J. Fluid Mech. (2013), *vol.* 722, *pp.* 461–495. © Cambridge University Press 2013 461
doi:10.1017/jfm.2013.57

Resonance theory of water waves in the long-wave limit

Takeshi Kataoka†

Department of Mechanical Engineering, Graduate School of Engineering, Kobe University, Rokkodai, Nada, Kobe 657-8501, Japan

(Received 19 March 2012; revised 19 January 2013; accepted 22 January 2013; first published online 28 March 2013)

The instability due to resonant interactions of finite-amplitude water waves is examined in the long-wave limit. In contrast to the well-known case of a small-amplitude limit in which the resonance is considered for a flat surface, here we consider a periodic approximate of the finite-amplitude solitary wave which is the long-wave limit of the periodic wave. The resonance conditions for the corresponding perturbations yield a new family of resonance curves that are totally different from those of the small-amplitude limit obtained by Phillips and Mclean. Under these resonance conditions, we conduct a systematic asymptotic analysis for small wavenumbers to obtain the growth rates of the perturbations explicitly and clarify whether each resonance curve is associated with instability. These results are verified numerically by showing that the instability bands for finite-amplitude periodic waves in shallow water are located along these unstable resonance curves.

Key words: surface gravity waves, waves/free-surface flows

1. Introduction

We are interested in the three-dimensional instability of periodic waves in water of finite depth. These waves are characterized by two dimensionless parameters, the wave amplitude and the wavelength (relative to the depth of the water), and the instability has been discussed in terms of resonant interactions in the small-amplitude limit, the theory for which was devised by Phillips (1960, 1967)) and Mclean (1982*a*,*b*). However, a recent numerical study by Francius & Kharif (2006) revealed that for periodic waves in shallow water, the agreement between the theoretical and numerical results is poor. In this study, we propose a new theory of resonance by considering resonant interactions not in the small-amplitude limit but in the long-wave (shallow-water) limit of finite-amplitude periodic waves. This theory can compensate for the weakness of the existing theory at the small-amplitude limit as it applies to finite-amplitude waves in shallow water.

Let us survey previous works on the stability of periodic waves. There are two principal types of two-dimensional instability. First, Benjamin (1967) and Benjamin & Feir (1967) found that periodic waves in deep water are subject to instability with respect to long-wavelength disturbances. This instability is called the Benjamin–Feir instability. Second, Longuet-Higgins (1978*a*) showed numerically that large-amplitude

† Email address for correspondence: kataoka@mech.kobe-u.ac.jp

waves can be strongly unstable with respect to disturbances that are periodic in one wavelength of the basic wave. This instability is called superharmonic instability. Many works have appeared since then: Zakharov (1968), Lake *et al.* (1977), Longuet-Higgins (1978*b*), Crawford *et al.* (1981), Melville (1982), Su (1982), Tulin & Waseda (1999) and Bridges & Dias (2007) treated the Benjamin–Feir instability, whereas Tanaka (1983, 1985), Saffman (1985), Kharif & Ramamonjiarisoa (1990), Longuet-Higgins & Tanaka (1997) and Kataoka (2006) investigated the superharmonic instability.

For three-dimensional instability, Phillips (1960, 1967) found that the resonance condition in the small-amplitude limit deduced from the linear dispersion relation can lead to instability. The corresponding resonance curve resembles the number 8 and is called Phillips' figure eight. Mclean (1982*a*,*b*) revealed that the resonance corresponding to a figure-eight curve is only the first of a family of resonance conditions corresponding to the lowest-order four-wave interactions. A family of resonance conditions is obtained by considering more than four-wave interactions; these higher-order conditions yield resonance curves that differ from that of the figure eight. These resonance instabilities are called class I if the number of interacting waves associated with the resonance condition is even and class II if it is odd. Numerical linear stability analysis on the basis of the full Euler set of equations showed that the bands of instability for small waves are located near these resonance curves.

The above classification of instability in the small-amplitude limit has been used extensively to discuss the three-dimensional instability of periodic waves so far (Su 1982; Su *et al.* 1982; Su & Green 1984; Kharif & Ramamonjiarisoa 1988; Fructus *et al.* 2005). The recent numerical work by Francius & Kharif (2006), however, revealed that the instability bands become very narrow for waves in shallow water, and these narrow bands are not located near the resonance curves of the small-amplitude limit. This fact indicates that the instability of periodic waves in shallow water cannot be described well by the resonance in the small-amplitude limit.

The present work therefore develops a new theory of resonance by considering resonant interactions not in the small-amplitude limit but in the long-wave limit of finite-amplitude periodic waves. The resonance theory is applied not to a flat surface but to a periodic approximate of the finite-amplitude solitary wave which is the long-wave limit of the periodic wave. The perturbations are composed not only of the usual linear waves but also of those associated with transversely distorted solitary wave (Bridges 2001; Kataoka & Tsutahara 2004). The resonance conditions for these two types of perturbations produce a new family of resonance curves that are totally different from those of the small-amplitude limit.

To determine whether these resonance curves are associated with instability, we also conduct an asymptotic analysis of the linear perturbation equations for small wavenumbers. The leading-order solution consists of perturbations that participate in the resonant interactions and have a common angular frequency. The next-order solution gives us their growth rates and clarify whether the corresponding resonance curve is associated with instability. Finally, we verify these theoretical results numerically.

This paper is structured as follows. We formulate the basic equations in § 2, where the problem of stability is reduced to a linear eigenvalue problem for the perturbation. The two types of resonance conditions, types I and II, are derived in the long-wave limit. In §§ 3 and 4, we conduct asymptotic analyses for small wavenumbers under resonance conditions of types I and II, respectively, to obtain the growth rates of the perturbations explicitly. We verify these theoretical results numerically in § 5, and

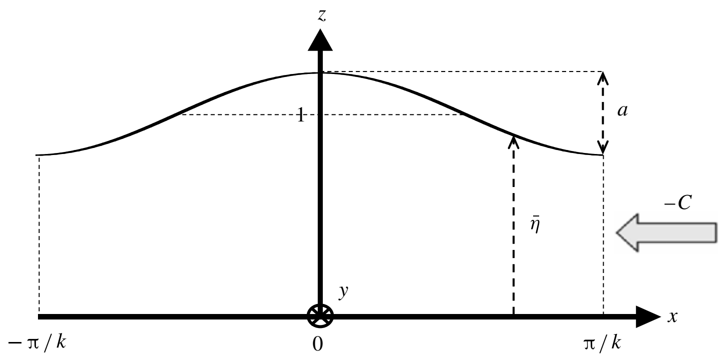

FIGURE 1. Geometry (non-dimensional system).

concluding remarks follow in § 6. Supplements to the asymptotic analyses in §§ 3 and 4 are provided in appendices A, B and C.

2. Basic equations

2.1. *Formulation*

Consider an incompressible fluid of finite depth with a free surface under uniform gravitational acceleration. The effects of viscosity and surface tension are neglected and a flow is supposed to be irrotational. All of the variables are made dimensionless by using appropriate combinations of the fluid density, gravitational acceleration g and a characteristic length L associated with the Bernoulli constant $(L = (|\boldsymbol{u}|^2 - |\boldsymbol{c}|^2)/(2g) + Z$, where $\boldsymbol{u}$ is the flow velocity on the surface, $\boldsymbol{c}$ is the average flow velocity at any horizontal level beneath the lowest surface and Z is the surface height). Let x, y and z be the Cartesian coordinates, with the z-axis pointing vertically upward and $z = 0$ at the bottom (figure 1). The motion of the fluid is governed by

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \quad \text{for } 0 < z < \eta, \tag{2.1}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial x}\frac{\partial \eta}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \eta}{\partial y} = \frac{\partial \phi}{\partial z} \quad \text{at } z = \eta, \tag{2.2}$$

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right] + \eta = f(t) \quad \text{at } z = \eta, \tag{2.3}$$

$$\frac{\partial \phi}{\partial z} = 0 \quad \text{at } z = 0, \tag{2.4}$$

where t is the time, ϕ is the velocity potential, η is the surface height from the bottom and $f(t)$ is a given function of time.

We first consider a two-dimensional periodic wave and seek a solution of (2.1)–(2.4) in the following form:

$$\phi = -Cx + \bar{\phi}(x, z), \qquad \eta = \bar{\eta}(x), \tag{2.5a,b}$$

with

$$f(t) = 1 + \tfrac{1}{2}C^2, \tag{2.5c}$$

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subject to the periodic condition at $x = \pm\pi/k$:

$$\bar{\phi}(-\pi/k, z) = \bar{\phi}(\pi/k, z), \quad \bar{\eta}(-\pi/k) = \bar{\eta}(\pi/k). \tag{2.6}$$

Solution (2.5) with (2.6) represents a two-dimensional periodic wave of wavenumber k that propagates steadily against a uniform stream of constant (dimensionless) velocity $-C$ in the x direction (figure 1). We consider the solution $(\bar{\phi}, \bar{\eta})$ of this class in which all of the crests have a uniform height and all of the troughs have a different uniform height. Substituting (2.5) into (2.1)–(2.4), we obtain a set of governing equations for $(\bar{\phi}, \bar{\eta})$:

$$\nabla_\perp^2 \bar{\phi} = 0 \quad \text{for } 0 < z < \bar{\eta}, \tag{2.7}$$

$$\left(-C + \frac{\partial \bar{\phi}}{\partial x}\right)\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}x} = \frac{\partial \bar{\phi}}{\partial z} \quad \text{at } z = \bar{\eta}, \tag{2.8}$$

$$-C\frac{\partial \bar{\phi}}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \bar{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \bar{\phi}}{\partial z}\right)^2\right] + \bar{\eta} = 1 \quad \text{at } z = \bar{\eta}, \tag{2.9}$$

$$\frac{\partial \bar{\phi}}{\partial z} = 0 \quad \text{at } z = 0, \tag{2.10}$$

and the periodic condition (2.6), where

$$\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}. \tag{2.11}$$

The existence of the above solution $(\bar{\phi}, \bar{\eta})$ was rigourously proved by Krasovskii (1961) and Keady & Norbury (1978), and its uniqueness was proved by Garabedian (1965). This solution is symmetric with respect to crests at $x = 2n\pi/k$ and troughs at $x = (2n+1)\pi/k$ (n is an integer); that is, $\bar{\phi}(x - n\pi/k, z)$ is odd and $\bar{\eta}(x - n\pi/k)$ is even in $x - n\pi/k$ (Garabedian 1965). Here we define the wave amplitude a as the dimensionless crest-to-trough height, i.e.

$$a = \bar{\eta}(0) - \bar{\eta}(\pi/k), \tag{2.12}$$

(figure 1); we use a and k as a set of two independent dimensionless parameters to characterize the solution (2.5). Computed wave forms $\bar{\eta}(x)$ for $a = 0.4$ are shown for three different values of k (=0.1, 0.5, 1) in figure 2. It is obvious from this figure that the solution (2.5) approaches a periodic approximate of the solitary wave as $k \to 0$. The convergence to the solitary wave in the limit $k \to 0$ was rigourously proved by Amick & Toland (1981*b*).

To examine the stability of the above two-dimensional unperturbed wave (2.5) with respect to an infinitesimal three-dimensional perturbation, let

$$\phi = -Cx + \bar{\phi}(x, z) + \hat{\phi}(x, z)\exp\left[\mathrm{i}k(-\sigma t + px + qy)\right], \tag{2.13a}$$

$$\eta = \bar{\eta}(x) + \hat{\eta}(x)\exp\left[\mathrm{i}k(-\sigma t + px + qy)\right], \tag{2.13b}$$

where (kp, kq) represents a wavenumber vector of perturbation in the (x, y) direction, σ is an unknown complex constant and $(\hat{\phi}, \hat{\eta})$ are unknown functions of the same periodicity as the unperturbed wave (2.5) (Floquet theory). That is,

$$\hat{\phi}(-\pi/k, z) = \hat{\phi}(\pi/k, z), \quad \hat{\eta}(-\pi/k) = \hat{\eta}(\pi/k). \tag{2.14}$$

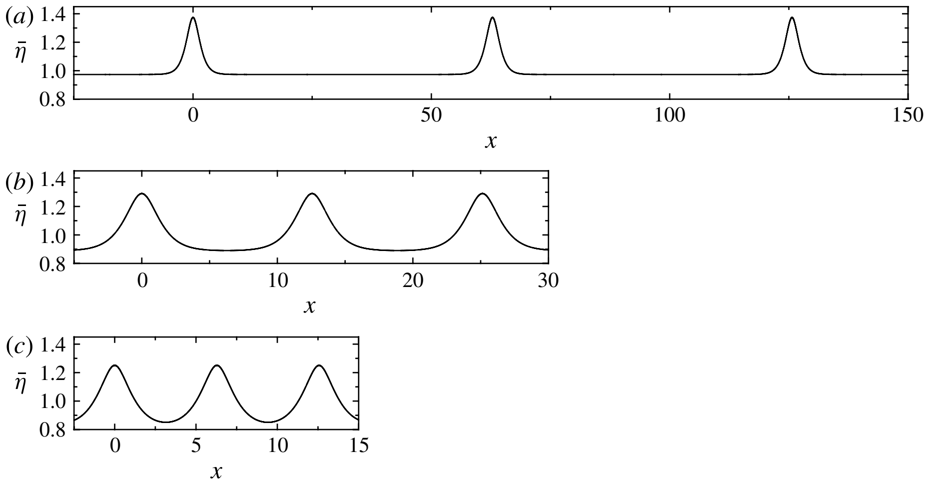

FIGURE 2. Wave forms $\bar{\eta}(x)$ of the unperturbed wave solution (2.5) for $a = 0.4$ with: (a) $k = 0.1$; (b) $k = 0.5$; (c) $k = 1$.

Substituting (2.13) into (2.1)–(2.4) and linearizing with respect to $(\hat{\phi}, \hat{\eta})$, we obtain

$$\nabla_\perp^2 \hat{\phi} = -2ikp\frac{\partial \hat{\phi}}{\partial x} + k^2(p^2+q^2)\hat{\phi} \quad \text{for } 0 < z < \bar{\eta}, \tag{2.15}$$

$$L_K[\hat{\phi}, \hat{\eta}] = ik\left\{\sigma\hat{\eta} + p\left[-\frac{d\bar{\eta}}{dx}\hat{\phi} + \left(C - \frac{\partial \bar{\phi}}{\partial x}\right)\hat{\eta}\right]\right\} \quad \text{at } z = \bar{\eta}, \tag{2.16}$$

$$L_D[\hat{\phi}, \hat{\eta}] = ik\left[\sigma\hat{\phi} + p\left(C - \frac{\partial \bar{\phi}}{\partial x}\right)\hat{\phi}\right] \quad \text{at } z = \bar{\eta}, \tag{2.17}$$

$$\frac{\partial \hat{\phi}}{\partial z} = 0 \quad \text{at } z = 0, \tag{2.18}$$

where L_K and L_D are the linear operators defined by

$$L_K[\hat{\phi}, \hat{\eta}] = \left(-\frac{\partial}{\partial z} + \frac{d\bar{\eta}}{dx}\frac{\partial}{\partial x}\right)\hat{\phi} + \left[\frac{\partial^2 \bar{\phi}}{\partial x^2} + \frac{\partial^2 \bar{\phi}}{\partial x \partial z}\frac{d\bar{\eta}}{dx} + \left(-C + \frac{\partial \bar{\phi}}{\partial x}\right)\frac{d}{dx}\right]\hat{\eta}, \tag{2.19a}$$

$$L_D[\hat{\phi}, \hat{\eta}] = \left[\left(-C + \frac{\partial \bar{\phi}}{\partial x}\right)\frac{\partial}{\partial x} + \frac{\partial \bar{\phi}}{\partial z}\frac{\partial}{\partial z}\right]\hat{\phi} + \left[\left(-C + \frac{\partial \bar{\phi}}{\partial x}\right)\frac{\partial^2 \bar{\phi}}{\partial x \partial z} + \frac{\partial \bar{\phi}}{\partial z}\frac{\partial^2 \bar{\phi}}{\partial z^2} + 1\right]\hat{\eta}. \tag{2.19b}$$

Equations (2.14)–(2.18) constitute an eigenvalue problem for $(\hat{\phi}, \hat{\eta})$ whose eigenvalue is σ. When this problem possesses a solution for which the eigenvalue σ has a positive imaginary part, the corresponding unperturbed wave (2.5) is unstable with respect to an infinitesimal perturbation of the wavenumber vector (kp, kq).

2.2. *Resonance theory*

Let us derive the resonance conditions and resonance curves in the long-wave limit $k \to 0$, keeping the wave amplitude finite ($a = O(1)$). As $k \to 0$, the unperturbed wave solution (2.5) for given a first becomes an approximation to the cnoidal wave (A 17) (see § A.3), and then converges to the solitary wave solution (Φ, H) with wave speed $v(a)$. (Amick & Toland (1981*b*) proved this convergence for a fixed mean depth h and steepest angle β between surface and horizontal, instead of L (see the first paragraph of § 2.1) and a in our case. One can obtain the same result that the periodic wave as $k \to 0$ converges to the solitary wave either for fixed (h, β) and (L, a) according to § A.3) So we here consider the resonance theory under the existence of the above finite-amplitude solitary wave. There are two typical perturbations. First, in association with flat region,

$$(\hat{\phi}, \hat{\eta}) = (1, 0), \quad \sigma = \sigma_L(p, q) \equiv -vp + s_1\sqrt{p^2 + q^2}, \tag{2.20a,b}$$

where $s_1 = -1$ or 1. This solution represents the usual long infinitesimal perturbation having the wavenumber vector (kp, kq) $(k|p|, k|q| \ll 1)$. Second, in association with a non-zero part of the solitary wave,

$$(\hat{\phi}, \hat{\eta}) = \left(\frac{\partial \Phi}{\partial x}, \frac{\mathrm{d}H}{\mathrm{d}x}\right), \quad \sigma = \sigma_S(q) \equiv s_2\sqrt{\frac{vEq^2}{\mathrm{d}E/\mathrm{d}v}} \quad \left(\text{for } 0 < a < 0.781 \text{ or } \frac{\mathrm{d}E}{\mathrm{d}v} > 0\right), \tag{2.21a,b}$$

(Kataoka & Tsutahara 2004) where $s_2 = -1$ or 1, and E is the energy of the solitary wave defined by (A 7*e*). The solution (2.21) represents oscillation of a transversely distorted solitary wave $(0 < a < 0.781)$ when it is distorted by a long-wavelength transverse perturbation $(k|q| \ll 1)$. For the larger amplitude a $(0.781 < a < 0.833)$, the solitary wave is longitudinally unstable (Tanaka 1986) and perturbation grows rapidly. Here, we are interested in the former case because the longitudinally unstable solitary wave is highly unstable and cannot persist for long (Tanaka *et al.* 1987).

A solution for $k > 0$ is obtained simply by connecting (2.20) or (2.21) in the region $-\pi/k < x < \pi/k$ successively every time x increases by $2\pi/k$. Such perturbations can amplify these modes if the eigenvalues σ agree for two perturbations whose normalized wavenumber vectors (p, q) are different by $(m, 0)$ for some integer m. In terms of σ defined by (2.20*b*) and (2.21*b*), the resonance conditions are arranged into the following two types.

The type I resonance,

$$\sigma_L(p, q) = \sigma_L(p - m, q) \quad (m = \pm 1, \pm 2, \ldots), \tag{2.22}$$

represents the resonance between two infinitesimal long waves whose wavenumber vectors differ by a factor of m with respect to that of the unperturbed wave. If s_1 of $\sigma_L(p, q)$ and that of $\sigma_L(p - m, q)$ in (2.20) are chosen to be 1 and -1 for $m > 0$ (or -1 and 1 for $m < 0$), (2.22) yields

$$\left(p - \frac{m}{2}\right)^2 + \frac{v^2}{v^2 - 1}q^2 = \frac{m^2 v^2}{4}, \quad \sigma_L = \frac{v^2 - 1}{v}\left(\frac{m}{2} - p\right). \tag{2.23a,b}$$

For the other choice of the signs s_1, (2.22) has no solution. The integer m is called the order of the resonance.

The type II resonance,

$$\sigma_L(p, q) = \sigma_S(q), \tag{2.24}$$

represents the resonance between an infinitesimal long wave and a transversely distorting perturbation of solitary wave both having the same transverse wavenumber.

Equation (2.24) leads to

$$p = \pm \frac{v+b}{\sqrt{(v^2-1)(b^2-1)}} q, \quad \sigma_L = \sigma_S = \frac{1-v^2}{v+b} p, \tag{2.25a,b}$$

where

$$b = \begin{cases} -\sqrt{1 + \dfrac{v^2-1}{vE}\dfrac{\mathrm{d}E}{\mathrm{d}v}} < 0 & \text{for } s_1 = s_2, \\ \sqrt{1 + \dfrac{v^2-1}{vE}\dfrac{\mathrm{d}E}{\mathrm{d}v}} > 0 & \text{for } s_1 = -s_2. \end{cases} \tag{2.26}$$

For the type I resonance, (2.23*a*) on the $p-q$ plane becomes an ellipse centred at $(p, q) = (m/2, 0)$ with a major axis of length $|m|v$ in the p direction and a minor axis of length $|m|\sqrt{v^2-1}$ in the q direction. These resonance curves for $m = \pm 1$, ± 2 and ± 3 are shown in figure 3 for $a = 0.1$, 0.4 and 0.7. (The resonance curves for $m > 0$ and those for $m < 0$ are physically the same. The difference is only an artefact of the choice of representation. To avoid duplicating these curves while maintaining a symmetric view with respect to $p = 0$, half of each curve in the larger $|p|$ region for given m is treated as original and represented by a solid line, whereas the remaining half in the smaller $|p|$ region is treated as a duplicate and denoted by a dotted line). For the type II resonance, (2.25*a*) on the $p-q$ plane becomes two straight lines representing $b < 0$ and $b > 0$ that cross each other diagonally. The resonance lines for $b < 0$ and $b > 0$ are shown by dashed and solid lines, respectively, in figure 3. We will find later (in §§ 3 and 4) that solid lines in figure 3 are associated with instability. These unstable resonance curves in the long-wave limit ($k \to 0$) are replotted in figure 4 and compared with those in the small-amplitude limit ($a \to 0$) obtained by Mclean (1982*a*,*b*). Here only the first quadrant on the $p-q$ plane is shown and duplication of resonance curves is avoided as stated in the parentheses above. We see that the unstable resonance curves of the two limits, the long-wave limit and the small-amplitude limit, bear some resemblance. A discussion of this point is given in § 5.2.

In §§ 3 and 4, the linear stability of finite-amplitude cnoidal wave solution (A 17), or the solution (2.5) with (2.6) for small k (with $a = O(1)$), is examined when the type I or II resonance condition is satisfied. Mackay & Saffman (1986) found that the necessary condition for losing stability in a Hamiltonian system is a collision of two eigenvalues of opposite signs (see also Cairns 1979 and Kharif & Ramamonjiarisoa 1990). Here the sign can be taken to be s_1 and s_2 for the eigenvalues (2.20*b*) and (2.21*b*), respectively. Thus, instability does not occur on the type II resonance lines with $b < 0$ because they represent a collision of eigenvalues of the same sign. In contrast, on the resonance lines of type I or II with $b > 0$, which represent a collision of eigenvalues of opposite signs, the above necessary condition is satisfied, and we do not know whether the instability occurs because the condition is not sufficient. Thus, the stability must be investigated in the latter case. Because removal of the former stable case does not greatly simplify the analysis, we will analyse the stability in both cases. The analyses are carried out separately for the two types of resonance: type I in § 3 and type II in § 4.

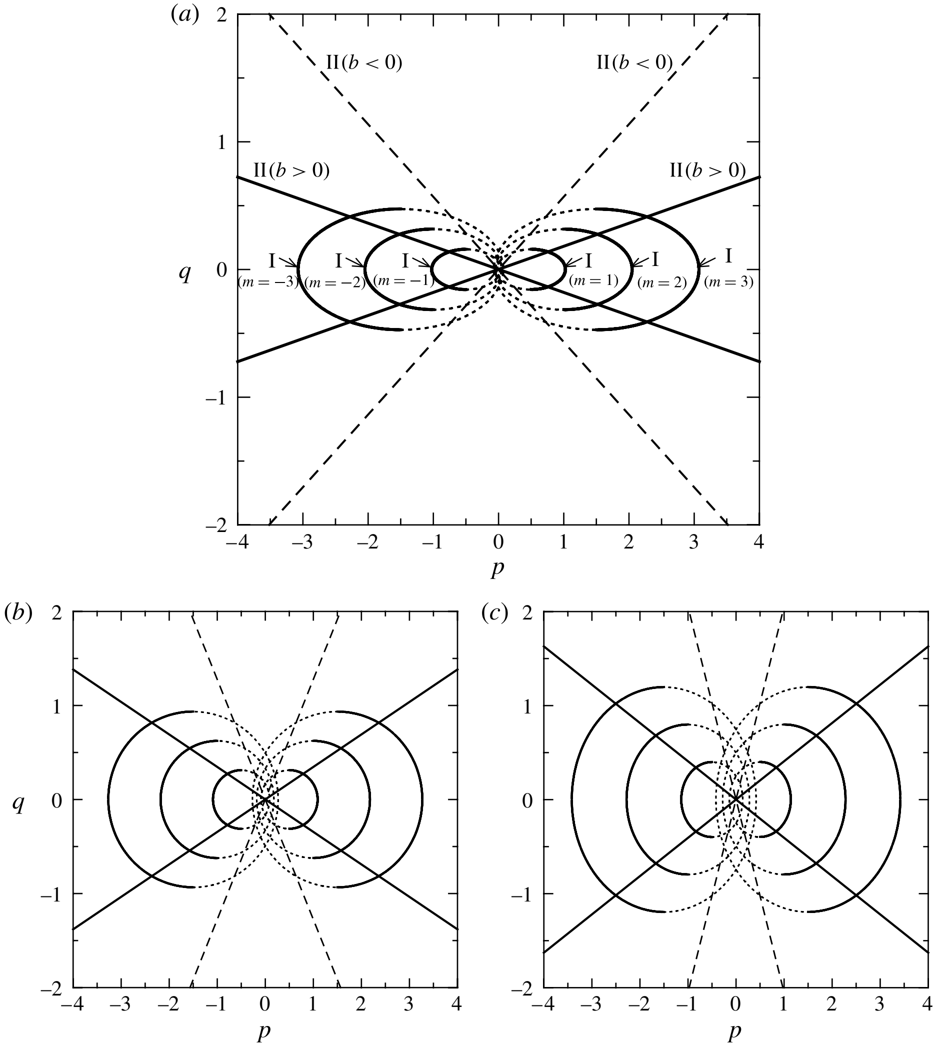

FIGURE 3. Resonance curves in the long-wave limit ($k \to 0$): (*a*) $a = 0.1$; (*b*) $a = 0.4$; (*c*) $a = 0.7$. The ellipses represent (2.23*a*) with $m = \pm 1$, ± 2 and ± 3 (denoted by the solid lines for $|p| > |m|/2$ and the dotted lines for $|p| < |m|/2$); straight diagonal lines represent (2.25*a*) with $b > 0$ (solid lines) and $b < 0$ (dashed lines). Part (*a*) shows, for all figures (*a*–*c*), the type of resonance (I or II) for each curve together with the order m of the resonance or the sign of b.

3. Asymptotic analysis (type I resonance)

We will prove in this section that the type I resonance (2.22) is associated with instability. To this end, we make an asymptotic analysis as $k \to 0$ (with $a = O(1)$) of (2.14)–(2.18). At the leading (zeroth) order in k, we have a solution of perturbation satisfying the type I resonance condition (2.22), and we obtain a result that this perturbation grows in time.

3.1. *Core solution*

Let us seek an asymptotic solution of (2.14)–(2.18) for small k when the type I resonance condition (2.22), which gives (2.23), is satisfied with a deviation of $O(k)$,

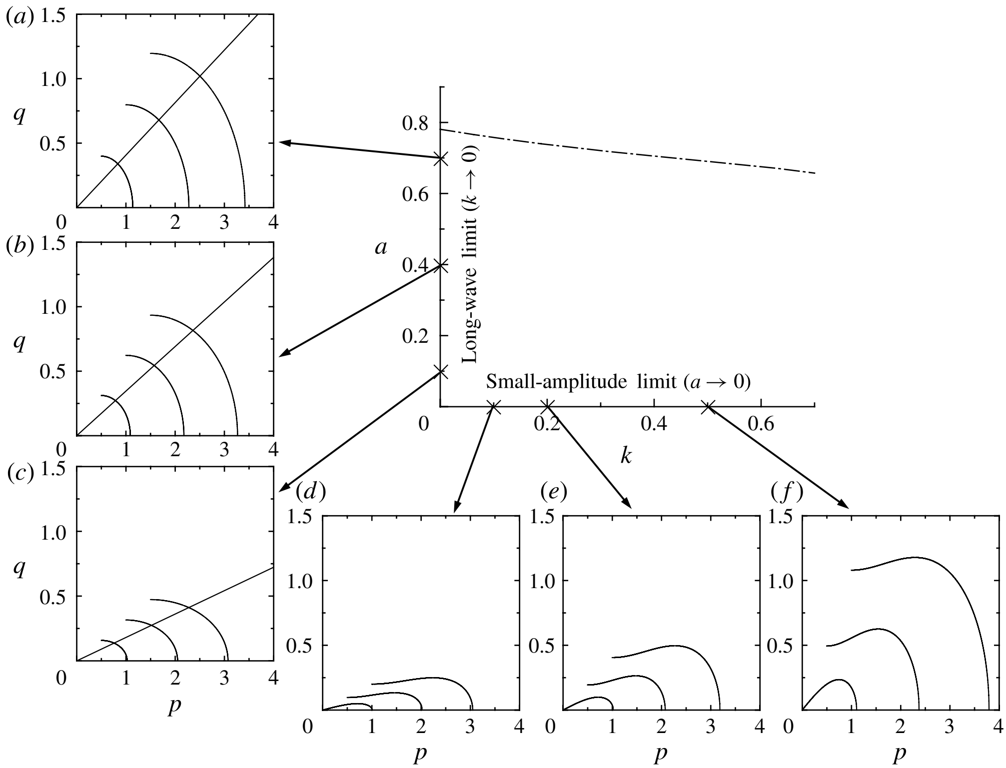

FIGURE 4. Unstable resonance curves in the long-wave limit ($k \rightarrow 0$) for (a) $a = 0.7$, (b) $a = 0.4$, (c) $a = 0.1$ and those in the small-amplitude limit ($a \rightarrow 0$) (reproduced from Mclean 1982*b*) for (d) $k = 0.1$, (e) $k = 0.2$, (f) $k = 0.5$. Only the first quadrant on the $p - q$ plane is shown, and the third resonance curve (the order of the interaction $N = 4$) in the small-amplitude limit is shifted in the positive p direction by unity from the conventional position used by Mclean. The figure on the upper-right corner is the parameter plane $a - k$ of the unperturbed wave solution (2.5) with (2.6), in which dash–dot line represents an upper limit for the superharmonically stable solution (Kataoka 2006).

i.e.

$$p = p_0 + kp_1, \quad q = q_0 + kq_1, \tag{3.1a,b}$$

where p_0 and q_0 are given constants of the order of unity satisfying (2.23*a*) or

$$\left(p_0 - \frac{m}{2}\right)^2 + \frac{v^2}{v^2 - 1} q_0^2 = \frac{m^2 v^2}{4}, \tag{3.1c}$$

and p_1 and q_1 are detuning parameters of the order of unity.

Putting aside the periodic condition (2.14), we first look for a solution $(\hat{\phi}, \hat{\eta})$ of (2.15)–(2.18) with moderate variation in x and z ($\partial\hat{\phi}/\partial x = O(\hat{\phi})$, $\partial\hat{\phi}/\partial z = O(\hat{\phi})$ and $\mathrm{d}\hat{\eta}/\mathrm{d}x = O(\hat{\eta})$) in the following power series of k:

$$\hat{\phi}_C = \hat{\phi}_{C0} + k\hat{\phi}_{C1} + k^2\hat{\phi}_{C2} + \cdots, \quad \hat{\eta}_C = \hat{\eta}_{C0} + k\hat{\eta}_{C1} + k^2\hat{\eta}_{C2} + \cdots, \tag{3.2a,b}$$

$$\sigma = \sigma_0 + k\sigma_1 + k^2\sigma_2 + \cdots, \tag{3.2c}$$

where the subscript C on $(\hat{\phi}, \hat{\eta})$ indicates the type of solution (core solution). Assuming that the two fundamental solutions (2.20*a*) and (2.21*a*) are of the same leading order, we can write the leading-order solution $(\hat{\phi}_{C0}, \hat{\eta}_{C0})$ as

$$(\hat{\phi}_{C0}, \hat{\eta}_{C0}) = \left(\frac{\partial \Phi}{\partial x} + \beta, \frac{\mathrm{d}H}{\mathrm{d}x}\right), \quad \sigma_0 = \frac{v^2 - 1}{v}\left(\frac{m}{2} - p_0\right), \tag{3.3a,b}$$

where β is an undetermined constant and σ_0 is obtained from (2.23*b*). Note that the unperturbed wave solution $(\bar{\phi}, \bar{\eta})$ is also expanded in k in (A 17) (cnoidal wave solution). Substituting (3.1)–(3.3) and (A 17) into (2.15)–(2.18) and arranging the same-order terms in k, we obtain a series of inhomogeneous equations for $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn})$ $(n = 1, 2 \ldots)$,

$$\nabla_\perp^2 \hat{\phi}_{Cn} = F_n \quad \text{for } 0 < z < H, \tag{3.4}$$

$$L_{K0}\left[\hat{\phi}_{Cn}, \hat{\eta}_{Cn}\right] = G_n \quad \text{at } z = H, \tag{3.5}$$

$$L_{D0}\left[\hat{\phi}_{Cn}, \hat{\eta}_{Cn}\right] = I_n \quad \text{at } z = H, \tag{3.6}$$

$$\frac{\partial \hat{\phi}_{Cn}}{\partial z} = 0 \quad \text{at } z = 0. \tag{3.7}$$

Here F_n, G_n, and I_n are the following inhomogeneous terms:

$$F_n = \begin{cases} -2\mathrm{i}p_0 \dfrac{\partial \hat{\phi}_{C0}}{\partial x} \quad (n = 1), \\ -2\mathrm{i}\left(p_0 \dfrac{\partial \hat{\phi}_{C1}}{\partial x} + p_1 \dfrac{\partial \hat{\phi}_{C0}}{\partial x}\right) + (p_0^2 + q_0^2)\hat{\phi}_{C0} \quad (n = 2), \ldots, \end{cases} \tag{3.8a}$$

$$G_n = \begin{cases} \mathrm{i}\sigma_0 \hat{\eta}_{C0} + \mathrm{i}p_0\left[-\dfrac{\mathrm{d}H}{\mathrm{d}x}\hat{\phi}_{C0} + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\hat{\eta}_{C0}\right] - L_{K1}\left[\hat{\phi}_{C0}, \hat{\eta}_{C0}\right] \quad (n = 1), \\ \mathrm{i}\left(\sigma_0 \hat{\eta}_{C1} + \sigma_1 \hat{\eta}_{C0}\right) + \mathrm{i}p_0\left[-\dfrac{\mathrm{d}H}{\mathrm{d}x}\left(\hat{\phi}_{C1} + \dfrac{\partial \hat{\phi}_{C0}}{\partial z}H_1\right) - \dfrac{\mathrm{d}H_1}{\mathrm{d}x}\hat{\phi}_{C0}\right. \\ \quad \left. + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\hat{\eta}_{C1} + \left(c_1 - \dfrac{\partial \Phi_1}{\partial x} - \dfrac{\partial^2 \Phi}{\partial x \partial z}H_1\right)\hat{\eta}_{C0}\right] \\ \quad + \mathrm{i}p_1\left[-\dfrac{\mathrm{d}H}{\mathrm{d}x}\hat{\phi}_{C0} + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\hat{\eta}_{C0}\right] - L_{K1}\left[\hat{\phi}_{C1}, \hat{\eta}_{C1}\right] \\ \quad - L_{K2}\left[\hat{\phi}_{C0}, \hat{\eta}_{C0}\right] \quad (n = 2), \ldots, \end{cases} \tag{3.8b}$$

$$I_n = \begin{cases} \mathrm{i}\sigma_0 \hat{\phi}_{C0} + \mathrm{i}p_0\left(v - \dfrac{\partial \Phi}{\partial x}\right)\hat{\phi}_{C0} - L_{D1}\left[\hat{\phi}_{C0}, \hat{\eta}_{C0}\right] \quad (n = 1), \\ \mathrm{i}\left[\sigma_0\left(\hat{\phi}_{C1} + \dfrac{\partial \hat{\phi}_{C0}}{\partial z}H_1\right) + \sigma_1 \hat{\phi}_{C0}\right] + \mathrm{i}p_0\left[\left(v - \dfrac{\partial \Phi}{\partial x}\right)\left(\hat{\phi}_{C1} + \dfrac{\partial \hat{\phi}_{C0}}{\partial z}H_1\right)\right. \\ \quad \left. + \left(c_1 - \dfrac{\partial \Phi_1}{\partial x} - \dfrac{\partial^2 \Phi}{\partial x \partial z}H_1\right)\hat{\phi}_{C0}\right] \\ \quad + \mathrm{i}p_1\left(v - \dfrac{\partial \Phi}{\partial x}\right)\hat{\phi}_{C0} - L_{D1}\left[\hat{\phi}_{C1}, \hat{\eta}_{C1}\right] - L_{D2}\left[\hat{\phi}_{C0}, \hat{\eta}_{C0}\right] (n = 2), \ldots, \end{cases} \tag{3.8c}$$

and L_{Kn} and L_{Dn} are the nth-order components in k of the linear operators L_K and L_D evaluated at $z=\bar{\eta}=H+kH_1+\cdots$. Specific forms of L_{Kn} and L_{Dn} for $n=0$ and 1 are presented in (B 4*a–d*).

The homogeneous part of the above inhomogeneous equations (3.4)–(3.7) satisfies

$$\int_{-\infty}^{\infty}\left\{\int_0^H \frac{\partial \Phi}{\partial x}\nabla_\perp^2\hat{\phi}\mathrm{d}z+\left[\frac{\partial \Phi}{\partial x}L_{K0}\left[\hat{\phi},\hat{\eta}\right]-\frac{\mathrm{d}H}{\mathrm{d}x}L_{D0}\left[\hat{\phi},\hat{\eta}\right]\right]_{z=H}\right\}\mathrm{d}x=0, \tag{3.9}$$

where the quantities in the square brackets with subscript $z=H$ are evaluated at $z=H$; therefore, the inhomogeneous terms, F_n, G_n and I_n on the right-hand sides of (3.4)–(3.7) must satisfy the solvability condition:

$$\int_{-\infty}^{\infty}\left\{\int_0^H \frac{\partial \Phi}{\partial x}F_n\mathrm{d}z+\left[\frac{\partial \Phi}{\partial x}G_n-\frac{\mathrm{d}H}{\mathrm{d}x}I_n\right]_{z=H}\right\}\mathrm{d}x=0 \quad (n=1,2,\ldots). \tag{3.10}$$

For $n=1$, (3.10) is identically satisfied and a solution of (3.4)–(3.7) is

$$\hat{\phi}_{C1}=\mathrm{i}\sigma_0\left(\frac{\partial \Phi}{\partial v}+\Phi_B\beta\right)-\mathrm{i}p_0x\hat{\phi}_{C0}+\frac{\partial \Phi_1}{\partial x}+\Phi_U\gamma, \tag{3.11a}$$

$$\hat{\eta}_{C1}=\mathrm{i}\sigma_0\left(\frac{\partial H}{\partial v}+H_B\beta\right)-\mathrm{i}p_0x\hat{\eta}_{C0}+\frac{\mathrm{d}H_1}{\mathrm{d}x}+H_U\gamma, \tag{3.11b}$$

where γ is a new undetermined constant and (B 1) is used to derive (3.11). Here $\partial(\Phi,H)/\partial v$ represents the derivative of (Φ,H) with respect to v while keeping x and z constant, and (Φ_B,H_B) and (Φ_U,H_U) are defined by (A 14*a,b*) and (A 15*a,b*), respectively. From (3.11*a*) and (A 7*a*), we find

$$\left[\hat{\phi}_{C1}-x\frac{\partial\hat{\phi}_{C1}}{\partial x}\right]_{x\to-\infty}=\left[\hat{\phi}_{C1}-x\frac{\partial\hat{\phi}_{C1}}{\partial x}\right]_{x\to\infty}+\mathrm{i}\sigma_0\frac{\mathrm{d}\Omega}{\mathrm{d}v}+\mathrm{i}\sigma_0\Omega_B\beta+\Omega_U\gamma, \tag{3.12}$$

$$\left[\frac{\partial\hat{\phi}_{C1}}{\partial x}\right]_{x\to\pm\infty}=-\mathrm{i}p_0\beta+\gamma, \tag{3.13}$$

where Ω, Ω_B and Ω_U are defined by (A 7*b*), (A 14*c*) and (A 15*c*), respectively, and the quantities in the square brackets with subscript $x\to-\infty$, ∞ and $\pm\infty$ are evaluated as $x\to-\infty$, ∞ and $\pm\infty$, respectively. These quantities in (3.12) and (3.13) are used in the matching procedure conducted in § 3.3.

For $n=2$, the solvability condition (3.10) becomes, with the aid of (3.9) and (B 1)–(B 3),

$$-\frac{\sigma_0^2}{v}\frac{\mathrm{d}E}{\mathrm{d}v}+q_0^2E=\left[\sigma_0^2\left(\frac{\mathrm{d}M}{\mathrm{d}v}+v\frac{\mathrm{d}\Omega}{\mathrm{d}v}\right)-vMq_0^2\right]\beta-\mathrm{i}(v^2-1)\sigma_0\frac{\mathrm{d}\Omega}{\mathrm{d}v}\gamma, \tag{3.14}$$

where M and E are defined by (A 7*c,e*). When (3.14) is satisfied, a solution for $n=2$ exists and satisfies

$$\left[\hat{u}_{C2}-x\frac{\partial\hat{u}_{C2}}{\partial x}\right]_{x\to-\infty}=\left[\hat{u}_{C2}-x\frac{\partial\hat{u}_{C2}}{\partial x}\right]_{x\to\infty}+\sigma_0^2\frac{\mathrm{d}M}{\mathrm{d}v}-q_0^2vM$$
$$+\left(\sigma_0^2M_B-q_0^2M\right)\beta-\mathrm{i}\sigma_0M_U\gamma, \tag{3.15}$$

where M_B and M_U are given by (A 14*d*) and (A 15*d*), respectively, and

$$\hat{u}_{Cn} = \int_0^H \left(\frac{\partial \hat{\phi}_{Cn}}{\partial x} + \mathrm{i}p_0 \hat{\phi}_{Cn-1} \right) \mathrm{d}z - \upsilon \hat{\eta}_{Cn}. \tag{3.16}$$

We can derive (3.15) by integrating (3.4) for $n = 1$ and 2 with respect to x and z and using (3.5) for $n = 1$ and 2, noting that the terms in the square brackets after $\mathrm{i}p_0$ for G_2 in (3.8*b*) are equal to $-L_{K0}[x\hat{\phi}_{C1}, x\hat{\eta}_{C1}] - L_{K1}[x\hat{\phi}_{C0}, x\hat{\eta}_{C0}] + x(L_{K0}[\hat{\phi}_{C1}, \hat{\eta}_{C1}] + L_{K1}[\hat{\phi}_{C0}, \hat{\eta}_{C0}])$.

Now it is evident from (3.13) that the above core solution becomes inappropriate in the far region $|x| = O(1/k)$, where the periodic condition (2.14) is imposed, because $\hat{\phi}_{C1}$ becomes $O(1/k)$, and $k\hat{\phi}_{C1}$ has the same order of magnitude as the lower-order solution $\hat{\phi}_{C0}$. This inappropriateness arises because we do not take into account the balance between the terms including $\mathrm{i}\sigma_0$ and those with $-\upsilon\partial/\partial x$ in (3.5) and (3.6). To achieve a balance between these terms, we need to introduce a reduced coordinate with respect to x and seek a solution whose variation is slow in x. This solution will be called a far-field solution. By matching the present core solution (3.2*a,b*), with the far-field solution, an overall solution of (2.15)–(2.18) that satisfies the periodic condition (2.14) can be constructed. Sections 3.2 and 3.3 treat the far-field solution and matching of the two solutions, respectively.

3.2. *Far-field solution*

We introduce a reduced coordinate with respect to x,

$$X = kx. \tag{3.17}$$

We then seek a solution $(\hat{\phi}, \hat{\eta})$ of (2.14)–(2.18) with moderate variation in X and z $[\partial\hat{\phi}/\partial X = O(\hat{\phi}),\ \partial\hat{\phi}/\partial z = O(\hat{\phi})$ and $\mathrm{d}\hat{\eta}/\mathrm{d}X = O(\hat{\eta})]$ in the following power series of k:

$$\hat{\phi}_F(X, z) = \hat{\phi}_{F0} + k\hat{\phi}_{F1} + k^2\hat{\phi}_{F2} + \cdots, \tag{3.18a}$$

$$\hat{\eta}_F(X) = k\hat{\eta}_{F1} + k^2\hat{\eta}_{F2} + \cdots, \tag{3.18b}$$

where the subscript F indicates the far-field solution. The series of (3.18) start at $O(1)$ and $O(k)$ for $\hat{\phi}_F$ and $\hat{\eta}_F$, respectively, because the core solution has non-zero values as $x \to \pm\infty$ from these orders (see (3.3*a*) and (3.11*b*)).

Substituting (3.1), (3.2*c*), (3.17)–(3.18) and (A 17) into (2.14)–(2.18) and arranging the same-order terms in k, we obtain a series of sets of equations for $\hat{\phi}_{Fn}$ $(n = 0, 1, 2, \ldots)$. For $n = 0$ and 1, they are homogeneous ($\partial^2\hat{\phi}_{Fn}/\partial z^2 = 0$ for $0 < z < 1$ and $\partial\hat{\phi}_{Fn}/\partial z = 0$ at $z = 0$ and 1), and have a solution independent of z satisfying the periodic condition at $X = \pm\pi$, i.e.

$$\hat{\phi}_{Fn} = \hat{\phi}_{Fn}(X) \quad (n = 0 \text{ and } 1), \tag{3.19}$$

where

$$\hat{\phi}_{Fn}(-\pi) = \hat{\phi}_{Fn}(\pi), \quad \frac{\mathrm{d}\hat{\phi}_{Fn}}{\mathrm{d}X}(-\pi) = \frac{\mathrm{d}\hat{\phi}_{Fn}}{\mathrm{d}X}(\pi). \tag{3.20}$$

For $n=2$ and 3, the equations are inhomogeneous, i.e.

$$\frac{\partial^2 \hat{\phi}_{Fn}}{\partial z^2} = J_n \equiv \left[q_0^2 - \left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)^2\right]\hat{\phi}_{Fn-2} + 2\left[q_0 q_1 - \mathrm{i}p_1\left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)\right]\hat{\phi}_{Fn-3} \quad \text{for } 0 < z < 1, \tag{3.21}$$

$$\frac{\partial \hat{\phi}_{Fn}}{\partial z} = 0 \quad \text{at } z = 0, \tag{3.22}$$

$$\frac{\partial \hat{\phi}_{Fn}}{\partial z} = L_n \equiv -\left[\mathrm{i}\sigma_0 + v\left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)\right]\hat{\eta}_{Fn-1} - \left[\mathrm{i}(\sigma_1 + vp_1) + \left(c_1 - \frac{\partial \Phi_1}{\partial x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)\right]\hat{\eta}_{Fn-2} \quad \text{at } z = 1, \tag{3.23}$$

$$\hat{\phi}_{Fn}(-\pi, z) = \hat{\phi}_{Fn}(\pi, z), \tag{3.24}$$

$$(n = 2 \text{ and } 3)$$

where

$$\hat{\eta}_{Fn-1} = \left[\mathrm{i}\sigma_0 + v\left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)\right]\hat{\phi}_{Fn-2} + \left[\mathrm{i}(\sigma_1 + vp_1) + \left(c_1 - \frac{\partial \Phi_1}{\partial x}\right)\left(\frac{\mathrm{d}}{\mathrm{d}X} + \mathrm{i}p_0\right)\right]\hat{\phi}_{Fn-3}, \tag{3.25}$$

and $\hat{\phi}_{F-1} = \hat{\eta}_{F0} = 0$. For (3.21)–(3.24) to have a solution, their inhomogeneous terms J_n and L_n on the right-hand sides of (3.21) and (3.23) must satisfy the solvability condition:

$$\int_0^1 J_n \, \mathrm{d}z = L_n. \tag{3.26}$$

This condition (3.26) for $n=2$, subject to the periodic condition (3.20) and matching with the leading-order core solution (3.3*a*) (which is made here in advance of § 3.3 for the sake of a concise analytical process), leads to

$$\hat{\phi}_{F0} = \beta - \bar{\beta} + \bar{\beta}\mathrm{e}^{-\mathrm{i}mX}, \tag{3.27}$$

where $\bar{\beta}$ is a new undetermined constant, and (3.1*c*) and (3.3*b*) are used to derive (3.27). The solution (3.27) represents two infinitesimal long waves whose wavenumber vectors differ by a factor of m with respect to that of the unperturbed wave. Here $\hat{\eta}_{F1}$ is obtained from (3.25) as

$$\hat{\eta}_{F1} = \mathrm{i}(\sigma_0 + vp_0)(\beta - \bar{\beta}) + \mathrm{i}\,[\sigma_0 + v(p_0 - m)]\,\bar{\beta}\mathrm{e}^{-\mathrm{i}mX}. \tag{3.28}$$

For $n=3$, the solvability condition (3.26), subject to the periodic condition (3.20), determines $\hat{\phi}_{F1}$ as

$$\hat{\phi}_{F1} = \left[\frac{\sigma_1 - \Delta_I}{c_x}(\beta - \bar{\beta}) + \frac{\sigma_1 - \bar{\Delta}_I}{\bar{c}_x}\bar{\beta}\mathrm{e}^{-\mathrm{i}mX}\right]\mathrm{i}\chi(X), \tag{3.29}$$

where $\chi(X)$, Δ_I and $\bar{\Delta}_I$ are defined by

$$\chi(X) = \begin{cases} X + \pi & \text{for } -\pi < X < 0, \\ X - \pi & \text{for } 0 < X < \pi, \end{cases} \tag{3.30}$$

$$\Delta_I = c_x p_1 + c_y q_1 - \left(c_1 - \frac{\Omega}{2\pi}\right) p_0, \quad \bar{\Delta}_I = \bar{c}_x p_1 + \bar{c}_y q_1 - \left(c_1 - \frac{\Omega}{2\pi}\right)(p_0 - m), \tag{3.31}$$

and (c_x, c_y) and $(\bar{c}_x, \bar{c}_y)$ are the group velocities in the (x, y) directions of linear long waves having the normalized wavenumber vectors (p_0, q_0) and (p_0-m, q_0), respectively. That is,

$$c_x \equiv \left[\frac{\partial \sigma_L(p, q)}{\partial p}\right]_{(p,q)=(p_0,q_0)} = -\frac{v\sigma_0 + (v^2-1)p_0}{\sigma_0 + vp_0}(<0), \tag{3.32a}$$

$$c_y \equiv \left[\frac{\partial \sigma_L(p, q)}{\partial q}\right]_{(p,q)=(p_0,q_0)} = \frac{q_0}{\sigma_0 + vp_0}, \tag{3.32b}$$

$$\bar{c}_x \equiv \left[\frac{\partial \sigma_L(p, q)}{\partial p}\right]_{(p,q)=(p_0-m,q_0)} = -\frac{v\sigma_0 + (v^2-1)(p_0-m)}{\sigma_0 + v(p_0-m)}(<0), \tag{3.32c}$$

$$\bar{c}_y \equiv \left[\frac{\partial \sigma_L(p, q)}{\partial q}\right]_{(p,q)=(p_0-m,q_0)} = \frac{q_0}{\sigma_0 + v(p_0-m)}. \tag{3.32d}$$

Here $\hat{\eta}_{F2}$ is obtained from (3.25) as

$$\begin{aligned}\hat{\eta}_{F2} = &-\left[(\sigma_0 + vp_0)(\sigma_1 - \Delta_I)\frac{\chi(X)}{c_x} + A_1(p_0)\right](\beta - \bar{\beta}) \\ &+ \left[-(\sigma_0 + v(p_0-m))\left(\sigma_1 - \bar{\Delta}_I\right)\frac{\chi(X)}{\bar{c}_x} + A_1(p_0-m)\right]\bar{\beta}\mathrm{e}^{-\mathrm{i}mX},\end{aligned} \tag{3.33a}$$

where

$$A_1(p) = \frac{2\mathrm{i}}{(v^2-1)m}\left\{p\left[\sigma_1 + \left(c_1 - \frac{\Omega}{2\pi}\right)p\right] - vq_0q_1\right\}. \tag{3.33b}$$

3.3. *Matching*

Let us connect the core solution $(\hat{\phi}_C, \hat{\eta}_C)$ and the far-field solution $(\hat{\phi}_F, \hat{\eta}_F)$. In the core region, the far-field solution $(\hat{\phi}_{Fn}, \hat{\eta}_{Fn})$ is expanded in power series of X (or kx):

$$\hat{h}_{Fn} = \left(\hat{h}_{Fn}\right)_0 + kx\left(\frac{\partial \hat{h}_{Fn}}{\partial X}\right)_0 + \frac{k^2x^2}{2}\left(\frac{\partial^2 \hat{h}_{Fn}}{\partial X^2}\right)_0 + \cdots, \tag{3.34}$$

where $\hat{h}$ represents $(\hat{\phi}, \hat{\eta})$, and the quantities in parentheses with subscript 0, or $(\cdots)_0$, are evaluated at $X = 0$. With this reordering, we collect the same orders of k and obtain a reordered form (say, $(\hat{\phi}^*_{Fn}, \hat{\eta}^*_{Fn})$) of $(\hat{\phi}_{Fn}, \hat{\eta}_{Fn})$. Matching is accomplished if

$$\left[\hat{\phi}_{Cn}\right]_{x\to\pm\infty} = \hat{\phi}^*_{Fn}, \quad \left[\hat{\eta}_{Cn}\right]_{x\to\pm\infty} = \hat{\eta}^*_{Fn} \tag{3.35}$$

are satisfied, with the differences being smaller than any inverse power of x.

For $n = 0$, matching is already achieved by (3.3*a*) and (3.27). For $n = 1$, because $\hat{\phi}^*_{F1} = (\hat{\phi}_{F1})_0 + x(\mathrm{d}\hat{\phi}_{F0}/\mathrm{d}X)_0$, the matching conditions consist of two different types of terms, those independent of x and those proportional to x. From those independent of x, we have

$$\left[\hat{\phi}_{C1} - x\frac{\partial \hat{\phi}_{C1}}{\partial x}\right]_{x\to\pm\infty} = \mp\mathrm{i}\pi\left[\frac{\sigma_1 - \Delta_I}{c_x}(\beta - \bar{\beta}) + \frac{\sigma_1 - \bar{\Delta}_I}{\bar{c}_x}\bar{\beta}\right], \tag{3.36}$$

where both upper or both lower signs should be chosen for the double signs of (3.36) (and (3.38*a*) below). From those proportional to x and (3.27), we find

$$\left[\frac{\partial \hat{\phi}_{C1}}{\partial x}\right]_{x\to\pm\infty} = -\mathrm{i}m\bar{\beta}. \tag{3.37}$$

For $n=2$, because $\hat{\phi}^*_{F2} = (\hat{\phi}_{F2})_0 + x\,(\mathrm{d}\hat{\phi}_{F1}/\mathrm{d}X)_0 + (x^2/2)\,(\mathrm{d}^2\hat{\phi}_{F0}/\mathrm{d}X^2)_0$, three different types of terms appear: those independent of x, those proportional to x and those proportional to x^2. The conditions among those proportional to x contribute to the determination of unknowns at this order and they become, in terms of $\hat{u}_{C2}$ as defined by (3.16),

$$\left[\hat{u}_{C2} - x\frac{\partial \hat{u}_{C2}}{\partial x}\right]_{x\to\pm\infty} = \left[-A_2(p_0) \pm \frac{\pi(1-v^2)m}{2}\frac{\sigma_1 - \Delta_I}{c_x}\right](\beta - \bar{\beta}) + \left[A_2(p_0 - m) \mp \frac{\pi(1-v^2)m}{2}\frac{\sigma_1 - \bar{\Delta}_I}{\bar{c}_x}\right]\bar{\beta}, \tag{3.38a}$$

where
$$A_2(p) = \frac{2\mathrm{i}}{m}\left\{\frac{\sigma_0}{v^2-1}\left[\sigma_1 + \left(c_1 - \frac{\Omega}{2\pi}\right)p\right] + q_0 q_1\right\} - \mathrm{i}p_1. \tag{3.38b}$$

The matching conditions for $\hat{\eta}$ are automatically satisfied if (3.37) and (3.38) are satisfied.

Now substituting (3.36)–(3.38) into (3.12), (3.13) and (3.15), we have four equations, (3.12)–(3.15), for β, $\bar{\beta}$, γ and σ_1. The solution for σ_1 is, with the aid of (B 6),

$$\sigma_1 = \begin{cases} \dfrac{1}{2}\left[\Delta_I + \bar{\Delta}_I + mR_- \pm \mathrm{i}\sqrt{-\left(\Delta_I - \bar{\Delta}_I + mR_+\right)^2 + m^2R_{cr}^2}\right] \\ \quad \text{for } \left|\dfrac{\Delta_I - \bar{\Delta}_I}{m} + R_+\right| < R_{cr}, \\ \dfrac{1}{2}\left[\Delta_I + \bar{\Delta}_I + mR_- \pm \sqrt{\left(\Delta_I - \bar{\Delta}_I + mR_+\right)^2 - m^2R_{cr}^2}\right] \\ \quad \text{for } \left|\dfrac{\Delta_I - \bar{\Delta}_I}{m} + R_+\right| > R_{cr}, \end{cases} \tag{3.39}$$

where Δ_I and $\bar{\Delta}_I$ are defined by (3.31). Here $R_\pm$ and R_{cr} are

$$R_\pm = c_x\left(\frac{S+\bar{S}}{2} - \Gamma^2 W\right) \pm \bar{c}_x\left[\frac{S+\bar{S}}{2} - \bar{\Gamma}^2 W + \frac{3\sigma_0}{2\pi(v^2-1)m}\left(\Omega - v\frac{\mathrm{d}\Omega}{\mathrm{d}v}\right)\right], \tag{3.40a}$$

$$R_{cr} = 2\sqrt{c_x\bar{c}_x}\left|\frac{S-\bar{S}}{2} + \Gamma\bar{\Gamma}W\right|, \tag{3.40b}$$

where both upper or both lower signs should be chosen for the double signs in (3.40*a*) and

$$S = \frac{1}{4\pi}\left[\frac{3\,(\sigma_0 + vp_0)}{m}\left(\Omega - v\frac{\mathrm{d}\Omega}{\mathrm{d}v}\right) + (v^2-1)\frac{\mathrm{d}\Omega}{\mathrm{d}v}\right], \tag{3.40c}$$

$$\bar{S} = \frac{1}{v(v^2-1)m^2\pi}\left[\sigma_0\,(\sigma_0 + vp_0)\left(2vM - \left(\frac{v^2}{2}+1\right)\frac{\mathrm{d}M}{\mathrm{d}v}\right) + \sigma_0^2\frac{\mathrm{d}M}{\mathrm{d}v} - q_0^2 vM\right], \tag{3.40d}$$

$$\Gamma = \frac{\sigma_0[v\sigma_0 + (v^2-1)p_0]}{q_0^2}\frac{\mathrm{d}\Omega}{\mathrm{d}v} + \frac{\sigma_0^2}{q_0^2}\frac{\mathrm{d}M}{\mathrm{d}v} - vM, \tag{3.40e}$$

$$\bar{\Gamma} = -\frac{\sigma_0[v\sigma_0 + (v^2-1)p_0]}{q_0^2}\frac{\mathrm{d}\Omega}{\mathrm{d}v} + \frac{\sigma_0^2}{q_0^2}\frac{\mathrm{d}M}{\mathrm{d}v} - vM, \tag{3.40f}$$

$$W = \frac{vq_0^4}{2\pi(v^2-1)m^2\left(\sigma_0^2 - \sigma_S(q_0)^2\right)\mathrm{d}E/\mathrm{d}v}, \tag{3.40g}$$

with $\sigma_S(q)$ in (3.40g) being defined by (2.21b).

From (3.39), a solution σ_1 with a positive imaginary part exists if

$$\left|\frac{\Delta_I - \bar{\Delta}_I}{m} + R_+\right| < R_{cr} \quad \left(\text{or}\ \left|\frac{(c_x-\bar{c}_x)p_1 + (c_y-\bar{c}_y)q_1}{m} + R_+ - \left(c_1 - \frac{\Omega}{2\pi}\right)\right| < R_{cr}\right). \tag{3.41}$$

Because $R_{cr} > 0$, an instability band of width $O(k)$ exists on a $p-q$ plane along all of the resonance curves (2.23a). These resonance curves are shown in figure 3 by the solid and dotted lines labelled 'I'. (See the statement in the parentheses after (2.26) for the meaning of the solid and dotted lines.) Thus, we have proved that the type I resonance (2.22) is associated with instability.

According to (3.39), the local maximum growth rate $\max(\mathrm{Im}[\sigma])$ across the instability band is $k|m|R_{cr}/2$, which is proportional to $|m|$. Thus, the dominant instability occurs at higher-order resonances of large $|m|$. This is in contrast to the case of small-amplitude waves, for which the dominant instability is always associated with the lowest-order resonance (Mclean 1982a,b). Our theoretical result is consistent with the numerical results of Francius & Kharif (2006), who found that in shallow water the dominant instability is often associated with higher-order resonances such as interactions among five, six or more waves. We also mention that this instability disappears in the small-amplitude limit $a \to 0$ because $R_{cr} \to 0$ as $a \to 0$. This result is consistent with previous work based on the Kadomtsev–Petviashvili (KP) equation (Kadomtsev & Petviashvili 1970) that the Korteweg–de Vries (KdV) cnoidal wave solution is stable (Kuznetsov, Spektor & Fal'kovich 1984; Spektor 1988).

4. Asymptotic analysis (type II resonance)

We consider the type II resonance (2.24). We will prove in this section that the type II resonance with $b > 0$ (see (2.26)) is associated with instability, whereas that with $b < 0$ is not. To this end, we make an asymptotic analysis of (2.14)–(2.18) for small k (with $a = O(1)$).

For the type II resonance $\sigma_L(p, q) = \sigma_S(q)$ to occur, $\sigma = \sigma_L(p, q)$ and $\sigma = \sigma_S(q)$ should be derived in the course of the analysis. According to (3.14), the latter condition $\sigma = \sigma_S(q)(= \pm\sqrt{vEq^2/(\mathrm{d}E/\mathrm{d}v)})$ appears when the fundamental homogeneous solution $\hat{\phi} = \beta$ in (3.3a) is $o(1)$ because the terms on the right-hand side of (3.14) including β or γ then vanish. The former condition, $\sigma = \sigma_L(p, q)$, arises when the fundamental homogeneous solution $\hat{\phi} = \beta$ in (3.3a) is larger than $O(k)$ because the core solution $\hat{\phi}_C$ then approaches a constant value β as $x \to \pm\infty$, and matching with the far-field solution yields a constant $\hat{\phi}_F$ whose linear dispersion relationship (obtained from the solvability condition (3.26)) is $\sigma = \sigma_L(p, q)$. Thus, the fundamental solution $\hat{\phi} = \beta$ in (3.3a) should be smaller than $O(1)$ and larger than $O(k)$. We choose this order to be $O(k^{1/2})$. The validity of this estimate is confirmed if the following analysis is consistent.

4.1. *Core solution*

Let us seek an asymptotic solution of (2.14)–(2.18) for small k when the type II resonance condition (2.24), which yields (2.25), is satisfied with a deviation of $O(k^{1/2})$, i.e.

$$p = p_0 + k^{1/2} p_{1/2}, \quad q = q_0 + k^{1/2} q_{1/2}, \tag{4.1a,b}$$

where p_0 and q_0 are given constants of the order of unity satisfying (2.25a), or

$$p_0 = \pm \frac{v+b}{\sqrt{(v^2-1)(b^2-1)}} q_0, \tag{4.1c}$$

where b is defined by (2.26), and $p_{1/2}$ and $q_{1/2}$ are detuning parameters of the order of unity.

As in § 3.1, putting aside the periodic condition (2.14), we first look for a solution $(\hat{\phi}, \hat{\eta})$ of (2.15)–(2.18) having moderate variation in x and z $[\partial\hat{\phi}/\partial x = O(\hat{\phi})$, $\partial\hat{\phi}/\partial z = O(\hat{\phi})$ and $\mathrm{d}\hat{\eta}/\mathrm{d}x = O(\hat{\eta})]$ in the following power series of $k^{1/2}$:

$$\hat{\phi}_C = \hat{\phi}_{C0} + k^{1/2}\hat{\phi}_{C1/2} + k\hat{\phi}_{C1} + \cdots, \quad \hat{\eta}_C = \hat{\eta}_{C0} + k^{1/2}\hat{\eta}_{C1/2} + k\hat{\eta}_{C1} + \cdots, \tag{4.2a,b}$$

$$\sigma = \sigma_0 + k^{1/2}\sigma_{1/2} + k\sigma_1 + \cdots, \tag{4.2c}$$

where the subscript C on $(\hat{\phi}, \hat{\eta})$ indicates the type of solution (core solution). According to the discussion in the first paragraph of § 4 and (2.25b),

$$(\hat{\phi}_{C0}, \hat{\eta}_{C0}) = \left(\frac{\partial \Phi}{\partial x}, \frac{\mathrm{d}H}{\mathrm{d}x}\right), \quad (\hat{\phi}_{C1/2}, \hat{\eta}_{C1/2}) = (\beta, 0), \quad \sigma_0 = \frac{1-v^2}{v+b} p_0, \tag{4.3a–c}$$

where β is an undetermined constant. Substituting (4.1)–(4.3) and (A 17) into (2.15)–(2.18) and arranging the same-order terms in $k^{1/2}$, we obtain a series of inhomogeneous equations (3.4)–(3.7) for $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn})$ $(n = 1, 3/2, 2, \ldots)$ whose inhomogeneous terms F_n, G_n and I_n are

$$F_n = \begin{cases} -2\dfrac{\partial \phi_{Pn-1}}{\partial x} \quad (n = 1 \text{ and } 3/2), \\ -2\dfrac{\partial \phi_{Pn-1}}{\partial x} + (p_0^2 + q_0^2)\hat{\phi}_{Cn-2} + 2(p_0 p_{1/2} + q_0 q_{1/2})\hat{\phi}_{Cn-5/2} \\ \qquad (n = 2 \text{ and } 5/2), \ldots, \end{cases} \tag{4.4a}$$

$$G_n = \begin{cases} \mathrm{i}(\sigma_0 \hat{\eta}_{Cn-1} + \sigma_{1/2}\hat{\eta}_{Cn-3/2}) - \dfrac{\mathrm{d}H}{\mathrm{d}x}\phi_{Pn-1} + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\eta_{Pn-1} \\ \quad -L_{K1}\left[\hat{\phi}_{Cn-1}, \hat{\eta}_{Cn-1}\right] \quad (n = 1 \text{ and } 3/2), \\ \mathrm{i}(\sigma_0 \hat{\eta}_{Cn-1} + \sigma_{1/2}\hat{\eta}_{Cn-3/2} + \sigma_1 \hat{\eta}_{Cn-2} + \sigma_{3/2}\hat{\eta}_{Cn-5/2}) \\ \quad -\dfrac{\mathrm{d}H}{\mathrm{d}x}\left(\phi_{Pn-1} + \dfrac{\partial \phi_{Pn-2}}{\partial z} H_1\right) - \dfrac{\mathrm{d}H_1}{\mathrm{d}x}\phi_{Pn-2} \\ \quad + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\eta_{Pn-1} + \left(c_1 - \dfrac{\partial \Phi_1}{\partial x} - \dfrac{\partial^2 \Phi}{\partial x \partial z} H_1\right)\eta_{Pn-2} \\ \quad -L_{K1}\left[\hat{\phi}_{Cn-1}, \hat{\eta}_{Cn-1}\right] - L_{K2}\left[\hat{\phi}_{Cn-2}, \hat{\eta}_{Cn-2}\right] \quad (n = 2 \text{ and } 5/2), \ldots, \end{cases} \tag{4.4b}$$

$$I_n = \begin{cases} \mathrm{i}(\sigma_0\hat{\phi}_{Cn-1} + \sigma_{1/2}\hat{\phi}_{Cn-3/2}) + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\phi_{Pn-1} \\ \quad - L_{D1}\left[\hat{\phi}_{Cn-1}, \hat{\eta}_{Cn-1}\right] \quad (n = 1 \text{ and } 3/2), \\ \mathrm{i}\left[\sigma_0\left(\hat{\phi}_{Cn-1} + \dfrac{\partial \hat{\phi}_{Cn-2}}{\partial z}H_1\right) + \sigma_{1/2}\left(\hat{\phi}_{Cn-3/2} + \dfrac{\partial \hat{\phi}_{Cn-5/2}}{\partial z}H_1\right)\right. \\ \quad \left. + \sigma_1\hat{\phi}_{Cn-2} + \sigma_{3/2}\hat{\phi}_{Cn-5/2}\right] \\ \quad + \left(v - \dfrac{\partial \Phi}{\partial x}\right)\left(\phi_{Pn-1} + \dfrac{\partial \phi_{Pn-2}}{\partial z}H_1\right) + \left(c_1 - \dfrac{\partial \Phi_1}{\partial x} - \dfrac{\partial^2 \Phi}{\partial x \partial z}H_1\right)\phi_{Pn-2} \\ \quad - L_{D1}\left[\hat{\phi}_{Cn-1}, \hat{\eta}_{Cn-1}\right] - L_{D2}\left[\hat{\phi}_{Cn-2}, \hat{\eta}_{Cn-2}\right] \quad (n = 2 \text{ and } 5/2), \ldots, \end{cases} \tag{4.4c}$$

where

$$\phi_{Pn} = \mathrm{i}(p_0\hat{\phi}_{Cn} + p_{1/2}\hat{\phi}_{Cn-1/2}), \quad \eta_{Pn} = \mathrm{i}(p_0\hat{\eta}_{Cn} + p_{1/2}\hat{\eta}_{Cn-1/2}), \tag{4.4d}$$

and $\hat{\phi}_{C-1/2} = \hat{\eta}_{C-1/2} = 0$. The solvability condition of the above inhomogeneous equations (3.4)–(3.7) with (4.4) is (3.10). For $n = 1$ and $3/2$, they are identically satisfied, and the solution $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn})$ is

$$\hat{\phi}_{C1} = \mathrm{i}\sigma_0\frac{\partial \Phi}{\partial v} - \mathrm{i}p_0 x\frac{\partial \Phi}{\partial x} + \frac{\partial \Phi_1}{\partial x}, \quad \hat{\eta}_{C1} = \mathrm{i}\sigma_0\frac{\partial H}{\partial v} - \mathrm{i}p_0 x\frac{\mathrm{d}H}{\mathrm{d}x} + \frac{\mathrm{d}H_1}{\mathrm{d}x}, \tag{4.5a,b}$$

$$\hat{\phi}_{C3/2} = \mathrm{i}\sigma_{1/2}\frac{\partial \Phi}{\partial v} + \mathrm{i}\sigma_0\Phi_B\beta - \mathrm{i}x\left(p_0\beta + p_{1/2}\frac{\partial \Phi}{\partial x}\right) + \Phi_U\gamma, \tag{4.6a}$$

$$\hat{\eta}_{C3/2} = \mathrm{i}\sigma_{1/2}\frac{\partial H}{\partial v} + \mathrm{i}\sigma_0 H_B\beta - \mathrm{i}p_{1/2}x\frac{\mathrm{d}H}{\mathrm{d}x} + H_U\gamma, \tag{4.6b}$$

where γ is a new undetermined constant, and (Φ_B, H_B) and (Φ_U, H_U) are defined by (A 14*a,b*) and (A 15*a,b*), respectively. From (4.5*a*) and (4.6*a*),

$$\left[\hat{\phi}_{C1}\right]_{x\to-\infty} = \left[\hat{\phi}_{C1}\right]_{x\to\infty} + \mathrm{i}\sigma_0\frac{\mathrm{d}\Omega}{\mathrm{d}v}, \tag{4.7}$$

$$\left[\frac{\partial \hat{\phi}_{C3/2}}{\partial x}\right]_{x\to\pm\infty} = -\mathrm{i}p_0\beta + \gamma, \tag{4.8}$$

where Ω is defined by (A 7*b*), and quantities in the square brackets with subscript $x \to -\infty$, ∞ and $\pm\infty$ are evaluated as $x \to -\infty$, ∞ and $\pm\infty$, respectively.

For $n = 2$, the solvability condition (3.10) is identically satisfied and for $n = 5/2$ it becomes

$$-\frac{2\sigma_0\sigma_{1/2}}{v}\frac{\mathrm{d}E}{\mathrm{d}v} + 2Eq_0q_{1/2} = \left[\sigma_0^2\left(\frac{\mathrm{d}M}{\mathrm{d}v} + v\frac{\mathrm{d}\Omega}{\mathrm{d}v}\right) - vMq_0^2\right]\beta - \mathrm{i}(v^2 - 1)\sigma_0\frac{\mathrm{d}\Omega}{\mathrm{d}v}\gamma, \tag{4.9}$$

where (3.9) and (B 1)–(B 3) are used, and M and E are defined by (A 7*c,e*). Integrating (3.4) for $n = 1$ and 2 with respect to x and z and using (3.5) for $n = 1$ and 2, we find

$$\left[\hat{u}_{C2}\right]_{x\to-\infty} = \left[\hat{u}_{C2}\right]_{x\to\infty} + \sigma_0^2\frac{\mathrm{d}M}{\mathrm{d}v} - vMq_0^2, \tag{4.10}$$

where $\hat{u}_{C2}$ is defined by (3.16).

Now it is evident from (4.5*a*) and (4.6*a*) that the above core solution becomes inappropriate in the far region where the periodic condition (2.14) is imposed, because $\hat{\phi}_{C1} = O(1)$ and $\hat{\phi}_{C3/2} = O(|x|)$ for large $|x|$, and $k^{3/2}\hat{\phi}_{C3/2}$ becomes greater than the lower-order solution $k\hat{\phi}_{C1}$ for $|x| \gg k^{-1/2}$. To avoid such inappropriateness, we need to introduce a far-field solution that varies slowly in x, as in § 3. Sections C.1 and C.2 consider the far-field solution and matching of the two solutions, respectively.

4.2. *Eigenvalues*

From the analyses in §§ C.1 and C.2, we find that the quantities as $x \to \pm\infty$ that appear in (4.7), (4.8) and (4.10), or $[\hat{\phi}_{C1}]_{x\to\pm\infty}$, $[\partial\hat{\phi}_{C3/2}/\partial x]_{x\to\pm\infty}$ and $[\hat{u}_{C2}]_{x\to\pm\infty}$, are expressed as (C 13)–(C 15). Substitution of these expressions into (4.7), (4.8) and (4.10) yields four equations, (4.7)–(4.10), for β, $\bar{\beta}$ (see (C 11)), γ and $\sigma_{1/2}$. The solution for $\sigma_{1/2}$ is

$$\sigma_{1/2} = \sigma_0 \frac{q_{1/2}}{q_0} + \begin{cases} \dfrac{1}{2}\left(\Delta_{II} \pm \sqrt{\Delta_{II}^2 + \Delta_{cr}^2}\right) & \text{for } b < 0, \\ \dfrac{1}{2}\left(\Delta_{II} \pm \mathrm{i}\sqrt{-\Delta_{II}^2 + \Delta_{cr}^2}\right) & \text{for } b > 0 \text{ and } |\Delta_{II}| < \Delta_{cr}, \\ \dfrac{1}{2}\left(\Delta_{II} \pm \sqrt{\Delta_{II}^2 - \Delta_{cr}^2}\right) & \text{for } b > 0 \text{ and } |\Delta_{II}| > \Delta_{cr}, \end{cases} \tag{4.11}$$

where

$$\Delta_{II} = c_x p_{1/2} + c_y q_{1/2} - \sigma_0 \frac{q_{1/2}}{q_0} = c_x\left(p_{1/2} - \frac{p_0}{q_0} q_{1/2}\right), \quad \Delta_{cr} = \sqrt{\frac{\Gamma^2 q_0^2}{2\pi E}\left|\frac{c_x}{b}\right|}, \tag{4.12a,b}$$

and (c_x, c_y), Γ and b are defined by (3.32*a*,*b*), (3.40*e*) and (2.26), respectively. From (4.11), a solution $\sigma_{1/2}$ with a positive imaginary part exists if

$$b > 0 \quad \text{and} \quad |\Delta_{II}| < \Delta_{cr} \left(\left| p_{1/2} - \frac{p_0}{q_0} q_{1/2} \right| < \sqrt{\frac{\Gamma^2 q_0^2}{2\pi E\,|b|}} \right). \tag{4.13}$$

This result shows that there is an instability band of width $O(k^{1/2})$ on a $p - q$ plane along all of the resonance lines (2.25*a*) with $b > 0$, as indicated by two solid lines labelled 'II ($b > 0$)' in figure 3 that cross each other diagonally. In contrast, the instability does not occur along the resonance lines with $b < 0$. Thus, we have proved that the type II resonance (2.24) with $b > 0$ leads to instability and that with $b < 0$ produces no instability. These analytical results are consistent with the necessary condition for losing stability identified by Mackay & Saffman (1986) as discussed in the last paragraph in § 2.

The solution (4.11) shows that the local maximum growth rate $\max(\mathrm{Im}[\sigma])$ across the instability band is $k^{1/2}\Delta_{cr}/2$, which is proportional to $|q_0|$ according to (4.12*b*). The dominant instability thus occurs at large values of $|q_0|$. This is in contrast to the case of small-amplitude waves, whose dominant instability is always associated with the lowest-order resonance, for which $|q_0|$ is small (Mclean 1982*a*,*b*). It also should be mentioned that the instability derived in this section disappears in the small-amplitude limit $a \to 0$ because $\Delta_{cr} \to 0$ as $a \to 0$.

5. Numerical verification

5.1. *Numerical method*

We verify the theoretical results numerically. The numerical procedure consists of two parts, computing the unperturbed wave solution (2.5) and solving the linear eigenvalue problem (2.14)–(2.18). To compute the unperturbed wave solution (2.5), we employ the method of Tanaka (1986) with some modifications. The variable $x - \bar{\phi}/C$ on the free surface is employed as an independent variable with its origin on the crest, and unknown variables are represented at discrete mesh points properly distributed along the free surface for the region $0 \leqslant x - \bar{\phi}/C \leqslant \pi/k$, which corresponds to a half wavelength. We introduce

$$x - \frac{\bar{\phi}}{C} = \frac{\pi}{k}\left[\alpha\gamma_j + (1-\alpha)\left(\gamma_j - \frac{\sin \pi\gamma_j}{\pi}\right)\right], \tag{5.1a}$$

where $N+1$ mesh points are distributed at

$$\gamma_j = \frac{j}{N} \quad (j = 0, 1, 2, \ldots, N), \tag{5.1b}$$

and α $(0 < \alpha < 1)$ is a small positive parameter concentrating the mesh points toward a wave crest at $x - \bar{\phi}/C = 0$. Then $\mathrm{d}\left(x - \bar{\phi}/C\right)/\mathrm{d}\gamma_j$ is periodic in γ_j and infinitely differentiable with respect to γ_j even at $x - \bar{\phi}/C = 0$ and π/k, where symmetric conditions are imposed. The remaining numerical procedure is the same as that explained in Tanaka (1986); the reader is referred to that paper for details. The computed results converge so rapidly that the surface height $\bar{\eta}$ converges up to $O(10^{-7})$ or more on a pointwise basis at only $N = 120$ for $k \geqslant 0.05$ and $N = 480$ for $k \geqslant 0.01$.

To solve the eigenvalue problem (2.14)–(2.18) for $(\hat{\phi}, \hat{\eta})$, we note that $\hat{\phi}\exp(\mathrm{i}kpx)$ is a solution of the modified Helmholtz equation $(\partial^2/\partial x^2 + \partial^2/\partial z^2 - k^2q^2)\hat{\phi}\exp(\mathrm{i}kpx) = 0$ with an impermeable condition at the bottom $(z = 0)$. Therefore, we introduce Green's function G for the modified Helmholtz equation satisfying $(\partial^2/\partial x^2 + \partial^2/\partial z^2 - k^2q^2)G = -2\pi\delta(x - x', z - z')$ (δ is Dirac's delta function) and $\partial G/\partial z = 0$ at the $z = 0$. That is,

$$G(x - x', z, z') = K_0(k|q|r) + K_0(k|q|\bar{r}), \tag{5.2}$$

where K_0 is the modified Bessel function of the second kind of order zero and

$$r = \sqrt{(x - x')^2 + (z - z')^2}, \quad \bar{r} = \sqrt{(x - x')^2 + (z + z')^2}. \tag{5.3}$$

We then set

$$\hat{\phi}\exp(\mathrm{i}kpx) = \int_{-\infty}^{\infty} \psi(s')\exp\left[\mathrm{i}kpx'(s')\right] G\left(x - x'(s'), z, z'(s')\right) \mathrm{d}s', \tag{5.4}$$

where $(x(s), z(s))$ is the spatial coordinate on the free surface $z = \bar{\eta}$ of the unperturbed wave, s is its arclength along the free surface and $\psi(s)$ is a new unknown function of s having the same periodicity as the unperturbed wave. The spatial derivatives of $\hat{\phi}\exp(\mathrm{i}kpx)$ along the surface and normal to it are evaluated as

$$\frac{\partial\hat{\phi}\exp(\mathrm{i}kpx)}{\partial s} = PV\int_{-\infty}^{\infty} \psi(s')\exp\left[\mathrm{i}kpx'(s')\right]\frac{\partial G}{\partial s}\mathrm{d}s', \tag{5.5a}$$

$$\frac{\partial\hat{\phi}\exp(\mathrm{i}kpx)}{\partial n} = \pi\psi\exp(\mathrm{i}kpx) + \int_{-\infty}^{\infty} \psi(s')\exp\left[\mathrm{i}kpx'(s')\right]\frac{\partial G}{\partial n}\mathrm{d}s', \tag{5.5b}$$

respectively, where PV denotes the principal value and n is the normal to the surface, pointing out of the fluid. The eigenvalue problem (2.14)–(2.18) for $\hat{\phi}(x,z)$ and $\hat{\eta}(x)$ then reduces to that of integro-differential equations for $\hat{\eta}(s)$ and $\psi(s)$:

$$-\mathrm{i}k\sigma\hat{\eta} = \frac{1}{\cos\theta}\left[\pi\psi + \int_{-\infty}^{\infty}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}\frac{\partial G}{\partial n}\mathrm{d}s' + \frac{\mathrm{d}(u\cos\theta\hat{\eta})}{\mathrm{d}s}\right] + \mathrm{i}kpu\hat{\eta}\cos\theta, \tag{5.6}$$

$$-\mathrm{i}k\sigma\int_{-\infty}^{\infty}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}G\mathrm{d}s' = u\times PV\int_{-\infty}^{\infty}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}\frac{\partial G}{\partial s}\mathrm{d}s' - \left[u\frac{\mathrm{d}(u\sin\theta)}{\mathrm{d}s}+1\right]\hat{\eta}, \tag{5.7}$$

where the periodic condition for $\hat{\eta}(s)$ and $\psi(s)$ is imposed at $x(s) = \pm\pi/k$. Here $u(s)(>0)$ is the flow speed of the unperturbed wave in a reference frame moving with it, and $\theta(s)$ is the angle between a tangent to the surface of the unperturbed wave and the x-axis, i.e.

$$u(s) = \left[\left(-C + \frac{\partial\bar{\phi}}{\partial x}\right)^2 + \left(\frac{\partial\bar{\phi}}{\partial z}\right)^2\right]^{1/2}_{(x,z)=(x(s),z(s))}, \quad \theta(s) = \arctan\left(\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}x}\right). \tag{5.8a,b}$$

The above eigenvalue problem (5.6) and (5.7) for $\hat{\eta}(s)$ and $\psi(s)$ is solved numerically on $2N$ mesh points (5.1) with $j = -N+1, \ldots, -1, 0, 1, \ldots, N$ for the range $-\pi/k < x - \bar{\phi}/C \leqslant \pi/k$.

The two integrals $PV\int_{-\infty}^{\infty}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}\partial G/\partial s\mathrm{d}s'$ and $\int_{-\infty}^{\infty}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}G\mathrm{d}s'$ in (5.7) contain an inverse-first-power singularity and a logarithmic singularity, respectively, at $s' = s$. The former integral is calculated according to the usual trapezoidal rule by evaluating the variables with s and those with s' at different sets of mesh points γ_j and $\gamma_{j-1/2} = (j-1/2)/N$ $(j = -N+1, \ldots, -1, 0, 1, \ldots, N)$, respectively. The latter integral is first divided into a singular part $-\int_{s(\gamma_{j-2})}^{s(\gamma_{j+2})}\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}\ln\left|s' - s(\gamma_j)\right|\mathrm{d}s'$ and the remaining non-singular part. The singular part is calculated analytically by approximating the function $\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]}$ in terms of a quartic function of s' as

$$\psi(s')\mathrm{e}^{\mathrm{i}kp[x'(s')-x(s)]} = a_4 s'^4 + a_3 s'^3 + a_2 s'^2 + a_1 s' + a_0, \tag{5.9}$$

where the coefficients a_0, a_1, a_2, a_3 and a_4 are determined by evaluation of (5.9) at five discrete mesh points $s' = s(\gamma_{j-2})$, $s(\gamma_{j-1})$, $s(\gamma_j)$, $s(\gamma_{j+1})$ and $s(\gamma_{j+2})$. The non-singular part is computed simply by Simpson's rule, as usual.

The derivative $\mathrm{d}\hat{\eta}/\mathrm{d}s$ in (5.6) is evaluated by the usual fourth-order centred finite-difference formula, and the periodic condition is evaluated as $\hat{\eta}(\gamma_j) = \hat{\eta}(\gamma_{j+2N})$ and $\psi(\gamma_j) = \psi(\gamma_{j+2N})$. We then obtain the following $4N$ algebraic equations for $\{\hat{\eta}, \psi\} \equiv \left(\hat{\eta}(\gamma_{-N+1}), \hat{\eta}(\gamma_{-N+2}), \ldots, \hat{\eta}(\gamma_N), \psi(\gamma_{-N+1}), \psi(\gamma_{-N+2}), \ldots, \psi(\gamma_N)\right)$ from (5.6) and (5.7):

$$\sigma[\boldsymbol{Y}]\begin{Bmatrix}\hat{\eta}\\ \psi\end{Bmatrix} = [\boldsymbol{Z}]\begin{Bmatrix}\hat{\eta}\\ \psi\end{Bmatrix}, \tag{5.10}$$

where $[\boldsymbol{Y}]$ and $[\boldsymbol{Z}]$ are the linear operators of a $4N\times 4N$ matrix form. Thus, the original eigenvalue problem (2.14)–(2.18) reduces to that of the $4N$ algebraic equations (5.10).

(a)		(b)	
N	σ	N	σ
60	No unstable solution	60	$-0.23240 + 0.00026967\mathrm{i}$
120	$-0.15571 + 0.00052120\mathrm{i}$	120	$-0.23238 + 0.00027228\mathrm{i}$
240	$-0.15571 + 0.00053661\mathrm{i}$	240	$-0.23238 + 0.00027229\mathrm{i}$
480	$-0.15571 + 0.00053658\mathrm{i}$	480	$-0.23238 + 0.00027229\mathrm{i}$
960	$-0.15571 + 0.00053657\mathrm{i}$	960	$-0.23238 + 0.00027229\mathrm{i}$

TABLE 1. Convergence of computed eigenvalues σ as half the number N of mesh points is increased: (*a*) $(a, k) = (0.4, 0.05)$ and $(p, q) = (0.9574, 0.2)$; (b) $(a, k) = (0.1, 0.1)$ and $(p, q) = (2.113, 0.258)$. The computational parameter α in (5.1*a*) is set to 0.04.

The eigenvalues σ of the $4N \times 4N$ matrix $[\boldsymbol{Y}^{-1}\boldsymbol{Z}]$ are obtained by a standard eigenvalue solver based on the QZ algorithm (Wilkinson 1965).

5.2. *Results*

Table 1 shows the convergence of the computed eigenvalues σ as half the number N of mesh points is increased. We see that convergence of up to three significant digits is achieved with $N = 240$ and 120 for $(a, k) = (0.4, 0.05)$ and $(0.1, 0.1)$, respectively. In this study, we sought convergence of up to three and two significant digits for any computed eigenvalues σ of $\mathrm{Im}[\sigma] \geqslant 10^{-4}$ and $\mathrm{Im}[\sigma] < 10^{-4}$, respectively.

Figure 5 shows the instability regions thus obtained for $k = 0.05$ and 0.2 when $a = 0.1$, 0.4 and 0.7. Only the first quadrant on the $p - q$ plane is shown. Owing to symmetry, the instability regions in the other quadrants are readily obtained by using a mirror image of all four quadrants. The dashed lines in figure 5 are the analytically obtained resonance curves in the long-wave limit ($k \to 0$) given by (2.23*a*) and (2.25*a*). The numerically obtained instability bands for periodic waves in shallow water are located near these resonance curves. The agreement between the analytical and numerical results becomes poorer for larger k and also larger $|p|$ and $|q|$ because $k|p|, k|q| \ll 1$ was assumed when deriving the resonance curves for the long-wave limit ($k \to 0$) in § 2.2.

Figure 5 also shows that as a decreases, the instability bands become thinner and are almost reduced to lines of infinitesimal thickness for $a = 0.1$ (figure 5*a,b*). This was first observed by Francius & Kharif (2006). Regarding the dependence on k, the width of the type I resonance instability band is $O(k)$, whereas that of type II is $O(k^{1/2})$ according to the asymptotic analyses in §§ 3 and 4 [see (3.41) and (4.13)]. The widths of the instability bands are generally proportional to the corresponding local maximum growth rates, which are $k|m|R_{cr}/2$ [from (3.39)] and $k^{1/2}\Delta_{cr}/2$ [from (4.11)] for types I and II, respectively.

Let us see how the numerical results of the local maximum growth rates $\max(\mathrm{Im}[\sigma])$ behave as functions of k. These are plotted by the symbols connected by solid lines in figure 6 for $a = 0.4$. The dashed lines are the corresponding results of the asymptotic solutions for small k presented above, which are straight lines of gradients 1 and 1/2 for types I and II, respectively, on a logarithmic scale. The numerical results uniformly approach those of the asymptotic solutions as k becomes smaller.

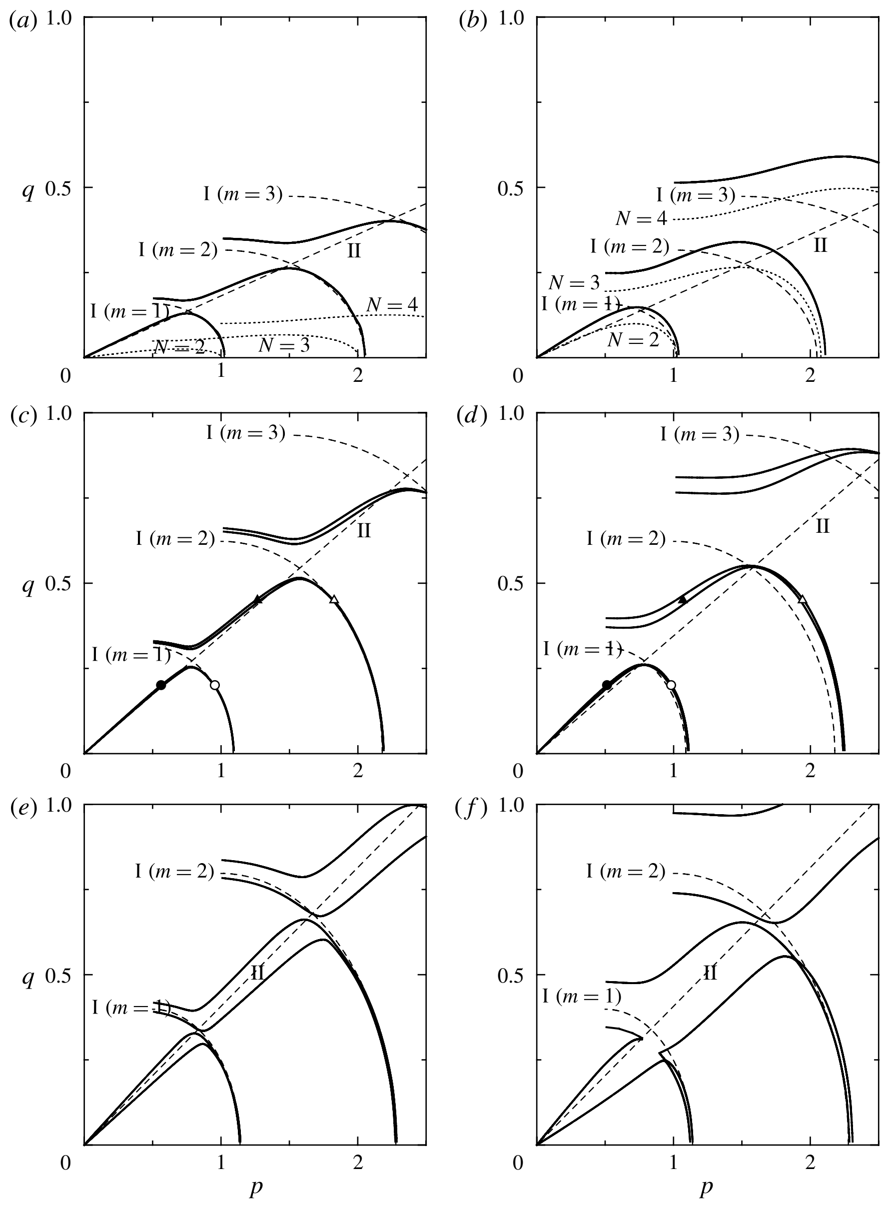

FIGURE 5. Numerically obtained instability bands (regions pinched by the two solid lines: for (*a, b*) they almost reduce to the solid lines themselves) for $k = 0.05$ and 0.2 when $a = 0.1$, 0.4 and 0.7: (*a*) $(a, k) = (0.1, 0.05)$; (*b*) $(a, k) = (0.1, 0.2)$; (*c*) $(a, k) = (0.4, 0.05)$; (*d*) $(a, k) = (0.4, 0.2)$; (*e*) $(a, k) = (0.7, 0.05)$; (*f*) $(a, k) = (0.7, 0.2)$. Dashed lines are resonance curves (2.23*a*) and (2.25*a*) in the long-wave limit ($k \to 0$), and the indices I and II label their resonance types with m being the order of the resonance. Dotted lines in (*a, b*) are the resonance curves in the small-amplitude limit ($a \to 0$) for given k with N representing the order of the interaction. Symbols (○, △, ●, ▲) in (*c,d*) are described in the caption for figure 6.

The relationship between the unstable resonance curves in the long-wave limit and those in the small-amplitude limit should be discussed. In the long-wave limit, the

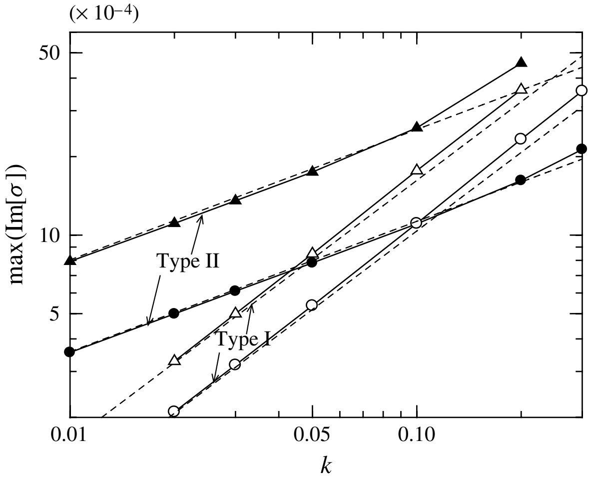

FIGURE 6. Local maximum growth rates $\max(\mathrm{Im}[\sigma])$ versus k for $a = 0.4$ on various instability bands: —○—, type I with $m = 1$ (along $q = 0.2$); —△—, type I with $m = 2$ (along $q = 0.45$); —●—, type II (along $q = 0.2$); —▲—, type II (along $q = 0.45$). Dashed lines are the corresponding results of the asymptotic solutions for small k given by $k\,|m|\,R_{cr}/2$ (from (3.39)) and $k^{1/2}\Delta_{cr}/2$ (from (4.11)) for types I and II, respectively. The points representing the achieved local maximum growth rates are plotted in figure 5(*c*,*d*) using the same symbols as used in this figure.

unstable resonance curves (2.23*a*) and (2.25*a*) of types I and II intersect at

$$(p, q) = \left(\frac{v+b}{2b} m, \pm \frac{\sqrt{(v^2-1)(b^2-1)}}{2b} m \right), \tag{5.11}$$

where m represents the order of the type I resonance and b is given here by (2.26) with positive sign. Figure 5 shows, however, that the corresponding instability bands do not intersect (note that they may touch if they are wide, as shown in figure 5*f*). In fact, the instability bands do not cross each other at the intersection point (5.11) because the type of resonance curve being tracked changes. A connected instability band is thus located along two or three different types of resonance curves: (i) along the type I resonance curve of $m = 1$ in the lower part and the type II resonance line between the origin and this type I curve; (ii) along two type I resonance curves of successive resonance orders, say $m = M$ in the upper part and $M + 1$ in the lower part ($M = 1, 2, \ldots$) and the type II resonance line between these two curves (the case of $M = 1$ can be clearly seen in figure 5*a,c,e*).

The above connected instability bands (i) and (ii) deform as a and k vary, and in the limit of $a \to 0$, they correspond to the resonance curves of the small-amplitude limit. Specifically, band (i) reduces to the resonance curve of the order of interaction $N = 2$ represented by dotted lines labelled '$N = 2$' in figure 5(*a,b*), which is nothing but a one-quarter figure-eight curve (Phillips 1960, 1967); and band (ii) reduces to the resonance curve in the small-amplitude limit of the order of interaction $N = M + 2$. These resonance curves in the small-amplitude limit are shifted in the positive p direction by an integer $N/2 - 1$ (for even N) or $(N - 3)/2$ (for odd N) from the conventional position (Mclean 1982*a*,*b*; Francius & Kharif 2006) because the solution of the form (2.13) is arbitrary under a shift in p by an integer.

Finally, note that this study treats the case in which the wavenumbers kp and kq of perturbation are of the same order of magnitude as k for the unperturbed wave, or $|p|$, $|q| = O(1)$, whereas for the (transverse) stability of a solitary wave, the wavenumber of perturbation is much larger than that of the unperturbed wave, or $|p|$, $|q| \gg 1$. Thus, the stability result in the present study will approach that of the solitary wave as $|p|$, $|q| \to \infty$: the instability disappears for $a < 0.713$ and arises for $0.713 < a < 0.781$ (Kataoka & Tsutahara 2004; Kataoka 2010).

6. Concluding remarks

We derived the resonance conditions and resonance curves of periodic waves not in the small-amplitude limit but in the long-wave limit $k \to 0$, keeping the wave amplitude finite ($a = O(1)$). Resonant interactions were considered under a periodic approximate of finite-amplitude solitary wave that corresponds to the long-wave limit of periodic wave. Two types of resonance conditions exist, type I and type II. The resonance curves for type I become ellipses, whereas those for type II consist of two sets of diagonally crossing straight lines on the perturbation wavenumber plane. A systematic asymptotic analysis for small wavenumbers ($k \to 0$) yielded the growth rates of the perturbations explicitly, and revealed that instability occurs along all the type I resonance curves and a single set of diagonally crossing type II resonance lines. We verified these theoretical results numerically by showing that the instability bands for finite-amplitude periodic waves in shallow water are located along these unstable resonance curves. Thus, we can conclude that our theory of resonance in the limit $k \to 0$ can describe the instability of periodic waves in shallow water well. This theory compensates for the weakness of the existing theory of small-amplitude limit as it applies to finite-amplitude waves in shallow water.

Acknowledgements

The author is grateful to Professor C. Kharif, Dr M. Francius and the staff of Institut de Recherche sur les Phénomènes Hors Équilibre and to Professor N. Peake of Cambridge University for their hospitality and fruitful discussions during the author's short stays in Marseille and Cambridge. Professor Kharif in particular made me aware of the importance of the work by Mackay & Saffman (1986). The author is also grateful to the referees for their valuable comments.

Appendix A. Finite-amplitude solitary wave solution, its extended version and cnoidal wave solution

A.1. *Finite-amplitude solitary wave solution*

Consider a solution of (2.1)–(2.4) in the following form:

$$\phi = -vx + \Phi(x, z; v), \quad \eta = H(x; v), \tag{A 1a,b}$$

with

$$f(t) = 1 + \tfrac{1}{2}v^2, \tag{A 1c}$$

and

$$\frac{\partial \Phi}{\partial x} \to 0, \quad \frac{\partial \Phi}{\partial z} \to 0, \quad H \to 1 \quad \text{as } x \to \pm\infty. \tag{A 2}$$

This solution represents a localized progressive wave that propagates steadily against a uniform stream of constant velocity $-v$ in the x direction. We call this solution a solitary wave solution. Substitution of (A 1) into (2.1)–(2.4) reveals that $\varPhi$ and H should satisfy

$$\frac{\partial^2 \varPhi}{\partial x^2} + \frac{\partial^2 \varPhi}{\partial z^2} = 0 \quad \text{for } 0 < z < H, \tag{A 3}$$

$$\left(-v + \frac{\partial \varPhi}{\partial x}\right)\frac{\mathrm{d}H}{\mathrm{d}x} = \frac{\partial \varPhi}{\partial z} \quad \text{at } z = H, \tag{A 4}$$

$$-v\frac{\partial \varPhi}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \varPhi}{\partial x}\right)^2 + \left(\frac{\partial \varPhi}{\partial z}\right)^2\right] + H = 1 \quad \text{at } z = H, \tag{A 5}$$

$$\frac{\partial \varPhi}{\partial z} = 0 \quad \text{at } z = 0, \tag{A 6}$$

subject to the boundary conditions (A 2) at infinity. The existence of the above solitary wave solution was rigourously proved by Amick & Toland (1981*a*). Numerical solutions were obtained by Hunter & Vanden-Broeck (1983) and many others (Byatt-Smith & Longuet-Higgins 1976; Tanaka 1986; Longuet-Higgins & Tanaka 1997). According to them, the solitary wave solution exists in the range $1 < v < 1.2942$ and has the property that the surface elevation $H - 1$ is always positive and possesses a single extremum called the crest. The solution is symmetric with respect to the crest; that is, $\varPhi(x, z)$ is odd and $H(x)$ is even with respect to x when the crest is located at $x = 0$ and the approach to the state at infinity described in (A 2) is exponentially fast. In terms of the maximum surface elevation $a = H(0) - 1$, the solitary wave solution exists for $0 < a < 0.83332$. In fact, $\varPhi$ approaches different constant values as $x \to \pm\infty$, i.e.

$$[\varPhi]_{x\to\pm\infty} = \mp\frac{\varOmega}{2}, \tag{A 7a}$$

where

$$\varOmega(v) = \frac{2T}{v} - vM, \tag{A 7b}$$

(McCowan 1891) and M and T are defined by

$$M(v) = \int_{-\infty}^{\infty}(H-1)\,\mathrm{d}x, \quad T(v) = \int_{-\infty}^{\infty}\mathrm{d}x\int_0^H \frac{1}{2}\left[\left(\frac{\partial \varPhi}{\partial x}\right)^2 + \left(\frac{\partial \varPhi}{\partial z}\right)^2\right]\mathrm{d}z. \tag{A 7c,d}$$

Here $\varOmega, M$ and T physically represent the circulation (in the clockwise direction), mass and kinetic energy of the solitary wave, respectively. The total energy E of the solitary wave is defined by

$$E(v) = T(v) + \frac{1}{2}\int_{-\infty}^{\infty}(H-1)^2\,\mathrm{d}x = \int_{-\infty}^{\infty}\mathrm{d}x\int_0^H\left(\frac{\partial \varPhi}{\partial x}\right)^2\mathrm{d}z, \tag{A 7e}$$

where the far right-hand side was first derived by Starr (1947) (see also Appendix A of Kataoka (2008).

A.2. *Extended solitary wave solution*

Let us extend the solitary wave solution (A 1) by adding two more parameters, B and U, representing the Bernoulli constant and average flow velocity of wave, as

$$\phi = -v^* x + \varPhi^*(x, z; v^*, B, U), \quad \eta = H^*(x; v^*, B, U), \tag{A 8a,b}$$

with

$$f(t) = B + \tfrac{1}{2}v^{*2}, \tag{A 8c}$$

where the boundary conditions at infinity are

$$\frac{\partial \Phi^*}{\partial x} \to U, \quad \frac{\partial \Phi^*}{\partial z} \to 0, \quad H^* \to h \quad \text{as } x \to \pm\infty, \tag{A 9}$$

with

$$h = B + v^* U - \frac{U^2}{2}. \tag{A 10}$$

Substitution of (A 8) into (2.1)–(2.4) reveals that Φ^* and H^* should satisfy

$$\frac{\partial^2 \Phi^*}{\partial x^2} + \frac{\partial^2 \Phi^*}{\partial z^2} = 0 \quad \text{for } 0 < z < H^*, \tag{A 3$'$}$$

$$\left(-v^* + \frac{\partial \Phi^*}{\partial x}\right)\frac{\mathrm{d}H^*}{\mathrm{d}x} = \frac{\partial \Phi^*}{\partial z} \quad \text{at } z = H^*, \tag{A 4$'$}$$

$$-v^* \frac{\partial \Phi^*}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \Phi^*}{\partial x}\right)^2 + \left(\frac{\partial \Phi^*}{\partial z}\right)^2\right] + H^* = B \quad \text{at } z = H^*, \tag{A 5$'$}$$

$$\frac{\partial \Phi^*}{\partial z} = 0 \quad \text{at } z = 0, \tag{A 6$'$}$$

subject to the boundary conditions (A 9) at infinity. For $B = 1$ and $U = 0$, the above extended solution (Φ^*, H^*) corresponds with the original solitary wave solution (Φ, H).

Substitution of the transformation

$$v \to \frac{v^* - U}{\sqrt{h}}, \quad (x, z) \to \frac{1}{h}(x, z), \quad \Phi \to \frac{\Phi^* - Ux}{h^{3/2}}, \quad H \to \frac{H^*}{h}, \tag{A 11}$$

into the set of basic equations (A 2)–(A 6) for the original solitary wave solution yields (A 3$'$)–(A 6$'$) and (A 9) for the extended solitary wave solution $(\Phi^*(x, z), H^*(x))$ as above. Thus, (Φ^*, H^*) is expressed in terms of (Φ, H) as

$$\Phi^*(x, z; v^*, B, U) = Ux + h^{3/2}\Phi\left(\frac{x}{h}, \frac{z}{h}; \frac{v^* - U}{\sqrt{h}}\right), \tag{A 12a}$$

$$H^*(x; v^*, B, U) = hH\left(\frac{x}{h}; \frac{v^* - U}{\sqrt{h}}\right), \tag{A 12b}$$

and exists for $1 < (v^* - U)/\sqrt{h} < 1.2942$. When the solution exists, its approach to the state at infinity in (A 9) is exponentially fast and $\Phi^* - Ux$ approaches different constant values as $x \to \pm\infty$. That is,

$$[\Phi^* - Ux]_{x\to\pm\infty} = \mp\frac{\Omega^*}{2}, \tag{A 13a}$$

where

$$\Omega^*(v^*, B, U) = \frac{1}{h}\left[\frac{2T^*}{v^* - U} - (v^* - U)M^*\right] = h^{3/2}\Omega\left(\frac{v^* - U}{\sqrt{h}}\right), \tag{A 13b}$$

and M^* and T^* are defined by

$$M^*(v^*, B, U) = \int_{-\infty}^{\infty} (H^* - h)\,\mathrm{d}x = h^2 M\left(\frac{v^* - U}{\sqrt{h}}\right), \tag{A 13c}$$

$$T^*(v^*, B, U) = \int_{-\infty}^{\infty} \mathrm{d}x \int_0^{H^*} \frac{1}{2}\left[\left(\frac{\partial \Phi^*}{\partial x} - U\right)^2 + \left(\frac{\partial \Phi^*}{\partial z}\right)^2\right] \mathrm{d}z = h^3 T\left(\frac{v^* - U}{\sqrt{h}}\right). \tag{A 13d}$$

By using (A 12) and (A 13), the partial derivatives of $(\Phi^*, H^*, \Omega^*, M^*, T^*)$ with respect to B (for fixed v^* and U) or U (for fixed v^* and B) evaluated at $B = 1$ and $U = 0$ can be expressed in terms of the original solitary wave solution (Φ, H) as

$$\Phi_B(v) \equiv \left[\frac{\partial \Phi^*}{\partial B}\right]_{\substack{B=1,\\ U=0}} = \frac{3}{2}\Phi - x\frac{\partial \Phi}{\partial x} - z\frac{\partial \Phi}{\partial z} - \frac{v}{2}\frac{\partial \Phi}{\partial v}, \tag{A 14a}$$

$$H_B(v) \equiv \left[\frac{\partial H^*}{\partial B}\right]_{\substack{B=1,\\ U=0}} = H - x\frac{\mathrm{d}H}{\mathrm{d}x} - \frac{v}{2}\frac{\partial H}{\partial v}, \tag{A 14b}$$

$$\Omega_B(v) \equiv \left[\frac{\partial \Omega^*}{\partial B}\right]_{\substack{B=1,\\ U=0}} = \frac{3}{2}\Omega - \frac{v}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}v}, \tag{A 14c}$$

$$M_B(v) \equiv \left[\frac{\partial M^*}{\partial B}\right]_{\substack{B=1,\\ U=0}} = 2M - \frac{v}{2}\frac{\mathrm{d}M}{\mathrm{d}v}, \tag{A 14d}$$

$$T_B(v) \equiv \left[\frac{\partial T^*}{\partial B}\right]_{\substack{B=1,\\ U=0}} = 3T - \frac{v}{2}\frac{\mathrm{d}T}{\mathrm{d}v}, \tag{A 14e}$$

$$\Phi_U(v) \equiv \left[\frac{\partial \Phi^*}{\partial U}\right]_{\substack{B=1,\\ U=0}} = x + v\left(\frac{3}{2}\Phi - x\frac{\partial \Phi}{\partial x} - z\frac{\partial \Phi}{\partial z}\right) - \left(\frac{v^2}{2} + 1\right)\frac{\partial \Phi}{\partial v}, \tag{A 15a}$$

$$H_U(v) \equiv \left[\frac{\partial H^*}{\partial U}\right]_{\substack{B=1,\\ U=0}} = v\left(H - x\frac{\mathrm{d}H}{\mathrm{d}x}\right) - \left(\frac{v^2}{2} + 1\right)\frac{\partial H}{\partial v}, \tag{A 15b}$$

$$\Omega_U(v) \equiv \left[\frac{\partial \Omega^*}{\partial U}\right]_{\substack{B=1,\\ U=0}} = \frac{3}{2}v\Omega - \left(\frac{v^2}{2} + 1\right)\frac{\mathrm{d}\Omega}{\mathrm{d}v}, \tag{A 15c}$$

$$M_U(v) \equiv \left[\frac{\partial M^*}{\partial U}\right]_{\substack{B=1,\\ U=0}} = 2vM - \left(\frac{v^2}{2} + 1\right)\frac{\mathrm{d}M}{\mathrm{d}v}, \tag{A 15d}$$

$$T_U(v) \equiv \left[\frac{\partial T^*}{\partial U}\right]_{\substack{B=1,\\ U=0}} = 3vT - \left(\frac{v^2}{2} + 1\right)\frac{\mathrm{d}T}{\mathrm{d}v}. \tag{A 15e}$$

Note that $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn}) = (\Phi_B, H_B)$ is a particular solution of the inhomogeneous equations (3.4)–(3.7) with $(F_n, G_n, I_n) = (0, 0, 1)$ and $(\hat{\phi}_{Cn}, \hat{\eta}_{Cn}) = (\Phi_U, H_U)$ is a solution of the homogeneous equations (3.4)–(3.7) with $(F_n, G_n, I_n) = (0, 0, 0)$.

A.3. *Finite-amplitude cnoidal wave solution*

Let us find a periodic solution of the form (2.5) with (2.6) when the wavelength is long compared with the depth of the fluid and call it a (finite-amplitude) cnoidal wave solution. Specifically, we seek the solution (2.5) that satisfies (2.6)–(2.10) asymptotically for small k with a given finite a. It is convenient to use the extended solitary wave solution $(\varPhi^*, H^*; v^*)$ for $B=1$ because it already satisfies (2.7)–(2.10). All we have to do here is to choose the remaining parameter U so as to satisfy the periodic condition (2.6) at $x=\pm\pi/k$. In fact, if we choose U such that $2\pi U/k=\varOmega^*$ (see (A 13*a*)), (2.6) is satisfied, and we have

$$U=U_0\equiv\sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{k}{2\pi}\right)^n\left[\frac{\partial^{n-1}\varOmega^{*n}}{\partial U|_a^{n-1}}\right]_{\substack{B=1\\U=0}}, \tag{A 16}$$

where $\partial/\partial U|_a$ represents the derivative with respect to U keeping a and B constant, and the quantities in the square brackets with subscript $B=1$, $U=0$ are evaluated at $B=1$ and $U=0$. Thus, the cnoidal wave solution $(\bar{\phi}, \bar{\eta}; C)$ is given by the extended solitary wave solution $(\varPhi^*, H^*; v^*)$ for $B=1$ and $U=U_0$. Thus, we have

$$\begin{cases}\bar{\phi}=\varPhi+k\varPhi_1+k^2\varPhi_2+\cdots\\ \bar{\eta}=H+kH_1+k^2H_2+\cdots\\ C=v(a)+kc_1+k^2c_2+\cdots\end{cases}\quad\text{for small } k, \tag{A 17a–c}$$

where the leading-order solution $(\varPhi, H; v)$ is the original solitary wave solution having the wave speed $v(a)$. The higher-order components are

$$(\varPhi_n, H_n; c_n)=\frac{1}{(2\pi)^n\,n!}\left[\frac{\partial^{n-1}}{\partial U|_a^{n-1}}\left(\varOmega^{*n}\frac{\partial(\varPhi^*, H^*; v^*)}{\partial U|_a}\right)\right]_{\substack{B=1\\U=0}}, \tag{A 18}$$

which can be expressed in terms of the solitary wave solution $(\varPhi, H; v)$ using (A 12). Specifically, for $n=1$,

$$\varPhi_1=\frac{\varOmega}{2\pi}\left[x+v\left(\frac{3}{2}\varPhi-x\frac{\partial\varPhi}{\partial x}-z\frac{\partial\varPhi}{\partial z}-a\frac{\partial\varPhi}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}a}\right)\right], \tag{A 19a}$$

$$H_1=\frac{\varOmega}{2\pi}v\left(H-x\frac{\mathrm{d}H}{\mathrm{d}x}-a\frac{\partial H}{\partial v}\frac{\mathrm{d}v}{\mathrm{d}a}\right), \tag{A 19b}$$

$$c_1=\frac{\varOmega}{2\pi}\left(\frac{v^2}{2}+1-va\frac{\mathrm{d}v}{\mathrm{d}a}\right), \tag{A 19c}$$

where $\varOmega$ is defined by (A 7*b*).

Appendix B. Some relations

Some relations used in §§ 3 and 4 are listed here. The first three are associated with the linear operators L_{Kn} and L_{Dn} introduced in (3.5) and (3.6):

$$L_{\{K,D\}n}\left[\frac{\partial\varPhi}{\partial x}, \frac{\mathrm{d}H}{\mathrm{d}x}\right]+\sum_{j=0}^{n-1}L_{\{K,D\}j}\left[\frac{\partial\varPhi_{n-j}}{\partial x}, \frac{\mathrm{d}H_{n-j}}{\mathrm{d}x}\right]=0\quad(n=1,2,\ldots), \tag{B 1}$$

$$\begin{aligned}&\int_{-\infty}^{\infty}\left[\frac{\partial\varPhi}{\partial x}\hat{\eta}-\frac{\mathrm{d}H}{\mathrm{d}x}\hat{\phi}\right]_{z=H}\mathrm{d}x\\ &=\int_{-\infty}^{\infty}\left\{-\int_0^H\frac{\partial\varPhi}{\partial v}\nabla_\perp^2\hat{\phi}\mathrm{d}z+\left[-\frac{\partial\varPhi}{\partial v}L_{K0}\left[\hat{\phi}, \hat{\eta}\right]+\frac{\partial H}{\partial v}L_{D0}\left[\hat{\phi}, \hat{\eta}\right]\right]_{z=H}\right\}\mathrm{d}x\\ &\quad-\left[\frac{\partial\varPhi}{\partial v}\left(\int_0^H\frac{\partial\hat{\phi}}{\partial x}\mathrm{d}z-v\hat{\eta}\right)\right]_{x\to-\infty}+\left[\frac{\partial\varPhi}{\partial v}\left(\int_0^H\frac{\partial\hat{\phi}}{\partial x}\mathrm{d}z-v\hat{\eta}\right)\right]_{x\to\infty},\end{aligned} \tag{B 2}$$

$$\int_{-\infty}^{\infty}\left[\frac{\partial \Phi}{\partial x}L_{K1}\left[\hat{\phi},\hat{\eta}\right]-\frac{\mathrm{d}H}{\mathrm{d}x}L_{D1}\left[\hat{\phi},\hat{\eta}\right]\right]_{z=H}\mathrm{d}x$$
$$=\int_{-\infty}^{\infty}\left\{-\int_{0}^{H}\left(\frac{\partial \Phi_1}{\partial x}-\frac{\Omega}{2\pi}\right)\nabla_{\perp}^{2}\hat{\phi}\mathrm{d}z\right.$$
$$+\left[-\frac{\partial \Phi}{\partial x}H_1\nabla_{\perp}^{2}\hat{\phi}-\left(\frac{\partial \Phi_1}{\partial x}-\frac{\Omega}{2\pi}+\frac{\partial^2 \Phi}{\partial x\partial z}H_1\right)L_{K0}\left[\hat{\phi},\hat{\eta}\right]\right.$$
$$\left.\left.+\frac{\mathrm{d}H_1}{\mathrm{d}x}L_{D0}\left[\hat{\phi},\hat{\eta}\right]\right]_{z=H}\right\}\mathrm{d}x, \tag{B 3}$$

where L_{Kn} and L_{Dn} are specifically given by

$$L_{K0}\left[\hat{\phi},\hat{\eta}\right]=\left(-\frac{\partial}{\partial z}+\frac{\mathrm{d}H}{\mathrm{d}x}\frac{\partial}{\partial x}\right)\hat{\phi}+\left(\frac{\partial}{\partial x}+\frac{\mathrm{d}H}{\mathrm{d}x}\frac{\partial}{\partial z}\right)\left[\left(-v+\frac{\partial \Phi}{\partial x}\right)\hat{\eta}\right], \tag{B 4a}$$

$$L_{D0}\left[\hat{\phi},\hat{\eta}\right]=\left[\left(-v+\frac{\partial \Phi}{\partial x}\right)\frac{\partial}{\partial x}+\frac{\partial \Phi}{\partial z}\frac{\partial}{\partial z}\right]\hat{\phi}+\left[\left(-v+\frac{\partial \Phi}{\partial x}\right)\frac{\partial^2 \Phi}{\partial x\partial z}+\frac{\partial \Phi}{\partial z}\frac{\partial^2 \Phi}{\partial z^2}+1\right]\hat{\eta}, \tag{B 4b}$$

$$L_{K1}\left[\hat{\phi},\hat{\eta}\right]=\left(\frac{\partial}{\partial x}+\frac{\mathrm{d}H}{\mathrm{d}x}\frac{\partial}{\partial z}\right)\left[H_1\frac{\partial\hat{\phi}}{\partial x}+\left(-c_1+\frac{\partial \Phi_1}{\partial x}+\frac{\partial^2 \Phi}{\partial x\partial z}H_1\right)\hat{\eta}\right]$$
$$-H_1\nabla_{\perp}^{2}\hat{\phi}, \tag{B 4c}$$

$$L_{D1}\left[\hat{\phi},\hat{\eta}\right]=\left[\left(-c_1+\frac{\partial \Phi_1}{\partial x}+\frac{\partial^2 \Phi}{\partial x\partial z}H_1\right)\frac{\partial}{\partial x}+\left(\frac{\partial \Phi_1}{\partial z}+\frac{\partial^2 \Phi}{\partial z^2}H_1\right)\frac{\partial}{\partial z}\right.$$
$$\left.+H_1\left(\left(-v+\frac{\partial \Phi}{\partial x}\right)\frac{\partial^2}{\partial x\partial z}+\frac{\partial \Phi}{\partial z}\frac{\partial^2}{\partial z^2}\right)\right]\hat{\phi}$$
$$+\hat{\eta}\frac{\partial}{\partial z}\left[\left(-v+\frac{\partial \Phi}{\partial x}\right)\left(-c_1+\frac{\partial \Phi_1}{\partial x}+\frac{\partial^2 \Phi}{\partial x\partial z}H_1\right)\right.$$
$$\left.+\left(\frac{\partial \Phi_1}{\partial z}+\frac{\partial^2 \Phi}{\partial z^2}H_1\right)\frac{\partial \Phi}{\partial z}\right],\ldots, \tag{B 4d}$$

with Φ_n, H_n and c_n being defined by (A 18). $\hat{\phi}(x,z)$ in (B 2) and (B 3) is an arbitrary function of x and z satisfying $\partial\hat{\phi}/\partial z=0$ at $z=0$, and $\hat{\eta}(x)$ is an arbitrary function of x. We can obtain (B 1) ($L_{\{K,D\}n}$ represents L_{Kn} or L_{Dn}) by differentiating (2.8) and (2.9) with respect to x and substituting (A 17). The right-hand side of (B 2) is derived by replacing $\mathrm{d}H/\mathrm{d}x$ and $\partial\Phi/\partial x$ on the left-hand side of (B 2) with $L_{K0}[\partial\Phi/\partial v,\partial H/\partial v]$ and $L_{D0}[\partial\Phi/\partial v,\partial H/\partial v]$ (which are obtained by differentiating (A 4) and (A 5) with respect to v), respectively, and integrating the result by parts. Equation (B 3) is obtained by collecting the first-order terms in k of

$$\int_{-\infty}^{\infty}\left\{\int_{0}^{\bar{\eta}}\left(\frac{\partial\bar{\phi}}{\partial x}-U_0\right)\nabla_{\perp}^{2}\hat{\phi}+\left[\left(\frac{\partial\bar{\phi}}{\partial x}-U_0\right)L_K\left[\hat{\phi},\hat{\eta}\right]-\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}x}L_D\left[\hat{\phi},\hat{\eta}\right]\right]_{z=\bar{\eta}}\right\}\mathrm{d}x=0, \tag{B 5}$$

where $(\bar{\phi}, \bar{\eta})$ is the finite-amplitude cnoidal wave solution given by (A 17*a,b*), and U_0 is defined by (A 16). We can check the equality (B 5) by applying integration by parts and using the x derivatives of (2.7)–(2.10).

The next relation is associated with the integral properties of the solitary wave:

$$\frac{3}{2}\left(-\varOmega + v\frac{\mathrm{d}\varOmega}{\mathrm{d}v}\right) = 2vM - \left(\frac{v^2}{2}+1\right)\frac{\mathrm{d}M}{\mathrm{d}v}, \tag{B 6}$$

where $\varOmega$ and M are defined by (A 7*b,c*). (B 6) can be divided into the following two equalities:

$$3(E-T) = (v^2-1)M, \quad \frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{2T}{v}\right) = \frac{1}{v}\frac{\mathrm{d}E}{\mathrm{d}v}, \tag{B 7\textit{a,b}}$$

where T and E are defined by (A 7*d,e*). The relation (B 7*a*) was derived by Starr (1947) (see also Longuet-Higgins 1974). We will derive (B 7*b*).

To derive (B 7*b*), we express the kinetic energy T as

$$T = -\frac{v}{2}\int_{-\infty}^{\infty}[\varPhi]_{z=H}\frac{\mathrm{d}H}{\mathrm{d}x}\mathrm{d}x,$$

where $[\varPhi]_{z=H}$ represents $\varPhi$ at $z=H$ and use is made of (A 2)–(A 4), (A 6) and Gauss's divergence theorem. We then obtain

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\frac{2T}{v}\right) = -\frac{\mathrm{d}}{\mathrm{d}v}\int_{-\infty}^{\infty}[\varPhi]_{z=H}\frac{\mathrm{d}H}{\mathrm{d}x}\mathrm{d}x = \int_{-\infty}^{\infty}\left[\frac{\partial\varPhi}{\partial x}\frac{\partial H}{\partial v} - \frac{\mathrm{d}H}{\mathrm{d}x}\frac{\partial\varPhi}{\partial v}\right]_{z=H}\mathrm{d}x = \frac{1}{v}\frac{\mathrm{d}E}{\mathrm{d}v}, \tag{B 8}$$

where (A 4) and (A 5) are substituted for $\mathrm{d}H/\mathrm{d}x$ and $\partial\varPhi/\partial x$, respectively, in the last relation.

Appendix C. Far-field solution and matching (type II resonance)

This section continues the asymptotic analysis in § 4.1.

C.1. *Far-field solution*

Introducing the reduced coordinate X defined by (3.17), we seek a solution $(\hat{\phi}, \hat{\eta})$ of (2.14)–(2.18) that produces moderate variation in X and z in the following power series of $k^{1/2}$:

$$\hat{\phi}_F(X,z) = k^{1/2}\hat{\phi}_{F1/2} + k\hat{\phi}_{F1} + k^{3/2}\hat{\phi}_{F3/2} + \cdots, \tag{C 1\textit{a}}$$

$$\hat{\eta}_F(X) = k^{3/2}\hat{\eta}_{F3/2} + k^2\hat{\eta}_{F2} + k^{5/2}\hat{\eta}_{F5/2} + \cdots, \tag{C 1\textit{b}}$$

where the series of (C 1) start at $O(k^{1/2})$ and $O(k^{3/2})$ for $\hat{\phi}_F$ and $\hat{\eta}_F$, respectively, because the core solution $(\hat{\phi}_C, \hat{\eta}_C)$ has non-zero values as $x \to \pm\infty$ from these orders (see (4.3*b*) and (4.6*b*)).

Substituting (3.17), (4.1), (4.2*c*), (A 17) and (C 1) into (2.14)–(2.18) and arranging the same-order terms in $k^{1/2}$, we obtain a series of sets of equations for $\hat{\phi}_{Fn}$ $(n = 1/2, 1, 3/2, \ldots)$. For $n = 1/2$, 1, 3/2 and 2, they are homogeneous ($\partial^2\hat{\phi}_{Fn}/\partial z^2 = 0$ for $0 < z < 1$ and $\partial\hat{\phi}_{Fn}/\partial z = 0$ at $z = 0$ and 1) and have a solution independent of z satisfying the periodic condition at $X = \pm\pi$, i.e.

$$\hat{\phi}_{Fn} = \hat{\phi}_{Fn}(X) \quad (n = 1/2, 1, 3/2, 2), \tag{C 2}$$

where

$$\hat{\phi}_{Fn}(-\pi)=\hat{\phi}_{Fn}(\pi), \quad \frac{d\hat{\phi}_{Fn}}{dX}(-\pi)=\frac{d\hat{\phi}_{Fn}}{dX}(\pi). \tag{C 3}$$

For $n=5/2$ and 3, the equations are inhomogeneous, i.e.

$$\frac{\partial^2\hat{\phi}_{Fn}}{\partial z^2}=J_n\equiv\left[q_0^2-\left(\frac{d}{dX}+ip_0\right)^2\right]\hat{\phi}_{Fn-2} + 2\left[q_0q_{1/2}-ip_{1/2}\left(\frac{d}{dX}+ip_0\right)\right]\hat{\phi}_{Fn-5/2} \quad \text{for } 0<z<1, \tag{C 4}$$

$$\frac{\partial\hat{\phi}_{Fn}}{\partial z}=0 \quad \text{at } z=0, \tag{C 5}$$

$$\frac{\partial\hat{\phi}_{Fn}}{\partial z}=L_n\equiv-\left[i\sigma_0+v\left(\frac{d}{dX}+ip_0\right)\right]\hat{\eta}_{Fn-1}-i\left(\sigma_{1/2}+vp_{1/2}\right)\hat{\eta}_{Fn-3/2} \quad \text{at } z=1, \tag{C 6}$$

$$\hat{\phi}_{Fn}(-\pi,z)=\hat{\phi}_{Fn}(\pi,z), \tag{C 7}$$

$$(n=5/2 \text{ and } 3)$$

where

$$\hat{\eta}_{Fn-1}=\left[i\sigma_0+v\left(\frac{d}{dX}+ip_0\right)\right]\hat{\phi}_{Fn-2}+i\left(\sigma_{1/2}+vp_{1/2}\right)\hat{\phi}_{Fn-5/2}, \tag{C 8}$$

and $\hat{\phi}_{F0}=\hat{\eta}_{F1}=0$. For (C 4)–(C 7) to have a solution, their inhomogeneous terms J_n and L_n on the right-hand sides of (C 4) and (C 6) must satisfy the solvability condition (3.26). This condition (3.26) for $n=5/2$, subject to the periodic condition (C 3) and matching with the core solution (4.3*b*) (which is made here in advance of § C.2 for the sake of concise analytical process), leads to

$$\hat{\phi}_{F1/2}=\beta. \tag{C 9}$$

Here $\hat{\eta}_{F3/2}$ is obtained from (C 8) as

$$\hat{\eta}_{F3/2}=i(\sigma_0+vp_0)\beta. \tag{C 10}$$

For $n=3$, the solvability condition (3.26), subject to (C 3), determines $\hat{\phi}_{F1}$ as

$$\hat{\phi}_{F1}=\frac{\sigma_{1/2}-c_xp_{1/2}-c_yq_{1/2}}{c_x}i\chi(X)\beta+\bar{\beta}\exp\left[\frac{2b\sigma_0}{v^2-1}i\chi(X)\right], \tag{C 11}$$

where $\bar{\beta}$ is a new undetermined constant, $\chi(X)$ is given by (3.30), and (c_x, c_y) are defined by (3.32*a*,*b*), in which σ_0, p_0 and q_0 satisfy (4.1*c*) and (4.3*c*). Here $\hat{\eta}_{F2}$ is obtained from (C 8) as

$$\hat{\eta}_{F2}=\left[(\sigma_0+vp_0)\frac{-\sigma_{1/2}+c_xp_{1/2}+c_yq_{1/2}}{c_x}\chi(X)+i\frac{p_0\sigma_{1/2}-vq_0q_{1/2}}{b\sigma_0}\right]\beta + i\left[\sigma_0+v\left(p_0+\frac{2b\sigma_0}{v^2-1}\right)\right]\bar{\beta}\exp\left[\frac{2b\sigma_0}{v^2-1}i\chi(X)\right]. \tag{C 12}$$

C.2. *Matching*

See the first paragraph of § 3.3 for the matching procedure. For $n = 1/2$, matching is already achieved by (4.3*b*) and (C 9). For $n = 1$, because $\hat{\phi}^*_{F1} = (\hat{\phi}_{F1})_0$ (the asterisk indicates a reordered form), we have

$$\left[\hat{\phi}_{C1}\right]_{x\to\pm\infty} = \mp \pi \mathrm{i} \frac{\sigma_{1/2} - c_x p_{1/2} - c_y q_{1/2}}{c_x} \beta + \bar{\beta} \exp\left(\mp \mathrm{i} \frac{2\pi b \sigma_0}{v^2 - 1}\right). \tag{C 13}$$

(Both upper or both lower signs should be chosen for the double signs in (C 13) and (C 15) below.)

For $n = 3/2$ and 2, because $\hat{\phi}^*_{Fn} = (\hat{\phi}_{Fn})_0 + x\,(\mathrm{d}\hat{\phi}_{Fn-1}/\mathrm{d}X)_0$, two different types of terms appear: those independent of x and those proportional to x. The conditions among those proportional to x contribute to the determination of unknowns at this order. For $n = 3/2$, they become

$$\left[\frac{\partial \hat{\phi}_{C3/2}}{\partial x}\right]_{x\to\pm\infty} = 0, \tag{C 14}$$

and for $n = 2$,

$$\begin{aligned}\left[\hat{u}_{C2}\right]_{x\to\pm\infty} = {} & \left[\mathrm{i}\left(\frac{\sigma_{1/2}}{b} - p_{1/2} + \frac{v^2 - 1}{b\sigma_0} q_0 q_{1/2}\right) \pm \pi b \sigma_0 \frac{\sigma_{1/2} - c_x p_{1/2} - c_y q_{1/2}}{c_x}\right] \beta \\ & - \mathrm{i} b \sigma_0 \bar{\beta} \exp\left(\mp \mathrm{i} \frac{2\pi b \sigma_0}{v^2 - 1}\right),\end{aligned} \tag{C 15}$$

where $\hat{u}_{C2}$ is defined by (3.16).

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