Robotica (2005) volume 23, pp. 607–614. © 2005 Cambridge University Press
doi:10.1017/S0263574704001201 Printed in the United Kingdom

Contact stability analysis of a one degree-of-freedom robot using hybrid system stability theory

Zoe Doulgeri* and George Iliadis

Department of Electrical & Computer Engineering, Aristotle University of Thessaloniki, Thessaloniki 54124 (Greece)

(Received in Final Form: October 8, 2004)

SUMMARY

In this work, the elementary task of controlling the contact of a one degree-of-freedom (dof) robot with a compliant surface is modeled as a switched system. A position controller is used for the free motion and a force controller for the contact task and the goal is to stabilize the robot in contact with the spring-like environment while exerting a desired force. As the robot makes or breaks contact, the control law switches accordingly and the aim is to examine the system's stability using ideas from hybrid stability theory. By considering typical candidate Lyapunov functions for each of the two discrete system states, conditions on feedback gains are derived that guarantee Lyapunov stability of the hybrid task. It is shown that conditions can be decoupled with respect to the discrete state only when the more general hybrid stability theorems are used.

KEYWORDS: Hybrid and switched systems; Multiple Lyapunov functions; Contact stability; Compliant environment; One D.O.F. robot.

I. INTRODUCTION

In the majority of the literature in robot control, the robot evolves either in free space (motion control), or remains in contact with a surface (hybrid force/position control). Each case has been traditionally considered separately and several controllers have been proposed. The contact surface is modeled as either an algebraic equation constraining the end effector or as a spring like environment with the assumption that contact is not lost during the task. Some of the most popular controllers suggested for the constrained or compliant contact case can be found in[1–6]. However, real robotic tasks include phases of transition between free motion and constrained motion. The transition phase appears to be in most cases crucial for the system's stability as for example are the cases of hopping robots, walking machines and robotic hand manipulations. Furthermore, it is possible to loose contact with the surface in a contact task as a result of external disturbances and/or uncertainties on the constraint surface. Contact stability problems have been investigated[7–10] while some theoretical and experimental results have been devoted to impact and force control.[11–14] Attempts have also been made to model and control the transition phase between free and constrained motion.[15–20]

In general, the force controller that is designed to achieve the contact task will be unsuitable when the robot is in free motion. One solution to this problem is to use a position controller for the free motion and a force controller for the contact task. When the robot makes or breaks contact with the environment the control law can be switched accordingly. This type of solution will be analyzed in this paper. An alternative solution proposed in the literature is to use a unique control scheme for both free-motion and contact mode (e.g. as in impedance control[5]). Most of the work that has been done in robotics in dealing with the problem of switching from free motion to contact is mainly experimental and the problem of contact stability has not been extensively analyzed yet. A possible explanation for this is the absence, up to relatively recently, of a "complete" theory for such systems. However, the well-established Lyapunov stability theory of non-linear systems has now been extended for the case of hybrid and switched systems and the idea of multiple Lyapunov functions has been introduced.[21]

This work deals with the problem of contact stability of a one degree-of-freedom manipulator coming into contact with a compliant surface and exerting a desired force, using the stability theory of hybrid systems. A simple PD controller is used for position control in the non-contact phase and a PI force controller when the system is in contact. It is shown that when stability theorems for hybrid systems, with strict conditions, are used, the analysis results in conditions that depend on the stiffness of the environment. On the other hand, when more general theorems are used, greater flexibility for the stability conditions can be obtained. The paper is organized as follows. In section 2, after a short review of the models used to describe hybrid and switched systems, the hybrid model of a one degree-of-freedom robot arm is given. In section 3, two important theorems for the stability of hybrid systems are presented. Section 4 is devoted to the stability analysis and the derivation of sufficient conditions for the one degree-of-freedom robot contact problem and last, in section 5, conclusions are drawn and future extensions are discussed.

II. HYBRID SYSTEM MODEL

A hybrid system can be described as a finite set of discrete states, with each discrete state corresponding to different continuous dynamics. The state of a hybrid system is

* Corresponding author. E-mail: doulgeri@eng.auth.gr

therefore composed of discrete and continuous components. Reviews of formal models for hybrid systems and stability results can be found in references [22–25]. Typically, a model is chosen depending on the problem that needs to be addressed. We will mainly follow the approach in references [22, 23] where the formal model of a hybrid system is given as:

$$\dot{x} = f(x, m) \tag{1}$$

where $x \in \Re^n$ is the continuous state and $m \in M = \{m_1, m_2, \ldots, m_N\}$ is the discrete state that in general and for an autonomous system depends on the continuous state x and the previous discrete state m^- i.e. $m = \varphi(x, m^-)$ where $\varphi : M \times \Re^n \to M$ is a discrete transition. Sometimes (1) is written in the form

$$\dot{x}(t) = \bar{f}_i(x(t)) \tag{2}$$

which is obtained if (1) is evaluated for $m_i \in M$ i.e. $\bar{f}_i(x(t)) = f(x, m_i)$.

The evolution of the continuous state of the hybrid system (1) can be described in the following way. Starting at (x_0, m_i) at time t_0, the continuous trajectory evolves according to $\dot{x} = f(x, m_i)$. Let us assume that at time t_1, x reaches a value x_1 that triggers a discrete state change from m_i to m_j; then, the process continues according to $\dot{x} = f(x, m_j)$. Here, we consider hybrid systems for which the continuous state does not change during switching and therefore the hybrid state (x_1, m_i) becomes (x_1, m_j) at switching. The changes of discrete states are formally described by switch sets

$$S_{i,j} = \{x \in \Re^n | m_j = \varphi(x, m_i)\} \tag{3}$$

that are typically given by hyper-surfaces of the state space e.g. $s_{ij}(x) = 0$ and result in a state space partition.

If for each $x \in \Re^n$, only one $m_i \in M$ is possible then the system is called a switched system. However if there are some $x \in \Re^n$ for which several discrete states are possible then the system is called a hybrid system.

The evolution of the discrete state of the hybrid system (1) from an initial state (x_0, m_i) can be described by a switching sequence of

$$\Delta_{(x_0, m_i)} = (\mu_0, t_0), (\mu_1, t_1), \ldots \mu_k \in M \tag{4}$$

where $t_k < t_{k+1}$ and $\mu_0 = m_i$. The notion (μ_k, t_k) means that $\dot{x}(t) = f(x, \mu_k)$ for $t_k \le t < t_{k+1}$.

An existence and uniqueness theorem for hybrid systems with a finite number of discrete states is given in references [26–27]. Each vector field is assumed to be globally Lipschitz and the switch sets $S_{i,j}$, are assumed to satisfy certain properties. For instance, it is assumed that $S_{i,j} \cap S_{j,i} = \emptyset, i \neq j$, which implies that no sliding motion occurs. Furthermore, it is assumed that $S_{i,j} \cap S_{i,k} = \emptyset$, $i \neq j \neq k$ such that the next discrete state is uniquely defined. Under these conditions, it is shown that there exist a unique continuous function x(t) and a switching sequence m(t) satisfying the dynamics (1) almost everywhere (except at switching points). Furthermore, it is also guaranteed that there will be finitely many switching points in finite time.

Similarly to nonlinear systems, an important concept in hybrid systems is that of equilibrium points. A hybrid state (x_{eq}, m_{eq}) is said to be a hybrid equilibrium of (1) (or simply an equilibrium state) if it has the property that whenever the hybrid state starts at (x_{eq}, m_{eq}) it will remain there for all future time. A continuous equilibrium state x_{eq} is also defined as the continuous state that whenever the hybrid system starts at x_{eq} for some discrete state, it will remain there for all future time. As in autonomous differential equations, the hybrid equilibrium points may be obtained by finding the states satisfying.

$$f(x, m) = 0 \tag{5}$$

Obviously, all continuous states satisfying (5) are continuous equilibrium points x_{eq}. However, not all solutions of (5) are hybrid equilibriums because there may be not possible hybrid states. For example, one solution of (5) (x_{eq}, m_i) may not be possible in the sense that x_{eq} is not contained in that region of the state space that is associated with the discrete state m_i. In some systems x_{eq} is possible for all discrete states in M but this is limited to special cases; for example in the linear hybrid system in which $f(x, m_i) = A(m)x = 0$ despite the value of m.

As for nonlinear systems, it is possible to transform hybrid systems (with nonlinear vector fields) such that a specific continuous equilibrium point x_{eq} is at the origin of the continuous state space.[28,29]

We will use hybrid system stability in order to investigate the contact stability problem in robotics. We will analyze a simple but significant example that can be considered as a simplified representation of the more complex mechanical system of a multi degree of freedom robot. Consider a simple one degree-of-freedom robot with rigid end-point that is initially at non-contact with a compliant surface. Our control objective is to establish contact with the surface in a stable way and apply a constant desired force. This objective implies robot motion both in free space while moving towards the surface and in contact with the surface while trying to push against it with the desired force.

If friction and gravity are neglected the equivalent mechanical model is that of a mass moving horizontally under the action of a command force u towards the surface that is here modeled by a spring (figure 1). The spring can be considered linear with stiffness $k > 0$ and being at rest at $x = 0$. The position of the mass is given by x(t). We distinguish two discrete states in this system. The contact state where the applied force is given by $f = kx$ for $x \ge 0$ and

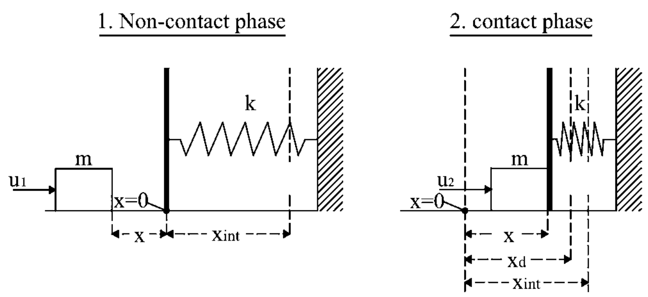

Fig. 1. Controlled mass establishing contact with an elastic surface.

https://doi.org/10.1017/S0263574704001201 Published online by Cambridge University Press

the non-contact state where $f = 0$ for $x < 0$. The system can therefore be cast as a hybrid system with two discrete states.

Many efficient control laws have been proposed in the past for each of the discrete states of free motion and force control. It is reasonable to use a position controller for the free space motion and a force controller for the contact problem, particularly if the surface compliance is uncertain. In our case we utilize a simple PD position controller for the non-contact state with a position reference inside the surface in order to achieve the contact, and a PI force controller for the contact state in order to apply the desired force.

The closed loop dynamics of the one dof robot for the non-contact case are

$$m\ddot{x} = u_1 \tag{6}$$

where

$$u_1 = -K_{v1}\dot{x} - K_{p1}\Delta x_{int}, \quad \Delta x_{int} = x - x_{int}, x_{int} > 0$$

is the PD controller used to establish contact with the surface. This can be achieved by choosing a reference position $x_{int} > 0$ i.e. any point inside the surface.

For the contact phase the closed loop dynamics of the robot are

$$m\ddot{x} = u_2 - f \tag{7}$$

where

$$u_2 = f_d - K_{v2}\dot{x} - K_f\Delta f - K_I \int_{t_k}^{t} \Delta f d\xi, \quad \Delta f = f - f_d$$

is a PI force controller with K_{v1}, K_{p1}, K_{v2}, K_f, K_I positive real gains, t_k the time instant where the robot's end-effector comes in contact with the elastic surface and f_d denotes the desired contact force that corresponds to the desired resting position $x_d = \frac{1}{k} f_d > 0$.

Compactly, the closed loop system dynamics at each discrete state can be written as

$$\begin{aligned} m\ddot{x} &= -K_{v1}\dot{x} - K_{p1}\Delta x_{int} && \text{for } x < 0 \\ m\ddot{x} &= -K_{v2}\dot{x} - K_f'\Delta f - K_I\Delta F && \text{for } x \geq 0 \end{aligned} \tag{8}$$

where

$$K_f' = K_f + 1 \quad \text{and} \quad \Delta F = \int_{t_k}^{t} \Delta f d\xi.$$

Since f is a function of x we can define $(x, \dot{x}) \in \Re^2$ as the state of (8). Furthermore let $M = \{m_{nc}, m_c\}$ be the set of the two distinct discrete states for non-contact and contact phase. Note that (8) is a switched system since for each $(x, \dot{x})$ only one $m_i \in M$ is possible. The continuous dynamics of the system described by (8) can be written in the form of (2) i.e. $\dot{x}(t) = \overline{f}_c(x(t))$ for the contact case and $\dot{x}(t) = \overline{f}_{nc}(x(t))$ for the non-contact case. The switching between the two discrete states occurs at the following two lines of the state space:

$$S_{nc,c} = \{(x, \dot{x}) \in \Re^2 : x = 0 \quad \text{and} \quad \dot{x} \geq 0\} \tag{9}$$

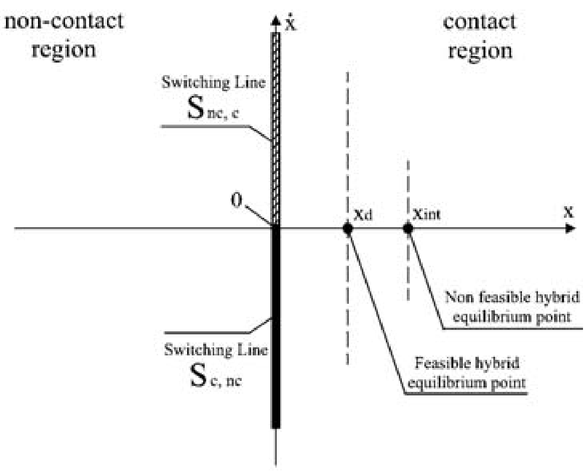

Fig. 2. Switching lines and system equilibrium.

and

$$S_{c,nc} = \{(x, \dot{x}) \in \Re^2 : x = 0 \quad \text{and} \quad \dot{x} < 0\}$$

which are the positive and negative y-axis of the phase plane. It is obvious that $S_{nc,c} \cap S_{c,nc} = \emptyset$, implying that sliding does not occur, the solution of the system is unique and that there are finite switches in finite time. Switch sets (9) partition the continuous state space in two regions (figure 2).

There are two continuous equilibrium points one for each discrete system state, $(x_{int}, 0)$ for the non-contact subsystem in (8) and $(f_d, 0)$ or $(x_d, 0)$ for the second subsystem. It is obvious that if $x_{int} > 0$, only the latter equilibrium point is possible thus the system possesses only one hybrid equilibrium point at $(x_d, 0)$, that can be easily shifted to the origin with a suitable change of variables if desired. The continuous equilibrium is the same for both discrete states in the special case when $x_{int} \equiv x_d$. This may be possible if the model of the surface compliance is fully known and the reference position is set to the desired resting point. Although it has been proved that each subsystem is asymptotically stable, there is no guarantee that the hybrid system (8) is stable. Our aim is to investigate the stability of the hybrid system (8). We will, first, state some of the known stability theorems for the hybrid systems in the next section.

III. STABILITY OF HYBRID SYSTEMS

Many stability results of switched and hybrid systems have been presented in the literature.[21–25] Most of the stability results require the existence of several auxiliary functions, often referred as multiple Lyapunov functions, or candidate Lyapunov functions with certain properties, which can be interpreted as a measure of the system energy, and are extension of Lyapunov functions used in classical Lyapunov theory. In order to show stability it is assumed that the solution of the hybrid system exists, is unique and that there are finite switches in finite time.

Next we state two basic stability theorems for hybrid systems. Their generalization can be found in references [21–23]. Without loss of generality the origin is assumed to be a continuous equilibrium for which stability is investigated.

As already mentioned, this does not necessarily imply that $f(0, m_i) = 0 \quad \forall m_i \in M$.

Assume that the hybrid state space H is partitioned into $\ell < \infty$ disjoint regions $\Omega_1 \ldots \Omega_\ell \subseteq H$ (i.e. $\Omega_i \cap \Omega_j = \emptyset, i \neq j$). Let us define a continuous scalar function with continuous partial derivatives $V_i : \Omega_i^x \to \Re$ used as a measure of system's energy in region Ω_i where by Ω_i^x we denote the continuous state that belongs to Ω_i. Let the overall energy be defined by the function $V(x) = \{V_i(x) : (x, m) \in \Omega_i\}$ which, in general, is discontinuous at switching times. Under these assumptions the following theorem holds.[22]

Theorem 1: Let the hybrid system (2). If there exist $V_i(x) : \Omega_i^x \to \Re$, and class K functions $\alpha : \Re^+ \to \Re^+$ and $\beta : \Re^+ \to \Re^+$ such that

(i). $x \in \Omega_i^x$, $\alpha(\|x\|) \leq V_i(x) \leq \beta(\|x\|)$
(ii). $(x, m) \in \Omega_i$, $\dot{V}_i \leq 0$
(iii). $V_j(x) \leq V_i(x)$ at $S_{i,j}$

then the equilibrium point 0 of (2) is stable in the sense of Lyapunov.

The lower restriction of the scalar functions V_i by the class K function is equivalent to saying that these functions are positive definite. A positive definite function is greater than zero in all points except at the origin (or another point if explicitly written), where it is zero.[28] This definition may be used also in the case when V_i is not defined at the origin i.e. when it is not associated with the state region containing the origin, with the obvious interpretation that the function only has to be strictly greater than zero in the set of states where it is defined.[22] Such an example is shown in figure 3, where V_1 is a function for which condition (i) holds although it has a nonzero minimum in Ω_2. Clearly, if V_1 minimum is contained in its definition region one could not find a class K function α to satisfy the lower bound of condition (i). Function β prevents the overall energy V(x) from being discontinuous at the origin in the radial direction.[22]

Condition (iii) of theorem 1 is a requirement for the system's energy V to decrease at switching points and concerns neighboring functions V_i, V_j; i.e. condition 3 requires that each time a new V_j is active its value must be smaller than the value of the previously active V_i. This is a conservative condition that can be relaxed in a more general hybrid stability theorem.[21,23]

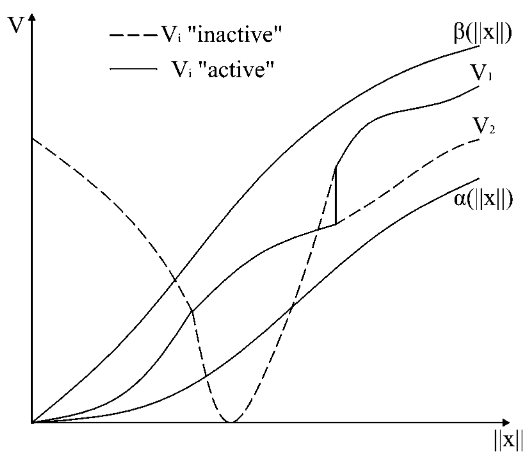

Fig. 3. Example of functions V_i satisfying the first condition in theorem 1.

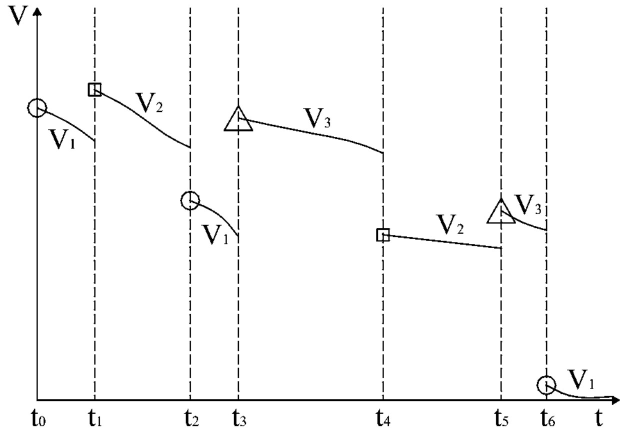

Fig. 4. Sequence of candidate Lyapunov functions V_i satisfying condition 3 of theorem 2.

Theorem 2: Let the hybrid system (2). If there exist continuous scalar functions with continuous partial derivatives $V_i(x) : \Omega_i^x \to \Re$, and class *K* functions $\alpha : \Re^+ \to \Re^+$ and $\beta : \Re^+ \to \Re^+$ such that

(i). $x \in \Omega_i^x$, $\alpha(\|x\|) \leq V_i(x) \leq \beta(\|x\|)$
(ii). $(x, m) \in \Omega_i$, $\dot{V}_i \leq 0$
(iii). $V_i(x(t_{k+1})) \leq V_i(x(t_k))$ where t_k, t_{k+1} are consequent times where the function V_i is "switched-in" then the equilibrium 0 of (2) is stable.

Condition (iii) concerns the value of each function V_i each time is "switched in". It means that the value of V_i at switching points (at $S_{k,i}$ for some k) should be smaller than that of the previous time it has become active or "switched in"; this may have happened at $S_{k,i}$ or at another switch set say $S_{\ell,i}$. Note that this condition is more relaxed than that of theorem 1 since it does not require the overall system's energy V to decrease at switching points (figure 4). In general, however, this condition requires knowledge of the continuous trajectory of the hybrid system, at times where there are switches of candidate Lyapunov functions, which is usually hard to find. However, for special cases of switched systems with state-dependent switching of the Lyapunov function, condition 3 can be checked more easily.

Theorems 1 and 2 refer to the stability of hybrid systems but it is possible to easily extend them for the case of asymptotic stability by strengthening their conditions. We can distinguish two cases depending on whether the switches of the Lyapunov functions are infinite or not. Infinitely many switches of Lyapunov functions are possible if, for example, the continuous equilibrium point is associated with all regions e.g. $f(0, m_i) = 0 \ \forall m_i$. If there are infinitely switches of the Lyapunov functions for which all the conditions hold (in either theorem) and they are strictly decreasing each time they "switched in" then the hybrid system is asymptotically stable. If there are not infinitely many, but the last active one is strictly decreasing, then the system is also asymptotically stable.[23]

https://doi.org/10.1017/S0263574704001201 Published online by Cambridge University Press

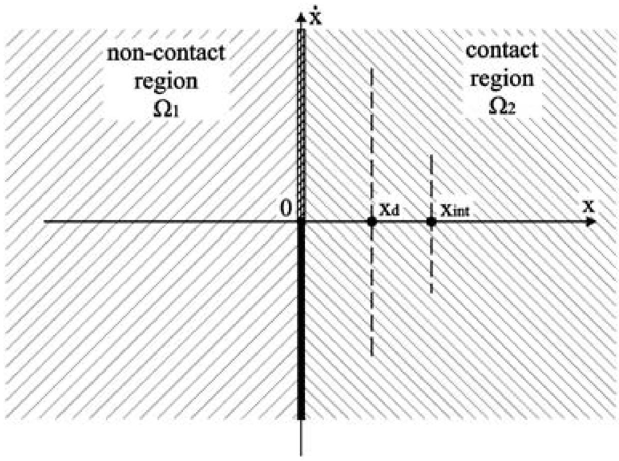

Fig. 5. State space partition in regions Ω_1 and Ω_2.

IV. STABILITY ANALYSIS FOR ONE DOF ROBOT SWITCHED SYSTEM

For the case of a simple one degree-of-freedom robot of this work, we can easily divide the state space in two disjoint regions Ω_1 and Ω_2 (figure 5) in a way that Ω_1^x is the non-contact part of the state-space (i.e. $\Omega_1^x = \{(x,\dot{x}) \in \Re^2 : x < 0\}$) and Ω_2^x the contact part (i.e. $\Omega_2^x = \{(x,\dot{x}) \in \Re^2 : x \geq 0\}$). It is important to note that, both the continuous equilibriums of system (8) belong to Ω_2^x.

Let us define two continuous scalar functions with continuous partial derivatives V_{nc} and V_c used as a measure of system's energy in region Ω_1 and Ω_2, respectively.

For the non-contact case, we choose

$$V_{nc} = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}K_{p1}\Delta x_{int}^2 \tag{10}$$

which is strictly positive in Ω_1^x, since the unique point $(\Delta x_{int}, 0)$ that minimizes V_{nc} does not belong to the region Ω_1 where V_{nc} is defined.

For the contact case a good choice of candidate Lyapunov function is

$$V_c = \frac{1}{2}m\dot{x}^2 + amk\Delta x\,\dot{x} + \frac{1}{2}K_f'k\Delta x^2 + \frac{1}{2}aK_{v2}k\Delta x^2 + K_I\Delta F\Delta x + \frac{1}{2}aK_I\Delta F^2 \tag{11}$$

that can also be written as

$$V_c = \frac{1}{4}m\dot{x}^2 + \frac{1}{4}m(\dot{x} + 2ak\Delta x)^2 - ma^2k^2\Delta x^2 + \frac{1}{2}K_f'k\Delta x^2 + \frac{1}{2}aK_{v2}k\Delta x^2 + K_I\Delta F\Delta x + \frac{1}{2}aK_I\Delta F^2$$

or equivalently

$$V_c = \frac{1}{4}m\dot{x}^2 + \frac{1}{4}m(\dot{x} + 2ak\Delta x)^2 + \frac{1}{2}\begin{bmatrix}\Delta x\\ \Delta F\end{bmatrix}^T \times \begin{bmatrix} k\left(K_f' + a\left(K_{v_2} - 2amk\right)\right) & K_I \\ K_I & aK_I\end{bmatrix}\begin{bmatrix}\Delta x\\ \Delta F\end{bmatrix}$$

where $a > 0$, $\Delta x = x - x_d$ and the positive controller gains are chosen to satisfy

$$\left(K_f' + a\left(K_{v_2} - 2amk\right)\right) > 0 \tag{12.1}$$

and

$$ak\left(K_f' + a\left(K_{v_2} - 2amk\right)\right) - K_I > 0 \tag{12.2}$$

so that V_c is positive definite in Ω_2^x meaning that V_c is strictly positive for all $(x,\dot{x}) \in \Omega_2^x$ except $(x_d, 0)$ where it becomes zero.

Using the fact that V_{nc} is strictly positive in Ω_1^x and V_c is positive definite in Ω_2^x, it can be shown similarly to [28], that α and β functions of K class exist, such that the first condition of theorems 1 and 2 is satisfied (see figure 3).

For the non-contact case the time derivative of (10) is

$$\dot{V}_{nc} = -K_{v1}\dot{x}^2 \tag{13}$$

which is negative semi-definite in Ω_1. It is easy to prove that if motion is free at the entire state space and hence there is no system switching, $(x_{int}, 0)$ would be the unique asymptotically stable equilibrium.

The derivative of (11) is

$$\dot{V}_c = (-K_{v2} + amk)\dot{x}^2 + k(K_I - aK_f'k)\Delta x^2 \tag{14}$$

and it is negative definite in Ω_2 if the controller gains satisfy inequalities $K_{v2} - amk > 0$ and $aK_f'k - K_I > 0$. Hence, if we assume that we do not loose contact with the environment then $(x_d, 0)$ will be the asymptotically stable equilibrium of the closed loop system.

A conservative choice for K_{v2} can be made to satisfy both (12.1) and the first of the above inequalities. This is if we choose K_{v2} to satisfy the inequality $K_{v2} - 2amk > 0$. Then, note that by satisfying the second of the above inequalities $aK_f'k - K_I > 0$ implies that (12.2) also holds. Hence, controller gains should be chosen to satisfy

$$\begin{aligned} K_{v2} &> 2amk \\ K_I &< akK_f' \end{aligned} \tag{15}$$

so that the second condition of stability theorems 1 and 2 is true.

Let us assume that the robot arm loses contact with the environment at $t = t_i$ and establishes contact at $t = t_{i+1} > t_i$. Then, free motion occurs on intervals $[t_{2i}, t_{2i+1})$ and the mass is in contact on intervals $[t_{2i+1}, t_{2i+2})$ (figure 6). At the contact surface $(x = 0)$ the velocity of the mass, is positive if the mass moves from non-contact phase to contact phase and negative for the reverse (figure 1). Since the non-contact closed loop system continuous equilibrium $(x_{int}, 0)$ would be asymptotically stable in free motion, we can easily deduce that each time the arm loses contact with the surface it will reestablish contact after some finite time (figure 6).

Proposition 1: Condition (iii) of theorem 1 is satisfied for system (8) if controller gain K_{p1} is chosen so as to

https://doi.org/10.1017/S0263574704001201 Published online by Cambridge University Press

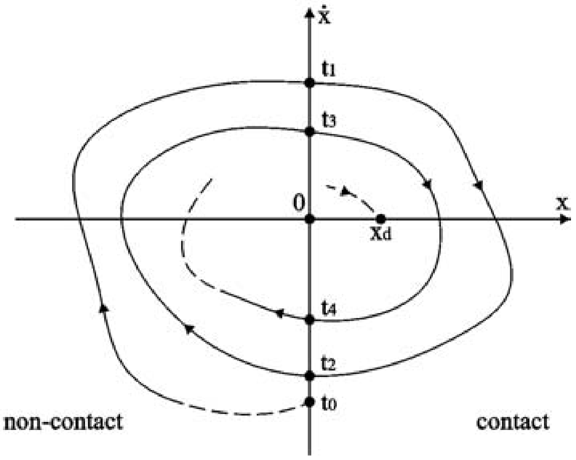

Fig. 6. A stable solution trajectory.

satisfy, $\frac{1}{x_{int}^2}(K_f' + aK_{v2})kx_d^2 \le K_{p1} \le \frac{1}{x_{int}^2}((K_f' + aK_{v2})kx_d^2 + \gamma)$ for some suitable choice of a for which $\gamma \ge 0$.

Proof: Let switching times $t = t_i$ when the arm moves from the contact phase to free motion and the state of the system reaches switching line $S_{c,nc}$ (figure 2). Then, $\dot{x} < 0$,

$$\Delta x = -x_d, \quad \Delta x_{int} = -x_{int}, \quad \Delta f = -f_d.$$

Hence

$$V_{nc}(x(t_i)) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}K_{p1}x_{int}^2$$

and

$$V_c(x(t_i)) = \frac{1}{2}m\dot{x}^2 - amkx_d\dot{x} + \frac{1}{2}K_f'kx_d^2 + \frac{1}{2}aK_{v2}kx_d^2 + \frac{1}{2}\gamma$$

where $\gamma = aK_I\Delta F_i^2 - 2K_I\Delta F_i x_d$ with $\Delta F_i = \int_{t_{i-1}}^{t_i} \Delta f d\xi < \infty$ and $0 < t_i - t_{i-1} < \infty$ the total time that the robot's end-effector stays in touch with the compliant surface before it loses contact at time t_i. We assume that the integrator in the PI controller resets each time the robot end effector loses contact with the surface. Hence, $\Delta F_i = 0$ at switching times when crossing $S_{nc,c}$ and then $\gamma = 0$. Otherwise, at $S_{c,nc}$ it is possible to find a suitable $a > 0$ such that $\gamma \ge 0$. Specifically, if $\Delta F_i \le 0$ then $\gamma \ge 0$ for any $a > 0$, or else, if $\Delta F_i > 0$ we can choose $a \ge \frac{2f_d}{k \min_i \Delta F_i} > 0$ so that $\gamma = aK_I\Delta F_i^2 - 2K_I\Delta F_i x_d \ge 0$ for all i.

Therefore,

$$V_{nc}(x(t_i)) - V_c(x(t_i)) = \frac{1}{2}K_{p1}x_{int}^2 + amkx_d\dot{x} - \frac{1}{2}(K_f' + aK_{v2})kx_d^2 - \frac{1}{2}\gamma,$$

and since $\dot{x} < 0$,

$$V_{nc}(x(t_i)) - V_c(x(t_i)) \le \frac{1}{2}K_{p1}x_{int}^2 - \frac{1}{2}(K_f' + aK_{v2})kx_d^2 - \frac{1}{2}\gamma \le 0$$

when

$$K_{p1} \le \frac{1}{x_{int}^2}\left((K_f' + aK_{v2})kx_d^2 + \gamma\right). \qquad (16)$$

At switching times $t = t_{i+1}$ when the mass moves from free motion to contact phase, the system reaches switching line $S_{nc,c}$. Then $\dot{x} > 0$, $\Delta x = -x_d$, $\Delta x_{int} = -x_{int}$, $\Delta f = -f_d$ and $\Delta F = 0$.

Hence

$$V_{nc}(x(t_{i+1})) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}K_{p1}x_{int}^2$$

$$V_c(x(t_{i+1})) = \frac{1}{2}m\dot{x}^2 - amkx_d\dot{x} + \frac{1}{2}K_f'kx_d^2 + \frac{1}{2}aK_{v2}kx_d^2$$

Therefore,

$$V_{nc}(x(t_{i+1})) - V_c(x(t_{i+1})) = \frac{1}{2}K_{p1}x_{int}^2 + amkx_d\dot{x} - \frac{1}{2}(K_f' + aK_{v2})kx_d^2.$$

Since $\dot{x} \ge 0$,

$$V_{nc}(x(t_{i+1})) - V_c(x(t_{i+1})) \ge \frac{1}{2}K_{p1}x_{int}^2 - \frac{1}{2}(K_f' + aK_{v2})kx_d^2 \ge 0$$

when

$$K_{p1} \ge \frac{1}{x_{int}^2}(K_f' + aK_{v2})kx_d^2 \qquad (17)$$

Q.E.D.

Control gain conditions (15), (16) and (17) are sufficient conditions for the system (8) to be Lyapunov stable according to theorem 1. Note however, that the proportional gain K_{p1} of the PD controller for the free robot motion has a lower bound that depends on the surface's stiffness, a result that becomes useless as soon as the stiffness of the surface k becomes too large. Moreover, gains of the two controllers cannot be chosen independently which is a significant drawback in the usefulness of this theorem and the stability analysis of the contact problem. To overcome this problem we next consider the third condition of the second, more general stability theorem.

Proposition 2: If $\gamma \ge 0$ then, the third condition (iii) of theorem 2 is satisfied for system (8)

Proof: V_{nc} is a non-increasing function along the system solution in Ω_1^x because of (13). Hence during the non-contact time interval $[t_{2i}, t_{2i+1})$, it is true that

$$V_{nc}(t_{2i}) \ge V_{nc}(t_{2i+1}), \qquad (18)$$

which implies

$$\frac{1}{2}m\dot{x}(t_{2i})^2 \ge \frac{1}{2}m\dot{x}(t_{2i+1})^2$$

https://doi.org/10.1017/S0263574704001201 Published online by Cambridge University Press

and therefore

$$|\dot{x}(t_{2i})| \geq |\dot{x}(t_{2i+1})|, \quad \forall i. \tag{19}$$

On the other hand, V_c is a strictly decreasing function in Ω_2^x along the system solution because of (14). Hence during the contact time interval $[t_{2i+1}, t_{2i+2})$, it is true that

$$V_c(t_{2i+1}) > V_c(t_{2i+2}). \tag{20}$$

Given that $\dot{x}(t_{2i+1})$ is the velocity of the arm when crossing $S_{nc,c}$ and $\dot{x}(t_{2i+2})$ is the velocity when crossing $S_{c,nc}$, $\dot{x}(t_{2i+1}) > 0$ and $\dot{x}(t_{2i+2}) < 0$. Hence, if $\gamma \geq 0$, (20) implies

$$\frac{1}{2}m\dot{x}(t_{2i+1})^2 > \frac{1}{2}m\dot{x}(t_{2i+2})^2.$$

Therefore,

$$|\dot{x}(t_{2i+1})| > |\dot{x}(t_{2i+2})|, \quad \forall i. \tag{21}$$

From (19) and (21) we deduce that

$$|\dot{x}(t_{2i})| > |\dot{x}(t_{2i+2})|, \quad \forall i \tag{22}$$

i.e. the velocity of the system at consequent times where the function V_{nc} is "switched-in" is decreasing.

It is easy to show that because of (22),

$$V_{nc}(t_{2i}) > V_{nc}(t_{2i+2}). \tag{23}$$

On the other hand,
at t_{2i+2}:

$$V_c(t_{2i+2}) = \frac{1}{2}m\dot{x}(t_{2i+2})^2 - amkx_d\,\dot{x}(t_{2i+2}) + \frac{1}{2}K_f'kx_d^2 + \frac{1}{2}aK_{v2}kx_d^2 + \frac{1}{2}\gamma$$

and at t_{2i+3}:

$$V_c(t_{2i+3}) = \frac{1}{2}m\dot{x}(t_{2i+3}) - amkx_d\dot{x}(t_{2i+3}) + \frac{1}{2}K_f'kx_d^2 + \frac{1}{2}aK_{v2}kx_d^2.$$

Note that in the second equation $\gamma = 0$ because during the non-contact phase $[t_{2i+2}, t_{2i+3})$ the integral of the force error has been reset to zero. Subtracting the above two functions and using the assumption that $\gamma \geq 0$ we find that

$$V_c(t_{2i+2}) - V_c(t_{2i+3}) \geq \frac{1}{2}m\dot{x}(t_{2i+2})^2 - amkx_d\,\dot{x}(t_{2i+2}) - \frac{1}{2}m\dot{x}(t_{2i+3})^2 + amkx_d\,\dot{x}(t_{2i+3}).$$

Since

$$\dot{x}(t_{2i+2}) < 0 \quad \text{and} \quad \dot{x}(t_{2i+3}) > 0,$$
$$V_c(t_{2i+2}) - V_c(t_{2i+3}) \geq \frac{1}{2}m\dot{x}(t_{2i+2})^2 - \frac{1}{2}m\dot{x}(t_{2i+3})^2.$$

Furthermore, due to (19), $|\dot{x}(t_{2i+2})| \geq |\dot{x}(t_{2i+3})|$ and therefore $V_c(t_{2i+2}) \geq V_c(t_{2i+3})$ and due to (20),

$$V_c(t_{2i+1}) > V_c(t_{2i+3}). \tag{24}$$

Taking (23) and (24) together imply that the third condition of theorem 2 holds. *Q.E.D.*

By assuming $\gamma \geq 0$ we mean that we can choose $a > 0$ so that $\gamma = aK_I\Delta F_i^2 - 2K_I\Delta F_i x_d \geq 0$ and as before $a \geq \frac{2f_d}{k\min_i \Delta F_i} > 0$.

Theorem 3: The hybrid equilibrium point $(x_d, 0)$ of system (8) is asymptotically stable if control gains are chosen to satisfy

$$K_{v2} > 2amk$$
$$K_I < akK_f'$$
$$\gamma = K_I\Delta F_i(a\Delta F_i - 2x_d) \geq 0.$$

Proof: We have proved that all three conditions of theorem 2 hold for the hybrid equilibrium point $(x_d, 0)$ of the switched system (8) and therefore $(x_d, 0)$ is stable. Furthermore, we can easily prove by contradiction that there is a finite number of Lyapunov function switches. The assumption of infinite switches results in an equilibrium point at the origin that contradicts the fact that $(x_d, 0)$ is the unique equilibrium of the system. Given that $x_{int} > 0$ the last switch would bring the system to the contact phase. As the contact Lyapunov function (V_c) is strictly decreasing, we can easily prove that $(x_d, 0)$ is asymptotic stable. *Q.E.D.*

Note that control gain conditions concern only the gains of the contact force controller. The surface stiffness affect the lower bound of the contact damping gain and consequently the speed of response when in contact. Moreover, although constant a is not directly involved in the control signals, it may further intensify the stiffness effect through a big lower bound. This can happen if $\min_i \Delta F_i$ is comparatively small that may be related to the system velocity at the time of establishing contact.

V. CONCLUSIONS

We have proved asymptotic stability of a simple one degree of freedom robot moving, without friction towards a compliant surface, using theorems of hybrid stability theory. The two controllers used for each phase are simple PD and PI control signals that have been successfully used at each case. We have shown that the contact stability problem can be formulated as a hybrid stability problem. Two hybrid stability theorems based on properties of multi-Lyapunov functions have been used to prove the stability of the hybrid system. The more conservative but easier to use theorem that involves values of successive Lyapunov functions at switch points result in stability conditions that unfortunately couple the gains of the two controllers. However, it is shown that the more general theorem that in general requires the solution of the system at switch times can be used in this case. Thus, we have proved the asymptotic stability of the hybrid equilibrium under conditions that involve only gains of the contact controller. It is anticipated that this analysis can be further extended to the more general case of a multi degree of freedom robot as well as to the consideration of nonlinear spring like environmental models.

References

1. T. Yoshikawa, T. Surgie and M. Tanaka, "Dynamic hybrid position/force control of robot manipulators – controller design and experiment," *IEEE Journal of Robotics and Automation* **4**, 699–705 (1988).
2. N. H. McClamroch and D. Wang, "Feedback stabilization and tracking of constrained robots," *IEEE Transactions on Automatic Control* **33**(5), 419–426 (1998).
3. S. Chiaverini and L. Sciavicco, "The Parallel Aproach to Force/Position Control of Robotic Manipulators," *IEEE Transactions on Robotics and Automation* **9**(4), 361–373 (1993).
4. S. Arimoto, "Joint space orthogonalization and passivity for physical interpretations of dexterous robot motions under geometric constraints," *Int. J. Robust and Nonlinear Control* **5**(4), 269–284 (1995).
5. N. Hogan, "Impedance control: An approach to manipulation: Part I-Theory, Part II-Implementation, Part III-Applications," *ASME J. Dyn. Sys Meas. Control* **107**, 1–24 (1985).
6. J. K. Mills, "Hybrid control: A constrained motion perspective," *J. Robotic Sys.* **8**(2), 135–158 (1991).
7. N. Hogan, "On the stability of manipulators performing contact tasks," *IEEE Journal of Robotics and Automation* **4**, 667–686 (1988).
8. J. E. Colgate, and N. Hogan, "Robust control of dynamically interacting systems," *International Journal of Control* **48**(1), 65–88 (1988).
9. H. Kazerooni, "Contact instability of the direct drive robot when constrained by a rigid environment," *IEEE Transactions on Automatic Control* **35**, 710–714 (1990).
10. H. Kazerooni, "Robust nonlinear impedance control for robotic manipulators" *Procs. IEEE International Conference on Robotics and Automation* (1991) pp. 741–750.
11. R. Volpe and P. Khosla, "A theoretical and experimental investigation of impact control for manipulators," *International Journal of Robotics Research* **12**(4), 351–365 (1993).
12. K. Youcef-Toumi, and D. A. Gutz, "Impact and force control: Modeling and experiments," *ASME Journal of Dynamic Systems, Measurement and Control* **116**, 89–98 (1994).
13. Z. C. Lin, R. V. Patel and C. A. Balafoutis, "Impact reductions for redundant manipulators using augmented impedance control," *Journal of Robotic Systems* **12**(5), 301–313 (1995).
14. J. K. Mills and D. M. Lokhorst, "Control of robotic manipulators during general task execution: a discontinuous control approach," *International Journal of Robotic Research* **12**(2), 146–163 (1993).
15. A. Tornambé, "Modeling and controlling one-degree-of-freedom impacts," *IEE Proceedings Control Theory and Applications* **143**(1), 85–90 (1996).
16. B. Brogliatio, S-I Niculescu and P. Orhant, "On the control of finite-dimensional mechanical systems with unilateral constraints," *IEEE Transactions on Automatic Control* **42**(2), 200–215 (1997).
17. A. Tornambé, "Modeling and control of impact in mechanical systems: theory and experimental results," *IEEE Transaction on Automatic Control* **44**(2), 294–309 (1999).
18. S. I. Niculescu and B. Brogliato, "Force measurement time-delays and contact instability phenomenon," *European Journal of Control* **5**(2), 275–289 (1999).
19. M. Indri and A. Tornambé, "Impact Control of a Single-link Robot Striking Different Environments: Theoretical and Experimental Investigation," *European Journal of Control* **6**, 322–337 (2000).
20. B. Brogliatio and P. Orhant, "Contact Stability Analysis of a One Degree-of-Freedom Robot," *Dynamics and Control* **8**, 37–53(1998).
21. M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Transactions on Automatic Control* **43**(4), 475–482 (1998).
22. S. Pettersson, "Analysis and Design of Hybrid Systems." *Ph.D. Thesis* (Department of Signals and Systems, Chalmers University of Technology, Göteborg, Sweden, 1999).
23. S. Pettersson and B. Lennartson, "Stability and Robustness for Hybrid Systems." *Procs. 35th IEEE Conference on Decision and Control*, Kobe (December, 1996) pp. 1202–1207.
24. G. N. Davrazos and N. T. Koussoulas, "A Review of Stability Results for Switched and Hybrid Systems." *Procs. 9th Mediterranean Conference on Control and Automation*, CD proceedings, file med 01-169.pdf (June 27–29, Dubrovnik, Croatia 2001).
25. R. A. Decarlo, M. S. Branicky, S. Pettersson and B. Lennartson, "Perspectives and Results on the Stability and Stabilizability of Hybrid Systems," *Proceedings of IEEE* (2000) **Vol. 88**(7), pp. 1069–1082.
26. L. Tavernini, "Differential automata and their discrete simulators," *Nonlinear Analysis, Theory, Methods & Applications* **11**(6), 665–683 (1987).
27. H. S. Witsenhausen, "A class of hybrid-state continuous-time dynamic systems," *IEEE Transactions on Automatic Control* **11**(2), 161–167 (1966).
28. H. K. Khalil, *Nonlinear Systems* (Prentice-Hall, 1996).
29. J. J. Slotine and W. Li, *Applied Nonlinear Control* (Prentice-Hall, 1999).