## 106.11 On a synthetic proof of Bottema's theorem

Bottema's theorem [1] concerns squares *BFGC* and *AEDC* which are constructed externally on the sides of a triangle *ABC*, as in Figure 1. It states that the midpoint of the segment *EF* is independent of the position of *C*.

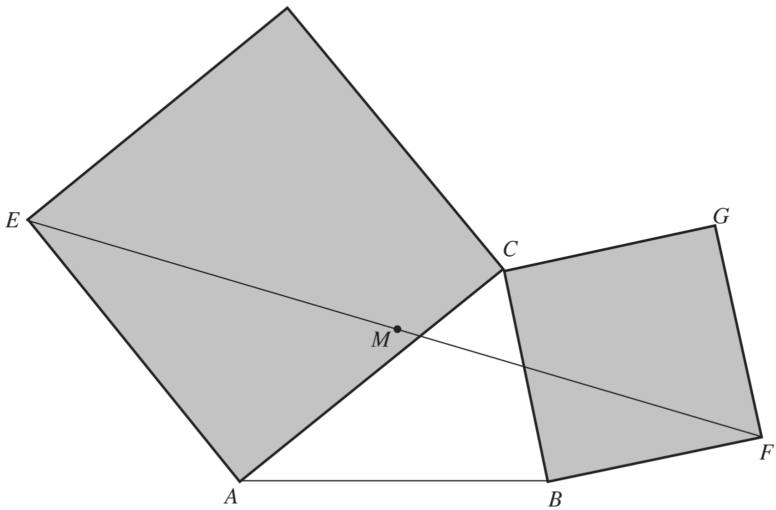


FIGURE 1: Bottema's theorem. The midpoint *M* of segment *EF* does not depend on the vertex *C* of the triangle.

This seems surprising at first sight but the following proof illustrates why point *C* is irrelevant.

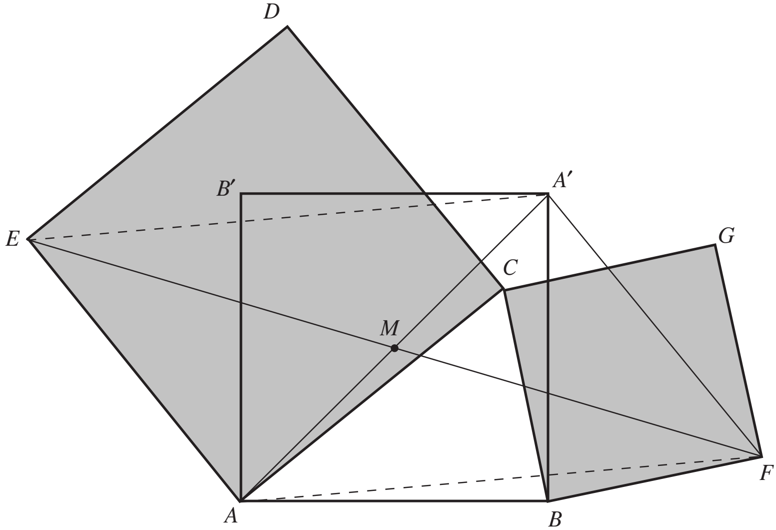


FIGURE 2. Sketch for a proof of Bottema's theorem

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Consider the square $ABA'B'$ constructed on the same side of $AB$ as $C$, as in Figure 2. Then, since they are copies of triangle $ABC$ under rotations of 90° (in opposite directions), the triangles $A'BF$ and $AB'E$ are congruent to one another, with $A'F$ parallel to $EA$. Hence $AFA'E$ is a parallelogram and its diagonals bisect one another at $M$. As this is the midpoint of $AA'$ as well as that of $EF$, it is independent of $C$.

It is clear that this argument works equally for squares which are constructed internally on the sides of the triangle $ABC$.

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*Reference*

1. A. Bogomolny, (1996). https://www.cut-the-knot.org.

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## 106.12 A new proof of the *n*-dimensional Pythagorean theorem

We shall use a useful tool of functional analysis, Parseval's identity, to give a new proof for the $n$-dimensional Pythagorean theorem in [1]. We recall Parseval's identity in $n$-dimensional Euclidean space $\mathbb{R}^n$ as follows.

*Theorem* 1: (Parseval's identity [2]). Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be an orthonormal basis of $n$-dimensional Euclidean space $\mathbb{R}^n$. Then for every vector $\vec{u} \in \mathbb{R}^n$, we have

$$|\vec{u}|^2 = (\vec{u} \cdot \vec{e}_1)^2 + (\vec{u} \cdot \vec{e}_2)^2 + \ldots + (\vec{u} \cdot \vec{e}_n)^2. \quad (1)$$

Throughout this Note, in $n$-dimensional Euclidean space $\mathbb{R}^n$, we denote by $XY$ the Euclidean distance between two points $X$ and $Y$, and $\overrightarrow{XY}$ the Euclidean vector connecting an initial point $X$ with a terminal point $Y$. We recall the $n$-dimensional Pythagorean theorem in [1].

*Theorem* 2: ($n$-dimensional Pythagorean theorem). In $n$-dimensional Euclidean space $\mathbb{R}^n$, if the edges $OP_1, OP_2, \ldots, OP_n$ of a simplex $OP_1P_2 \ldots P_n$ are all perpendicular, and if the bounding simplexes opposite to the vertices $O, P_1, P_2, \ldots, P_n$ have $(n-1)$-dimensional contents $A, A_1, A_2, \ldots, A_n$ respectively, then

$$A^2 = A_1^2 + A_2^2 + \ldots + A_n^2. \quad (2)$$

*Proof*: Let $H$ be the orthogonal projection of $O$ on hyperplane $(P_1P_2 \ldots P_n)$, and let V be volume of the simplex $OP_1P_2 \ldots P_n$. Since the edges