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NOTES

E-STABILITY DOES NOT IMPLY LEARNABILITY

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The concept of E-stability is widely used as a learnability criterion in studies of macroeconomic dynamics with adaptive learning. In this paper, it is demonstrated, via a counterexample, that E-stability generally does not imply learnability of rational expectations equilibria. The result indicates that E-stability may not be a robust device for equilibrium selection.

Keywords: Adaptive Learning, E-Stability, Learnability, Rational Expectations Equilibria

1. INTRODUCTION AND BACKGROUND

In recent years, there has been an explosion in research that studies macroeconomic dynamics with adaptive learning.[1] A key question in this literature is whether it is possible for agents that update their expectations using econometric algorithms (i.e., learn adaptively) to learn the rational expectations equilibrium (REE) as the sample size of their data set increases. This is also known as *learnability* of an REE. Since conditions for learnability typically are hard to pin down in a direct way, researchers have been looking for (and have often successfully developed) indirect ways for identifying them. One such popular approach is the E-stability criterion.

In this paper, I revisit the concept of E-stability and demonstrate, via a counterexample, that E-stability does not always imply learnability of an REE. The example used is a generic reduced-form model with expectations dated at time t and a lag of the endogenous variable. In particular, I show that, for certain parameter regions for which E-stability holds for a minimal state variable (MSV) solution, there is a learning algorithm (namely, stochastic gradient) that does not converge to the solution; that is, the REE is not learnable. Furthermore, I discuss some examples of economic models that can be expressed in this reduced form.

The fact that E-stability may not always be an appropriate learnability criterion is not entirely new to the literature. This possibility has been pointed out by Barucci

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and Landi (1997) and further explored by Heinemann (2000). Nevertheless, in the former article, there is no example confirming the assertion. The latter article provides an example for which numerical simulations indicate that stability or instability under stochastic gradient learning is independent of the E-stability conditions; in particular, it is shown that the stochastic gradient algorithm converges to an E-unstable solution, that is, that E-stability is *not a necessary* condition for learnability. In the present paper, I further show that there may exist E-stable equilibria that are not learnable by a stochastic gradient algorithm or, in other words, that E-stability is *not a sufficient* condition for learnability.

2. MODEL

Suppose that the reduced form of the model is

$$y_t = \lambda y_{t-1} + \alpha E_t^* y_{t+1} + \gamma w_t, \tag{1}$$

$$w_t = \rho w_{t-1} + u_t, \tag{2}$$

where $\{w_t\}$ is an AR(1) exogenous variable with $|\rho| < 1$ and $u_t \sim \text{i.i.d.}(0, \sigma_u^2)$. When writing the expectations, the asterisk is used to denote that they are not necessarily the expectations in the statistical sense.

Let $x_t = (y_t,\ w_t)'$ and $\Phi = (\phi_1, \phi_2)'$. If agents perceive the law of motion of y_t to be

$$y_t = x_{t-1}'\Phi + \eta_t \Rightarrow E_t^* y_{t+1} = x_t'\Phi, \tag{3}$$

then the true law of motion of y_t is

$$y_t = T(\Phi)x_{t-1} + V(\Phi)u_t, \tag{4}$$

where

$$T(\Phi) \equiv (T_1(\Phi), T_2(\Phi))' = ((1 - \alpha\phi_1)^{-1}\lambda, (1 - \alpha\phi_1)^{-1}(\rho\alpha\phi_2 + \gamma\rho))' \tag{5}$$

is a mapping from the perceived law of motion to the true law of motion and

$$V(\Phi) = (1 - \alpha\phi_1)^{-1}(\alpha\phi_2 + \gamma). \tag{6}$$

Solving the fixed-point problem $T(\bar{\Phi}) = \bar{\Phi}$ yields the MSV rational expectations solutions

$$\bar{\Phi}_{+,-} = (\bar{\phi}_1, \bar{\phi}_2)' = \left(\frac{1}{2\alpha}(1 \pm \sqrt{1 - 4\alpha\lambda}), \frac{\rho\gamma}{1 - \alpha(\rho + \bar{\phi}_1)}\right)'. \tag{7}$$

It can be shown that for $|\alpha + \lambda| < 1$, the only stationary solution is $\bar{\Phi}_-$. Furthermore, for α and λ such that $|\alpha + \lambda| > 1$, $|\alpha\lambda| < \frac{1}{4}$ and $|\alpha| > \frac{1}{2}$ both solutions are stationary. For the remaining combinations of α and λ, both solutions are either nonstationary or nonreal.[2]

3. E-STABILITY AND LEARNABILITY

An REE is expectationally stable or *E-stable* if it is a locally asymptotically stable equilibrium of the ordinary differential equation

$$\frac{d\Phi}{d\tau} = T(\Phi) - \Phi. \tag{8}$$

Equivalently, an REE is E-stable if the Jacobian of $T(\Phi) - \Phi$ evaluated at the REE, that is,

$$J(\bar{\Phi}) = \left.\frac{d[T(\Phi) - \Phi]}{d\Phi}\right|_{\Phi=\bar{\Phi}} = \left.\frac{dT(\Phi)}{d\Phi}\right|_{\Phi=\bar{\Phi}} - I, \tag{9}$$

is a stable matrix (i.e., it has eigenvalues with strictly negative real parts).[3]

A widely used learning algorithm is the recursive least squares, which is expressed as

$$R_t = R_{t-1} + \frac{1}{t}(x_{t-1}x'_{t-1} - R_{t-1}), \tag{10}$$

$$\Phi_t = \Phi_{t-1} + \frac{1}{t}R_t^{-1}x_{t-1}[x'_{t-1}(T(\Phi_{t-1}) - \Phi_{t-1}) + V(\Phi_{t-1})u_t]. \tag{11}$$

It has been shown by Marcet and Sargent (1989) that E-stability is a necessary and sufficient condition for convergence of the recursive least-squares algorithm. Because of this property and because least squares is a simple and natural choice for estimating parameters of linear models, E-stability has been widely used as a learnability criterion.

Nevertheless, E-stability does not always imply learnability when the estimation algorithm is stochastic gradient. This learning algorithm is expressed recursively as

$$\Phi_t = \Phi_{t-1} + (1/t)x_{t-1}[x'_{t-1}(T(\Phi_{t-1}) - \Phi_{t-1}) + V(\Phi_{t-1})u_t]. \tag{12}$$

The algorithm differs from least squares in that it ignores the second moment matrix when updating.[4] Barucci and Landi (1997) show that an REE is learnable under stochastic gradient if it is a locally asymptotically stable equilibrium of the ordinary differential equation

$$\frac{d\Phi}{d\tau} = [M(\Phi)(T(\Phi) - \Phi)], \tag{13}$$

where $M(\Phi) = \lim_{t\to\infty} E[x_t(\Phi)x_t(\Phi)']$. The local asymptotic stability of an REE $\bar{\Phi}$ under stochastic gradient learning is again determined by the stability of the Jacobian matrix $J^{SG}(\Phi) = d[M(\Phi)(T(\Phi) - \Phi)]/d\Phi$, evaluated at $\bar{\Phi}$. The stochastic gradient algorithm converges locally to the REE if and only if the real parts of the eigenvalues of $J^{SG}(\bar{\Phi})$ are strictly negative.

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Using the reduced-form model introduced in the preceding section, it is possible to demonstrate that E-stability, that is, stability of $J(\bar{\Phi})$, does not necessarily imply stability of $J^{SG}(\bar{\Phi})$. The E-stability conditions [by imposing that the eigenvalues of $J(\bar{\Phi})$ have strictly negative real parts] are

$$E1: \frac{\alpha\lambda}{(1-\alpha\bar{\phi}_1)^2} < 1 \quad \text{and} \quad E2: \frac{\rho\alpha}{1-\alpha\bar{\phi}_1} < 1. \tag{14}$$

For $|\alpha+\lambda| < 1$, the unique stationary solution $\bar{\Phi}_-$ is always E-stable. For the areas defined by $|\alpha+\lambda| > 1$, $|\alpha\lambda| < 1/4$, and $|\alpha| < 1/2$, it is possible to find regions for which both solutions are E-stable, only one solution is E-stable, or no solution is E-stable. For example, if $\rho > 0$, in the area defined by $\alpha, \lambda > 0$, $\lambda < -\rho^2\alpha + \rho$, and $\alpha + \lambda > 1$ (shown in Figure 1 as region II, including the shaded area), both solutions are stationary, but only $\bar{\Phi}_-$ is E-stable, whereas in the area defined by $\alpha, \lambda > 0$, $\lambda > -\rho^2\alpha + \rho$ and $\alpha\lambda < 1/4$ (shown in Figure 1 as region III), both solutions are stationary, but none is E-stable.

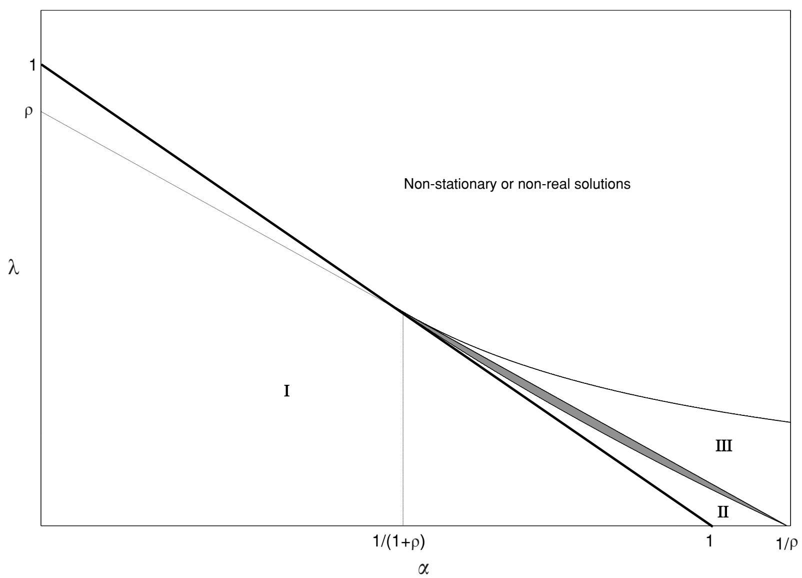

FIGURE 1. E-stability and learnability regions. In region I, defined by the thick solid line, the solution $\bar{\Phi}_-$ is E-stable and learnable with stochastic gradient. In region III, the solution is neither E-stable nor learnable with stochastic gradient. In region II (which includes the shaded area) the solution is E-stable, but is learnable with stochastic gradient only for some parameters. The shaded area corresponds to combinations of α and λ where the solution is E-stable but not learnable with stochastic gradient.

Next, the conditions for learnability under stochastic gradient are determined by the stability of the matrix

$$J^{SG}(\bar{\Phi}) = M(\bar{\Phi}) \begin{pmatrix} \dfrac{\alpha\lambda}{(1-\alpha\bar{\phi}_1)^2} - 1 & 0 \\ \dfrac{\alpha\gamma\rho}{(1-\alpha\bar{\phi}_1)(1-\alpha\rho-\alpha\bar{\phi}_1)} & \dfrac{\alpha\rho}{1-\alpha\bar{\phi}_1} - 1 \end{pmatrix}, \quad \textbf{(15)}$$

where the matrix $M(\bar{\Phi})$ as well as all the derivations of the stability properties are given in Appendixes A and B.

Although it is not possible to present the learnability conditions under stochastic gradient learning in an elegant way, it is straightforward to explore these numerically, by using a finely defined grid of the regions of interest. In this manner, it can be shown, for example, that $\bar{\Phi}_-$ is stable under stochastic gradient only for part of region II of Figure 1. In particular, in the shaded part of region II, the E-stable solution $\bar{\Phi}_-$ is not learnable by stochastic gradient learning.[5] Furthermore, in region I the solution is always learnable with stochastic gradient, whereas in region III the solution is not learnable with stochastic gradient. Note that for different parameter values of σ_u^2 the stability properties of stochastic gradient remain unchanged. However, as ρ decreases, the shaded area of Figure 1, where $\bar{\Phi}_-$ is not learnable under stochastic gradient, becomes larger.

Models that can be expressed in the reduced form analyzed here are the loglinearized versions of the real business cycle (RBC) model and, in general, several dynamic stochastic general equilibrium models. An example of how such models can be written in this reduced form can be found in Giannitsarou (2004). Note that for standard versions of the RBC model [e.g., Hansen (1985)], the reduced-form coefficients are in the zone defined by $|\alpha + \lambda| < 1$; that is, the model is regular or saddle-point stable, for all reasonable calibrations of the model parameters. In this zone, the unique stationary equilibrium is E-stable and always learnable under stochastic gradient learning. However, for irregular models (i.e., models with indeterminacies), reasonable calibrations can yield solutions in the shaded parameter region of interest shown in Figure 1. In the next section, I provide an example that fits in this category, namely, the model of Schmitt-Grohé and Uribe (1997). Thus the counterexample provided here cannot be characterized as pathological.

Furthermore, the result presented here has implications for the learnability of REEs under heterogeneous learning, in particular, for the case where some proportion of the population estimates using least squares and the rest of the agents use stochastic gradient. This issue is covered extensively by Giannitsarou (2003).

Finally, turning to the general issue of equilibrium selection, the present result is also related to the recent work of McCallum (2002a,b). Basing his analysis on a similar reduced-form model (the difference is that it includes an intercept), McCallum first demonstrates how to select a unique MSV solution and then shows that this solution is always E-stable, thus concluding that it is always (least squares) learnable in real time. This argument is then used to reinforce the appropriateness of his equilibrium selection device. However, as illustrated here, the unique MSV solution (as defined by McCallum) may fail to be learnable if the adaptive algorithm is other than least-squares learning.

4. EXAMPLE

In this section, the model of Schmitt-Grohé and Uribe (1997) is employed as an example for demonstrating how a plausible RBC model can be mapped into the reduced form (1)–(2). This model is irregular; that is, for certain parameter values, it has an indeterminate steady state, and therefore it is a potential candidate for having parameter values that bring the coefficients α and λ in the shaded area of Figure 1. The model is an extension of the Hansen (1985) model, augmented with a government that maintains a balanced budget and finances its constant expenditures by taxing labor income at an endogenously determined rate.[6] The representative agent solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - N_t) \tag{16}$$

$$\text{s.t. } K_t = Y_t + (1-\delta)K_{t-1} - C_t - G, \tag{17}$$

$$Y_t = Z_t K_{t-1}^{s_k} N_t^{s_n}, \tag{18}$$

$$G = \Theta_t (s_n Y_t), \tag{19}$$

$$\log Z_t = \rho \log Z_{t-1} + u_t,\, u_t \sim \text{i.i.d.}\left(0, \sigma_u^2\right), \tag{20}$$

where $K_t, Y_t, C_t, N_t,\ \Theta_t$, and G denote capital, output, consumption, labor, labor income tax rate, and government expenditures, respectively. It is assumed that $\delta, \beta, \Theta_t \in (0,1)$. Furthermore, the production function has constant returns to scale, $s_k + s_n = 1$, and $s_k,\ s_n > 0$. The loglinearized equilibrium conditions are

$$c_t = -\lambda_t, \tag{21}$$

$$0 = \lambda_t - \frac{\Theta}{1-\Theta}\theta_t + s_k(k_{t-1} - n_t) + z_t, \tag{22}$$

$$0 = E_t\lambda_{t+1} - \lambda_t + (1-\beta+\beta\delta)[s_n(E_t n_{t+1} - k_t) + \rho z_t], \tag{23}$$

$$k_t - k_{t-1} = \left(\frac{1-\beta}{\beta}\right)k_{t-1} + \frac{\delta s_n}{s_i}n_t + \frac{\delta s_c}{s_i}\lambda_t + \frac{\delta}{s_i}z_t, \tag{24}$$

$$0 = \theta_t + s_k(k_{t-1} - n_t) + n_t + z_t, \tag{25}$$

$$z_t = \rho z_{t-t} + u_t, \tag{26}$$

where $s_i = \delta K/Y$, $s_c = C/Y$, and λ_t is the Lagrange multiplier. Uppercase letters denote the steady-state values of the variables; lowercase letters denote the loglinear variables, that is, $x_t = \log X_t - \log X$ for any variable X_t of the

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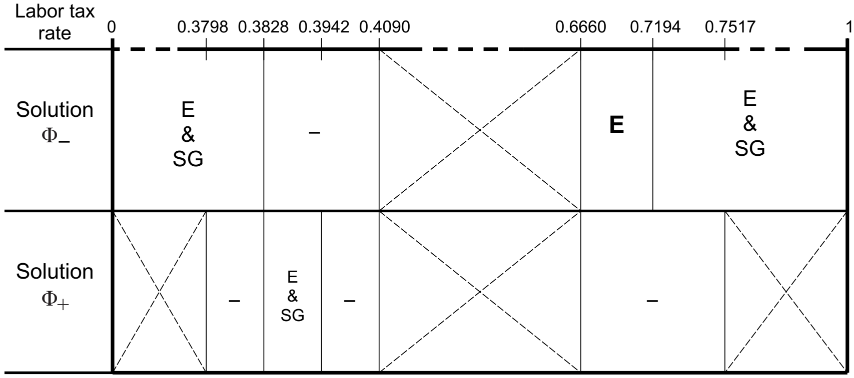

FIGURE 2. E-stability and learnability for the model of Schmitt-Grohé and Uribe (1997). The left column shows the values of the steady-state tax rate Θ that determine the stability properties of the two solutions. For the crossed-out areas, the corresponding solutions are either nonstationary or nonreal. *E & SG* denotes a solution that is E-stable and learnable with stochastic gradient. A hyphen is used to denote a solution that is neither E-stable nor learnable with stochastic gradient.

model. Equations (22)–(25) correspond to equations (12)–(15) of Schmitt-Grohé and Uribe (1997). Using (22) and (25), one can write n_t, by eliminating θ_t, as

$$n_t = \frac{1-\Theta}{s_k - \Theta}\lambda_t + \frac{s_k}{s_k - \Theta}k_{t-1} + \frac{1}{s_k - \Theta}z_t. \qquad \textbf{(27)}$$

Next, replace n_t in (23) and (24) to obtain

$$\lambda_t = \kappa_1 E_t \lambda_{t+1} + \kappa_2 k_t + \kappa_3 \rho z_t, \qquad \textbf{(28)}$$

$$\lambda_t = \mu_1 k_t + \mu_2 k_{t-1} + \mu_3 z_t. \qquad \textbf{(29)}$$

The coefficients $\kappa_i,\ \mu_i,\ i = 1, 2, 3$ are stated in Appendix C. Using the above equations, one can eliminate λ_t to obtain the reduced form (1), with

$$\alpha = \kappa_1 \mu_1 / d, \qquad \lambda = -\mu_2/d, \qquad \gamma = (\kappa_1 \mu_3 \rho + \kappa_3 \rho - \mu_3 \rho)/d, \qquad \textbf{(30)}$$

where $d = \mu_1 + \kappa_1 \mu_2 - \kappa_2$. Using the calibration of Schmitt-Grohé and Uribe (1997) for yearly data ($\delta = 0.1$, $\beta = 1/1.04$, $s_k = 0.3$) and setting $\rho = 0.9$ and $\sigma = 0.01$, it can easily be shown that for labor tax rates in the range (0.6660, 0.7194) the solution $\bar{\Phi}_-$ is E-stable but not learnable with stochastic gradient learning. Figure 2 summarizes the stability properties for both solutions, for the above calibration and for all steady-state labor tax rates $\Theta \in (0, 1)$.[7]

5. CONCLUDING REMARKS

The idea behind the E-stability conditions goes far back to the work of DeCanio (1979) and Evans (1985) and it was originally viewed as a kind of learning

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process taking place in notional time, without the need of any associated real-time adaptive algorithm. Originally, its appeal drew from explaining how economic agents come to possess rational expectations, thereby justifying the existence of rational expectations equilibria. After Marcet and Sargent (1989) showed the equivalence of E-stability with convergence of real-time adaptive least-squares learning, the E-stability conditions have been extensively used as a criterion of learnability of REEs.

Motivated by this fact, I provided a counterexample that shows that E-stability generally does not imply learnability. Naturally, this result raises an important issue: How robust is adaptive learning as an equilibrium selection device? So far, the literature has been heavily relying on E-stability to make inferences about which equilibria are learnable or not; it appears, however, that which equilibria are learned depends on the selected learning algorithm. This is a somewhat alarming result, given that the suggested alternative algorithm (stochastic gradient) is just a small deviation from the commonly used least-squares algorithm. It would thus be useful to identify general criteria under which E-stability implies or does not imply stability under algorithms other than the least squares.

NOTES

1. A detailed literature review of the topic is beyond the scope of this paper. A comprehensive reading on adaptive learning is Evans and Honkapohja (2001).

2. The model studied here is the similar to the one in Sect. 8.6.2 of Evans and Honkapohja (2001). The only difference is that the present model does include an intercept. Although this alteration does not have an effect on the stationarity properties of the solutions $\bar{\Phi}_{+,-}$, it will imply different E-stability conditions, which will now depend on the persistence ρ of the exogenous variable.

Apart from the MSV solutions, the model may also have other solutions, for example, sunspot equilibria. The analysis of E-stability or instability of such solutions is beyond the scope of this paper.

3. The concept of E-stability is extensively discussed by Evans and Honkapohja (2001).

4. For more details on the two algorithms, see Marcet and Sargent (1989), Barucci and Landi (1997), and Evans and Honkapohja (2001).

5. The analysis was done by calculating the eigenvalues of the relevant Jacobian matrices for a thin grid of 2,000 points of the region defined by $\alpha, \lambda \in (0, 1/\rho)$. To produce Figure 1, the parameters γ and σ were set to 0.5, and ρ was set to 0.9. The codes were written using Matlab 6.0 and are available at http://wueconb.wustl.edu/jda/md/contents.html.

6. Schmitt-Grohé and Uribe (1997) present their model in continuous time, without uncertainty. Here, I include an exogenous technological shock in the production function and the model is presented in discrete time.

7. The computations were done with Matlab 6.0, by calculating the relevant eigenvalues of $J(\bar{\Phi}_-)$ and $J^{SG}(\bar{\Phi}_-)$. The code is available at http://wueconb.wustl.edu/jda/md/contents.html.

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APPENDIX A: E-STABILITY AND STOCHASTIC GRADIENT LEARNABILITY CONDITIONS

The matrix $dT(\Phi)/d\Phi$ is

$$\frac{dT(\Phi)}{d\Phi} = \begin{pmatrix} \dfrac{\alpha\lambda}{(1-\alpha\phi_1)^2} & 0 \\ \dfrac{\alpha\rho(\alpha\phi_2+\gamma)}{(1-\alpha\phi_1)^2} & \dfrac{\alpha\rho}{1-\alpha\phi_1} \end{pmatrix}. \tag{A.1}$$

Noting that $(1-\alpha\bar{\phi}_1)^{-1}(\alpha\bar{\phi}_2+\gamma) = (1-\alpha\rho-\alpha\bar{\phi}_1)^{-1}\gamma$, the matrix $J(\bar{\Phi})$ that determines the E-stability conditions reduces to

$$J(\bar{\Phi}) = \begin{pmatrix} \dfrac{\alpha\lambda}{(1-\alpha\bar{\phi}_1)^2} - 1 & 0 \\ \dfrac{\alpha\gamma\rho}{(1-\alpha\bar{\phi}_1)(1-\alpha\rho-\alpha\bar{\phi}_1)} & \dfrac{\alpha\rho}{1-\alpha\bar{\phi}_1} - 1 \end{pmatrix}. \tag{A.2}$$

This has eigenvalues

$$\frac{\alpha\lambda}{(1-\alpha\bar{\phi}_1)^2} - 1 \quad \text{and} \quad \frac{\alpha\rho}{1-\alpha\bar{\phi}_1} - 1. \tag{A.3}$$

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Furthermore, the second moment matrix $M(\Phi)$ is defined by

$$M(\Phi) = \lim_{t\to\infty}\begin{bmatrix} E\left(y_t^2\right) & E(y_t w_t) \\ E(y_t w_t) & E\left(w_t^2\right)\end{bmatrix} \equiv \begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22}\end{bmatrix}. \quad \textbf{(A.4)}$$

Since w_t is a stationary AR(1), $m_{22} = \sigma_u^2/(1-\rho^2)$. Moreover, $y_t = T_1(\Phi)y_{t-1} + T_2(\Phi)w_{t-1} + V(\Phi)u_t$, and in the limit y_t will be stationary; thus,

$$E(y_t w_t) = E\{[T_1(\Phi)y_{t-1} + T_2(\Phi)w_{t-1} + V(\Phi)u_t](\rho w_{t-1} + u_t)\} \quad \textbf{(A.5)}$$

$$= \rho T_1(\Phi)E(y_{t-1}w_{t-1}) + \rho T_2(\Phi)m_{22} + V(\Phi)\sigma_u^2 \Rightarrow \quad \textbf{(A.6)}$$

$$m_{12} = \frac{\rho T_2(\Phi)m_{22} + V(\Phi)\sigma_u^2}{1-\rho T_1(\Phi)} \quad \textbf{(A.7)}$$

and

$$E\left(y_t^2\right) = E[(T_1(\Phi)y_{t-1} + T_2(\Phi)w_{t-1} + V(\Phi)u_t)^2] \quad \textbf{(A.8)}$$

$$= T_1^2(\Phi)E\left(y_{t-1}^2\right) + T_2^2(\Phi)m_{22} + V^2(\Phi)\sigma_u^2 + 2T_1(\Phi)T_2(\Phi)m_{12} \Rightarrow \quad \textbf{(A.9)}$$

$$m_{11} = \frac{T_2^2(\Phi)m_{22} + V^2(\Phi)\sigma_u^2 + 2T_1(\Phi)T_2(\Phi)m_{12}}{1-T_1^2(\Phi)}. \quad \textbf{(A.10)}$$

At the two solutions, $T(\bar{\Phi}) = \bar{\Phi}$ and $V(\bar{\Phi}) = (1-\alpha\bar{\phi}_1)^{-1}(\alpha\bar{\phi}_2+\gamma) = (1-\alpha\rho-\alpha\bar{\phi}_1)^{-1}\gamma$; thus, the second moment matrix simplifies to

$$M(\bar{\Phi}) = \frac{\sigma_u^2}{1-\rho^2}\begin{bmatrix} \frac{\gamma^2(1+\rho\bar{\phi}_1)}{(1-\alpha\rho-\alpha\bar{\phi}_1)^2(1-\rho\bar{\phi}_1)\left(1-\bar{\phi}_1^2\right)} & \frac{\gamma}{(1-\alpha\rho-\alpha\bar{\phi}_1)(1-\rho\bar{\phi}_1)} \\ \frac{\gamma}{(1-\alpha\rho-\alpha\bar{\phi}_1)(1-\rho\bar{\phi}_1)} & 1 \end{bmatrix}. \quad \textbf{(A.11)}$$

The Jacobian that determines learnability conditions under stochastic gradient is

$$J^{SG}(\bar{\Phi}) = M(\bar{\Phi})\begin{bmatrix} \frac{\alpha\lambda}{(1-\alpha\bar{\phi}_1)^2} - 1 & 0 \\ \frac{\alpha\gamma\rho}{(1-\alpha\bar{\phi}_1)(1-\alpha\rho-\alpha\bar{\phi}_1)} & \frac{\alpha\rho}{1-\alpha\bar{\phi}_1} - 1 \end{bmatrix}. \quad \textbf{(A.12)}$$

Although there are analytic expressions for the eigenvalues of $J^{SG}(\bar{\Phi})$, it is not possible to present them in an elegant way, and they are therefore omitted.

APPENDIX B: E-STABILITY PARAMETER REGIONS

To find the regions where the solutions are E-stable, it is assumed for simplicity that $\rho > 0$. Analogous analysis can be done for $\rho < 0$, but the regions are different.

Solution $\bar{\Phi}_-$. Assuming that $1 - 4\alpha\lambda > 0$, it follows that

$$\begin{aligned} 1 - 4\alpha\lambda > 0 &\Rightarrow -(1 - 4\alpha\lambda) < \sqrt{1 - 4\alpha\lambda} \Rightarrow 8\alpha\lambda - 2 < 2\sqrt{1 - 4\alpha\lambda} \\ &\Rightarrow 4\alpha\lambda < 2 - 4\alpha\lambda + 2\sqrt{1 - 4\alpha\lambda} \Rightarrow 4\alpha\lambda < (1 + \sqrt{1 - 4\alpha\lambda})^2 \\ &\Rightarrow \frac{4\alpha\lambda}{(1 + \sqrt{1 - 4\alpha\lambda})^2} < 1 \Rightarrow \frac{\alpha\lambda}{(1 - \alpha\bar{\phi}_{1,-})^2} < 1. \end{aligned} \quad \textbf{(B.1)}$$

Thus, the first E-stability condition (E1) is always satisfied for $\bar{\Phi}_-$.

Next, for the second E-stability condition (E2), assuming that $1 - 4\alpha\lambda > 0$, it is required that

$$\frac{\rho\gamma}{1 - \alpha(\rho + \bar{\phi}_{1,-})} < 1 \Leftrightarrow \frac{2\alpha\rho}{1 + \sqrt{1 - 4\alpha\lambda}} < 1 \Leftrightarrow 2\alpha\rho - 1 < \sqrt{1 - 4\alpha\lambda}. \quad \textbf{(B.2)}$$

If $2\alpha\rho - 1 < 0$, or equivalently, $\alpha < 1/(2\rho)$, then the above condition is satisfied. If $2\alpha\rho - 1 > 0$, or equivalently, $0 < 1/(2\rho) < \alpha$, then

$$2\alpha\rho - 1 < \sqrt{1 - 4\alpha\lambda} \Leftrightarrow 4\alpha^2\rho^2 + 1 - 4\alpha\rho < 1 - 4\alpha\lambda \Leftrightarrow \lambda < \rho - \alpha\rho^2. \quad \textbf{(B.3)}$$

Thus, (E2) is satisfied for all λ if $\alpha < 1/(2\rho)$, and for all $\lambda < \rho - \alpha\rho^2$ if $\alpha > 1/(2\rho)$. To conclude, whenever $\bar{\Phi}_-$ is stationary, it is E-stable for all λ if $\alpha < 1/(2\rho)$, and for all $\lambda < \rho - \alpha\rho^2$ if $\alpha > 1/(2\rho)$.

Solution $\bar{\Phi}_+$. Assuming that $1 - 4\alpha\lambda > 0$, (E1) is satisfied if

$$\begin{aligned} \frac{\alpha\lambda}{(1 - \alpha\bar{\phi}_{1,+})^2} < 1 &\Leftrightarrow \frac{4\alpha\lambda}{(1 - \sqrt{1 - 4\alpha\lambda})^2} < 1 \Leftrightarrow 4\alpha\lambda < (1 - \sqrt{1 - 4\alpha\lambda})^2 \\ &\Leftrightarrow 4\alpha\lambda < 2 - 4\alpha\lambda - 2\sqrt{1 - 4\alpha\lambda} \Leftrightarrow 1 - 4\alpha\lambda > \sqrt{1 - 4\alpha\lambda}. \end{aligned} \quad \textbf{(B.4)}$$

This condition is only satisfied if $1 - 4\alpha\lambda > 1$ or, equivalently, if $\alpha\lambda < 0$, that is, only when α and λ have opposite signs. Thus, the first E-stability condition (E1) is satisfied for all $\lambda < 0$ if $\alpha > 0$, and for all $\lambda > 0$ if $\alpha < 0$.

Next, for the second E-stability condition (E2), I restrict attention to the areas where (E1) is satisfied. Note that $\alpha\lambda < 0$ implies

$$\alpha\lambda < 0 \Rightarrow -4\alpha\lambda > 0 \Rightarrow 1 - 4\alpha\lambda > 1 \Rightarrow \sqrt{1 - 4\alpha\lambda} > 1 \Rightarrow 1 - \sqrt{1 - 4\alpha\lambda} < 0. \quad \textbf{(B.5)}$$

Thus, if $\alpha > 0$, (E2) is always satisfied because

$$\frac{\rho\gamma}{1 - \alpha(\rho + \bar{\phi}_{1,+})} < 1 \Leftrightarrow \frac{2\alpha\rho}{1 - \sqrt{1 - 4\alpha\lambda}} < 0 < 1. \quad \textbf{(B.6)}$$

If $\alpha < 0$, (E2) is equivalent to

$$\begin{aligned} \frac{2\alpha\rho}{1 - \sqrt{1 - 4\alpha\lambda}} < 1 &\Leftrightarrow \frac{-2\alpha\rho}{-1 + \sqrt{1 - 4\alpha\lambda}} < 1 \Leftrightarrow -2\alpha\rho < -1 + \sqrt{1 - 4\alpha\lambda} \\ &\Leftrightarrow 1 - 2\alpha\rho < \sqrt{1 - 4\alpha\lambda} \Leftrightarrow 4\alpha^2\rho^2 + 1 - 4\alpha\rho < 1 - 4\alpha\lambda \\ &\Leftrightarrow \lambda > \rho - \alpha\rho^2. \end{aligned} \quad \textbf{(B.7)}$$

Therefore, whenever α and λ have opposite signs, the second E-stability condition (E2) holds for all $\lambda < 0$ if $\alpha > 0$, and for $\lambda > \rho - \alpha\rho^2$. To conclude, whenever solution $\bar{\Phi}_+$ is stationary, it is E-stable for all λ and α such that $\lambda < 0$ and $\alpha > 0$, and for all λ and α such that $\alpha > 0$ and $\lambda > \rho - \alpha\rho^2$.

APPENDIX C: COEFFICIENTS FOR THE EXAMPLE

The coefficients of (28) and (29) are

$$\begin{aligned}
\kappa_1 &= 1 + \frac{(1-\beta+\beta\delta)s_n(1-\Theta)}{s_k - \Theta}, \\
\kappa_2 &= \frac{(1-\beta+\beta\delta)s_n\Theta}{s_k - \Theta}, \\
\kappa_3 &= (1-\beta+\beta\delta)\left(\frac{s_n}{s_k - \Theta} + 1\right)
\end{aligned} \tag{C.1}$$

$$\mu_1 = \left[\frac{\delta}{s_i}\left(s_c + \frac{s_n(1-\Theta)}{s_k - \Theta}\right)\right]^{-1} \tag{C.2}$$

$$\mu_2 = -\left[1 + \frac{1-\beta}{\beta} + \frac{\delta s_n s_k}{s_i(s_k - \Theta)}\right]\mu_1 \tag{C.3}$$

$$\mu_3 = -\left[\frac{\delta s_n}{s_i(s_k - \Theta)} + \frac{\delta}{s_i}\right]\mu_1 \tag{C.4}$$

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