grounds ground the grounded or symbolically: if $\Gamma < \phi$, then $\Gamma < (\Gamma < \phi)$. This claim is derivable in the system of (Litland, 2017b).[9]

2.3. Problems with Litland's account. I believe this picture is basically correct, but the development in (Litland, 2017b) had several shortcomings.

The most serious shortcoming is that what the above picture justifies is not just that true nonfactive grounding claims like $\Gamma \Rightarrow \phi$ are zero-grounded, what it justifies is that they are *solely* zero-grounded. In other words, if Δ grounds $\Gamma \Rightarrow \phi$, then Δ in fact has to be equivalent to the empty ground. The full justification for this claim has to wait until we have presented the introduction rules for $\Rightarrow$ and $<$. For now, just note that the system of (Litland, 2017b) is incapable of even *expressing* that claims of the form $\Gamma \Rightarrow \phi$ are solely zero-grounded.

First, the language employed there had no way of expressing identity between propositions. To remedy this, we introduce a sentential operator $\approx$ such that if $p \approx q$ then p can be substituted for q in any context.[10]

Second, in order to express that a proposition is identical to the empty ground we need a way of expressing the empty ground itself. In (Litland, 2017b) the only place the empty ground could appear was on the left of the grounding operators $\Rightarrow$ and $<$. (One could speak of zero-ground*ing*, but not of the empty *ground*.) In order to be able to express that a proposition is equivalent to the empty ground we allow the grounding operators $\Rightarrow$ and $<$ to have zero arguments on the right as well as on the left. For notational purposes we adopt the following grammar for the sentential operators $\Rightarrow$, $<$, and $\approx$. If Γ is a set of sentences and Δ, Σ are sets of cardinality at most 1 then $\Gamma \Rightarrow \Delta$, $\Gamma < \Delta$, and $\Delta \approx \Sigma$ are sentences. This formalization allows the empty ground to occur both on the left and on the right of the grounding operators.

Once we allow the empty ground to figure on the right of the grounding operators we have to make sure that it behaves as intended—in particular, it must be "minimal" in the sense that for no Γ do we have $\Gamma \Rightarrow \emptyset$. A convenient way of ensuring this is to allow arguments not just with empty premisses but also with an empty conclusion. Technically, we deal with this by having arguments be trees where the nodes are labeled with sets of sentences (the sets being of cardinality at most 1).[11] We then impose constraints that ensure that there are no explanatory arguments with the empty conclusion.

[9] It remains derivable in the logic presented here, see Proposition 7.7.4.

[10] Since $\approx$ is a sentential operator it is misleading to speak of it as an identity relation between propositions. But the rules governing $\approx$ ensure that it behaves like a higher-order analogue of identity.

[11] If one wanted to model the bicollective notion of ground mentioned in footnote 3 one would allow sets of any cardinality.

§3. Arguments explanatory and plain. The proof system distinguishes between explanatory and plain arguments. If Γ are the premisses of an explanatory argument with conclusion ϕ then Γ explains ϕ. If the argument from Γ to ϕ is merely plain, on the other hand, Γ need not in any way explain ϕ, though it is the case that if (each member of) Γ is true, then ϕ is true.

While we will not attempt to give a noncircular definition of "explanatory argument" it might help to think of explanatory arguments as composed of basic explanatory *inferences*. Finding out which particular inferences are explanatory is a concern of metaphysics. Logic is only concerned with which further explanatory arguments are generated from a given collection of explanatory inferences. As far as the pure logic of ground is concerned there might be no basic explanatory inferences. This would be no problem for the logic of ground—though it would obviously detract from its interest.

That being said, the following are plausible cases of explanatory inference: conjunction-introduction, disjunction-introduction (though note (Fine, 2010; Krämer, 2013; Litland, 2015)), and the inference from a is F to a is G where F is a determinate of the determinable G. In general, if one thinks that Γ immediately strictly grounds ϕ, we would hold that the inference from Γ to ϕ is an explanatory inference.[12]

Let us make this precise. An *argument* is a rooted labeled tree (of height at most ω) that is equipped with a discharge function. More specifically, an argument is a quadruple $\langle T, L, \leq, D\rangle$. Here T is the set of nodes of the tree and $\leq$ is the tree-order. We write

$$\frac{t_0 \quad t_1 \quad \dots}{s}$$

if $t_0, t_1, \dots$ are all and only the nodes immediately above s. L is a function assigning *sets* of formulae to the nodes of T. We demand that $L(s)$ is of cardinality at most 1 and allow $L(s) = \emptyset$. $D\colon T \to \mathcal{P}(T)$ is a function taking a $s \in T$ and giving us a set of nodes. We use the function D to keep track of which premisses are discharged in the course of an argument. Intuitively, the propositions labeling $D(s)$ are the propositions upon which $L(s)$ depends. The discharge function satisfies the following natural constraints.

12 I would be remiss if I did not say something about the recent attempt by Poggiolesi (2016, 2018) at defining a notion immediate ground. Central to her approach is a syntactically defined notion of complexity ("g-complexity"). Roughly, she says that Γ grounds ϕ iff ϕ is derivable from Γ and Γ is immediately less g-complex than ϕ. (Her actual account is more involved, but the same points will apply.) I believe that no attempt along the lines she proposes can work. Any attempt at defining complexity *syntactically* will founder on cases where we have ground but no increase—or even: decrease—in complexity. Cases like these are the grounding of facts involving determinables in facts involve determinates—e.g., the ball's being red is grounded in its being crimson—or the grounding of the fact that the sentence denoted by S is true in the fact that the sentence denoted by S means that there are ω-many Woodin cardinals with a measurable above, together with the fact that there are ω-many Woodin cardinals with a measurable above. (The point here is just that, syntactically, the claim that the sentence S is true is *atomic*; the set-theoretic statement, on the other hand, is very complicated.) One could argue that this criticism is unfair since Poggiolesi is only interested in capturing the notion of "formal" (or "logical") ground. (She follows Correia (2014) in drawing a distinction between *logical*, *conceptual*, and *metaphysical* ground.) One could then claim that both the determinate/determinable case and the truth-ascription cases are not cases of formal ground. I am skeptical whether there is an interesting distinction between formal and metaphysical ground—as opposed to just a distinction in subject matter—but even assuming that there is an interesting notion of formal ground, it is doubtful whether she succeeds in defining it. The problem is that she does not give a general definition, but only gives a definition for a particular choice of logical constants: at best she defines the notion of formal ground for propositions formed using that particular collection of constants.

1. If $s \in T$ is a top node in T then $D(s) = \{s\}$.
2. If $\dfrac{t_0 \quad t_0 \cdots}{s}$ then $D(s) \subseteq D(t_0) \cup D(t_1) \cup \cdots$.

If $s \leq t$ and t is a top node, but $t \notin D(s)$ we say that t has been discharged.

We use $\mathcal{E}, \mathcal{D}, \ldots$ (possibly with subscripts) as variables for arguments. When we write

$$\begin{array}{c} \Gamma \\ \mathcal{E} \\ \phi \end{array}$$

we normally mean that $\mathcal{E}$ is an argument where the undischarged top nodes are labeled with exactly the formulae in Γ and the conclusion node is labeled with ϕ. We occasionally use this notation also to mean that Γ are amongst the labels of the undischarged top nodes. (What we mean will be clear from context.) When ϕ labels the root node s, we will typically write $D(\phi)$ instead of $D(s)$.

We use the following notation to indicate discharge.

$$\begin{array}{c} \overline{\phi_0}\,0 \quad \overline{\phi_1}\,1 \quad \overline{\phi_2 \ldots}\,2 \\ \mathcal{E} \\ \dfrac{\phi}{\psi}\,0, 1, 2, \ldots \end{array}$$

This is to be read as follows. The conclusion ϕ of the subargument $\mathcal{E}$ depends on $\phi_0, \phi_1, \ldots$. In passing to the conclusion ψ we can discharge the top nodes that are labeled with the ϕ_i.

When $\Gamma = \{\gamma_0, \gamma_1, \ldots\}$ and we are considering an inference where all the γ_i are discharged, we typically write:

$$\begin{array}{c} \overline{\Gamma}\,1 \\ \mathcal{E} \\ \dfrac{\psi}{\theta}\,1 \end{array}$$

For any sets of arguments $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ such that $\mathfrak{E}_e \subseteq \mathfrak{E}_p$ we define the collection of explanatory and plain arguments over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ as the least $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ such that $\mathfrak{E}_e \subseteq \mathfrak{E}'_e$ and $\mathfrak{E}_p \subseteq \mathfrak{E}'_p$ and such that $\langle \mathfrak{E}'_e, \mathfrak{E}'_p \rangle$ is closed under the rules in Figure 1. It is helpful to think of $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ as the basic explanatory and plain arguments.

The constraints in this figure require explanation. In what follows we will use $\mathcal{E}(e)$, and $\mathcal{E}(p)$ to indicate that the argument $\mathcal{E}$ is, respectively, explanatory or plain.

(Assumption) is straightforward. We get to write down any assumption ϕ we like. The result is a plain argument with conclusion ϕ, where the conclusion depends on ϕ. As a degenerate case we allow the empty argument that has one node labeled with the empty set.

(Inclusion) works in the obvious way, ensuring that explanatory arguments are also plain.

The two chaining principles tell us how grounding works. Clearly, chaining together plain arguments will result in a plain argument. It is more contentious that the result of chaining together explanatory arguments yields an explanatory argument. In what follows I will simply assume that since we are dealing with *full* ground there is no problem here.[13]

[13] We thus follow Litland (2013) and Raven (2013) in rejecting the counterexamples to transitivity proposed by Schaffer (2012).

(Assumption) Δ is a plain argument for all Δ of cardinality ≤ 1.

(Inclusion) If $\mathcal{E}$ is an explanatory argument, then $\mathcal{E}$ is a plain argument.

(Chaining) If $\begin{array}{c}\phi_0, \phi_1, \ldots, \Sigma \\ \mathcal{E} \\ \phi\end{array}$ is an explanatory argument and $\begin{array}{c}\Delta_i \\ \mathcal{E}_i \\ \phi_i\end{array}$ is an explanatory argument for each i, then

$$\cfrac{\begin{array}{cccc}\Delta_0 & \Delta_1 & & \\ \mathcal{E}_0 & \mathcal{E}_1 & & \\ \phi_0 & \phi_1 & \vdots & \ldots\Sigma\end{array}}{\begin{array}{c}\mathcal{E} \\ \phi\end{array}}$$

is an explanatory argument.

(Plain Chaining) If $\begin{array}{c}\phi_0, \phi_1, \ldots \\ \mathcal{E} \\ \phi\end{array}$ is a plain argument and $\begin{array}{c}\Delta_i \\ \mathcal{E}_i \\ \phi_i\end{array}$ is a plain argument for each i, then

$$\cfrac{\begin{array}{cccc}\Delta_0 & \Delta_1 & & \\ \mathcal{E}_0 & \mathcal{E}_1 & & \\ \phi_0 & \phi_1 & \vdots & \ldots\end{array}}{\begin{array}{c}\mathcal{E} \\ \phi\end{array}}$$

is a plain argument.

(Noncircularity) If $\mathcal{E}$ is an explanatory argument from premisses $\delta_0, \delta_1, \ldots$ to $\Delta \subseteq \{\delta_0, \delta_1, \ldots\}$, and $\mathcal{D}_i$ is a plain argument from Δ_i to δ_i for each i then the following is a plain argument for any ψ

$$\cfrac{\begin{array}{ccc}\Delta_0 & \Delta_1 & \\ \delta_0 & \delta_1 & \ldots \\ & \mathcal{E} & \\ & \Delta & \end{array}}{\psi}$$

Fig. 1. Explanatory and plain arguments.

3.1. Noncircularity. (Noncircularity) requires more extensive comment. The main idea is that explanatory arguments correspond to strict ground in the sense that $\Gamma < \phi$ is the case iff Γ is the case and there is an explanatory argument from Γ to ϕ. Since strict ground is

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irreflexive in the sense noted in §2.1 there can be no Δ such that ϕ, Δ, Γ are all the case and there is an explanatory argument from ϕ, Δ to ϕ.[14]

However, it will not do simply to say that there are no explanatory arguments from Δ, ϕ to ϕ, for any Δ: this tells us nothing about what happens under the *supposition* that ϕ (partly) strictly grounds itself and we need to know what follows from this supposition. This no idle worry: in the course of reasoning about what grounds what we often end up with subordinate derivations where we have an explanatory argument from ϕ (and some other premisses) to ϕ itself.[15]

(Noncircularity) gets around this problem by expressing the asymmetry of ground as a closure-condition on the classes of explanatory and plain arguments. If, *per impossibile*, ϕ did contribute to explaining ϕ, then we can conclude anything whatsoever—albeit only plainly.

Note that (Noncircularity) does not simply take the form:

$$\begin{array}{c} \Gamma, \phi \\ \mathcal{E}(e) \\ \dfrac{\phi}{\psi}\ \text{Noncircularity*} \end{array}$$

For suppose there is an explanatory argument from ϕ (and some further premisses to ϕ itself). Suppose further that ϕ follows plainly from Δ. We ought to be able to conclude anything from Δ, Γ. But since the argument from Δ to ϕ is merely plain if we first use (Plain Chaining) to get the argument

$$\begin{array}{c} \Delta \\ \phi \qquad \Gamma \\ \mathcal{E}(e) \\ \phi \end{array}$$

we only get a plain argument and so we cannot apply Noncircularity*. The rule of (Noncircularity) gets around this by building some chaining into the rule of (Noncircularity).

Since arguments with the empty conclusion are allowed we count the following as an instance of (Noncircularity):

$$\begin{array}{c} \Delta \\ \mathcal{E}(e) \\ \dfrac{\emptyset}{\psi}\ \text{Noncircularity} \end{array}$$

In Proposition 7.7.2 we rely on such instances of (Noncircularity) to establish that the empty ground is minimal.

One might worry about this reliance on arguments with the empty conclusion: can we really make sense of this? We can, of course, always fall back on treating this as a merely

[14] It is, of course, true that the idea that ground has to be irreflexive has come under attack on various grounds—see, e.g., (Jenkins, 2011; Bliss, 2014; Wilson, 2014; Krämer, 2013; Correia, 2014). Here we will just assume that none of the reasons for rejecting irreflexivity are compelling. And even if irreflexivity fails for some notion of ground it is in any case possible to *introduce* a natural notion of ground that does satisfy (Noncircularity). (Fine, 2010, 105) indicates how this might be done for a notion of partial ground; and (Litland, 2015) shows how it can be done for full ground.

[15] Compare: it does not suffice to say that there are no derivations of the absurdity constant $\bot$. In subordinate derivations we often end up deriving $\bot$: we need to know what we can conclude in those circumstances.

technical trick that is required to make the empty ground behave properly. But I think we can do a bit better than that. First, there is no problem making sense of a plain argument with the empty conclusion. If we read the conclusion of an argument *conjunctively* what a plain argument with conclusion $\emptyset$ and premisses Γ commits us to is that if all the $\psi \in \Gamma$ are true then all the $\delta \in \emptyset$ are true. We have no problem making sense of this.

What we should say about explanatory arguments with the empty conclusion is this: *there are no such arguments*. But—just as it is not enough to say that there are no explanatory arguments from ϕ (and some other premisses) to ϕ itself—saying just this is not enough since this does not tell us what happens under the supposition that we have an explanatory argument with the empty conclusion. By formulating (Noncircularity) so that we can conclude anything from an explanatory argument with the empty conclusion we get around this problem.

3.1.1. Nonfactive noncircularity? There is a final, more philosophical, issue with the formulation of (Noncircularity). Note that we do not demand that if Γ grounds ϕ then it is impossible for ϕ to contribute to grounding some $\gamma \in \Gamma$. The reason we do not demand this is cases like the following. Suppose that a is part of b but it is possible that b instead is part of a. (Imagine, e.g., that a and b are two organisms and that a came to enter b's body and that a now plays an important role in keeping b alive; but if things had gone ever so slightly differently it would be b that had entered a's body and b that would have played an important role in keeping a alive.) Then, while the existence of a plausibly (partly) strictly grounds the existence of b, it is possible that the existence of b (partly) strictly grounds the existence of a. Cases like this show that (partial) nonfactive ground is not—in general—asymmetric.[16]

Given the connection between ground and explanatory arguments there are, then, some propositions Γ such that there is an explanatory argument from the proposition that a exists together with Γ to the conclusion that b exists. There are also some propositions Δ such that there is an explanatory argument from Δ together with the proposition that b exists to the conclusion that a exists. In other words, there are explanatory arguments from some proposition ϕ (together with some auxiliary premisses) to the proposition ϕ itself. The strictest we can expect ground to be, then, is that if there is an explanatory argument from Γ, ϕ to ϕ, then it is impossible for Γ, ϕ to be jointly true. This is exactly what (Noncircularity) ensures.

I should note that if one is not convinced by the above case one can strengthen (Noncircularity) to allow discharge of any of the premisses on which ϕ depends. We thus get:

$$
\begin{array}{ccc}
\overline{\Delta_0}\ 0 & \overline{\Delta_1}\ 1 & \\
\mathcal{D}_0 & \mathcal{D}_1 & \\
\delta_0 & \delta_1 & \dots \\
 & \mathcal{E} & \\
 & \dfrac{\Delta}{\psi}\ 0, 1, \dots \text{ Nonfactive noncircularity} &
\end{array}
$$

§4. Introduction rules. The introduction rules for nonfactive ($\Rightarrow$) and factive ground ($<$) as well as for propositional equivalence ($\approx$) are depicted in Figure 2. The introduction

[16] If such cases are possible then the logic for nonfactive ground proposed by Correia (2017) is incorrect.

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$$
\begin{array}{c}
\overline{\;\Gamma\;}^{\,1} \\
\mathcal{E} \\
\phi \\
\hline
\Gamma \Rightarrow \phi
\end{array}\ 1, \Rightarrow\text{-I}
\qquad\qquad
\frac{\Gamma \qquad \Gamma \Rightarrow \phi}{\Gamma < \phi}\ <\text{-I}
\qquad\qquad
\frac{}{\phi \approx \phi}\ \approx\text{-I}
$$

Fig. 2. Introduction rules for $\Rightarrow$, $<$, and $\approx$.

rule for $<$ says that we can infer $\Gamma < \phi$ from Γ together with $\Gamma \Rightarrow \phi$. The introduction rule for $\Rightarrow$ says that we can infer $\Gamma \Rightarrow \phi$ when we have an explanatory argument from premisses exactly Γ to conclusion ϕ. Since we distinguish between explanatory and plain arguments we have to specify whether the resulting arguments are explanatory or merely plain. We take the introduction rules for $\Rightarrow$ and $<$ to result in explanatory arguments. (We justify this in §5.) We also have to specify how the discharge function works. For the $<$-I rule we set $D(\{\gamma_0, \gamma_1, \dots\} < \phi) = D(\{\gamma_0, \gamma_1, \dots\} \Rightarrow \phi) \cup D(\gamma_0) \cup D(\gamma_1) \cup \cdots$. For the $\Rightarrow$-rule we set $D(\Gamma \Rightarrow \phi) = \emptyset$.

The rules for $\approx$ must ensure that $\approx$ behaves like identity. The introduction rule is then clear enough: we can assert every instance of the reflexivity of identity: $\phi \approx \phi$. I have written down the reflexivity principle not as an axiom but rather as a rule of inference, where we can infer $\phi \approx \phi$ from the empty collection of premisses. It is tempting to take this inference to be explanatory, in which case all true (propositional) identities are zero-grounded (see Proposition 7.3.4). But I will not argue explicitly for this view.

Since we have taken the introduction rules for $\Rightarrow$ and $<$ to generate explanatory arguments we now know what grounds claims of ground. We illustrate this for the case of nonfactive ground. Let $\mathcal{E}$ be an explanatory argument from Γ to ϕ. Then the following argument shows that $\Gamma \Rightarrow \phi$ is zero-grounded:

$$
\begin{array}{cl}
\overline{\;\Gamma\;}^{\,1} & \\
\mathcal{E}(e) & \\
\phi & \\
\hline
\Gamma \Rightarrow \phi & 1, \Rightarrow\text{-I} \\
\hline
\Rightarrow (\Gamma \Rightarrow \phi) & \Rightarrow\text{-I} \\
\hline
<(\Gamma \Rightarrow \phi) & <\text{-I}
\end{array}
$$

Introduction rules are not enough, of course: we need to find matching elimination rules. But before we do that a philosophical interlude is in order.

§5. Philosophical interlude: Real definition of operators. We stipulated that introduction rules result in explanatory arguments: what justifies this? For the purposes of defining and studying a logic there is, of course, no need to answer this question. But if PLIFG is to have any philosophical significance—in particular, if it is to help with the problem of "grounding ground"—this needs to be justified.

In my view we can justify this by holding that the explanatoriness of the rules in Figure 2 are definitional of the operators $\Rightarrow$ and $<$. The operator $\Rightarrow$ is that operator such that arguments ending with $\Rightarrow$-I are explanatory. This is reminiscent of Gentzen's famous remark:

> the introductions represent, as it were, the "definitions" of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. (Gentzen 1969, 80)

This "inferentialist" idea is often given both a *meaning-theoretic* and an *epistemic* spin. (One *knows* the *meaning* of a sentential operator when one knows in what epistemic positions one can canonically *verify* sentences with that operator dominant, taking the introduction rules to specify the shape of canonical verifications.) But the idea can also be given a *metaphysical* spin: the introduction rules for an operator specify the immediate strict full (nonfactive) grounds for any proposition with that operator dominant.

We are claiming, then, that the rules define the operators themselves and not the signs denoting the operators. The introduction rules provide a *real* and not merely a nominal definition. In the recent literature on real definition the focus has been on *explicit* real definition,[17] but to hold that the rules governing the grounding operators provide real definitions we have to countenance *implicit* real definition. Very little has been said about this (though see Yablo & Rosen, forthcoming). While this is not the place to flesh out and defend an account of implicit real definition, some points should be made.

First, one becomes, in effect, committed to a comprehension principle for operators. We can loosely put this as follows.[18]

(Comprehension) For any set of introduction rules R there is an operator λ_R such that for all propositions $\phi_0, \phi_1, \ldots, \ldots$ if $\{\phi_0, \phi_1, \ldots\}$ are of the right cardinality for λ_R then there exists a unique proposition $\lambda_R(\phi_0, \phi_1, \ldots)$ such that the immediate strict full grounds of $\lambda_R(\phi_0, \phi_1, \ldots)$ are exactly the undischarged premisses of an application of rules in R.

Second, and relatedly, the questions "are the above rules for the grounding operators correct?" is in a sense misguided. The above rules define *some* operators—some operators, moreover, that can be used to express claims of ground. Are there different operators that can be used to express claims of ground? There are clearly operators for various notions of partial ground; but there are also alternative ways of dealing with full factive ground. For instance, one might give introduction rules for a notion of full factive mediate ground $<^*$ as follows:

$$\dfrac{\begin{matrix}\Gamma \\ \mathcal{E} \\ \phi\end{matrix}}{\Gamma <^* \phi}\ {<^*}\text{-I}$$

This rule is like the rule for $\Rightarrow$ except that Γ is not discharged. The difference between $\Gamma < \phi$ and $\Gamma <^* \phi$ is that $\Gamma \Rightarrow \phi$ is a partial ground for the former but not for the latter.[19]

Third, if we are to define λ_R by giving its introduction rules no prior assumptions can be made about propositions of the form $\lambda_R(\phi_0, \phi_1, \ldots)$. To see this, consider a simple case. Suppose that we instead of trying to define an operator are trying to define a property. We could try to give a real definition of the property *being a brother* by stipulating what are to be the immediate strict grounds for propositions of the form *x is a brother*. If we assume that propositions of the form *x is a brother* already stand in some grounding relationships it is not safe to stipulate that a proposition of the form *x is a brother* is to have certain immediate grounds: if we stipulate the wrong grounds we might end up with a circle of ground. For instance, if we stipulate that a proposition *x is brother* is to be immediately

17 For some examples see (Rosen, 2015), (Fine, 1994a,b), (Rayo, 2013) and (Dorr, 2016).

18 "Loosely". In order to state such a principle rigorously one would have to develop a higher-order logic of ground: in stating the principle one would have to quantify over both propositions and operators. (Or perhaps better: one would have to quantify into both sentence and sentential operator position.)

19 The rules for the $<^*$-operator results in the view of iterated ground advocated by deRosset (2013a) and Bennett (2011).

strictly fully grounded in the propositions *x is a sibling*, *x is male* this would lead to a circle of ground if there is a prior assumption that propositions of the form *x is a brother* (partly) ground propositions of the form *x is a sibling*.

Fourth, one might worry that while it is plausible to take the introduction rules for conjunction, disjunction and the existential quantifier to result in explanatory arguments, it is not at all plausible to say the same about the (standard) introduction rules for the conditional, negation and (arguably) the universal quantifier. Is this not a serious problem for the present approach? No: for while we are committed to holding that by taking a set of introduction rules as explanatory we thereby define *some* operator, this does not commit us to holding that by taking the standard introduction rules for, say, the conditional to be explanatory we thereby have succeeded in defining the *conditional*.

(Admittedly, this leaves us with a challenge. For if the standard introduction rules for the conditional—taken as explanatory—do not specify the immediate strict full grounds of a conditional proposition one had better be able to find some explanatory introduction rules that do specify the immediate strict full grounds of conditional propositions. The challenge is serious. To meet it one would have to show that the present framework can be used to develop an impure logic of ground. While I am hopeful that this can be done this is not the place to go into detail.[20])

Fifth, for any of this to work we have to be able to find appropriate elimination rules. I have been somewhat rash in saying that a set of introduction rules R suffices to define an operator λ_R. A set of introduction rules R defines an operator λ_R when the set R tells us how propositions of the form $\lambda_R(\phi_0, \phi_1, \dots)$ are (immediately) grounded. But no set of introduction rules R can do that by itself: we need in addition that R are *all* the introduction rules that govern λ_R. For if there is a way of explanatorily inferring a proposition of the form $\lambda_R(\phi_0, \phi_1, \dots)$ that is not captured by the rules R then there is a way for propositions of the form $\lambda_R(\phi_0, \phi_1, \dots)$ to be grounded that is not captured by R—in which case R does not define λ_R.

It is the elimination rules for λ_R that ensure that the introduction rules for λ_R specify the only ways in which propositions of the form $\lambda_R(\phi_0, \phi_1, \dots)$ can be grounded. It is here that the treatment in (Litland, 2017b) is deficient: the elimination rules given there fail to ensure that the introduction rules specify the sole ways in which propositions formed using $\Rightarrow$ and $<$ are grounded.

§6. Elimination rules.

6.1. Inversion. We want the elimination rules to ensure that the introduction rules for λ_R represent all and only the ways of explanatorily inferring propositions of the form $\lambda_R(\phi_0, \phi_1, \dots)$. To find elimination rules doing this we turn to a proof-theoretic inversion principle.[21] The rough idea is that the elimination rule(s) for an operator λ_R should be such that, for all propositions of the form $\lambda_R(\psi_0, \psi_1, \dots)$, if θ follows from each of the immediate grounds of $\lambda_R(\psi_0, \psi_1, \dots)$, then θ should follow from $\lambda_R(\psi_0, \psi_1, \dots)$ by an elimination rule. And conversely, if θ follows from $\lambda(\psi_0, \psi_1, \dots)$ by an elimination rule, then θ has to follow from each of the immediate grounds for $\lambda_R(\psi_0, \psi_1, \dots)$.

Let us see how this plays out in the case of $<$.

[20] Thanks to an anonymous reviewer for probing comments on this issue.

[21] For a statement of the inversion principle, see, e.g., (Read, 2010).

6.2. ***$<$-Elimination.*** Applied to $<$ one might think that the inversion principle gives us something similar to the (generalized) elimination rule for conjunction.

$$\dfrac{\Gamma < \phi \qquad \begin{matrix} \overline{\Gamma}^{\,1} & \overline{\Gamma \Rightarrow \phi}^{\,2} \\ & \mathcal{E} \\ & \theta \end{matrix}}{\theta}\ 1{,}2\text{: } <\text{-Elimination}$$

This is to be read as follows. If $\mathcal{E}$ is an argument to conclusion θ and we, in the course of $\mathcal{E}$, have used the assumptions Γ and $\Gamma \Rightarrow \phi$ some number of times, we can conclude θ from $\Gamma < \phi$, discharging any number of the assumptions Γ and $\Gamma \Rightarrow \phi$.

Note here that we allow both *vacuous* and *multiple* discharge. That is, one can apply $<$-elimination even if one has relied on no occurrences of (say) $\Gamma \Rightarrow \phi$ in the subordinate derivation; and if one has used more than one application of $\Gamma \Rightarrow \phi$ one can discharge any number of those occurrences.

These were the elimination rules given in (Litland, 2017b). Unfortunately, the above rules for $<$ do not ensure that the *only* (immediate) grounds for $\Gamma < \phi$ are $\Gamma, (\Gamma \Rightarrow \phi)$ (taken together). The rules ensure that *if* $\Gamma < \phi$ is the case then Γ and $\Gamma \Rightarrow \phi$ are the case; but it is left open that $\Gamma < \phi$ might have other, different immediate grounds. Since the introduction rules for $<$ are meant to specify all and only the immediate grounds for propositions formed using $<$ the elimination rules have to ensure that the only immediate grounds for $\Gamma < \phi$ are Γ and $\Gamma \Rightarrow \phi$ taken together.

For now, note the problem and consider what the elimination rule for $\Rightarrow$ should look like.

6.3. ***$\Rightarrow$-Elimination: Hypothetical arguments.*** The introduction rule for $\Rightarrow$ tells us that we are entitled to assert $\Gamma \Rightarrow \phi$ if there is an explanatory argument with premisses Γ and conclusion ϕ. So anything which follows from the *existence* of such an argument should follow from $\Gamma \Rightarrow \phi$. How can we make sense of assuming and discharging arguments?

In (Litland, 2017b) I introduced the notion of a hypothetical argument to deal with this.[22] Let us use $\Gamma \Vdash_{e,p} \phi$ to refer to the argument $\dfrac{\Gamma}{\phi}$, where the subscript indicates whether we think of the argument as explanatory or merely plain. We will call expressions of the form $\Gamma \Vdash_{e,p} \phi$ “hypothetical arguments”. Recall that we have defined the explanatory and plain arguments over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ for any arguments $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ such that $\mathfrak{E}_e \subseteq \mathfrak{E}_p$. In *assuming* the hypothetical argument $\Gamma \Vdash_e \phi$ we consider the explanatory and plain arguments over $\langle \mathfrak{E}_e \cup \{\Gamma \Vdash_e \phi\}, \mathfrak{E}_p \cup \left\{\Gamma \Vdash_p \phi\right\} \rangle$. (And analogously for $\Gamma \Vdash_p \phi$.)

Discharge of hypothetical arguments is understood as follows. An inference rule that discharges a hypothetical argument $\Gamma \Vdash_e \phi$ is of the form:

$$\dfrac{\Delta \qquad \begin{matrix} \overline{\Gamma \Vdash_e \phi}^{\,1} \\ \mathcal{D} \\ \theta \end{matrix}}{\sigma}\ 1$$

What this means is that for all $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$, if the subordinate argument $\mathcal{D}$ is amongst the explanatory and plain arguments over $\langle \mathfrak{E}_e \cup \{\Gamma \Vdash_e \phi\}, \mathfrak{E}_p \cup \left\{\Gamma \Vdash_p \phi\right\} \rangle$, then the whole

[22] A more general notion of “rules which discharge rules” is developed by Schroeder-Heister (1984).

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argument is amongst the explanatory and plain arguments over $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$. (And similarly for $\Gamma \Vdash_p \phi$.)

An intuitive way of thinking about this is to think of $\Gamma \Vdash_{e,p} \phi$ as a rule of inference. We can then introduce the following piece of notation: $\dfrac{\Gamma}{\phi}\, \Gamma \Vdash_{e,p} \phi$. Think of this as saying that by using the rule of inference $\Gamma \Vdash_{e,p} \phi$ we infer ϕ (explanatorily or plainly) from Γ. (Officially, what is said is that $\dfrac{\Gamma}{\phi}$ is amongst the explanatory (plain) arguments over $\langle \mathfrak{E}_e \cup \{\Gamma \Vdash_e \phi\}, \mathfrak{E}_p \cup \left\{\Gamma \Vdash_p \phi\right\} \rangle$.)

Using the notion of a hypothetical explanatory argument, (Litland, 2017b) offered the following elimination rule for $\Rightarrow$.

$$\dfrac{\Gamma \Rightarrow \phi \qquad \begin{array}{c}\overline{\Gamma \Vdash_e \phi}^{\;1}\\ \mathcal{E}\\ \theta\end{array}}{\theta}\;1, \Rightarrow\text{-E}$$

This elimination rule allows us to prove that the nonfactive grounding claims are zero-grounded (if true).

$$\dfrac{\Gamma \Rightarrow \phi \qquad \dfrac{\dfrac{\dfrac{\dfrac{\overline{\Gamma}^{\;1}}{\phi}\,[\Gamma \Vdash \phi]^2}{\Gamma \Rightarrow \phi}\;1, \Rightarrow\text{-I}}{\emptyset \Rightarrow (\Gamma \Rightarrow \phi)}\;\Rightarrow\text{-I}}{\emptyset < \Gamma \Rightarrow \phi}\;<\text{-I}}{\emptyset < (\Gamma \Rightarrow \phi)}\;2, \Rightarrow\text{-E}$$

The problem with this rule is the same as with the $<$-rule. If the $\Rightarrow$-I rule is to define the $\Rightarrow$-operator the only ground for (true) propositions of the form $\Gamma \Rightarrow \phi$ should be the empty ground; but this version of the elimination rule is not strong enough to ensure this.

6.4. $\approx$-Elimination. Before we show how to remedy this problem let us show how hypothetical arguments allow us to give elimination rules for the $\approx$-operator. What is characteristic of identity is that it satisfies *Leibniz's Law*. If $\phi \approx \psi$ then any role that is played by ϕ can be played by ψ (and *vice versa*). In particular, we should be able to substitute ϕ for ψ in any argument, preserving the explanatory status of the argument. If θ is any formula let us, ambiguously, write $\theta[\psi/\phi]$ for any result of replacing some occurrences of ϕ in θ with ψ. (We do not demand that we replace every occurrence of ϕ with ψ.) Similarly, we write $\Gamma[\psi/\phi]$ for the result of replacing some occurrences of ϕ in the $\gamma \in \Gamma$ with ψ.

What we want to express, then, is that if $\phi \approx \psi$ and we have an argument $\mathcal{E}$ from some premisses Γ to θ then we can assume a hypothetical argument $\Gamma[\psi/\phi] \Vdash_{e,p} \theta[\psi/\phi]$, where the hypothetical argument is explanatory if $\mathcal{E}$ is explanatory, otherwise it is plain. We thus arrive at the following elimination rule for $\approx$.

$$\dfrac{\phi \approx \psi \qquad \begin{array}{c}\overline{\Gamma}^{\;1}\\ \mathcal{E}\\ \theta\end{array} \qquad \begin{array}{c}\overline{\Gamma[\psi/\phi] \Vdash_{e,p} \theta[\psi/\phi]}^{\;2}\\ \mathcal{D}\\ \sigma\end{array}}{\sigma}\;1,2, \approx\text{-E}$$

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6.5. Explanatory elimination. The key to finding sufficiently strong elimination rules for $\Rightarrow$ and $<$ is to observe that if the only way of arriving at propositions of the forms $\Delta \Rightarrow \phi$ and $\Delta < \phi$ is by means of arguments ending with $\Rightarrow$-I or $<$-I, then we should have two types of elimination rule. In one—given above in 6.2, 6.3, and 6.4—the formula to be eliminated is the conclusion of a merely plain argument; in the other, the formula to be eliminated is the conclusion of an explanatory argument. Let us call the former "plain elimination" and the latter "explanatory elimination".

To see how this works consider first the case of $\Rightarrow$-E. Suppose we have an explanatory argument $\mathcal{E}$ from $\Gamma = \{\gamma_0, \gamma_1, \dots\}$ to $\Delta \Rightarrow \phi$. Then this argument $\mathcal{E}$ must end with an application of $\Rightarrow$-I. Since $\Rightarrow$-I discharges all the premisses on which ϕ depends, Γ has to be identical to the empty ground. The explanatory elimination rule then has to look like this:

$$\dfrac{\begin{array}{c}\overline{\gamma_0, \gamma_1, \dots}^{\,a}\\ \mathcal{E}(e)\\ \Delta \Rightarrow \phi\end{array} \qquad \overline{\gamma_0 \approx \emptyset}^{\,0} \qquad \begin{array}{c}\overline{\gamma_1 \approx \emptyset \quad \dots}^{\,1} \qquad \overline{\Delta \Vdash_e \emptyset}^{\,b}\\ \mathcal{D}\\ \sigma\end{array}}{\sigma}\; a, b, 0, 1, \dots, \Rightarrow\text{-E}$$

Note that $\{\gamma_0, \gamma_1, \dots\}$ itself is discharged. In proving that any ground for $\Delta \Rightarrow \phi$ is identical to the empty ground we will need this feature (see Proposition 7.7.2). But allowing each of $\gamma_0, \gamma_1, \dots$ to be discharged is well-motivated: if each of the $\gamma \in \{\gamma_0, \gamma_1, \dots\}$ is identical to the empty ground, then—since the empty ground always obtains—each $\gamma \in \{\gamma_0, \gamma_1, \dots\}$ always obtains.

The $<$-E rule is more cumbersome to state. Let Γ, Δ, and $\Delta \Rightarrow \phi$ be given. Write $\{\Delta \Rightarrow \phi\} \cup \Delta = \left\{\sigma_j : j \in J\right\}$. A *$J$-cover* of Γ is a collection $C = \left\{\Gamma_j : j \in J\right\}$ of subsets of Γ such that $\bigcup_{j \in J} \Gamma_j = \Gamma$. (Note that several of the Γ_j can be $\emptyset$.) When J is clear from context we just refer to the covers of Γ. We can enumerate the covers of Γ as $\{C_i : i \in I\}$ where each C_i is of the form $\left\{\Gamma^i_j : j \in J\right\}$. For each $i \in I$, an *i-cover-story* is a set $\left\{\Gamma^i_J R^i_j \sigma_j : j \in J\right\}$ where each R^i_j is either $\Vdash_e$ or $\approx$. For each i, we can enumerate the i-cover-stories as $\left\{C_{i,k} : k \in K_i\right\}$. The totality of ways in which Γ can be the premisses of an explanatory argument with conclusion $\Delta < \phi$ is then represented by $\left\{C_{i,k} : i \in I, k \in K_i\right\}$—the set of cover stories.

If Γ are the premisses of an explanatory argument with conclusion $\Delta < \phi$, and σ is a consequence of any cover story $C_{i,k}$, then σ should follow from $\Delta < \phi$ by the explanatory elimination rule. We are thus led to the following rule.

$$\dfrac{\begin{array}{c}\Gamma\\ \mathcal{E}\\ \Delta < \phi\end{array} \qquad \begin{array}{c}\overline{\dots \quad C_{i,k} \quad \dots}^{\,i,k}\\ \mathcal{D}_{i,k}\\ \dots \quad \sigma \quad \dots\end{array}}{\sigma}\; i \in I, k \in K_i$$

Finally, if we take the $\approx$-I rule to be explanatory we must also give an explanatory $\approx$-E rule. By now this is straightforward:

$$\dfrac{\begin{array}{c}\overline{\Gamma}^{\,1}\\ \mathcal{E}\\ \phi \approx \psi\end{array} \qquad \begin{array}{c}\overline{\Delta}^{\,2}\\ \mathcal{D}\\ \theta\end{array} \qquad \begin{array}{c}\overline{\Delta[\psi/\phi] \Vdash_{e,p} \theta[\psi/\phi]}^{\,3} \qquad \overline{\Gamma \approx \emptyset}^{\,4}\\ \mathcal{F}\\ \sigma\end{array}}{\sigma}\; 1,2,3,4, \approx\text{-E}$$

This rule is exactly like the plain elimination rule except that we can assume that Γ (that is, each $\gamma \in \Gamma$) is equivalent to the empty ground.

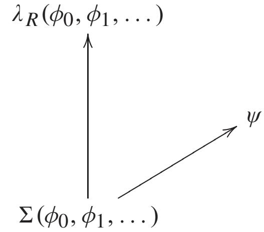

Fig. 3. Elimination rules are plain.

6.6. E-rules are plain. Just as we have to decide whether the introduction rules result in explanatory or merely plain arguments we have to decide whether applications of the E-rules result in explanatory or plain arguments. Intuitively, it is clear that the E-rules result in plain arguments: it is not plausible that the elimination rules given above specify the conditions that make it the case that their conclusions obtain.[23]

While this is no doubt correct it is problematic to rely on this if we take the introduction rules to define the grounding operators: it should then *follow* from the operators being so defined that the elimination rules lead to merely plain arguments. Fortunately, we can argue that this is indeed so.

Explanatory arguments induce an explanatory order, with the premisses of explanatory arguments being strictly lower in the explanatory order than their conclusions. Let λ_R be an operator defined by the introduction rules R. The introduction rules R for λ_R allow us to conclude $\lambda_R(\phi_0, \phi_1, \ldots)$ when one of the conditions $\Sigma_i(\phi_0, \phi_1, \ldots)$ on $\phi_0, \phi_1, \ldots, i \in I$ is met. Since the introduction rules are explanatory $\lambda_R(\phi_0, \phi_1, \ldots)$ is strictly higher than each $\Sigma_i(\phi_0, \phi_1, \ldots)$ in the explanatory order. The elimination rules for λ_R allow us to conclude ψ if ψ follows from every condition Σ_i from which we can conclude $\lambda_R(\phi_0, \phi_1, \ldots)$ by a rule in R. Precisely because $\lambda_R(\phi_0, \phi_1, \ldots)$ is strictly higher than each such condition Σ_i we simply have no information about how ψ stands to $\lambda_R(\phi_0, \phi_1, \ldots)$ in the explanatory order. Figure 3 makes this clear (here the arrows indicate location in the explanatory order): even if ψ is strictly above $\Sigma(\phi_0, \phi_1, \ldots)$ this gives us no information about its relation to $\lambda_R(\phi_0, \phi_1, \ldots)$.[24]

§7. The pure logic of iterated full ground. We can finally define the Pure Logic of Full Ground (PLIFG). For definiteness, the rules governing the operators $<$, $\Rightarrow$, and $\approx$ are repeated in Figure 4. (For simplicity, we only list the explanatory elimination rules.)

We need one final piece in order to define PLIFG: the infinitary discharge convention. This convention allows us to apply infinitely many elimination rules simultaneously.

DEFINITION 7.1 (Discharge Convention). *Let $\phi_0, \phi_1, \ldots$ be some formulae. Suppose that*

$$\begin{array}{c} \Theta_0^0, \Theta_1^0, \ldots \Theta_0^1, \Theta_1^1, \ldots \\ \mathcal{E} \\ \psi \end{array}$$

is an argument such that for each i

[23] Thanks to an anonymous reviewer on this point.

[24] As an example, the immediate ground of both $\phi \vee \phi$ and $\phi \wedge \phi$ is ϕ, but, plausibly, $\phi \wedge \phi$ and $\phi \vee \phi$ are not comparable in terms of ground.

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$$\dfrac{\begin{array}{c}\overline{\Gamma}^{\,1}\\ \mathcal{E}\\ \phi\end{array}}{\Gamma \Rightarrow \phi}\ 1, \Rightarrow\text{-I} \qquad \dfrac{\begin{array}{c}\overline{\gamma_0, \gamma_1, \dots}^{\,a}\\ \mathcal{E}(e)\\ \Delta \Rightarrow \phi\end{array} \quad \overline{\gamma_0 \approx \emptyset}^{\,0} \quad \begin{array}{c}\overline{\gamma_1 \approx \emptyset}^{\,1} \ \dots \quad \overline{\Delta \Vdash_e \emptyset}^{\,b}\\ \mathcal{D}\\ \sigma\end{array}}{\sigma}\ a, b, 0, 1, \dots, \Rightarrow\text{-E}$$

$$\dfrac{\Gamma \qquad \Gamma \Rightarrow \phi}{\Gamma < \phi}\ <\text{-I} \qquad \dfrac{\begin{array}{c}\Gamma\\ \mathcal{E}\\ \Delta < \phi\end{array} \quad \begin{array}{c}\overline{\dots \ C_{i,k} \ \dots}^{\,i,k}\\ \mathcal{D}_{i,k}\\ \dots \ \sigma \ \dots\end{array}}{\sigma}\ i \in I, k \in K_i$$

$$\overline{\phi \approx \phi}\ \approx\text{-I} \qquad \dfrac{\begin{array}{c}\overline{\Gamma}^{\,1}\\ \mathcal{E}\\ \phi \approx \psi\end{array} \quad \begin{array}{c}\overline{\Delta}^{\,2}\\ \mathcal{D}\\ \sigma\end{array} \quad \begin{array}{c}\overline{\Delta[\psi/\phi] \Vdash_{e,p} \theta[\psi/\phi]}^{\,3} \quad \overline{\Gamma \approx \emptyset}^{\,4}\\ \mathcal{F}\\ \sigma\end{array}}{\sigma}\ 1,2,3,4, \approx\text{-E}$$

Fig. 4. Rules for PLIFG.

$$\dfrac{\phi_i \qquad \begin{array}{c}[\Theta_0]^i, [\Theta_1]^i, \dots\\ \mathcal{E}\\ \psi\end{array}}{\psi}\ i,\ \phi_i\text{-elimination}$$

is a valid argument. (ϕ_i-elimination is λ_i-elimination where λ_i is the dominant operator in ϕ_i.) We then allow simultaneous *application of all the elimination rules for the principal operator in each ϕ_i. That is,*

$$\dfrac{\phi_0 \quad \phi_1 \quad \dots \qquad \begin{array}{c}[\Theta^0_0]^{(0,0)}, [\Theta^0_1]^{(0,1)}, \dots, [\Theta^1_0]^{(1,0)}, [\Theta^1_1]^{(1,1)}, \dots\\ \mathcal{E}\\ \psi\end{array}}{\psi}\ (0,0), (0,1), \dots, (1,0), (1,1), \dots;\ \phi_0\text{-}E,\ \phi_1\text{-}E, \dots$$

is to be a valid argument.

We need this convention because there are situations where we must apply infinitely many elimination rules. Since the argument trees are converse well-founded this cannot be done in succession but rather has to be done simultaneously.

Let $\mathfrak{E}_e, \mathfrak{E}_p$ be some arguments. The *arguments over* $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ is the closure of $\langle \mathfrak{E}_e, \mathfrak{E}_p \rangle$ under the rules in Figures 1 and 4 in accordance with the Discharge Convention. *The arguments of* PLIFG are the arguments over $\langle \emptyset, \emptyset \rangle$. The provability relation $\vdash$ of PLIFG is defined as follows. $\Gamma \vdash \phi$ iff there is a PLIFG-argument $\mathcal{E}$ with conclusion ϕ such that the undischarged premisses of $\mathcal{E}$ are a subset of Γ.

We now prove some basic results in PLIFG.

7.1. Basic results in PLIFG.

PROPOSITION 7.2 (Left and right factivity).

1. $\Delta < \phi \vdash \phi$.
2. $\Delta < \phi \vdash \delta$ *for each* $\delta \in \Delta$.

Proof. We prove Proposition 7.2.2. Note that we require vacuous discharge.

$$\dfrac{\delta_0, \delta_1, \dots < \phi \qquad \overline{\delta_i}^{\,i}}{\delta_i}\ i, <\text{-E}$$

□

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The following results ensure that $\approx$ behaves like identity.

PROPOSITION 7.3.

1. $\vdash \phi \approx \phi$, *for each* ϕ.
2. $\phi \approx \psi \vdash \psi \approx \phi$.
3. $\phi \approx \psi, \psi \approx \theta \vdash \phi \approx \theta$.
4. *If the* $\approx$*-I rule is explanatory,* $\phi \approx \psi \vdash \emptyset \Rightarrow (\phi \approx \psi)$.
5. *If we have the explanatory* $\approx$*-I and* $\approx$*-E rules,* $\Gamma \Rightarrow (\phi \approx \psi) \vdash \Gamma \approx \emptyset$.

Proof. We prove the middle three claims. The following establishes symmetry.

$$\dfrac{\phi \approx \psi \qquad \dfrac{}{\phi \approx \phi}\,{\approx}\text{-I} \qquad \dfrac{}{\psi \approx \phi}\,[\emptyset \Vdash \psi \approx \phi]^1}{\psi \approx \phi}\,1, {\approx}\text{-E}$$

The third-most subargument is obtained by replacing the left occurrence of ϕ in $\phi \approx \phi$ with ψ.

Transitivity is established similarly.

$$\dfrac{\phi \approx \psi \qquad \dfrac{}{\phi \approx \phi}\,{\approx}\text{-I} \qquad \dfrac{\psi \approx \theta \qquad \dfrac{}{\phi \approx \psi}\,[\emptyset \Vdash \phi \approx \psi]^1 \qquad \dfrac{}{\phi \approx \theta}\,[\emptyset \Vdash \psi \approx \theta]^2}{\phi \approx \theta}\,2, {\approx}\text{-E}}{\phi \approx \theta}\,1, {\approx}\text{-E}$$

The following establishes that all propositional equivalences are zero-grounded.

$$\dfrac{\phi \approx \psi \qquad \dfrac{}{\phi \approx \phi}\,{\approx}\text{-I} \qquad \dfrac{\dfrac{}{\phi \approx \psi}\,[\emptyset \Vdash_e \phi \approx \psi]^1}{\emptyset \Rightarrow (\phi \approx \psi)}\,{\Rightarrow}\text{-I}}{\emptyset \Rightarrow (\phi \approx \psi)}\,1, {\approx}\text{-E}$$

□

PROPOSITION 7.4 (Noncircularity). $\Delta, \phi < \phi \vdash \psi$, *for all* ψ.

Proof.

$$\dfrac{\Delta, \phi < \phi \qquad \dfrac{\dfrac{}{\Delta, \phi \Rightarrow \phi}\,0 \qquad \dfrac{\dfrac{\dfrac{}{\Delta, \phi}\,1}{\phi}\,[\Delta, \phi \Vdash_e \phi]^2}{\psi}\,\text{Noncircularity}}{\psi}\,2, {\Rightarrow}\text{-E}}{\psi}\,0{,}1, {<}\text{-E}$$

□

As a special case of Proposition 7.4 we have that the null ground is minimal in the sense that

$$\phi_0, \phi_1, \ldots < \emptyset \vdash \psi, \text{ for each } \psi.$$

We can however establish that the null ground is minimal in a stronger sense:

PROPOSITION 7.5. $\Gamma \Rightarrow \emptyset \vdash \psi$, *for all* ψ.

Proof. We first establish that $\Gamma \Rightarrow \emptyset \vdash \Gamma \approx \emptyset$.

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$$\dfrac{\Gamma \Rightarrow \emptyset \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma}\,1}{\emptyset}\,[\Gamma \Vdash_e \emptyset]^2}{\Gamma \Rightarrow \emptyset}\,1, \Rightarrow\text{-I}}{\emptyset \Rightarrow (\Gamma \Rightarrow \emptyset)}\,\Rightarrow\text{-I} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma}\,3}{\emptyset}\,[\Gamma \Vdash_e \emptyset]^2}{\Gamma \Rightarrow \emptyset}\,[\emptyset \Vdash_e \Gamma \Rightarrow \emptyset]^4 \qquad \dfrac{}{\Gamma \approx \emptyset}\,5}{\Gamma \approx \emptyset}\,3, 5, \Rightarrow\text{-E}}{\Gamma \approx \emptyset}\,4, \Rightarrow\text{-E}}{\Gamma \approx \emptyset}\,2, \Rightarrow\text{-E}$$

To establish Proposition 7.5 it then suffices to show that $\Gamma \Rightarrow \emptyset, \Gamma \approx \emptyset \Vdash \psi$, for all ψ. The following proof witnesses that

$$\dfrac{\Gamma \Rightarrow \emptyset \qquad \dfrac{\Gamma \approx \emptyset \qquad \dfrac{\dfrac{\dfrac{}{\Gamma}\,1}{\emptyset}\,[\Gamma \Vdash_e \emptyset]^2}{\psi}\,\text{Noncircularity} \qquad \dfrac{}{\psi}\,[\emptyset \Vdash_p \psi]^3}{\psi}\,1{,}3, \approx\text{-E}}{\psi}\,2, \Rightarrow\text{-E}$$

□

We have the following Cut-Principles.

PROPOSITION 7.6.

1. $\Delta_0 \Rightarrow \phi_0, \Delta_1 \Rightarrow \phi_1, \ldots, (\phi_0, \phi_1, \ldots, \Sigma \Rightarrow \phi) \vdash \Delta_0, \Delta_1, \ldots, \Sigma \Rightarrow \phi$.
2. $\Delta_0 < \phi_0, \Delta_1 < \phi_1, \ldots, (\phi_0, \phi_1, \ldots, \Sigma < \phi) \vdash \Delta_0, \Delta_1, \ldots, \Sigma < \phi$.

Proof. We prove Proposition 7.6.1; Proposition 7.6.2 follows easily. Note the need for applying the infinitary discharge convention.

$$\dfrac{\Delta_0 \Rightarrow \phi_0, \Delta_1 \Rightarrow \phi_1, \ldots, (\phi_0, \phi_1, \ldots, \Sigma \Rightarrow \phi) \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Delta_0}\,0a}{\phi_0}\,[\Delta_0 \Vdash_e \phi_0]^{0b} \qquad \dfrac{\dfrac{}{\Delta_1}\,1a}{\phi_1}\,[\Delta_1 \Vdash_e \phi_1]^{1b} \qquad \ldots \qquad \dfrac{}{\Sigma}\,c}{\phi}\,[\phi_0, \phi_1, \ldots, \Sigma \Vdash_e \phi]^d}{\Delta_0, \Delta_1, \ldots, \Sigma \Rightarrow \phi}\,0a, 1a, \ldots, c, \Rightarrow\text{-I}}{\Delta_0, \Delta_1, \ldots, \Sigma \Rightarrow \phi}\,0b, 1b, \ldots, d, \Rightarrow\text{-E}$$

□

The next results establish how iterated ground works; the key result is that the empty ground is the unique (immediate) full ground of the nonfactive grounding claims. Note how we require the explanatory elimination rules.

PROPOSITION 7.7.

1. $\Delta \Rightarrow \phi \vdash \emptyset \Rightarrow (\Delta \Rightarrow \phi)$.
2. $\Gamma \Rightarrow (\Delta \Rightarrow \phi) \vdash \emptyset \approx \Gamma$.
3. $\Gamma < \phi \vdash (\Gamma, (\Gamma \Rightarrow \phi)) < (\Gamma < \phi)$.
4. $\Gamma < \phi \vdash \Gamma < (\Gamma < \phi)$.
5. $(\Gamma \Rightarrow \psi) \Rightarrow (\Delta \Rightarrow \phi) \vdash \theta$*, for any* θ.

Proof. We only prove a few of the cases. The following establishes Proposition 7.7.2. Note how we first use the plain $\Rightarrow$-E rule to get a hypothetical explanatory argument from Γ to $\Delta \Rightarrow \phi$. We then use the explanatory elimination rule to conclude that $\Gamma \approx \emptyset$.

$$\dfrac{\Gamma \Rightarrow (\Delta \Rightarrow \phi) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma}\,1}{\Delta \Rightarrow \phi}\,[\Gamma \Vdash_e (\Delta \Rightarrow \phi)]^3 \qquad \dfrac{}{\Gamma \approx \emptyset}\,2}{\Gamma \approx \emptyset}\,1{,}2, \Rightarrow\text{-E}}{\Gamma \approx \emptyset}\,3, \Rightarrow\text{-E}$$

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The following establishes Proposition 7.7.4.

$$\dfrac{\Gamma < \phi \qquad \dfrac{\dfrac{}{\Gamma \Rightarrow \phi}\,1 \qquad \dfrac{\dfrac{}{\Gamma}\,2 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma}\,3 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma}\,4}{\phi}\,[\Gamma \Vdash_e \phi]^5}{\Gamma \Rightarrow \phi}\,4, \Rightarrow\text{-I}}{\Gamma < \phi}\,<\text{-I}}{\Gamma \Rightarrow (\Gamma < \phi)}\,3, \Rightarrow\text{-I}}{\Gamma < (\Gamma < \phi)}\,<\text{-I}}{\Gamma < (\Gamma < \phi)}\,5, \Rightarrow\text{-E}}{\Gamma < (\Gamma < \phi)}\,1{,}2, <\text{-E}$$

The following establishes Proposition 7.7.5.

$$\dfrac{(\Gamma \Rightarrow \psi) \Rightarrow (\Delta \Rightarrow \phi) \qquad \dfrac{\dfrac{\dfrac{}{\Gamma \Rightarrow \psi}\,1}{\Delta \Rightarrow \phi}\,[(\Gamma \Rightarrow \psi) \Vdash_e \Delta \Rightarrow \phi]^0 \qquad \dfrac{\dfrac{}{\Gamma \Rightarrow \psi}\,1 \qquad \dfrac{\dfrac{}{\emptyset \approx (\Gamma \Rightarrow \psi)}\,2 \qquad \dfrac{\dfrac{\dfrac{}{\Gamma}\,3}{\psi}\,[\Gamma \Vdash_e \psi]^4}{\Gamma \Rightarrow \psi}\,3, \Rightarrow\text{-I} \qquad \dfrac{\dfrac{}{\emptyset}\,[\emptyset \Vdash_e \emptyset]^5}{\theta}\,\text{Noncircularity}}{\theta}\,5, \approx\text{-E}}{\theta}\,4, \Rightarrow\text{-E}}{\theta}\,1{,}2, \Rightarrow\text{-E}}{\theta}\,0, \Rightarrow\text{-E}$$

□

REMARK 7.8. *Proposition 7.7.4 is the superinternality claim defended by deRosset and Bennett. The interpretation of this claim is, however, different in* PLIFG. *It is natural to take them to be committed to holding that the* immediate *ground for* $\Delta < \phi$ *is just* Δ.[25] *But that is not correct according to* PLIFG: *the immediate grounds for* $\Delta < \phi$ *are* Δ *and* $\Delta \Rightarrow \phi$ (*taken together*).

REMARK 7.9. *Proposition 7.7.5 is of some philosophical interest. In connection with the problem of grounding ground several authors* (*e.g., Dasgupta 2014b, 566n18*) *have suggested that if* ϕ *grounds* ψ *and* ψ *grounds* θ, *then* ϕ*'s grounding* ψ *together with* ψ*'s grounding* θ *might ground* ϕ*'s grounding* θ. *Proposition 7.7.5 rules this out.*

§8. Amalgamation, weak ground, and the pure logic of ground. In this section we compare PLIFG with Fine's Pure Logic of Ground (PLG). Since PLG contains various operators for partial ground the right system to which to compare PLIFG is the full fragment PLFG—the rules of which are depicted Figure 5. In PLFG the notion of weak full ground plays an important role. Unlike strict ground weak ground is reflexive. While we have not employed a primitive notion of weak ground, in the present framework it is easy to introduce such a notion.[26]

Say that Δ weakly nonfactively grounds ϕ—$\Delta \Rightarrow_w \phi$—iff either $\Delta \approx \phi$ (where this means that $\delta \approx \phi$, for each $\delta \in \Delta$)—or Δ strictly fully nonfactively grounds ϕ. We

25 I hasten to add that they do not explicitly claim this; but this does seem to be the natural way of developing their views.

26 This, I hasten to add, is not the notion of weak ground that Fine works with. Fine's notion of weak ground has come in for a fair amount of criticism—see, e.g., deRosset (2013b, 2014). To my mind Fine's notion arises naturally in the framework of truthmaker semantics, but that framework yields a more coarse-grained notion of ground than what we are interested in here. Our definition of weak ground also differs from the one given by Correia (2017); what corresponds to his notion is our notion of distributive weak ground.

$$\frac{\Delta < \phi}{\Delta \leq \phi}\text{ Subsumption} \qquad \frac{}{\phi \leq \phi}\text{ Identity} \qquad \frac{\Delta, \phi < \phi}{\bot}\text{ Noncircularity}$$

$$\frac{\Delta_0 \leq \phi_0 \quad \Delta_1 \leq \phi_1 \quad \dots \qquad \phi_0, \phi_1, \dots \leq \phi}{\Delta_0, \Delta_1, \dots \leq \phi}\text{ Cut}$$

$$\frac{\phi_0, \phi_1, \dots \leq \phi \qquad (\phi_0, \Sigma_0 < \psi_0), (\psi_0, \Theta_0 \leq \phi) \quad (\phi_1, \Sigma_1 < \psi_1), (\psi_1, \Theta_1 \leq \phi) \quad \dots}{\phi_0, \phi_1 \dots < \phi}\text{ Reverse Subsumption}$$

Fig. 5. Pure logic of ground.

introduce a factive weak ground operator $\leq$ in the obvious way. The (plain) introduction and elimination rules for $\Rightarrow_w$ and $\leq$ are given in Figure 6.

With this treatment of weak ground, it turns out that the Cut principle for weak ground is not valid in PLIFG. For suppose there is an explanatory argument from Δ to ϕ, and suppose Δ is the case. Then the introduction rules for $\Rightarrow_w$ and $\leq$ ensure that $\phi \leq \phi$ and $\Delta \leq \phi$. But we have no guarantee that $\Delta, \phi \leq \phi$. Since $\phi, \phi \leq \phi$ is the same formula as $\phi \leq \phi$ this gives us a counterexample to Cut (for $\leq$).

Another difference is that the principle of (Strict) Amalgamation is valid in PLFG but not in PLIFG.

$$\frac{\Delta_0 < \phi \qquad \Delta_1 < \phi \qquad \dots}{\Delta_0, \Delta_1, \dots < \phi}\text{ Amalgamation}$$

We give an informal counterexample. Suppose disjuncts ground disjunctions, and that disjunctions have no other immediate grounds. Then we have an explanatory argument from p to $p \vee q$ and an explanatory argument from $p \vee q$ to $(p \vee q) \vee r$. But we do not have an explanatory argument from p together with $p \vee q$ to $(p \vee q) \vee r$, and so strict Amalgamation fails.

To get around this problem we introduce the notions of distributive weak and strict ground.[27] We say that $\Gamma \leq_d \{\delta_i : i \in I\}$ iff $\Gamma = \bigcup_{i \in I} \Gamma_i$ and $\Gamma_i \leq \delta_i$ for each $i \in I$. (We give the analogous definition for $<_d$—strict distributive ground.) The introduction rule for weak distributive ground is given in Figure 6. The elimination rule is to be understood as follows. Let $\{\phi_i : i \in I\}$ be an indexed collection of formulae. An I-covering of Γ is a collection of sets of formulae $\{\Gamma_i \leq \phi_i : i \in I\}$ such that $\bigcup_{i \in I} \Gamma_i = \Gamma$. Let J index the I-coverings of Γ. We then write $\left\{\Gamma_i^j \leq \phi_i^j : i \in I_j\right\}$, for the I-covering with index $j \in J$. The elimination rule for $\leq_d$ then says that if θ follows no matter which I-covering of Γ we choose, then θ follows from $\Gamma \leq_d \{\phi_i : i \in I\}$. The introduction and elimination rules for distributive strict ground are exactly parallel.

Let PLIFG+W be PLIFG augmented with the rules for weak ground and distributive weak and strict ground. One reason for introducing PLIFG+W is that it allows one to state important facts about iterated factive ground, like the following:[28]

$$\Gamma < (\Delta < \phi) \vdash \Gamma \leq_d \{\Delta \Rightarrow \phi\} \cup \Delta$$

[27] Another, perhaps more satisfactory, way around the problem would be to let the grounding operators take *multisets* on the left (and right).

[28] Compare the elimination rules for conjunction and disjunction in (Fine, 2012a, 63–67).

$$\dfrac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow_w \phi}\ \Rightarrow_w\text{-I} \qquad\qquad \dfrac{\Gamma \approx \phi}{\Gamma \Rightarrow_w \phi}\ \Rightarrow_w\text{-I}$$

$$\dfrac{\Gamma \Rightarrow_w \phi \qquad \begin{array}{c}\overline{\Gamma \Rightarrow \phi}^{\,1}\\ \mathcal{E}\\ \theta\end{array} \qquad \begin{array}{c}\overline{\Gamma \approx \phi}^{\,2}\\ \mathcal{D}\\ \theta\end{array}}{\theta}\ 1,2,\Rightarrow_w\text{-E}$$

$$\dfrac{\Gamma \Rightarrow_w \phi \qquad \Gamma}{\Gamma \leq \phi}\ \leq\text{-I} \qquad\qquad \dfrac{\Gamma \leq \phi \qquad \begin{array}{c}\overline{\Gamma \Rightarrow_w \phi}^{\,1} \quad \overline{\Gamma}^{\,2}\\ \mathcal{E}\\ \theta\end{array}}{\theta}\ 1,2,\leq\text{-E}$$

$$\dfrac{\Delta_0 \leq \phi_0 \qquad \Delta_1 \leq \phi_1 \qquad \ldots}{\Delta_0, \Delta_1, \ldots, \leq_d \{\phi_0, \phi_1, \ldots\}}\ \leq_d\text{-I}$$

$$\dfrac{\Delta \leq_d \{\phi_i : i \in I\} \qquad \begin{array}{c}\overline{\left\{\Delta_i^{j_0} \leq \phi_i : i \in I\right\}}^{\,j_0}\\ \mathcal{E}_{j_0}\\ \theta\end{array} \qquad \begin{array}{c}\overline{\left\{\Delta_i^{j_1} \leq \phi_i : i \in I\right\}}^{\,j_1}\\ \mathcal{E}_{j_1}\\ \theta\end{array} \qquad \ldots}{\theta}\ j_0, j_1, \ldots, \leq_d\text{-E}$$

Fig. 6. Weak ground rules.

But the main point of introducing the notions of distributive ground is that all principles of PLFG hold in PLIFG+W for distributive ground. Consider the following translation of the language of PLFG into the language of PLIFG+W. We translate a sequent $\Gamma \leq \phi$ of PLFG by the formula $\Gamma \leq_d \phi$ of PLIFG+W; and we translate a sequent $\Gamma < \phi$ by the formula $\Gamma <_d \phi$. We then have the following.

THEOREM 8.1. *The translation of every theorem of* PLFG *is a theorem of* PLIFG+W.

Proof. We prove the stronger claim that there is a valid PLIFG+W argument from the (translation of the) premisses of every PLFG rule of inference to the (translation of the) conclusion of that inference. The only interesting case is Reverse Subsumption. We reason informally; the reader should have no problem transforming the informal reasoning into a proof in PLIFG+W. Suppose we have $\left\{\phi_j : j \in J\right\} \leq_d \phi$. Then one possibility is that we have $\left\{\phi_j : j \in J\right\} \leq_d \phi$ because $\left\{\phi_j : j \in J\right\} = \bigcup_{i \in I} \Gamma_i$ and $\Gamma_i <_d \phi$ for each $i \in I$. In this case clearly $\left\{\phi_j : j \in J\right\} <_d \phi$.

The only other possibility is that $\phi_{j_0} \approx \phi$, for some $j_0 \in J$. The application of Reverse Subsumption then has some premisses $\phi_{j_0}, \Sigma_{j_0} <_d \psi_{j_0}$ and $\psi_{j_0}, \Theta_{j_0} \leq_d \phi$. We then have $\phi_{j_0}, \Sigma'_{j_0} < \psi_{j_0}$ for some $\Sigma'_{j_0} \subseteq \Sigma_{j_0}$. We also have $\psi_{j_0}, \Theta'_{j_0} \leq \phi$, for some $\Theta'_{j_0} \subseteq \Theta_{j_0}$. One possibility is that $\Theta'_{j_0}, \psi_{j_0} \approx \phi$. In that case, application of the $\approx$-rules give us $\phi_{j_0}, \Sigma'_{j_0} < \phi$. The other possibility is that $\psi_{j_0}, \Theta'_{j_0} < \phi$. In that case Cut gives us $\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} < \phi$.

So it suffices to show that we can derive $\left\{\phi_j : j \in J\right\} <_d \phi$ from $\phi \approx \phi_{j_0}$ together with $\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} < \phi$ and also from $\phi \approx \phi_{j_0}$ together with $\phi_{j_0}, \Sigma'_{j_0} < \phi$. The following argument establishes the former claim. (The other claim is established similarly.)

$$\dfrac{\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} < \phi \qquad \dfrac{\overline{\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} \Rightarrow \phi}^{\,1} \qquad \dfrac{\phi \approx \phi_{j_0} \qquad \dfrac{\overline{\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0}}^{\,2}}{\phi}\,[\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} \Vdash_e \phi]^3 \qquad \dfrac{\dfrac{\overline{\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0}}^{\,4}}{\phi_{j_0}}\,[\phi_{j_0}, \Sigma'_{j_0}, \Theta'_{j_0} \Vdash_e \phi_{j_0}]^5}{\left\{\phi_j : j \in J\right\} <_d \phi}\,\text{Noncircularity}}{\left\{\phi_j : j \in J\right\} <_d \phi}\,2,5,\approx\text{-E}}{\left\{\phi_j : j \in J\right\} <_d \phi}\,3,\Rightarrow\text{-E}}{\left\{\phi_j : j \in J\right\} <_d \phi}\ 1,4,<\text{-E}$$

□

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§9. Conclusion: Open questions and methodological remarks. This article has shown how one, by tying ground to explanation, can rigorously develop a pure logic of iterated ground. There is obviously much more to be done—of both a philosophical and of a technical nature—fully to defend this conception of ground.

Perhaps the most important task is to extend the framework to give a logic of immediate ground. The natural thought is that just as mediate ground corresponds to explanatory arguments, immediate ground corresponds to explanatory *inferences*. It is then natural to take explanatory inference as a primitive notion. But this raises a technical difficulty. It is natural to think that mediate ground is the closure of immediate ground. If that is so an explanatory argument must be the result of chaining together some explanatory inferences. The technical problem is how to capture this in the formal system. Somehow we have to be able to represent that when we have a hypothetical explanatory argument from Γ to ϕ, then this explanatory argument is the composition of some explanatory inferences.

A second important question is whether there is a natural semantics with respect to which PLIFG is sound and complete. If one is just interested in PLIFG it is routine to show that the graph-theoretical semantics sketched in Litland (2017b) can be extended to do the work. But that semantics is somewhat artificial, and it is not obvious how to make it work for iterated immediate ground.

Moving beyond the pure logic of ground, one key question is to characterize the explanatory arguments ending with logically complex propositions. The difficult cases here are negation and the conditional. As one deals with the impure logic of ground one eventually has to deal with the "puzzles of ground" introduced in (Fine, 2010), and one has to determine whether the present framework deals with the puzzles in an adequate way.

Let us end on some methodological remarks.

Ground is all the rage; but why care about the *logic* of ground? Apart from its inherent interest, an important reason for being interested in the logic of ground is the still significant skepticism about ground (see, e.g., Daly, 2012; Hofweber, 2009; Wilson, 2014, 2016). Some of this skepticism is driven by a suspicion that there is not *one* thing that talk about ground latches onto. (Such worries are hardly allayed by the tendency to take ground as a primitive!) Developing logics of ground provides one way of allaying such fears. However, the logics of ground require certain features if they are to play this role.

One should certainly not subscribe to the view that merely by creating a consistent formal system to govern a primitive notion one dispels the worry that there is no unique notion of ground. After all, even if there is a multiplicity of different notions of ground, it may be that no contradiction arises from the supposition that there is a unique one. (This is one reason why just axiomatizing a grounding *relation*, in the obvious way, is so unsatisfactory.)

A logical system plays an important role in making us comfortable with a primitive notion when (some of) the following desiderata are met. 1. The principles—or most of the principles—of the logic flow naturally from a guiding conception; 2. the guiding conception allows us to discover new—and somewhat surprising—principles governing the primitive notion; 3. the logic allows us to draw distinctions that we were previously unable to draw but that appear correct on reflection.

PLIFG meets these desiderata to a considerable degree. First, the principles of PLIFG flow naturally from the conception of ground as metaphysical explanation. Second, taking seriously the idea of ground as explanation gives us principles for iterated ground. Third, the notion of the zero-grounding is demystified by its connection with explanatory arguments from no premisses and it turns out that there is a wide class of zero-grounded truths.

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UNIVERSITY OF TEXAS AT AUSTIN
STOP C3500
TX 78712, AUSTIN
E-mail: jon.litland@austin.utexas.edu