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SPIN CHAINS WITH TWISTED MONODROMY

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Abstract The integrable model corresponding to the $\mathcal{N} = 2$ supersymmetric SU(N) gauge theory with matter in the symmetric representation is constructed. It is a spin chain model, whose key feature is a new twisted monodromy condition.

Keywords: supersymmetric gauge theories; spectral curves; symplectic forms

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1. Introduction

The 1994 work of Seiberg and Witten [**14**, **15**] revealed the existence of a deep correspondence between supersymmetric gauge theories and integrable models [**5**, **7**, **12**, **13**]. However, the specific list of correspondences is still far from complete at the present time (cf. [**1**–**4**] and references therein). In particular, it is still not known how to construct the integrable model which corresponds to the $\mathcal{N} = 2$, SU(N) gauge theory with matter in the symmetric representation, although the spectral curve has been identified by Landsteiner and Lopez using M Theory [**6**, **11**].

The purpose of this paper is to solve this problem. In [**10**], we had solved a similar, but simpler problem, which is to construct the integrable model corresponding to the SU(N) gauge theory with matter in the antisymmetric representation. The main new difficulty is the asymmetry between the orders of the zero and the pole of the eigenvalues of the monodromy operator at the two compactification points of the spectral curve. The desired integrable model turns out to be still a spin chain p_n, q_n, but whose main feature is a new periodicity condition linking p_{n+N+2}, q_{n+N+2} to p_n, q_n through a twisted monodromy operator. Such periodicity conditions have not appeared before in the literature, and we take the opportunity to discuss them in some detail. An earlier proposal of how they can be used to construct other integrable models is in [**8**], but not pursued further there.

The mathematical problem can be formulated very simply. It is to find an integrable Hamiltonian system with spectral parameter x, spectral curve

$$R(x, y) \equiv y^3 + f_N(x)y^2 + f_N(-x)x^2y + x^6 = 0 \tag{1.1}$$

and symplectic form

$$\omega = \sum_{i=1}^{2N-2} \delta x(z_i) \wedge \frac{\delta y}{y}(z_i).$$

Here $f_N(x) = \sum_{i=0}^{N} u_i x^i$ is a generic polynomial of degree N, and the parameters u_i can be viewed as the moduli of the spectral curve. A system with the desired properties can be obtained as follows. Let q_n, p_n be complex three-dimensional (column) vectors satisfying $q_n^{\mathrm{T}} p_n = 0$ and the reflexivity condition $p_n = hp_{-n-1}$, $q_n = hq_{-n-1}$, where h is the 3×3 matrix whose only non-zero entries are $h_{31} = h_{22} = h_{13} = 1$. Let a, b, c be 3×3 matrices satisfying

$$a^2 = 1, \qquad ab = ba, \qquad b^2 = ac + ca, \qquad bc = cb, \qquad c^2 = 0. \tag{1.2}$$

Consider the dynamical system

$$\left.\begin{aligned} \dot{p}_n &= \frac{p_{n+1}}{p_{n+1}^{\mathrm{T}} q_n} + \frac{p_{n-1}}{p_{n-1}^{\mathrm{T}} q_n} + \mu_n p_n, \qquad \dot{q}_n = -\frac{q_{n+1}}{p_n^{\mathrm{T}} q_{n+1}} - \frac{q_{n-1}}{p_n^{\mathrm{T}} q_{n-1}} - \mu_n p_n, \\ \dot{a} &= \left\{ \frac{q_{m-1} p_m^{\mathrm{T}}}{p_m^{\mathrm{T}} q_{m-1}} - \frac{q_m p_{m-1}^{\mathrm{T}}}{p_{m-1}^{\mathrm{T}} q_m}, b \right\}, \qquad \dot{b} = \left\{ \frac{q_{m-1} p_m^{\mathrm{T}}}{p_m^{\mathrm{T}} q_{m-1}} - \frac{q_m p_{m-1}^{\mathrm{T}}}{p_{m-1}^{\mathrm{T}} q_m}, c \right\}, \qquad \dot{c} = 0, \end{aligned}\right\} \tag{1.3}$$

where $\mu_n(t)$ is an arbitrary scalar function, and we have set $m = -\frac{1}{2}N + 1$ for N even and $m = -\frac{1}{2}N + \frac{1}{2}$ for N odd. The system (1.3) appears uncoupled, but it will not be after imposing twisted monodromy conditions. More precisely, we have the following.

Main Theorem. *Let x be an external parameter, and set $L_n(x) = 1 + x q_n p_n^{\mathrm{T}}$. Then the following holds.*

(a) *There are unique 3×3 matrices $g_n(x) = a_n x^2 + b_n x + c_n$ which satisfy the periodicity condition*

$$g_{n+1} L_{n+N-2} = L_n g_n \tag{1.4}$$

for any fixed data a_r, b_r, c_r, $(p_n, q_n)_{n=r}^{n=r+N-3}$ with the constraint $q_n^{\mathrm{T}} p_n = 0$.

(b) *Consider the dynamical system (1.3) with $a_m = ah$, $b_m = bh$, $c_m = ch$. Then the system is integrable in the sense that it is equivalent to the following Lax equation*

$$\dot{L}_n = M_{n+1} L_n - L_n M_n, \tag{1.5}$$

where $M_n(x)$ is the 3×3 matrix defined by

$$M_n(x) = x \left(\frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n} \right). \tag{1.6}$$

(c) *The spectral curve $\Gamma = \{(x, y); \det(yI - g_n(x) L_{n+N-3}(x) \cdots L_n(x)) = 0\}$ for the Lax equation (1.5) is independent of n. It coincides with the Landsteiner–Lopez curve (1.1), and the system (1.3) is Hamiltonian with respect to the symplectic form ω on the reduced phase space $u_N = 1$, $u_{N-1} = 0$. The Hamiltonian is $H = u_{N-2}$.*

2. The Landsteiner–Lopez curve for the symmetric representation

It is convenient to list here the main geometric properties of the curve (1.1), which will henceforth be referred to as the LL (Landsteiner–Lopez) curve. It admits the following involution

$$\sigma : (x, y) \to \left(-x, \frac{x^4}{y}\right). \tag{2.1}$$

Points above $x = \infty$ and $x = 0$

Above $x = \infty$, there are three distinct solutions of the LL equation, given by $y \sim x^N$, $y \sim x^{-N+4}$ and $y \sim x^2$. The involution σ interchanges the first two, leaving the third one fixed.

Above $x = 0$, there are also three solutions, given by $y \sim x^2$, $y \sim x^4$ and $y \sim 1$. The involution interchanges the last two points, while the first one is left fixed. The points $y \sim x^2$ and $y \sim x^4$ cross each other, but they are not branching points.

Genus of the LL curve

The Riemann–Hurwitz formula says that the genus g of the LL curve is given by $2g - 2 = -6 + \nu$, where ν is the number of branching points (for generic moduli, the branching index is 2, which we assume). The branching points correspond to zeros of $\partial_y R(x, y)$

$$3y^2 + 2y f_N(x) + f_N(-x)x^2 = 0.$$

To determine their number, we determine the number of poles of $\partial_y R(x, y)$. These occur at $x = \infty$. At $x = \infty$, the three solutions $y \sim x^N$, $y \sim x^2$ and $y \sim x^{-N+4}$ contribute, respectively, $2N$, $N + 2$ and $N + 2$ poles, for a total of $4N + 4$ poles. Thus there are also $4N + 4$ zeros. At $x = 0$, there are two zeros $y \sim x^2$ and $y \sim x^4$, at each of which $\partial_y R(x, y)$ vanishes of second order. Thus the number ν of branching points is given by $\nu = 4N + 4 - 2 - 2 = 4N$, and consequently

$$\text{genus}(\Gamma) = 2N - 2.$$

Genus of the quotient curve

Let g_0 be the genus of $\Gamma_0 = \Gamma/\sigma$. Since Γ has two branch points over Γ_0, namely $y \sim x^2$ at $x = \infty$ and $y = x^2$ at $x = 0$, the Riemann–Hurwitz formula applies and gives $2g - 2 = 2(2g_0 - 2) + 2$, from which it follows that

$$g_0 = N - 1. \tag{2.2}$$

General case

The LL curve can be seen as a special case of a general family of curves defined by the equation

$$R(x, y) \equiv y^3 + f_N(x)y^2 + g_{N+2}(x)y + r_6(x) = 0, \tag{2.3}$$

where g_{N+2} and $r_6(x)$ are polynomials of degree $N+2$ and 6, respectively. This family has $2N+11$ moduli. The genus of a curve Γ defined by this equation can be found as before. It equals

$$\mathrm{genus}(\Gamma) = 2N. \tag{2.4}$$

3. Construction of the spin chain

We now give the proof of the Main Theorem. Since finding the desirable integrable model is an essential component of our result, we construct the model gradually instead of proceeding from its final description. It is natural to look for a spin chain of three-dimensional vectors with a period of $N+2$ spins, in order to arrive at a spectral curve of the form (2.3). The difficult steps are to create an involution of the form (2.1) and to obtain the correct number of degrees of freedom.

The spin chain system

We look for a spin chain system of the form

$$\psi_{n+1} = L_n(x)\psi_n \tag{3.1}$$

with the operators $L_n(x)$ given by $L_n(x) = 1 + xq_np_n^{\mathrm{T}}$, where x is an external variable, and q_n, p_n are three-dimensional complex vectors satisfying the condition $q_n^{\mathrm{T}}p_n = 0$. The vectors q_n and p_n should be viewed as column vectors, so that $q_n^{\mathrm{T}}p_n$ is a scalar, while $q_np_n^{\mathrm{T}}$ is a 3×3 matrix.

Twisted monodromy conditions

The key feature of the construction is the imposition of suitable twisted boundary conditions. Now the usual periodicity condition $L_{n+N-2}(x) = L_n(x)$ can be expressed as $TL = LT$, if we define the monodromy operator to be $(T\psi)_n = \psi_{n+N-2}$. For the Landsteiner–Lopez curve, we require a twisted periodicity condition of the form

$$g_{n+1}(x)L_{n+N-2}(x) = L_n(x)g_n(x), \tag{3.2}$$

with the $g_n(x)$'s suitable 3×3 matrices to be chosen later. This requires in turn the following more subtle choice of monodromy operator $T_n(x)$

$$T_n(x) = g_n(x)\prod_{k=0}^{N-3} L_{k+n}(x), \tag{3.3}$$

where, by convention, the indices in the product of the L_{k+n}s are in decreasing order as we move from left to right. The twisted periodicity condition (3.2) is then equivalent to

$$T_{n+1}L_n = L_nT_n, \tag{3.4}$$

which implies that the eigenvalues of T_n are independent of n. We may thus define the spectral curve of the system L_n by

$$\Gamma = \{(x,y); \det(yI - T_n(x)) = 0\}. \tag{3.5}$$

Construction of $g_n(x)$

We look for $g_n(x)$ under the form $g_n(x) = a_n x^2 + b_n x + c_n$, where a_n, b_n, c_n are 3×3 matrices. The periodicity condition $g_{n+1} L_{n+N-2} = L_n g_n$ is equivalent to the following system of equations

$$\left.\begin{aligned} c_{n+1} &= c_n, \\ a_{n+1} q_{n+N-2} p^{\mathrm{T}}_{n+N-2} &= q_n p^{\mathrm{T}}_n a_n, \\ a_{n+1} + b_{n+1} q_{n+N-2} p^{\mathrm{T}}_{n+N-2} &= a_n + q_n p^{\mathrm{T}}_n b_n, \\ b_{n+1} + c_{n+1} q_{n+N-2} p^{\mathrm{T}}_{n+N-2} &= b_n + q_n p^{\mathrm{T}}_n c_n. \end{aligned}\right\} \tag{3.6}$$

We claim that this system can be solved completely in terms of the following parameters

$$\left.\begin{aligned} &a_r,\ b_r,\ c_r, \\ &(p_r, q_r), \dots, (p_{r+N-3}, q_{r+N-3}), \quad q^{\mathrm{T}}_n p_n = 0, \quad r \leqslant n \leqslant r+N-3, \end{aligned}\right\} \tag{3.7}$$

for any choice of initial index r. To see this, define p_{n+N-2}, q_{n+N-2} by

$$p^{\mathrm{T}}_{n+N-2} = p^{\mathrm{T}}_n a_n, \qquad q_{n+N-2} = \lambda_n^{-1} a_n^{-1} q_n, \tag{3.8}$$

with λ_n a scalar yet to be determined. Then $p^{\mathrm{T}}_{n+N-2} q_{n+N-2} = \lambda_n^{-1} q^{\mathrm{T}}_n p_n = 0$ and orthogonality is preserved. With $c_n = c_0$ for all n and p_{N-2+n}, q_{N-2+n} defined already as indicated, the last two equations in (3.6) can be viewed as recursion relations defining a_{n+1} and b_{n+1}. Our task is to show now that λ_n can be chosen so as to satisfy the second equation in (3.6), which we rewrite as

$$\lambda_n^{-1} a_{n+1} a_n^{-1} q_n p^{\mathrm{T}}_n a_n = q_n p^{\mathrm{T}}_n a_n.$$

Now the recursive equation for a_{n+1} implies that

$$a_{n+1} a_n^{-1} q_n + b_{n+1} \lambda_n^{-1} a_n^{-1} q_n p^{\mathrm{T}}_n a_n a_n^{-1} q_n = q_n + q_n p^{\mathrm{T}}_n b_n a_n^{-1} q_n.$$

The second term on the left-hand side vanishes since $p^{\mathrm{T}}_n q_n = 0$. Furthermore, the term $p^{\mathrm{T}}_n b_n a_n^{-1} q_n$ on the right-hand side is a scalar, so that the preceding equation implies that q_n is an eigenvector for the operator $a_{n+1} a_n^{-1}$. Thus the second equation in (3.6) is satisfied by choosing λ_n to be the corresponding eigenvalue

$$a_{n+1} a_n^{-1} q_n = \lambda_n q_n, \qquad \lambda_n = 1 + p^{\mathrm{T}}_n b_n a_n^{-1} q_n, \tag{3.9}$$

completing the recursive construction.

Note that the spectral curve corresponding to generic chain constructed above has the form (2.3). The dimension of the phase space equals $D = 27 + 6(N-2) - (N-2) - (N-2) - 8 = 4N + 11$, which is equal to the dimension of the Jacobian bundle over the family of curves defined by (2.3).

Involution on the spectral curve

We turn to the task of choosing the twisted monodromy so that the spectral curve admits the desired involution (2.1). Recall that the matrix h is given by $h_{ij} = 0$, except for $h_{13} = h_{22} = h_{31} = 1$. In particular, $h^2 = 1$. Let us impose the following constraints on the spin chain and the twisted monodromy conditions

$$p_n = hp_{-n-1}, \qquad q_n = hq_{-n-1}, \qquad g_n(-x)hg_{-n-N+2}(x)h = x^4. \tag{3.10}$$

The first two constraints imply

$$T_n(-x) = g_n(-x)h\left(\prod_{k=0}^{N-3} L^{-1}_{-k-n-1}(x)\right)h = g_n(-x)hT^{-1}_{-n-N+2}(x)g_{-n-N+2}(x)h.$$

Therefore, the last constraint implies that the spectral curve Γ admits the involution $(x, y) \to (-x, (x^4/y))$. Here we made use of the fact that $L_n(-x) = L_n(x)^{-1}$, which follows at once from the orthogonality condition $q_n^{\mathrm{T}} p_n = 0$. For generic choice of initial index m in (3.10) the second constraint is non-local in term of the corresponding parameters (3.7). It becomes local for a special choice of m. Let us assume for simplicity that N is even. Then we choose $m = -\frac{1}{2}N + 1$. The constraint (3.10) for $n = m$ is the following equation for g_m

$$g_m(-x)hg_m(x)h = x^4.$$

Then the last matrix equation in (3.10) is equivalent to the system of equations (1.2) for the matrices $a \equiv a_m h$, $b \equiv b_m h$ and $c \equiv c_m h$. The last equation in (1.2) implies that c is a traceless rank one matrix. Hence, it can be written in the form

$$c = \alpha\beta^{\mathrm{T}}, \qquad \beta^{\mathrm{T}}\alpha = 0,$$

where α, β are orthogonal three-dimensional vectors. The third equation can be solved for b in the form

$$b = \mu(a\alpha\beta^{\mathrm{T}} + \alpha\beta^{\mathrm{T}}a), \qquad \mu^2(\beta^{\mathrm{T}}a\alpha) = 1.$$

All the equations are satisfied for any α, β, and any choice of a such that $a^2 = 1$. We obtain in this way the crucial fact that the dimension of the admissible set of initial data $g_m \leftrightarrow (a_m, b_m, c_m)$ is equal to $8 = 4 + 4$. The first term is the dimension of matrices a and the second term is the dimension of orthogonal vectors α, β modulo transformation $\alpha \to \kappa\alpha$, $\beta \to \kappa^{-1}\beta$.

Degrees of freedom of the system

The system of vectors q_n, p_n with the orthogonality constraint has $5(N-2)$ degrees of freedom. The symmetry condition (3.10) reduces it to $\frac{5}{2}(N-2)$ (for say, N even). Now the system has the following gauge invariances.

- $(q_n, p_n) \to (\mu_n q_n, \mu_n^{-1} p_n)$. This removes $\frac{1}{2}(N-2)$ degrees of freedom.
- A global invariance $(q_n, p_n) \to (W^{-1} q_n, p_n)$ under 3×3 matrices W satisfying $Wh = hW$. Such matrices W are of the form

$$W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{21} \\ w_{13} & w_{12} & w_{11} \end{pmatrix}.$$

  Their space is five dimensional. However, one degree of freedom has already been accounted for since diagonal matrices are of the form of the preceding gauge invariance.

Thus the total number of gauge invariances is $\frac{1}{2}(N-2)+4$, and the number of degrees of freedom for the variables q_n, p_n is $2N-8$. A similar counting also produces the same number $2N-8$ of degrees of freedom for the system when N is odd. Now, as we saw in the previous section, the system a_m, b_m, c_m has eight degrees of freedom. Altogether, the number of degrees of freedom of our dynamical system is then

$$\#\,\text{degrees}\{a_0, b_0, c_0; (q_0, p_0), \ldots, (q_{N-3}, p_{N-3})\} = 2N, \tag{3.11}$$

which is the same as the dimension of the geometric phase space constructed out of the curve Γ/σ and its Jacobian.

The dynamical equations of motion for q_n, p_n

The equations of motion are determined by the matrix M_n completing L_n into a Lax pair with equations of motion $\dot{L}_n = M_{n+1} L_n - L_n M_n$. In this case, they are given by the matrices $M_n(x)$ in (1.6). We claim that the matrices M_n satisfy the following periodicity condition

$$M_{n+N-2}(x) = a_n^{-1} M_n(x) a_n. \tag{3.12}$$

In fact, the periodicity conditions for q_n and p_n imply that

$$M_{n+N-2}(x) = x\left(\frac{a_{n-1}^{-1} q_{n-1} p_n^{\mathrm{T}} a_n}{p_n^{\mathrm{T}} a_n a_{n-1}^{-1} q_{n-1}} - \frac{a_n^{-1} q_n p_{n-1}^{\mathrm{T}} a_{n-1}}{p_{n-1}^{\mathrm{T}} a_{n-1} a_n^{-1} q_n}\right).$$

Using the fact that q_{n-1} is an eigenvector of $a_n a_{n-1}^{-1}$, the first term on the right-hand side can be easily recognized as

$$a_n^{-1} \frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} a_n.$$

Similarly, the second term can also be rewritten as

$$a_n^{-1} \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n} a_n,$$

using the fact that p_n^{T} is an eigenvector (on the left) of the matrix $a_n a_{n+1}^{-1}$

$$p_n^{\mathrm{T}} a_n a_{n+1}^{-1} = \lambda_n p_n^{\mathrm{T}}. \tag{3.13}$$

To prove this identity, we use first the recursive relation defining a_{n+1} and obtain

$$p_n^{\mathrm{T}} a_{n+1} + \frac{1}{\lambda_n} p_n^{\mathrm{T}} b_{n+1} a_n^{-1} q_n p_n^{\mathrm{T}} a_n = p_n^{\mathrm{T}} a_n.$$

This implies already that p_n^{T} is an eigenvector

$$p_n^{\mathrm{T}} a_n a_{n+1}^{-1} = \left(1 - \frac{p_n^{\mathrm{T}} b_{n+1} a_n^{-1} q_n}{\lambda_n}\right)^{-1} p_n^{\mathrm{T}}.$$

It remains only to simplify the expression for the eigenvalue. This is done using the recurrence relation defining b_{n+1}:

$$\begin{aligned}\lambda_n - p_n^{\mathrm{T}} b_{n+1} a_n^{-1} q_n &= 1 + p_n^{\mathrm{T}} (b_n - b_{n+1}) a_n^{-1} q_n \\ &= 1 + p_n^{\mathrm{T}} \left(\frac{1}{\lambda_n} c_{n+1} a_n^{-1} q_n p_n^{\mathrm{T}} a_n - q_n p_n^{\mathrm{T}} c_n\right) a_n^{-1} q_n = 1.\end{aligned}$$

The proof of the relation (3.13) and hence of the periodicity relations for M_n is complete.

Equations of motion for a_m, b_m, c_m

Let Ψ_n be the solution of the equations $\Psi_n = L_n \Psi_n$, $\partial_t \Psi_n = M_n \Psi_n$, which is the eigenvector for the monodromy matrix

$$y \Psi_n = g_n \Psi_{n+N-2}.$$

Taking the derivative of the last equation we obtain

$$\dot{g}_n = M_n g_n - g_n M_{n+N-2}, \tag{3.14}$$

which is equivalent to the equations

$$\left.\begin{aligned}\dot{a}_n &= \left(\frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n}\right) b_n - b_n a_n^{-1} \left(\frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n}\right) a_n, \\ \dot{b}_n &= \left(\frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n}\right) c_n - c_n a_n^{-1} \left(\frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n}\right) a_n, \\ \dot{c}_n &= 0.\end{aligned}\right\} \tag{3.15}$$

The key consistency condition which has to be verified is that for $m = -\frac{1}{2}N + 1$, this dynamical system restricts to the variety of matrices $a = a_m h$, $b = b_m h$, $c = c_m h$ defined by the equation (1.2). Among these, the difficult equation to check is $b^2 = ac + ca$, and we turn to this next. Here we have assumed to be specific that N is even. The case of N odd is similar.

Relations at $m = -\frac{1}{2}N + 1$

We claim that the periodicity of the system together with their involution relations imply the following relations at $m = -\frac{1}{2}N + 1$:

$$\left.\begin{aligned} a_m^{-1}q_{m-1} &= hq_m, & p_m^{\mathrm{T}}a_m &= p_{m-1}^{\mathrm{T}}h, \\ a_m^{-1}q_m &= \lambda_m hq_{m-1}, & \lambda_{m-1}p_{m-1}^{\mathrm{T}}a_m &= p_m^{\mathrm{T}}h. \end{aligned}\right\} \tag{3.16}$$

To see this, we note that the periodicity conditions with $n = m$ and $n = m-1$ give, respectively,

$$\begin{aligned} q_{-m} &= \frac{1}{\lambda_m}a_m^{-1}q_m, & p_{-m}^{\mathrm{T}} &= p_m^{\mathrm{T}}a_m, \\ q_{-m-1} &= \frac{1}{\lambda_{m-1}}a_{m-1}^{-1}q_{m-1}, & p_{-m-1}^{\mathrm{T}} &= p_{m-1}^{\mathrm{T}}a_{m-1}. \end{aligned}$$

On the other hand, the involution relation with $n = -m$ and $n = -m-1$ gives

$$\begin{aligned} q_{-m} &= hq_{m-1}, & p_{-m} &= hp_{m-1}, \\ q_{-m-1} &= hq_m, & p_{-m-1} &= hp_m. \end{aligned}$$

Eliminating q_{-m}, p_{-m}, q_{-m-1}, p_{-m-1} between these relations, and applying the relation $a_n^{-1}q_n = \lambda_n a_{n+1}^{-1}q_n$, $p_n^{\mathrm{T}}a_n = \lambda_n p_n^{\mathrm{T}}a_{n+1}$, we obtain the desired relations.

In terms of a, b, c, the above relations imply in particular

$$p_{m-1}^{\mathrm{T}}a = \frac{1}{\lambda_{m-1}}p_m^{\mathrm{T}}, \qquad a^{-1}q_{m-1} = q_m. \tag{3.17}$$

We now claim that

$$\lambda_m = \lambda_{m-1} = 1. \tag{3.18}$$

In fact, the first and third relations in (3.16) imply at once $q_{m-1} = \lambda_{m-1}(a_m h)^2 q_{m-1}$, and hence $\lambda_{m-1} = 1$. Next we show that $\lambda_m = 1$. Recalling the expression (3.9) for λ_{m-1}, we may also write

$$\lambda_{m-1} = 1 + p_{m-1}^{\mathrm{T}}b_{m-1}a_{m-1}^{-1}q_{m-1} = 1 + p_m^{\mathrm{T}}a(b_{m-1}h)q_m,$$

using the facts that $p_{m-1}^{\mathrm{T}} = (1/\lambda_{m-1})p_m^{\mathrm{T}}a$ and $a_{m-1}^{-1}q_{m-1} = \lambda_{m-1}hq_{m-1}$. We use now the inductive relation on the b_ns:

$$b + ca^{-1}q_{m-1}p_{m-1}^{\mathrm{T}}a = (b_{m-1}h) + q_{m-1}p_{m-1}^{\mathrm{T}}c.$$

Substituting in the previous formula for λ_{m-1} gives

$$\lambda_{m-1} = 1 + p_m^{\mathrm{T}}a(b + ca^{-1}q_{m-1}p_{m-1}^{\mathrm{T}}a - q_{m-1}p_{m-1}^{\mathrm{T}}c)q_m = 1 + p_m^{\mathrm{T}}abq_m$$

since $p_m^{\mathrm{T}}a$ and aq_m are proportional to p_{m-1}^{T} and q_{m-1}, respectively, and p_n and q_n are orthogonal. Since we also know that $\lambda_{m-1} = 1$, we deduce that $p_m^{\mathrm{T}}abq_m = 0$. Now the relation (3.9) applies to λ_m itself, giving

$$\lambda_m = 1 + p_m^{\mathrm{T}}b_m a_m^{-1}q_m = 1 + p_m^{\mathrm{T}}ba^{-1}q_m = 1 + p_m^{\mathrm{T}}baq_m = 1 + p_m^{\mathrm{T}}abq_m,$$

where we have used the equations (1.2) for a, b, c. Since $p_m^{\mathrm{T}} abq_m$ is known to vanish, it follows that $\lambda_m = 1$.

We can now return to the equations of motion for a_m, b_m, c_m. The equations (3.16) imply

$$p_{m-1}^{\mathrm{T}} q_m = \frac{1}{\lambda_{m-1}} p_m^{\mathrm{T}} a q_m = \frac{1}{\lambda_{m-1}} p_m^{\mathrm{T}} q_{m-1}. \tag{3.19}$$

Using (3.16) and (3.19), it is now easy to recast the equations of motion (3.15) for a_m, b_m, c_m in terms of the equations of motion (1.2) for a, b, c

$$\dot{a} = Qb + bQ, \qquad \dot{b} = Qc + cQ, \qquad Q = \frac{q_{n-1} p_n^{\mathrm{T}}}{p_n^{\mathrm{T}} q_{n-1}} - \frac{q_n p_{n-1}^{\mathrm{T}}}{p_{n-1}^{\mathrm{T}} q_n}.$$

Now the compatibility condition for b is $b^2 = ac + ca$, which implies $\dot{b}b + b\dot{b} = \dot{a}c + c\dot{a}$. Substituting in the previous formulae show that this is verified.

4. The symplectic form

We turn now to the third statement in the Main Theorem, which concerns the Hamiltonian structure of our dynamical system. Since the arguments here are very close to the ones in our earlier work [**10**], except for corrections due to the twisted monodromy conditions, we shall be very brief.

As in [**10**], our approach is based on the universal symplectic forms obtained in [**8**, **9**] in terms of Lax pairs. Although our main interest is the symplectic form ω defined in § 1, there are other symplectic forms and flows which can be treated at the same stroke. Thus we define the following symplectic forms $\omega_{(\ell)}$

$$\omega_{(\ell)} = \frac{1}{2} \sum_{\alpha=1}^{3} \operatorname{Res}_{P_\alpha} \Omega_{(\ell)}, \tag{4.1}$$

where

$$\Omega_{(\ell)} = (\langle \psi_{n+1}^*(Q) \delta L_n(x) \wedge \delta\psi_n(Q)\rangle_k + \psi_k^*(\delta g_k g_k^{-1}) \wedge \delta\psi_k) \frac{\mathrm{d}x}{x^\ell}. \tag{4.2}$$

The various expressions in this equation are defined as follows. The notation $\langle f_n \rangle_k$ stands for the sum:

$$\langle f_n \rangle_k = \sum_{n=k}^{k+N-3} f_n. \tag{4.3}$$

The expression $\psi_n^*(Q)$ is the dual Baker–Akhiezer function, which is the row-vector solution of the equation

$$\psi_{n+1}^*(Q) L_n(z) = \psi_n^*(Q), \qquad \psi_{k+N-2}^* g_k^{-1}(Q) = y^{-1} \psi_k^*(Q), \tag{4.4}$$

normalized by the condition

$$\psi_k^*(Q)\psi_k(Q) = 1. \tag{4.5}$$

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Note that the last term in the definition of the symplectic form reflects the twisted boundary conditions. As we shall see, that makes the form independent of the choice of the initial index $n = k$.

We show now that the symplectic form $\omega_{(0)}$ coincides with ω. In fact, more generally,

$$\omega_{(\ell)} = -\sum_{i=1}^{2N-2} \delta \ln y(z_i) \wedge \frac{\delta x}{x^\ell}(z_i). \tag{4.6}$$

The expression $\Omega_{(\ell)}$ is a meromorphic differential on the spectral curve Γ. Therefore, the sum of its residues at the punctures P_α is equal to the opposite of the sum of the other residues on Γ. For $\ell \leqslant 2$, the differential $\Omega_{(\ell)}$ is regular at the points situated over $x = 0$, thanks to the normalization (4.5), which insures that $\delta\psi_n(Q) = O(x)$. Otherwise, it has poles at the poles z_i of $\psi_n(Q)$ and at the branch points s_i, where we have seen that $\psi^*_{n+1}(Q)$ has poles. We analyse in turn the residues at each of these two types of poles.

First, we consider the poles z_i of $\psi_n(Q)$. By genericity, these poles are all distinct and of first order, and we may write

$$\operatorname{Res}_{z_i} \Omega_{(\ell)} = (\langle \psi^*_{n+1} \delta L_n \psi_n \rangle_k + \psi^*_k (\delta g_k g_k^{-1}) \psi_k) \wedge \frac{\delta x}{x^\ell}(z_i). \tag{4.7}$$

The key observation now is that the right-hand side can be rewritten in terms of the monodromy matrix $T_n(x)$. In fact, the recursive relations $\psi_{n+1} = L_n \psi_n$ and $\psi^*_{n+1} L_n = \psi^*_n$ imply that

$$\begin{aligned}
&\langle \psi^*_{n+1} \delta L_n \psi_n \rangle_k + \psi_k (\delta g_k g_k^{-1}) \psi_k \\
&\quad = \sum_{n=k}^{k+N-2} \psi^*_{k+N-2} \left(\prod_{p=n+1}^{k+N-2} L_p \right) \delta L_n \left(\prod_{p=k}^{n-1} L_p \right) \psi_k + \psi_{k+N-2} (g_k^{-1} \delta g_k) \psi_{k+N-2} \\
&\quad = \psi^*_{k+N-2} g_k^{-1} \delta T_k \psi_k = \delta \ln y.
\end{aligned}$$

In the last equality, we have used the standard formula for the variation of the eigenvalue of an operator, $\psi^*_k \delta T_k \psi_k = \psi^*_k (\delta y) \psi_k$. Altogether, we have found that

$$\operatorname{Res}_{z_i} \Omega_{(\ell)} = \delta \ln y(z_i) \wedge \frac{\delta x}{x^\ell}(z_i).$$

The second set of poles of $\Omega_{(\ell)}$ is the set of branching points s_i of the cover. Arguing as in [**10**, p. 563], we find

$$\operatorname{Res}_{s_i} \Omega_{(\ell)} = \operatorname{Res}_{s_i} [\langle \psi^*_{n+1} \delta L_n d\psi_n \rangle_k + \psi^*_k (\delta g_k g_k^{-1}) \, \mathrm{d}\psi_k] \wedge \frac{\delta y \, \mathrm{d}x}{x^\ell \, \mathrm{d}y}.$$

Due to the identities $\mathrm{d}L(s_i) = \mathrm{d}g_k(s_i) = 0$, this can be rewritten as

$$\operatorname{Res}_{s_i} \Omega_{(\ell)} = \operatorname{Res}_{s_i} \left[(\psi^*_{k+N-2} g_k^{-1} \delta T_k \, \mathrm{d}\psi_k) \wedge \frac{\delta y \, \mathrm{d}x}{x^\ell \, \mathrm{d}y} \right]. \tag{4.8}$$

Next, exploiting the antisymmetry of the wedge product, we may replace δT_k in (4.8) by $(\delta T_k - \delta y)$. Then using the identities

$$\psi^*_{k+N-2} g_k^{-1}(\delta T_k - \delta y) = \delta(\psi^*_{k+N-2} g_k^{-1})(y - T_k),$$
$$(y - T_k)\,\mathrm{d}\psi_k = (\mathrm{d}T_k - \mathrm{d}y)\psi_k,$$

which result from $\psi^*_{k+N-2} g_k^{-1}(T_k - y) = (T_k - y)\psi_k = 0$, we obtain

$$\mathrm{Res}_{s_i}\, \Omega_{(\ell)} = \mathrm{Res}_{s_i}(\delta(\psi^*_{k+N-2} g_k^{-1})(\mathrm{d}L - \mathrm{d}y)\psi_k) \wedge \frac{\delta y\, \mathrm{d}x}{x^\ell\, \mathrm{d}y}.$$

Arguing as before we arrive at

$$\mathrm{Res}_{s_i}\, \Omega_{(\ell)} = \mathrm{Res}_{s_i}(\psi^*_{k+N-2} g_k^{-1} \delta\psi_k) \wedge \delta y \frac{\mathrm{d}x}{x^\ell}.$$

The differential form

$$(\psi^*_{k+N-2}\delta\psi_k) \wedge \delta y \frac{\mathrm{d}x}{x^\ell}$$

is holomorphic at $x = 0$ for $0 \leqslant \ell \leqslant 2$. Therefore,

$$\sum_{s_i} \mathrm{Res}_{s_i}(\psi^*_{k+N-2} g_k^{-1}\delta\psi_k) \wedge \delta y \frac{\mathrm{d}x}{x^\ell} = -\sum_{i=1}^{2N-2} \mathrm{Res}_{z_i}(\psi^*_{k+N-2} g_k^{-1}\delta\psi_k) \wedge \delta y \frac{\mathrm{d}x}{x^\ell}.$$

Using again the fact that $\psi^*_{N+k-2} g_k^{-1} = y^{-1}\psi^*_k$, the right-hand side of the last equation can be recognized as

$$\sum_{i=1}^{2N-2} \delta \ln y(z_i) \wedge \frac{\delta x(z_i)}{x^\ell(z_i)}.$$

Finally, we obtain

$$2\omega_{(\ell)} = -\sum_{i=1}^{2N} \mathrm{Res}_{z_i}\, \Omega_{(\ell)} - \sum_{s_i} \mathrm{Res}_{s_i}\, \Omega_{(\ell)} = -2 \sum_{i=1}^{2N-2} \delta \ln y(z_i) \wedge \frac{\delta x(z_i)}{x^\ell(z_i)}.$$

The identity (4.6) is proved.

The Hamiltonian of the flow

Let $\mathcal{M}_{(\ell)}$ be the reduced phase space defined by the following constraints

$$\mathcal{M}_{(0)} = \{(q_n, p_n; a_n, b_n, c_n);\ u_N = \alpha_0,\ u_{N-1} = \alpha_1\}/G,$$
$$\mathcal{M}_{(2)} = \{(q_n, p_n; a_n, b_n, c_n);\ u_0 = \alpha_0,\ u_1 = \alpha_1\}/G,$$

where $(q_n, p_n, a_n, b_n, c_n)$ satisfy the conditions of the previous sections, G is the group of all allowable gauge transformations, and α_0, α_1 are fixed constants.

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Lemma 4.1. *Let ℓ be either* 0 *or* 2. *Then the equations (1.3) restricted on $\mathcal{M}_{(\ell)}$ are Hamiltonian with respect to the symplectic form $\omega_{(\ell)}$ given by (4.6). The Hamiltonians $H_{(\ell)}$ are given by*

$$H_{(0)} = u_{N-2}, \qquad H_{(2)} = \ln u_N.$$

Proof. By definition, a vector field ∂_t on a symplectic manifold is Hamiltonian, if its contraction $i_{\partial_t}\omega(X) = \omega(X, \partial_t)$ with the symplectic form is an exact one-form $\delta H(X)$. The function H is the Hamiltonian corresponding to the vector field ∂_t. Thus

$$i_{\partial_t}\omega_{(\ell)} = \frac{1}{2}\sum_{\alpha} \mathrm{Res}_{P_\alpha}\big(\langle\psi^*_{n+1}\delta L_n\dot{\psi}_n\rangle_k - \langle\psi^*_{n+1}\dot{L}_n\delta\psi_n\rangle_k + \psi^*_k(\delta g_k g_k^{-1})\dot{\psi}_k - \psi^*_k(\dot{g}_k g_k^{-1})\delta\psi_k\big)\frac{\mathrm{d}x}{x^\ell}.$$

The equation of motion for $\dot{\psi}_n = (M_n + \mu)\psi_n$ implies

$$\begin{aligned}\sum_\alpha \mathrm{Res}_{P_\alpha}\langle\psi^*_{n+1}\delta L_n\dot{\psi}_n\rangle_k\frac{\mathrm{d}x}{x^\ell} &= \sum_\alpha \mathrm{Res}_{P_\alpha}\langle\psi^*_{n+1}\delta L_n(M_n+\mu)\psi_n\rangle_k\frac{\mathrm{d}x}{x^\ell}\\ &= \sum_\alpha \mathrm{Res}_{P_\alpha}\langle\psi^*_{n+1}\delta L_n\psi_n\rangle_k\frac{\mu\,\mathrm{d}x}{x^\ell}.\end{aligned}$$

We used here the equation

$$\sum_\alpha \mathrm{Res}_{P_\alpha}\langle\psi^*_{n+1}\delta L_n M_n\psi_n\rangle_k\frac{\mathrm{d}x}{x^\ell} = 0,$$

which is valid because the corresponding differential is holomorphic everywhere except at the punctures. We will drop similar terms in all consequent equations. The equation of motion (1.5) for L_n implies

$$\begin{aligned}\langle\psi^*\dot{L}\delta\psi_n\rangle_k &= \langle\psi^*_{n+1}(M_{n+1}L_n - L_nM_n)\delta\psi_n\rangle_k\\ &= \langle\psi^*_{n+1}M_{n+1}\delta\psi_{n+1} - \psi_nM_n\delta\psi_n\rangle_k - \langle\psi^*_{n+1}M_{n+1}\delta L_n\psi_n\rangle_k\\ &= \psi^*_{k+N-2}M_{k+N-2}\delta\psi_{k+N-2} - \psi_kM_k\delta\psi_k - \langle\psi^*_{n+1}M_{n+1}\delta L_n\psi_n\rangle_k.\end{aligned}$$

Again the last term does not contribute to the sum of residues.

Using the equation of motion for g_k and the equation

$$y\delta\psi_k = g_k\delta\psi_{k+N-2} + \delta g_k\psi_{k+N-2} - \delta y\psi_k,$$

we obtain

$$\begin{aligned}\psi^*_k(\dot{g}_kg_k^{-1})\delta\psi_k &= \psi^*_kM_k\delta\psi_k - y\psi^*_{k+N-2}M_{k+N-2}g_k^{-1}\delta\psi_k\\ &= \psi^*_kM_k\delta\psi_k - \psi^*_{k+N-2}M_{k+N-2}\delta\psi_{k+N-2}\\ &\quad - \psi^*_{k+N-2}M_{k+N-2}(g_k^{-1}\delta g_k)\psi_{k+N-2}\\ &\quad + \psi^*_{k+N-2}M_{k+N-2}\psi_{k+N-2}\delta\ln y.\end{aligned}$$

The last term does not contribute to a sum of the residues due to the constraints $\delta \ln y = O(x^{-2})$ for $\ell = 0$ and $\delta \ln y = O(x)$ for $\ell = 2$.

The expression for $i_{\partial_t}\omega_{(\ell)}$ reduces to

$$\begin{aligned} i_{\partial_t}\omega_{(\ell)} &= \frac{1}{2}\sum_\alpha \mathrm{Res}_{P_\alpha}(\langle\psi^*_{n+1}\delta L_n\psi_n\rangle_k + \psi_k(\delta g_k g_k^{-1})\psi_k)\frac{\mu(Q,t)\,\mathrm{d}x}{x^\ell} \\ &= \frac{1}{2}\sum_\alpha \mathrm{Res}_{P_\alpha}\,\delta(\ln y)\mu(t,Q)\frac{\mathrm{d}x}{x^\ell}. \end{aligned}$$

The proof can now be completed as in [**10**, p. 567]. □

5. θ-function solutions

Since the system (1.3) is completely integrable, we can obtain exact solutions in terms of θ-functions associated to the spectral curve. We give these formulae here without details, since their derivation is entirely similar to the one in [**10**, pp. 557–560], taking into account the twisted monodromy.

Let ψ_n be the Baker–Akhiezer function, which solves the simultaneous equations $\psi_{n+1} = L_n\psi_n$, $\partial_t\psi_n = M_n\psi_n$. Its components $\psi_{n\alpha}$, $1 \leqslant \alpha = 3$, are given by

$$\begin{aligned} \psi_{n,\alpha}(t,Q) &= \phi_{n,\alpha}(t,Q)\exp\biggl(\int_{Q_\alpha}^Q n\,\mathrm{d}\Omega_0 + t\,\mathrm{d}\Omega^+\biggr), \\ \phi_{n,\alpha}(t,Q) &= r_\alpha(Q)\frac{\theta(A(Q)+tU^+ + nV + Z_\alpha)\theta(Z_0)}{\theta(A(Q)+Z_\alpha)\theta(tU^+ + nV + Z_0)}. \end{aligned}$$

Here $\theta(Z)$ if the Riemann-theta function associated to the period matrix of the spectral curve; $A(Q)$ is the Abel map; V, U^+ are the vectors of B-periods of the meromorphic differentials $\mathrm{d}\Omega_0$, $\mathrm{d}\Omega^+$ defined by the following requirements. The differentials $\mathrm{d}\Omega_0$ and $\mathrm{d}\Omega_+$ have zero A-periods, they are holomorphic outside the two points P_1, P_3 above ∞ interchanged by the involution σ, with $\mathrm{d}\Omega_0$ having simple poles and residues ± 1, while $\mathrm{d}\Omega^+$ is of the form $\mathrm{d}\Omega^+ = \pm\mathrm{d}x(1+O(x^{-2}))$ at these two points. The $r_\alpha(Q_\beta)$ are meromorphic functions satisfying the normalization condition $r_\alpha(Q_\beta) = \delta_{\alpha\beta}$ and the condition that their divisor of poles Z_0 correspond to the initial data $q_n(0)$, $p_n(0)$ of the dynamical system. Let P_2 be the point above ∞ fixed by the involution, and let $\mathrm{d}\Omega_1$ be the meromorphic form satisfying $\mathrm{d}\Omega_1 + \mathrm{d}\Omega_1^\sigma = \mathrm{d}\Omega^+$, where $\mathrm{d}\Omega_1^\sigma$ is the image of $\mathrm{d}\Omega_1$ under the involution σ.

The Laurent expansion of the last factor as $Q \to P_i$ defines constants $v_{i\alpha}$, $w_{i\alpha}$, which depend only on the curve

$$\begin{aligned} v_{2\alpha} &= \int_{Q_\alpha}^{P_2}\mathrm{d}\Omega_0, \quad v_{i\alpha} = \lim_{x\to P_i}\biggl(\int_{Q_\alpha}^x \mathrm{d}\Omega_0 \mp \ln x\biggr), \quad i = 1,3, \\ w_{2\alpha} &= \int_{Q_\alpha}^{P_2}\mathrm{d}\Omega_1, \quad w_{i\alpha} = \lim_{x\to P_i}\biggl(\int_{Q_\alpha}^x \mathrm{d}\Omega_1 \mp x\biggr), \quad i = 1,3. \end{aligned}$$

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Let $\Phi_n^{(i)}(t)$ be vectors with coordinates

$$\Phi_{n,\alpha}^{(i)}(t) = \phi_{n,\alpha}(t, P_i)\mathrm{e}^{nv_{i\alpha}+tw_{i\alpha}}.$$

Then the vector p_n of the spin chain is the unique (up to multiplication; different choices lead to different gauge choices $\nu_n(t)$ in our dynamical system (1.3)) three-dimensional vector that is orthogonal to $\Phi_n^{(i)}$, $i = 2, 3$, i.e.

$$p_n^{\mathrm{T}}\Phi_n^{(2)} = p_n^{\mathrm{T}}\Phi_n^{(3)} = 0,$$

and the vector q_n is given by the formula

$$q_n = \frac{\Phi_n^{(1)}}{p_n^{\mathrm{T}}\Phi_{n-1}^{(1)}}.$$

The leading coefficients of the expansion of the Baker–Akhiezer function provide also the expression for the variables a_n. In the normalization $c = (c_{ij})$ with $c_{13} = 1$ and $c_{ij} = 0$ for all other i, j, we find

$$a_n = \hat{\Phi}_{N+n-2}\hat{\Phi}_n^{-1},$$

where $\hat{\Phi}_n$ is the (3×3) matrix with columns $\Phi_n^{(i)}$.

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